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V-sufficiency from the weighted point of view

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Two germs of functions $f, g: (\boldsymbol{R}^n, 0)\to(\boldsymbol{R}^p, 0)$ are said to have the same (local) v-type at 0 (v stands for variety), if the germs at 0 of $f^{-1}(0)$ and $g^{-1}(0)$ are homeomorphic. Let $f: (\boldsymbol{R}^n, 0)\to(\boldsymbol{R}^p, 0)$ be a C^k-function. A very interesting problem is to determine what terms from the Taylor expansion at 0, may be omitted without changing the v-type determined by f. For a solution of this problem see [$\mathbf{K}_1$].

In this paper we shall consider the weighted analogue to this problem, and using a new singular Riemannian metric on $\boldsymbol{R}^n$(introduced in [**P**]) we shall give a characterization of v-sufficiency (Theorem A and Theorem B below). Moreover we shall give a geometric corollary for functions whose components are the sum of at most two weighted homogeneous polynomials (generalizing the case with nondegenerate weighted homogeneous components), and also we give a generalization of a well-known inequality due to Bochnak and Lojasiewicz. The use of singular Riemannian metrics seems to be quite useful, see for instance [**Y**], [**P**].

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§ 1. The results.

Let us denote by $\boldsymbol{E}(n, p)$ the set of all germs of functions $f: (\boldsymbol{R}^n, 0)\to(\boldsymbol{R}^p, 0)$ which are C^2 in a punctured neighbourhood of the origin. From now on we shall fix a system of positive numbers $w=(w_1, \cdots, w_n)$, the weights of variables x_i, $w(x_i)=w_i$, $1\leqq i\leqq n$, and a positive number d. For any positive number q we may introduce (see [**P**]) the function $\rho=\rho(x)=(\sum_{i=1}^n x_i^{2q_i})^{1/2q}$, where $q_i=q/w_i$, $1\leqq i\leqq n$. This is a w-form of degree one with respect to w, and if $q_i\geqq 1$, $1\leqq i\leqq n$, then $\rho\in\boldsymbol{E}(n, 1)$. We also consider the spheres associated to this ρ

$$S_r=\{x\in\boldsymbol{R}^n \mid \rho(x)=r\}, \qquad r>0.$$

DEFINITION 1. We define a singular Riemannian metric on $\boldsymbol{R}^n$ by the following bilinear form

$$\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i} \right\rangle = \rho^{-2w_i}, \quad \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle = 0, \quad 1 \leqq i, j \leqq n, \quad i \neq j.$$

We shall denote by ∇_w, $\| \ \|_w$, the corresponding gradient and norm associated to this Riemannian metric (for more details about these see [**P**]).

In order to state our results (they are similar to those in [$\mathbf{K}_1$]) we need to introduce the weighted horn-neighbourhood, of degree d and width $c>0$, of a variety $f^{-1}(0)$, $f \in \boldsymbol{E}(n, p)$. This is by definition

$$H_d(f, c) = \{x \in \boldsymbol{R}^n \mid |f(x)| \leqq c\rho^d\}.$$

DEFINITION 2. We say that $f, g \in \boldsymbol{E}(n, p)$ are w-weighted d-equivalent or simply d-equivalent, if there exist $a>0$ and a neighbourhood U of 0 such that

(1) $|f_j(x) - g_j(x)| \leqq a\rho^d$

(2) $\left| \frac{\partial f_j}{\partial x_i}(x) - \frac{\partial g_j}{\partial x_i}(x) \right| \leqq a\rho^{d-w_i}$, $1 \leqq j \leqq p$, $1 \leqq i \leqq n$ and $x \in U$

(these f_j, g_j are the components of f and g respectively).

It is not hard to see that this is an equivalence relation.

DEFINITION 3. A given $f \in \boldsymbol{E}(n, p)$ is said to be w-weighted v-sufficient at degree d, or simply d-sufficient if for any $P \in \boldsymbol{E}(n, p)$ such that f and $f+P$ are d-equivalent then f and $f+P$ have the same v-type at 0.

REMARK 1. If f is d-sufficient then f is d_1-sufficient for any $d_1 > d$.

These are clearly weighted generalizations of the corresponding homogeneous notions (see for instance [$\mathbf{K}_1$]). For any $f \in \boldsymbol{E}(n, p)$ we shall consider $N(f, i, w, x)$, or simply $N(f, i, x)$, to be the vector $\nabla_w f_i(x) - p_i(x)$, $1 \leqq i \leqq p$, where $p_i(x)$ is the projection of $\nabla_w f_i(x)$, with respect to our metric, onto the subspace generated by $\nabla_w f_j(x)$, $1 \leqq j \leqq p$, $j \neq i$. Then $\|N(f, i, x)\|_w$ will represent the distance from the end of $\nabla_w f_i(x)$ to the subspace spanned by $\nabla_w f_j(x)$, $1 \leqq j \leqq p$, $j \neq i$. We shall denote by $d_w(\nabla_w f_1(x), \cdots, \nabla_w f_p(x))$ the minimum $\min_{1 \leq i \leq p} \|N(f, i, x)\|_w$.

Now we can state our results.

THEOREM A. *If for any $g \in \boldsymbol{E}(n, p)$ d-equivalent to f, there are positive numbers c, ε, δ, and a neighbourhood U of* 0, *all depending on g, such that the following inequality*

$$d_w(\nabla_w f_1(x), \cdots, \nabla f_p(x)) \geqq \varepsilon\rho^{d-\delta} \tag{A}$$

holds for $x \in H_d(g, c) \cap U$, then f is d-sufficient.

COROLLARY 1. *A sufficient condition for $f\in \boldsymbol{E}(n, p)$ to be d-sufficient is that there exist $\varepsilon>0$, $c>0$, $\delta>0$ for which $d_w(\nabla_w f_1(x), \cdots, \nabla_w f_p(x))\geqq\varepsilon\rho^{d-\delta}$ is satisfied for all $x\in H_{d-\delta}(f, c)$, x near 0.*

This is an easy consequence of Theorem A, because for any $g\in \boldsymbol{E}(n, p)$, g d-equivalent to f, then $H_d(g, c)\subseteqq H_{d-\delta}(f, c)$ in a sufficiently small neighbourhood of 0.

REMARK 2. When $p=1$, this corollary actually represents Theorem A from [**P**]. This can be shown using a generalization of an inequality due to Bochnak-Lojasiewicz [**B-L**].

PROPOSITION. *Let $f: (K^n, 0)\rightarrow(K, 0)$ be an analytic function ($K=\boldsymbol{C}$ or $\boldsymbol{R}$). Then for a given $0<c<1$ there exists a neighbourhood U of $0\in K^n$, such that the following inequality holds*

$$\sum_{i=1}^{n}|x_i|\left|\frac{\partial f}{\partial x_i}(x)\right|\geqq c|f(x)|, \qquad x\in U.$$

Indeed if we assume this proposition (it will be proved latter) then one can see that in order to have an inequality $\|\nabla_w f(x)\|_w\geqq c\rho^d$ it is enough to ask it only for all $x\in H_d(f, c)$. This is because outside this horn-neighbourhood (in a small neighbourhood of 0) we have $\|\nabla_w f\|_w\geqq(1/n)\sum_{i=1}^{n}\rho^{w_i}|\partial f/\partial x_i|\geqq(1/n)\sum_{i=1}^{n}|x_i|$ $|\partial f/\partial x_i|\geqq L|f(x)|$ so if $|f(x)|\geqq c\rho^d$ then automatically $\|\nabla_w f(x)\|_w\geqq c_1\rho^d$.

In the case when $f\in \boldsymbol{E}(n, p)$ is analytic we have the following theorem.

THEOREM B. *If $f\in \boldsymbol{E}(n, p)$ is an analytic function, and $d\geqq 3\sup\{w_1, \cdots, w_n\}$, the following are equivalent:*

(1) *f is d-sufficient.*

(2) *The hypothesis of Theorem A hold.*

(3) *For any $g\in \boldsymbol{E}(n, p)$, g d-equivalent to f, the variety $g^{-1}(0)$ admits 0 as a topologically isolated singularity ($\nabla g_i(x)$, $1\leqq i\leqq p$, $x\in g^{-1}(0)$, are linearly independent near 0, $x\neq 0$).*

REMARK 3. We can also prove a component-wise variant of our Theorem A. We shall do this considering instead of the positive number d, a positive p-tuple $\underline{d}=(d_1, \cdots, d_p)$.

DEFINITION 2′. We say that f, $g\in \boldsymbol{E}(n, p)$ are w-weighted $\underline{d}$-equivalent or simply $\underline{d}$-equivalent if there exists a neighbourhood U of 0 such that

(1) $f_j(x)-g_j(x)=0(\rho^{d_j})$

(2) $\dfrac{\partial f_j}{\partial x_k}(x)-\dfrac{\partial g_j}{\partial x_k}(x)=0(\rho^{d_j-w_k})$, $1\leqq k\leqq n$, $1\leqq j\leqq p$, $x\in U$.

Then we can introduce the corresponding horn-neighbourhood $H_{\underline{d}}(f, c)=$

$\{x\in\boldsymbol{R}^n/|f_j(x)|\leqq c\rho^{d_j},\ 1\leqq j\leqq\rho\}$ and the corresponding notion of $\underline{d}$-sufficiency.

We can state the following theorem.

THEOREM A′. *Let $f\in\boldsymbol{E}(n, p)$ be such that there exist positive numbers ε, c, such that in a small neighbourhood of* 0 *the following inequalities hold*:

$$\|N(f, i, x)\|_w \geqq \varepsilon\rho^{d_i},\quad 1\leqq i\leqq p,\quad x\in H_{\underline{d}}(f, c).$$

Then f is $\underline{d}$-sufficient.

The proof is similar to the proof of Theorem A and it will be omitted.

For a given $f\in\boldsymbol{E}(n, p)$ such that any component f_j has the form $f_j=\sum_{i=1}^{r_j}u_{ij}(r_j$ can be ∞ if f_j is analytic), where u_{ij} are w-forms of degree d_{ij}, $d_{ij}<d_{i+1j}$, $1\leqq j\leqq p$, we can write

$$\begin{aligned}\nabla_w f_j(x) &= \sum_{k=1}^{n}\Big(\sum_{i=1}^{r_j}\rho^{w_k}\frac{\partial u_{ij}}{\partial x_k}(x)\Big)\rho^{w_k}\frac{\partial}{\partial x_k}\\ &= \rho^{d_{2j}}\sum_{k=1}^{n}\Big(\sum_{i=1}^{r_j}\frac{1}{\rho^{d_{2j}-d_{ij}}}\frac{\partial u_{ij}}{\partial x_k}\Big(\frac{1}{\rho}\cdot x\Big)\Big)\rho^{w_k}\frac{\partial}{\partial x_k}\\ &= \rho^{d_{2j}}\sum_{k=1}^{n}L_{kj}\rho^{w_k}\frac{\partial}{\partial x_k},\qquad\text{where}\end{aligned}$$

$$L_{kj}(x)=\sum_{i=1}^{r_j}\frac{1}{\rho^{d_{2j}-d_{ij}}}\frac{\partial u_{ij}}{\partial x_k}\Big(\frac{1}{\rho}\cdot x\Big)=\frac{1}{\rho^{d_{2j}-d_{1j}}}\frac{\partial u_{1j}}{\partial x_k}\Big(\frac{1}{\rho}\cdot x\Big)+\frac{\partial u_{2j}}{\partial x_k}\Big(\frac{1}{\rho}\cdot x\Big)+0(\rho).$$

We denote by $L_j=\sum_{k=1}^{n}L_{kj}\partial/\partial x_k=(1/\rho^{d_{2j}-d_{1j}})\nabla u_{1j}((1/\rho)\cdot x)+\nabla u_{2j}((1/\rho)\cdot x)+0(\rho)$ and one can see that

$$\langle\nabla_w f_i,\ \nabla_w f_j\rangle_w=\rho^{d_{2i}+d_{2j}}\langle L_i,\ L_j\rangle.$$

The Gram determinant $\det(\langle\nabla_w f_j,\ \nabla_w f_i\rangle_w)_{1\leq j,i\leq p}$ can be computed in terms of $D_j=L_j/\|L_j\|$, namely

$$\det(\langle\nabla_w f_j,\ \nabla_w f_i\rangle_w)=\rho^{2(d_{21}+\cdots+d_{2p})}\|L_1\|^2\cdots\|L_p\|^2\det(\langle D_i,\ D_j\rangle)$$

and therefore we have the following formula for $\|N(f, i, x)\|_w$

$$\begin{aligned}\|N(f, i, x)\|_w &= \rho^{d_{2i}}\|L_i\|\Big[\frac{\det(\langle D_j,\ D_k\rangle)_{1\leq j,k\leq p}}{\det(\langle D_j,\ D_k\rangle)_{1\leq j,k\leq p,\,j\neq i\neq k}}\Big]^{1/2}=\rho^{d_{2i}}\|L_i\|h_i(x)\\ &= \|\nabla_w f_i(x)\|_w h_i(x),\end{aligned}$$

where $h_i(x)=[\det(\langle D_j,\ D_k\rangle)_{1\leq j,k\leq p}/\det(\langle D_j,\ D_k\rangle)_{1\leq j,k\leq p,\,j\neq i\neq k}]^{1/2}$ denotes the distance from $D_i(x)$ to the subspace spanned by the other $D_j(x)$'s.

Now let α be an analytic arc, $\alpha(0)=0$ and $\alpha(t)\in H_d(f, c)$, $t\in[0, \varepsilon)$. Let us consider the arc $\beta(t)=(1/\rho(\alpha(t)))\cdot\alpha(t)$, $t\geqq 0$. This arc is analytic because $|x_i|\leqq\rho^{w_i}(x)$, $1\leqq i\leqq n$, so it determines a well defined point $\beta(0)\in S_1$ (here $\cdot$ means the weighted action).

We have $L_j(\alpha(t))=(1/\rho^{d_{2j}-d_{1j}})\nabla u_{1j}(\beta(t))+\nabla u_{2j}(\beta(t))+0(\rho)$ and we can observe that the possible limits of $D_j(\alpha(t))$ as t tends to 0 are given by $\nabla u_{1j}(\beta(0))/\|\nabla u_{1j}(\beta(0))\|$ if $\nabla u_{1j}(\beta(0))\neq 0$ and by $(aL_j+\nabla u_{2j}(\beta(0)))/\|aL_j+\nabla u_{2j}(\beta(0))\|$ if $\nabla u_{1j}(\beta(0))=0$ and L_j is a limit direction of ∇u_{1j} at $\beta(0)$, $a\in \boldsymbol{R}$, provided that $aL_j+\nabla u_{2j}(\beta(0))\neq 0$. (We shall consider only these cases.)

We shall denote this directions, obtained along α, by $D(j, \alpha)$, $1\leqq j\leqq p$.

If we ask that any f_j, $1\leqq j\leqq p$, is such that $\|\nabla_w f_j\|_w\geqq c\rho^{d_j}$ in a small horn-neighbourhood $H_{\underline{d}}(f, c)$, and $D(j, \alpha)$, $1\leqq j\leqq p$, are linearly independent for any α as above, then we can apply Theorem A$'$ to conclude that f is $\underline{d}$-sufficient ($\underline{d}=(d_1, \cdots, d_p)$). In particular we have the following corollary.

COROLLARY 2. *If $f\in E(n, p)$ is such that $f_j=u_{1j}+u_{2j}$, and $D(j, \alpha)$ are linearly independent on $\bigcap_{j=1}^{p}\{u_{1j}=0\}\setminus\{0\}$, for any α in a horn-neighbourhood $H_{\underline{d}}(f, c)$, $\underline{d}=(d_{21}, d_{22}, \cdots, d_{2p})$, d_{2j} the weighted degree of u_{2j}, $1\leqq j\leqq p$, then f is $\underline{d}$-sufficient.*

Note. If $u_{1j}=0$, for some j, then we replace $\{u_{1j}=0\}$ by $\{u_{2j}=0\}$.

COROLLARY 3. *If $f\in \boldsymbol{E}(n, p)$ is such that $f_j=\sum_{i=1}^{r_j}u_{ij}$ and ∇u_{1j} are linearly independent on $\bigcap_{j=1}^{p}\{u_{1j}=0\}\setminus\{0\}$, then f is $\underline{d}$-sufficient, where $\underline{d}=(d_{11}, d_{12}, \cdots, d_{1p})$, d_{1j} the degree of u_{1j}, $1\leqq j\leqq p$.*

This result can be found in a nice paper of Buchner and Kucharz [**Bu-Kuc**]. Actually their result is given for slightly different conditions and for $t\in \boldsymbol{R}^k$, but this does not change the proof.

Examples (see [**W**]).

1) $f(x, y, z)=(xy+z^3, xz+y^4)$, ($FW_{13}$).

If $w(x)=2$, $w(y)=w(z)=1$, then $u_1=f_1=xy+z^3$ has the quasihomogeneous degree 3, and $f_2=xz+y^4$ can be written as $f_2=u_2+v_2$ where $u_2=xz$ and $v_2=y^4$, u_1 is nondegenerate and $\{u_1=0\}\cap\{u_2=0\}=\{x=z=0\}\cup\{y=z=0\}$.

On the set $\{x=z=0\}$ we have $\nabla u_1=(y, 0, 0)$ and $\nabla v_2=(0, 4y^3, 0)$.

Moreover $\nabla u_2(x, y, z)=(z, 0, x)$ and therefore for any limit direction l for ∇u_2 at $(0, y, 0)$ we cannot have $al+\nabla v_2=0$, and we can see that $al+\nabla v_2$, ∇u_1 are linearly independent. The same argument works on the set $\{y=z=0\}$ and therefore we may conclude that f is (3, 4)-sufficient with respect to this system of weights (see Corollary 2).

However if we use $w(x)=11/5$, $w(y)=4/5$, $w(z)=1$, then both f_1 and f_2 are nondegenerate quasihomogeneous polynomials of degree 3 and 16/5 respectively, and therefore f is (3, 16/5)-sufficient with respect to this system of weights.

2) $f(x, y, z)=(xy+z^3, x^2+z^3+y^5)$, ($HC_{15}$). If $w(x)=w(y)=1$ and $w(z)=2/3$ one can see, using $f_1=u_1=xy+z^3$, $f_2=u_2+v_2$, where $u_2=x^2+z^3$ and $v_2=y^5$, that f is (2, 5)-sufficient with respect to this system of weights.

3) $f(x, y, z)=(xy+z^3, xz+zy^4)$, (FW_{18}). If $w(x)=12$, $w(y)=3$, $w(z)=5$, one can see that f_1 and f_2 are quasihomogeneous of degree 15, 17 respectively and that the limit directions $D(1, \alpha)$, $D(2, \alpha)$ are independent and therefore it comes out that f is (15, 17)-sufficient with respect to this system of weights.

We can also state the following corollary.

COROLLARY 4. *Let $f\in \boldsymbol{E}(n, p)$ be an analytic map. If $f^{-1}(0)$ has 0 as a topologically isolated singularity then for all large d, f is d-sufficient.*

§ 2. Proofs.

PROOF OF THEOREM A.

The proof follows the proof given by Kuo [$\mathbf{K}_1$]. Let us consider any $P\in \boldsymbol{E}(n, p)$ with the property that f and $f+P$ are d-equivalent. We want to prove that f and $f+P$ have the same v-type at 0. In order to prove this we shall consider a new function $F(x, t)=f(x)+tP(x)$, $F\in \boldsymbol{E}(n+1, p)$, and in addition to the bilinear form from Definition 1, we define a new metric by

$$\left\langle\frac{\partial}{\partial x_i}, \frac{\partial}{\partial t}\right\rangle = 0, \quad 1\leqq i\leqq n, \quad \left\langle\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right\rangle = 1\,.$$

With respect to this singular Riemannian metric we have

$$\nabla_w F_i(x, t) = \sum_{j=1}^{n} \rho^{w_j}\left(\frac{\partial f_i}{\partial x_j}(x)+t\frac{\partial P_i}{\partial x_j}(x)\right)\rho^{w_j}\frac{\partial}{\partial x_j}+P_i(x)\frac{\partial}{\partial t}$$

(here f_i, P_j are the corresponding components of f, P respectively).

We shall show that any $t_0\in \boldsymbol{R}$ has a neighbourhood T such that for any $t_1, t_2\in T$ the germs $F(x, t_1)=0$ and $F(x, t_2)=0$ are homeomorphic and due to the fact that $I=[0, 1]$ is compact it will follow that the germs $f(x)=F(x, 0)=0$ and $f(x)+P(x)=F(x, 1)=0$ are homeomorphic, hence f is d-sufficient.

If we denote by $g(x)=f(x)+t_0P(x)$, $t_0\in \boldsymbol{R}^n$, then $|F_j(x, t)-g_j(x)|=|t-t_0|\,|P_j(x)|$, $1\leqq j\leqq p$. Because f and $f+P$ are d-equivalent we can choose a neighbourhood T of t_0 and a neighbourhood U of $0\in \boldsymbol{R}^n$, such that $|F_j(x, t)-g_j(x)| \leqq c\rho^d$, c as small as we want, $(x, t)\in U\times T$, $1\leqq j\leqq p$.

This shows that the variety $F(x, t)=0$ for $(x, t)\in U\times T$ is contained in $H_d(g, c)\times T$. (This is one reason for we are restricting our attention to this kind of sets.) We have the following lemma.

LEMMA 1. *$\|N(F, i, (x, t))\|_w \geqq (\varepsilon/2)\rho^{d-\delta}$, $(x, t)\in H_d(g, c)\times T$, x near 0, $1\leqq i \leqq p$.*

PROOF.

$$\|\nabla_w F_i(x, t)-\nabla_w f_i(x)\|_w = \|\nabla_w(tP_i(x))\|_w$$

$$= \left\| t\sum_{j=1}^{n} \rho^{w_j}\frac{\partial P_i}{\partial x_j}(x)\rho^{w_j}\frac{\partial}{\partial x_j}+P_i(x)\frac{\partial}{\partial t}\right\|_w = \left(t^2\sum_{j=1}^{n}\rho^{2w_j}\left(\frac{\partial P_i}{\partial x_j}(x)\right)^2+P_i^2(x)\right)^{1/2}$$

$$\leqq |t|\sum_{j=1}^{n}\rho^{w_j}\left|\frac{\partial P_i}{\partial x_j}(x)\right|+|P_i| \leqq c_1\rho^d,$$

for some constant $c_1>0$ and x in a small neighbourhood of 0, $t\in I$.

Now let us consider the following inequality

$$\left\|\sum_{i=1}^{p}\lambda_i\nabla_w F_i\right\|_w \geqq \left\|\sum_{i=1}^{p}\lambda_i\nabla_w f_i\right\|_w - \left\|\sum_{i=1}^{p}\lambda_i(\nabla_w F_i-\nabla_w f_i)\right\|_w.$$

If for example $\lambda_k\neq 0$ then

$$\frac{\|\lambda_k(\nabla_w F_k-\nabla_w f_k)\|_w}{\|\sum_{i=1}^{p}\lambda_i(\nabla_w f_i)\|_w} = \frac{\|\nabla_w F_k-\nabla_w f_k\|_w}{\|\nabla_w f_k+\sum_{i=1,\, i\neq k}^{p}(\lambda_i/\lambda_k)\nabla_w f_i\|_w}$$

$$\leqq \frac{c_1\rho^d}{\|N(f, k, x)\|_w} \leqq \frac{c_1\rho^d}{\varepsilon\rho^{d-\delta}} = \frac{c_1}{\varepsilon}\rho^\delta,$$

where $t\in I$ and $x\in H_d(g, c)$ near 0.

Let $\lambda_k=1$ and $\lambda_j(j\neq k)$ be numbers which satisfy

$$N(F, k, (x, t)) = \sum_{i=1}^{p}\lambda_i\nabla_w F_i.$$

Then we have

$$\|N(F, k, (x, t))\|_w = \left\|\sum_{i=1}^{p}\lambda_i\nabla_w F_i\right\|_w \geqq \frac{1}{2}\left\|\sum_{i=1}^{p}\lambda_i\nabla_w f_i\right\|_w$$

$$\geqq \frac{1}{2}\|N(f, k, x)\|_w \geqq \frac{1}{2}d_w(\nabla_w f_1(x), \cdots, \nabla_w f_p(x))$$

and this implies the required inequality.

Now we can introduce the Kuo vector field (see [**Y**], [$\mathbf{K}_1$], [**P**]) determined by $N(F, i, (x, t))$, $1\leqq i\leqq p$, (we shall use a shorter notation N_i for $N(F, i, (x, t))$):

$$K(x, t) = \frac{\partial}{\partial t}-\sum_{i=1}^{p}\frac{P_i(x)}{\|N_i\|_w^2}N_i \text{ if } x\neq 0 \text{ and } K(0, t)=\frac{\partial}{\partial t}.$$

By construction $K(x, t)$ satisfies the following

1) K is C^1 outside $x=0$ and continuous everywhere in $H_d(g, c)\times T$

2) At any (x, t), $x\neq 0$, $K(x, t)$ is tangent to the level $F=0$ (F is singular only along the t-axis in $H_d(g, c)\times T$).

One can write $N_i=\sum_{j=1}^{n}\rho^{w_j}C_{ji}(x, t)\rho^{w_j}(\partial/\partial x_j)+L_i(x, t)(\partial/\partial t)$, where C_{ji}, L_i are C^1 functions in a punctured horn-neighbourhood of 0 and then K can be written as

$$K(x, t)=\Big(1-\sum_{i=1}^{p}\frac{L_iP_i}{\|N_i\|_w^2}\Big)\frac{\partial}{\partial t}-\sum_{j=1}^{n}\Big(\sum_{i=1}^{p}\frac{P_iC_{ji}}{\|N_i\|_w^2}\Big)\rho^{2w_j}\frac{\partial}{\partial x_j}$$

$$=X\frac{\partial}{\partial t}-\sum_{j=1}^{n}X_j\frac{\partial}{\partial x_j}.$$

Moreover because $|L_i|\leqq\|N_i\|_w$ and $P_i/\|N_i\|_w$ tends to zero (uniformly for $t\in T$, see Lemma 1) it follows that X tends to 1 as x tends to 0 and X_j tends to 0 as x tends to 0. Actually we have the following inequalities

$$\frac{|P_i|}{\|N_i\|_w}\leqq\frac{a\rho^d}{\varepsilon\rho^{d-\delta}/2}\quad\text{and}\quad|X_j|\leqq\sum_{i=1}^{p}\frac{|P_i|}{\|N_i\|_w}\frac{|C_{ji}\rho^{w_j}|}{\|N_i\|_w}\rho^{w_j}\leqq c_j\rho^{w_j}$$

in a small horn-neighbourhood of 0, $c_j>0$, $1\leqq j\leqq n$, $1\leqq i\leqq p$.

In order to show that the integration of this vector field gives us the homeomorphism we need we are going to use two Liapunov functions

$$U(x, t)=e^{2Lt}\rho^2\quad\text{and}\quad V(x, t)=e^{-2Lt}\rho^2.$$

The computation shows that

$$\nabla U(x, t)\cdot K(x, t)=2e^{Lt}\rho\Big(L\rho X+\sum_{i=1}^{n}\frac{\partial\rho}{\partial x_i}X_i\Big)$$

$$\geqq 2e^{Lt}\rho\Big(L\rho X-\sum_{i=1}^{n}\Big|\frac{\partial\rho}{\partial x_i}\Big||X_i|\Big)\geqq 2e^{Lt}\rho\Big(L\rho X-\sum_{i=1}^{n}\Big|\frac{\partial\rho}{\partial x_i}\Big|c_i\rho^{w_i}\Big).$$

Because $c_i\rho^{w_i}|\partial p/\partial x_i|\leqq M\rho/n$, some $M>0$, we can find L big enough such that $\nabla U(x, t)\cdot K(x, t)>0$, $x\neq 0$. In a similar way we can show that there exists $L>0$ such that $\nabla V(x, t)\cdot K(x, t)<0$. The rest of the proof is as for the homogeneous case (see [$\mathbf{K}_1$]).

PROOF OF THEOREM B.

2) → 1) is just Theorem A. We shall prove that 2) ↔ 3) and 1) → 2). In order to prove 2) → 3) we observe that if f and g are d-equivalent then $|\partial g_j/\partial x_i-\partial f_j/\partial x_i|\leqq a\rho^{d-w_i}$, $1\leqq i\leqq n$, $1\leqq j\leqq p$, in a small neighbourhood of 0 and this implies that $\|\nabla_w g_j(x)-\nabla_w f_j(x)\|_w\leqq a\rho^d$, $1\leqq j\leqq p$, and therefore

$$\Big\|\sum_{j=1}^{p}\lambda_j\nabla_w g_j(x)\Big\|_w\geqq\Big\|\sum_{j=1}^{p}\lambda_j\nabla_w f_j(x)\Big\|_w-\Big\|\sum_{j=1}^{p}\lambda_j(\nabla_w f_j(x)-\nabla_w g_j(x))\Big\|_w\geqq\varepsilon_1\rho^{d-\delta}$$

any $(\lambda_1, \cdots, \lambda_n)\neq(0, \cdots, 0)$, for $x\in H_d(g, c)$, x near 0, and this implies that $\nabla_w g_j(x)$ are linearly independent (same for $\nabla g_i(x)$, $1\leqq i\leqq p$), on $g^{-1}(0)\subseteqq H_d(g, c)$, $x\neq 0$, (for this implication we do not need the fact f is analytic). In order to prove 3) → 2) we are going to assume 2) false and then to construct a function $\tilde f\in\boldsymbol{E}(n, p)$ such that f and $\tilde f$ are d-equivalent but $\nabla\tilde f_j$, $1\leqq j\leqq p$, are linearly dependent along an analytic arc in $\tilde f^{-1}(0)$.

We can replace "any $g\in\boldsymbol{E}(n, p)$ d-equivalent to f" by "any analytic $g\in\boldsymbol{E}(n, p)$ d-equivalent to f" in Theorem A.

Therefore let $g\in \boldsymbol{E}(n, p)$ be an analytic map d-equivalent with f and such that for any positive numbers c, ε, δ and any neighbourhood U of 0, the inequality (A) fails. Let E be the following sub-analytic set

$$E=\{x\in H_d(g, 1) \mid d_w(\nabla_w f_1(x), \cdots, \nabla_w f_p(x))=\min_{\substack{\rho(x)=\rho(y)\\ y\in H_d(g,1)}} d_w(\nabla_w f_1(y), \cdots, \nabla_w f_p(y))\}.$$

We can select an analytic arc β: $[0, \eta]\rightarrow E$ (see [**H**]) such that $\beta(0)=0$, $\beta(t)\neq 0$ for $t>0$.

Moreover modulo a permutation, we can choose this arc such that along β,

$$\begin{aligned} d_w(\nabla_w f_1(\beta(t)), \cdots, \nabla_w f_p(\beta(t))) &= \|N(f, 1, \beta(t))\|_w \\ &= \|\nabla_w f_1(\beta(t))-\sum_{k=2}^{p}\lambda_k(t)\nabla_w f_k(\beta(t))\|_w, \end{aligned}$$

where λ_k are analytic and $|\lambda_k(t)|\leqq 1$, $2\leqq k\leqq p$.

By the notation $A(t)\sim B(t)$ we shall understand that A/B lies between two positive constants for $t>0$ and t small.

If $\rho(\beta(t))\sim t^r$ then $r=\min_{1\leq i\leq n} s_i/w_i$ where $\beta_i(t)\sim t^{s_i}$, $1\leqq i\leqq n$, and modulo a permutation we may assume that $r=s_1/w_1\leqq s_i/w_i$, $1\leqq i\leqq n$, and $\beta_1(t)=t^{s_1}$.

Moreover if $\|N(f, 1, \beta(t))\|_w\sim t^{\mu}$ then due to the fact that (A) fails we have that $\mu/r\geqq d$.

Since $\|N(f, 1, \beta(t))\|_w=\sum_{i=1}^{n}\rho^{w_i}|\partial f_1/\partial x_i-\sum_{k=2}^{p}\lambda_k\partial f_k/\lambda x_i|$ then necessarily the order of any $\rho^{w_1}|\partial f_1/\partial x_i-\sum_{k=2}^{p}\lambda_k\partial f_k/\partial x_i|$ (along β) is at least μ.

If we consider also $f_i(\beta(t))\sim t^{l_i}$, $1\leqq i\leqq p$, we can say using the fact that $|f_i-g_i|\leqq a\rho^d$, $1\leqq i\leqq p$, that $l_i\geqq rd$ for any i, $1\leqq i\leqq p$ (this is because along β, $|g_i(\beta(t))|\leqq c\rho^d$ so $g_i(\beta(t))\sim t^{r_i}$ with $r_i\geqq rd$).

We can introduce the following function

$$\begin{aligned} P(x) = f_1(\beta(|x_1|^{1/s_1}))&+\sum_{i=2}^{n}\Big(\frac{\partial f_1}{\partial x_i}(\beta(|x_1|^{1/s_1})) \\ &-\sum_{k=2}^{p}\lambda_k(|x_1|^{1/s_1})\frac{\partial f_k}{\partial x_i}(\beta(|x_1|^{1/s_1}))\Big)(x_i-\beta_i(|x_1|^{1/s_1})) \end{aligned}$$

and then we define $\tilde{f}$: $(\boldsymbol{R}^n, 0)\rightarrow(\boldsymbol{R}^p, 0)$ by

$$\begin{aligned} \tilde{f}_1(x) &= f_1(x)-P(x) \\ \tilde{f}_k(x) &= f_k(x)-f_k(\beta(|x_1|^{1/s_1})), \qquad 2\leqq k\leqq p. \end{aligned}$$

One can check that $\tilde{f}\in \boldsymbol{E}(n, p)$ and the weighted order of $\tilde{f}-f$ is greater than d which shows, due to the particular form of $\tilde{f}$ and the fact that f is analytic, that f and $\tilde{f}$ are d-equivalent.

Moreover on $\beta(t)$, $\tilde{f}(\beta(t))=0$, and a simple computation shows that $\nabla\tilde{f}_1(\beta(t))-\sum_{k=2}^{p}\lambda_k(t)\nabla\tilde{f}_k(\beta(t))=0$. The rest of the proof is just as in [$\mathbf{K}_1$]. Using this

$\tilde{f}$ one can prove (just as in [$\mathbf{K}_1$]) that non 2) $\rightarrow$ non 1), and therefore the proof of Theorem B is complete.

PROOF OF PROPOSITION.

A similar inequality has been obtained by S. Koike [**Ko**] and the proof, using the curve selection lemma [**M**], is similar to Koike's one and therefore we shall omit it.

REMARK 5. Actually the proof shows that actually one can take $c=1$ if there exists at least one i such that $\partial f(0)/\partial x_i=0$.

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