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Holomorphic maps of projective algebraic manifolds into compact C-hyperbolic manifolds

Dedicated to Professor Nobuyuki Suita on his 60th birthday

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Introduction.

Let $\mathrm{Hol}(M, N)$ be the Douady space of compact complex manifolds M and N, that is $\mathrm{Hol}(M, N)$ is the set of all holomorphic maps of M into N. Then $\mathrm{Hol}(M, N)$ has a complex analytic space structure whose underlying topology is the compact-open topology. Moreover, the evaluation map of $\mathrm{Hol}(M, N)\times N$ into N sending (f, p) to $f(p)$ is holomorphic. (See Douady [**2**].)

The main purpose of this paper is to study concretely the structure of $\mathrm{Hol}(M, N)$ for a projective algebraic manifold M and a compact C-hyperbolic manifold N. A complex manifold N is said to be C-hyperbolic or Carathéodory hyperbolic if there exists a regular covering $\tilde{N}$ of N whose Carathéodory pseudo-distance is actually a distance (see Kobayashi [**12**], p. 129). A typical example of C-hyperbolic manifolds is a quotient space $N=\Omega/\Gamma$, where Ω is a bounded domain in the n-dimensional complex Euclidean space $\boldsymbol{C}^n$ and Γ is a fixed-point-free discrete subgroup of the analytic automorphism group $\mathrm{Aut}(\Omega)$ of Ω. Every submanifold of a C-hyperbolic manifold is also C-hyperbolic.

Throughout this paper, we assume that M is a projective algebraic manifold over the complex number field $\boldsymbol{C}$, and N is a compact C-hyperbolic manifold. (By Noguchi and Sunada [**19**], Lemma 2.3, for a C-hyperbolic projective algebraic manifold N, it is sufficient to only assume that M is a compact complex space.) Since a compact C-hyperbolic manifold N is complete hyperbolic, $\mathrm{Hol}(M, N)$ is a compact complex analytic space with finitely many irreducible components (see Kobayashi [**12**], Theorem 3.2 in Chap. V).

In Section 1, we obtain the following main result:

THEOREM 1. *Let M be a projective algebraic manifold with universal covering transformation group G, and let N be a compact C-hyperbolic manifold with universal covering transformation group Γ. If holomorphic maps $f_1, f_2: M\rightarrow N$ induces the same surjective monodromy $(\tilde{f}_1)_*=(\tilde{f}_2)_*: G\rightarrow\Gamma$ and if $f_1(M)\cap f_2(M)$*

$\neq \emptyset$, *then* $f_1=f_2$.

In Section 2, we shall study the structure of the Douady space $\mathrm{Hol}(M, N)$. In order to state our results, we fix the following notations. Let $\Pi: \tilde{N}\to N$ be a covering such that the Carathéodory pseudo-distance on $\tilde{N}$ is actually a distance. Denote by Γ the covering transformation group of $\Pi: \tilde{N}\to N$.

Let X be an irreducible component of $\mathrm{Hol}(M, N)$. Take the universal covering $\rho: \tilde{X}\to X$ of X with covering transformation group H and the universal covering $\pi: \tilde{M}\to M$ of M with covering transformation group G. We set

$$F(f, p)=f(p)$$

for all $(f, p)\in X\times M$. Then $F: X\times M\to N$ is a holomorphic map, which is lifted to a holomorphic map $\tilde{F}: \tilde{X}\times\tilde{M}\to\tilde{N}$. Let $\tilde{F}_*: H\times G\to\Gamma$ be a homomorphism such that

$$\tilde{F}\circ(h, g)=\tilde{F}_*(h, g)\circ\tilde{F}$$

for all $(h, g)\in H\times G$. We put

$$\hat{\Gamma}=\tilde{F}_*(H\times G) \quad\text{and}\quad \hat{N}=\tilde{N}/\hat{\Gamma}.$$

Since $\hat{\Gamma}$ is a subgroup of Γ, the quotient space $\hat{N}$ is a complex manifold. Note that $\hat{N}$ is not necessarily compact.

Let $\hat{\Pi}: \tilde{N}\to\hat{N}$ be the canonical projection, and let $\hat{F}: X\times M\to\hat{N}$ be the holomorphic map satisfying

$$\hat{F}\circ(\rho, \pi)=\hat{\Pi}\circ\tilde{F}.$$

For any $f\in X$, we obtain a holomorphic map $\hat{f}: M\to\hat{N}$ given by

$$\hat{f}(\cdot)=\hat{F}(f, \cdot).$$

For every point $p\in M$, we define the holomorphic map $\hat{p}: X\to\hat{N}$ by

$$\hat{p}(f)=\hat{f}(p)=\hat{F}(f, p)$$

for all $f\in X$.

Now we have the following assertions:

PROPOSITION 1. *For any point $p\in M$, the holomorphic map $\hat{p}: X\to\hat{p}(M)$ is c-biholomorphic.*

THEOREM 2. *For any fixed point f_0 of a component X of* $\mathrm{Hol}(M, N)$, *complex spaces $\hat{F}(X\times M)$ and $X\times\hat{f}_0(M)$ are c-biholomorphically equivalent.*

THEOREM 3. *For any component X of* $\mathrm{Hol}(M, N)$,

$$\dim X+\operatorname{rank} X\leqq\dim N.$$

If $\dim X+\operatorname{rank} X=\dim N$, *then* $X\times\hat{f}_0(M)$ *is biholomorphically equivalent to* $\hat{N}$, *where* $f_0\in X$. *In particular,* X *and* $\hat{f}_0(M)$ *are nonsingular, and* $\hat{\Gamma}$ *is a finite index subgroup of* Γ *so that the canonical projection* $\hat{\Pi}_0: \hat{N}=\tilde{N}/\hat{\Gamma}\to N=\tilde{N}/\Gamma$ *is a surjective finite holomorphic map.*

THEOREM 4. *Let* N *be a compact complex manifold represented by a quotient space* Ω/Γ *such that* Ω *is a bounded domain in* $\boldsymbol{C}^n$ *and* Γ *is a fixed-point-free discrete subgroup of* $\operatorname{Aut}(\Omega)$. *Let* $\ell(\Omega)$ *be the maximum dimension of all complex spaces included in the boundary* $\partial\Omega$ *of* Ω. *If a holomorphic map* $f: M\to N$ *is of* $\operatorname{rank}>\ell(\Omega)$, *then* f *is rigid. Moreover,* $\dim X\leqq\ell(\Omega)$ *for any component* X *of* $\operatorname{Hol}(M, N)$ *with* $X\neq Const$, *the component of* $\operatorname{Hol}(M, N)$ *whose elements are constant maps.*

These assertions are analogous to Noguchi's results [18] for the case where $N=\Gamma\backslash D$, D is a symmetric bounded domain, and Γ is a torsion-free discrete subgroup of $\operatorname{Aut}(D)$ such that either Γ is co-compact or an arithmetic discrete subgroup of the identity component of $\operatorname{Aut}(D)$. In this case, every component X of $\operatorname{Hol}(M, N)$ is smooth. However, in our case, X may have singular points. Actually, in Section 3, using a Kodaira surface, we construct a C-hyperbolic projective algebraic manifold N of dimension 3 such that N is not biholomorphically equivalent to a product of complex manifolds and such that for a certain compact Riemann surface C the Douady space $\operatorname{Hol}(C, N)$ has a 1-dimensional component with singular points.

§1. Rigidity of holomorphic maps in Hol(M, N).

Let M be a projective algebraic manifold over the complex number field $\boldsymbol{C}$. Denote by $\pi: \tilde{M}\to M$ the universal covering of M whose covering transformation group is G.

Let N be a compact C-hyperbolic manifold. Take a covering $\Pi: \tilde{N}\to N$ such that the Carathéodory pseudo-distance on $\tilde{N}$ is actually a distance. Denote by Γ the covering transformation group of $\Pi: \tilde{N}\to N$.

For any holomorphic map $f: M\to N$, the Monodromy Theorem implies that there exists a holomorphic map $\tilde{f}: \tilde{M}\to\tilde{N}$ with $f\circ\pi=\Pi\circ\tilde{f}$. We have the following commutative diagram:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\ \tilde{f}\ } & \tilde{N} \\ \pi\downarrow G & & \Pi\downarrow\Gamma \\ M & \xrightarrow{\ f\ } & N. \end{array}$$

This holomorphic map $\tilde{f}$ is called a lift of f. Note that $\tilde{f}$ is not unique, for we may replace $\tilde{f}$ by $\tilde{f}_1=\gamma_0\circ\tilde{f}\circ g_0$, where $g_0\in G$ and $\gamma_0\in\Gamma$. A lift $\tilde{f}$ of f induces a group homomorphism $\tilde{f}_*: G\to\Gamma$ such that

$$\tilde{f}\circ g=\tilde{f}_*(g)\circ\tilde{f}$$

for any $g\in G$. We call $\tilde{f}_*$ a monodromy of f. If a lift $\tilde{f}$ is replaced by $\tilde{f}_1=\gamma_0\circ\tilde{f}\circ g_0$, then $\tilde{f}_1$ induces a homomorphism $(\tilde{f}_1)_*$ such that

$$(\tilde{f}_1)_*(g)=\gamma_0\circ\tilde{f}_*(g_0\circ g\circ g_0^{-1})\circ\gamma_0^{-1}$$

for each $g\in G$.

Now, we have the following rigidity theorem of holomorphic maps.

THEOREM 1. *Let M be a projective algebraic manifold with universal covering transformation group G, and let N be a compact C-hyperbolic manifold with covering transformation group Γ. If holomorphic maps $f_1, f_2: M\to N$ induces the same surjective monodromy $(\tilde{f}_1)_*=(\tilde{f}_2)_*: G\to\Gamma$ and if $f_1(M)\cap f_2(M)\neq\emptyset$, then $f_1=f_2$.*

REMARK. In Theorem 1, the surjectivity of $(\tilde{f}_i)_*: G\to\Gamma$ is necessary. For example, take a compact Riemann surface R of genus $g\,(\geqq 2)$ such that R has a biholomorphic involution T with no fixed points. Then T acts on $N_0=R\times R$ by

$$T(p, q)=(T(p), T(q))$$

for any $p, q\in R$. Let $\langle T\rangle$ be the subgroup of $\mathrm{Aut}(N_0)$ generated by T. Set $N=N_0/\langle T\rangle$. Then N is a 2-dimensional compact complex manifold and the canonical projection $P: N_0\to N$ is an unramified finite covering. Let $\pi: \Delta\to R$ be the universal covering with covering transformation group G, where Δ is the unit disc in the complex plane. Then $\Pi_0=(\pi, \pi): \tilde{N}=\Delta\times\Delta\to N_0$ is the universal covering with covering transformation group $\Gamma_0=G\times G$. Let Γ be the covering transformation group of $\Pi: \tilde{N}\to N$.

Fix a point $p_0\in R$ and set $\varphi_1(p)=(p_0, p)$, and $\varphi_2(p)=(T(p_0), p)$ for any $p\in R$. Then φ_1, φ_2 are distinct holomorphic maps of R into N_0. Take $z_1, z_2\in\Delta$ with $\pi(z_1)=p_0$ and $\pi(z_2)=T(p_0)$. Then the holomorphic map $\tilde{\varphi}_i: \Delta\to\tilde{N}$ sending z to (z_i, z) is a lift of φ_i for $i=1, 2$. Moreover, they induce the same homomorphism $\chi=(\tilde{\varphi}_i)_*: G\to\Gamma_0$ such that $\chi(g)=(1, g)$ for each $g\in G$, where 1 is the unit of G. Since $\Gamma_0=G\times G$, the homomorphism χ is not surjective.

We set $f_i=P\circ\varphi_i$ for $i=1, 2$. Then $\tilde{\varphi}_i$ is a lift of f_i for each $i=1, 2$ and $\chi=(\tilde{\varphi}_1)_*=(\tilde{\varphi}_2)_*$. On the other hand, f_1, f_2 are distinct holomorphic maps R into N and $f_1(R)=f_2(R)$. Hence, this example shows that the assumption of surjectivity of $(\tilde{f}_i)_*$ in Theorem 1 is necessary.

By using the following two lemmas, the proof of Theorem 1 is reduced to the case where M is a projective algebraic curve, that is, a compact Riemann surface.

LEMMA 1. *Let R be a compact Riemann surface with universal covering transformation group G, and let N be a compact C-hyperbolic manifold with covering transformation group Γ. If holomorphic maps $f_1, f_2: R \to N$ induce the same monodromy $(\tilde{f}_1)_* = (\tilde{f}_2)_*: G \to \Gamma$ and if $\tilde{f}_1(\tilde{R}) \cap \tilde{f}_2(\tilde{R}) \neq \emptyset$ for the universal covering surface $\tilde{R}$ of R, then $f_1 = f_2$.*

PROOF. The Uniformization Theorem for Riemann surfaces implies that the universal covering surface $\tilde{R}$ of R is biholomorphically equivalent to the Riemann sphere $\hat{\boldsymbol{C}}$, the complex plane $\boldsymbol{C}$, or the unit disc $\Delta = \{z \in \boldsymbol{C} \mid |z| < 1\}$. If $\tilde{R}$ is biholomorphically equivalent to $\hat{\boldsymbol{C}}$ or $\boldsymbol{C}$, then $\tilde{f}_1, \tilde{f}_2$ are constant maps, for $\tilde{N}$ is C-hyperbolic. Since $\tilde{f}_1(\tilde{R}) \cap \tilde{f}_2(\tilde{R}) \neq \emptyset$, we have $\tilde{f}_1 = \tilde{f}_2$, and hence $f_1 = f_2$.

Now, assume that $\tilde{R}$ is biholomorphically equivalent to Δ. In this case, we can use the same method as the proof of Theorem 1 in [**6**]. Take two points $z_1, z_2 \in \Delta$ with $\tilde{f}_1(z_1) = \tilde{f}_2(z_2)$. Let $\chi = (\tilde{f}_1)_* = (\tilde{f}_2)_*$. We obtain

$$\begin{cases} \tilde{f}_1 \circ g = \chi(g) \circ \tilde{f}_1 \\ \tilde{f}_2 \circ g = \chi(g) \circ \tilde{f}_2 \end{cases} \tag{1}$$

for all $g \in G$.

Assume that $\tilde{f}_1 \neq \tilde{f}_2$ on Δ. Since $\tilde{N}$ is C-hyperbolic, there exists a bounded holomorphic function α on $\tilde{N}$ such that $\alpha \circ \tilde{f}_1 \neq \alpha \circ \tilde{f}_2$ on Δ. Then $A_1 = \alpha \circ \tilde{f}_1$ and $A_2 = \alpha \circ \tilde{f}_2$ are bounded holomorphic functions on Δ.

Since G is of divergence type, almost every boundary point $\zeta \in \partial\Delta$ is an angular limit point of G, that is, there exists a sequence $\{g_n\}_{n=1}^{\infty}$ of G such that $g_n(z) \to \zeta$ through a Stolz domain with vertex at ζ as $n \to \infty$ for each $z \in K$, where K is a compact subset in Δ. Apply Fatou's theorem to bounded holomorphic functions A_1, A_2. For almost every boundary point $\zeta \in \partial\Delta$, there exist two complex numbers $b_1(\zeta), b_2(\zeta)$ such that $A_1(z)$ and $A_2(z)$ converge uniformly to $b_1(\zeta)$ and $b_2(\zeta)$, respectively as $z \to \zeta$ through a fixed Stolz domain with vertex at ζ. Hence, for almost every boundary point ζ, there exists a sequence $\{g_n\}_{n=1}^{\infty}$ of G such that

$$\lim_{n\to\infty} A_1 \circ g_n(z_1) = b_1(\zeta),$$
$$\lim_{n\to\infty} A_2 \circ g_n(z_2) = b_2(\zeta).$$

By relation (1) and $\tilde{f}_1(z_1) = \tilde{f}_2(z_2)$, we have $b_1(\zeta) = b_2(\zeta)$. Hence, A_1 and A_2 have the same boundary value for almost all $\zeta \in \partial\Delta$, which implies $A_1 = A_2$ on Δ. This contradicts $A_1 \neq A_2$ on Δ. Therefore, $f_1 = f_2$ on R. ∎

LEMMA 2. *Let M be a projective algebraic manifold. For any two points*

p_1, $p_2 \in M$, there exists a non-singular, connected, 1-dimensional analytic subset R of M such that R contains p_1 and p_2, and such that R is biholomorphically equivalent to the Riemann sphere $\hat{\boldsymbol{C}}$ or the inclusion map $\iota: R \hookrightarrow M$ induces a surjective homomorphism $\iota_*: \pi_1(R) \to \pi_1(M)$ between the fundamental groups of R and M.

PROOF. We will prove this lemma by the induction of dimension of M. If M is of 1-dimension, then we can take $R=M$. Assume that M is of dimension $n(\geqq 2)$ and embedded in a complex projective space $\boldsymbol{P}^k(\boldsymbol{C})$ with $k>n$. We may assume that $p_1 \neq p_2$. Then we can take $p_1=[1, 0, 0, \cdots, 0]$ and $p_2=[0, 1, 0, \cdots, 0]$ in $\boldsymbol{P}^k(\boldsymbol{C})$. If M contains the complex projective line L_0 in $\boldsymbol{P}^k(\boldsymbol{C})$ which meets p_1 and p_2, then we take $R=L_0$.

Now, assume that M contains no complex projective lines meeting p_1 and p_2. Let $\mathcal{L}$ be the set of all hyperplanes H in $\boldsymbol{P}^k(\boldsymbol{C})$ such that H contains p_1 and p_2. Then any element H of $\mathcal{L}$ is represented by an equation

$$a_2 z_2 + \cdots + a_k z_k = 0$$

in $\boldsymbol{P}^k(\boldsymbol{C})$, where $[a_2, \cdots, a_k] \in \boldsymbol{P}^{k-2}(\boldsymbol{C})$. Since M does not contain L_0, the intersection $B=M \cap L_0$ consists of finite points. By Bertini's Theorem, the intersection $M \cap H$ is smooth away from B for the generic element H of $\mathcal{L}$. (See Griffiths and Harris [**5**], p. 137.) We shall prove that $M \cap H$ is also smooth at every point of $M \cap H \cap B$ for the generic element H of $\mathcal{L}$.

Take any point $x=[x_0, x_1, 0, \cdots, 0] \in B$. We may assume that $x_0 \neq 0$. Then $\zeta = z_i/z_0$ $(i=1, \cdots, k)$ are local coordinates at $x \in \boldsymbol{P}^k(\boldsymbol{C})$. There exist holomorphic functions f_i $(i=1, \cdots, k-n)$ defined on a neighborhood U of $x \in \boldsymbol{P}^k(\boldsymbol{C})$ such that $M \cap U = \{\zeta \in U \mid f_i(\zeta)=0\ (i=1, \cdots, k-n)\}$ and $\det(\partial f_i/\partial \zeta_j)_{1 \leq i, j \leq k-n}$ does not vanish on U. Let $f_{k-n+1}(\zeta) = a_2\zeta_2 + \cdots + a_k\zeta_k$. Then

$$M \cap H \cap U = \{\zeta \in U \mid f_i(\zeta) = 0,\ i = 1, \cdots, k-n+1\}$$

and

$$\det(\partial f_i/\partial \zeta_j)_{1 \leq i, j \leq k-n+1} = \sum_{j=2}^{k-n+1} a_j \Delta_j(\zeta),$$

where each Δ_j is a holomorphic function on U.

Since $\Delta_{k-n+1} = \det(\partial f_i/\partial \zeta_j)_{1 \leq i, j \leq k-n}$, the function Δ_{k-n+1} does not vanish on U. Hence

$$\left\{[a_2, \cdots, a_k] \in \boldsymbol{P}^{k-2}(\boldsymbol{C}) \mid \sum_{j=2}^{k-n+1} a_j \Delta_j(\zeta) = 0\right\}$$

is of dimension $<k-2$. Therefore, $M \cap H$ is smooth at x for the generic element H of $\mathcal{L}$. This implies that $M \cap H$ is non-singular for the generic element H of $\mathcal{L}$.

Take an element H of $\mathcal{L}$ such that $M \cap H$ is non-singular. By Lefschetz's Theorem, the pair $(M, M \cap H)$ is $(n-1)$-connected. (See Milnor [**16**], Theorem

7.4, or Lamotke [**15**], Theorem (8.11).) Hence, $M\cap H$ is connected and the inclusion map $\iota: M\cap H\hookrightarrow M$ induces a surjection $\iota_*: \pi_1(M\cap H)\to\pi_1(M)$ providing that $n=\dim M\geqq 2$. Therefore, for the generic element H of $\mathcal{L}$, the hyperplane section $M\cap H$ is a non-singular, connected, $(n-1)$-dimensional analytic subset of M such that $\iota_*: \pi_1(M\cap H)\to\pi_1(M)$ is surjective. This completes the proof of Lemma 2. ■

REMARK. This lemma is proved in Zaïdenberg and Lin [**22**], Lemma 1, p. 130.

LEMMA 3. *Let M be a projective algebraic manifold with universal covering transformation group G, and let N be a compact C-hyperbolic manifold with covering transformation group Γ. If holomorphic maps $f_1, f_2: M\to N$ induce the same monodromy $(\tilde{f}_1)_*=(\tilde{f}_2)_*: G\to\Gamma$ and $\tilde{f}_1(\tilde{M})\cap\tilde{f}_2(\tilde{M})\neq\emptyset$, then $f_1=f_2$.*

PROOF. Take two points $\tilde{p}_1, \tilde{p}_2\in\tilde{M}$ with $\tilde{f}_1(\tilde{p}_1)=\tilde{f}_2(\tilde{p}_2)$, and set $p_1=\pi(\tilde{p}_1)$, $p_2=\pi(\tilde{p}_2)$. Let R be a compact Riemann surface as in Lemma 2. If R is of genus $\leqq 1$, then f_1, f_2 are constant on R. Since R contains p_1 and p_2, we have $f_1=f_2$ on R.

Now, assume that R is of genus >1. Then the inclusion map $\iota: R\hookrightarrow M$ induces a surjection $\iota_*: \pi_1(R)\to\pi_1(M)$. We set $\hat{R}=\pi^{-1}(R)$, which is a non-singular, connected, 1-dimensional analytic subset of $\tilde{M}$. By definition, $\hat{R}$ is invariant under G and the quotient space $\hat{R}/G$ is biholomorphically equivalent to R. Let $\pi_0: \Delta\to\hat{R}$ be the universal covering of $\hat{R}$, where Δ is the unit disc. Then $\hat{\pi}=\pi\circ\pi_0: \Delta\to R$ is the universal covering of R with covering transformation group H. Note that

$$H=\{h\in\mathrm{Aut}(\Delta)\,|\,\pi_0\circ h=g\circ\pi_0 \text{ for some } g\in G\},$$

and that π_0 induces a surjective homomorphism $(\pi_0)_*: H\to G$ sending $h\in H$ into $(\pi_0)_*(h)=g\in G$, where g is uniquely determined by the relation $\pi_0\circ h=g\circ\pi_0$.

Set $\varphi_i=\tilde{f}_i\circ\pi_0$ for $i=1, 2$. Then $\varphi_i: \Delta\to\tilde{N}$ is a lift of $f_i|R: R\to M$ and satisfies

$$\varphi_i\circ h=(\tilde{f}_i)_*((\pi_0)_*(h))\circ\varphi_i$$

for all $h\in H$. Let $(\varphi_i)_*=(\tilde{f}_i)_*\circ(\pi_0)_*$. Holomorphic maps $\varphi_1, \varphi_2: \Delta\to\tilde{N}$ induce the same homomorphism $(\varphi_1)_*, (\varphi_2)_*: H\to\Gamma$, and $\varphi_1(\Delta)\cap\varphi_2(\Delta)\neq\emptyset$. Hence, Lemma 1 implies that $\tilde{f}_1=\tilde{f}_2$ on $\hat{R}$ and $f_1=f_2$ on R.

Let p be an arbitrary point on M. By Lemma 2, we can take a non-singular, connected, 1-dimensional analytic subset R' of M such that R' contains p and p_1, and such that R' is biholomorphically equivalent to $\hat{C}$ or the inclusion map $\iota: R'\hookrightarrow M$ induces a surjective homomorphism $\iota_*: \pi_1(R')\to\pi_1(M)$. If R' is of genus $\leqq 1$, then f_1, f_2 are constant on R'. Hence, $f_1(p)=f_1(p_1)=f_2(p_2)=f_2(p)$.

Now, assume that R' is of genus >1. Since $\tilde{p}_1 \in \hat{R}' = \pi^{-1}(R')$ and $\tilde{f}_1(\tilde{p}_1) = \tilde{f}_2(\tilde{p}_1)$, the same reasoning as above implies that $f_1 = f_2$ on R'. In particular, $f_1(p) = f_2(p)$. Since p is arbitrary, we have $f_1 = f_2$ on M. ■

PROOF OF THEOREM 1. Take two points $p_1, p_2 \in M$ with $f_1(p_1) = f_2(p_2)$. Let $\tilde{p}_i \in \tilde{M}$ such that $\pi(\tilde{p}_i) = p_i$ for $i=1, 2$. There exists an element $\gamma_0 \in \Gamma$ with $\tilde{f}_2(\tilde{p}_2) = \gamma_0 \circ \tilde{f}_1(\tilde{p}_1)$. Since $(\tilde{f}_1)_*: G \to \Gamma$ is surjective, we find an element $g_0 \in G$ with $\gamma_0 = (\tilde{f}_1)_*(g_0)$. We have $\tilde{f}_2(\tilde{p}_2) = \tilde{f}_1 \circ g_0(\tilde{p}_1)$. Hence, $\tilde{f}_1(\tilde{M}) \cap \tilde{f}_2(\tilde{M}) \neq \emptyset$, and Lemma 3 implies that $f_1 = f_2$ on M. This completes the proof of Theorem 1.

COROLLARY 1. *Let M be a projective algebraic manifold with universal covering transformation group G, and let N be a compact C-hyperbolic manifold with covering transformation group Γ. If surjective holomorphic maps f_1, f_2: $M \to N$ induce the same monodromy $(\tilde{f}_1)_* = (\tilde{f}_2)_*: G \to \Gamma$, then $f_1 = f_2$. In particular, if surjective holomorphic maps f_1, f_2: $M \to N$ are homotopic, then $f_1 = f_2$.*

PROOF. Let $\Gamma_0 = (\tilde{f}_1)_*(G)$ and $N_0 = \tilde{N}/\Gamma_0$. Let $\Pi_0: \tilde{N} \to N_0$ be the canonical projection. Take holomorphic maps $\varphi_i: M \to N_0$ satisfying $\varphi_i \circ \pi = \Pi_0 \circ \tilde{f}_i$ for $i=1$, 2. Since M is compact, the Proper Mapping Theorem implies that $\varphi_1(M)$ is a compact analytic subset of N_0. Since f_1 is surjective, $\dim \varphi_1(M) = \dim f_1(M) = \dim N_0$. Hence, $\varphi_1(M) = N_0$, and N_0 is compact. Similarly, we have $\varphi_2(M) = N_0$. Then φ_1, φ_2: $M \to N_0$ are surjective holomorphic maps which induce the same surjective homomorphism $(\tilde{f}_1)_* = (\tilde{f}_2)_*: G \to \Gamma_0$. By Theorem 1, we have $\varphi_1 = \varphi_2$, and hence $f_1 = f_2$. ■

Now, we have the following well-known finiteness Theorem of surjective holomorphic maps of M to N. (Cf. Kalka, Shiffman and Wong [**9**], Kobayashi and Ochiai [**13**], Noguchi and Sunada [**19**], and Urata [**20**].)

COROLLARY 2. *Let M be a projective algebraic manifold, and let N be a compact C-hyperbolic manifold. Then there exist finitely many surjective holomorphic maps of M to N.*

PROOF. Assume that there exist infinitely many distinct surjective holomorphic maps $\{f_n\}_{n=1}^{\infty}$ of M to N. Fix a point $\tilde{x}_0 \in \tilde{M}$ and take a relatively compact fundamental set K_0 for Γ. There exists a unique lift $\tilde{f}_n: \tilde{M} \to \tilde{N}$ of f_n with $\tilde{f}_n(\tilde{x}_0) \in K_0$ for each n. We may assume that $\tilde{f}_n(\tilde{x}_0) \to \tilde{y}_0 \in \tilde{N}$ as $n \to \infty$.

Let $d_{\tilde{M}}$ be the Kobayashi pseudo-distance of $\tilde{M}$ and $d_{\tilde{N}}$ the Kobayashi pseudo-distance of $\tilde{N}$. Since $\tilde{N}$ is C-hyperbolic and $N = \tilde{N}/\Gamma$ is compact, $d_{\tilde{N}}$ is a complete distance. (See Kobayashi [**12**], Theorem 4.7 in Chap. 4.)

Let $\{g_1, \cdots, g_\ell\}$ be a finite system of generators of G. We set

$$\delta = \max_{1 \le i \le \ell} d_{\tilde{M}}(g_i(\tilde{x}_0), \tilde{x}_0).$$

The distance decreasing property of Kobayashi pseudo-distances gives

$$d_{\tilde{N}}(\tilde{f}_n \circ g_i(\tilde{x}_0),\ \tilde{f}_n(\tilde{x}_0)) \leqq d_{\tilde{M}}(g_i(\tilde{x}_0),\ \tilde{x}_0) \leqq \delta$$

for all $n=1, 2, \cdots$, and $i=1, 2, \cdots, \ell$. Since $d_{\tilde{N}}$ is a complete distance on $\tilde{N}$, the set K_0 is a relatively compact subset of $\tilde{N}$, and $\tilde{f}_n(\tilde{x}_0) \in K_0$ for each n, we may assume that $\{\tilde{f}_n \circ g_i(\tilde{x}_0)\}_{n=1}^{\infty}$ converges to a point $z_i \in \tilde{N}$ as $n \to \infty$ for each $i=1, 2, \cdots, \ell$.

Since $\tilde{N}$ is complete hyperbolic, $\tilde{N}$ is taut (see Eisenman [3] or Kiernan [**11**]). Hence, from the relation $\tilde{f}_n \circ g_i = (\tilde{f}_n)_*(g_i) \circ \tilde{f}_n$, we may assume that $\{(\tilde{f}_n)_*(g_i)\}_{n=1}^{\infty}$ converges uniformly to a holomorphic map γ_i defined in $\tilde{N}$ on compact subsets of $\tilde{N}$ as $n \to \infty$. Therefore, H. Cartan's Theorem implies that $\gamma_i \in \mathrm{Aut}(\tilde{N})$ for each $i=1, 2, \cdots, \ell$ (cf. Narasimhan [**17**], Chap. 5, Theorem 4). Since Γ is discrete, there exists a positive integer n_0 such that $(\tilde{f}_n)_*(g_i) = \gamma_i$ for all $n \geqq n_0$ and $i=1, 2, \cdots, \ell$. Hence, $(\tilde{f}_n)_* = (\tilde{f}_{n_0})_*$ for all $n \geqq n_0$. By Corollary 1, we have $f_n = f_{n_0}$ for all $n \geqq n_0$. This is a contradiction. ■

§2. The structure of Hol(M, N).

We shall study concretely the structure of the Douady space Hol(M, N) of a projective algebraic manifold M and a compact C-hyperbolic manifold N.

Let *Const* be the set of all constant maps of M into N. Then *Const* is an irreducible component of Hol(M, N) and it is biholomorphically equivalent to N.

Take a covering $\Pi: \tilde{N} \to N$ such that the Carathéodory pseudo-distance on $\tilde{N}$ is actually a distance. Denote by Γ the covering transformation group of $\Pi: \tilde{N} \to N$.

Let X be an irreducible component of Hol(M, N) which is distinct from *Const*. We may assume that X is reduced. (See Grauert and Remmert [**4**], pp. 20-21.) Take the universal covering $\rho: \tilde{X} \to X$ of X with covering transformation group H, and the universal covering $\pi: \tilde{M} \to M$ of M with covering transformation group G. Then $(\rho, \pi): \tilde{X} \times \tilde{M} \to X \times M$ is the universal covering of $X \times M$ with covering transformation group $H \times G$, where $H \times G$ is the direct product of H and G. We set

$$F(f, p) = f(p)$$

for all $(f, p) \in X \times M$. Then $F: X \times M \to N$ is a holomorphic map, which is lifted to a holomorphic map $\tilde{F}: \tilde{X} \times \tilde{M} \to \tilde{N}$. We obtain the following commutative diagram:

$$\begin{array}{ccc} \tilde{X} \times \tilde{M} & \xrightarrow{\ \tilde{F}\ } & \tilde{N} \\ {\scriptstyle (\rho,\, \pi)}\downarrow {\scriptstyle H \times G} & & {\scriptstyle \Pi}\downarrow {\scriptstyle \Gamma} \\ X \times M & \xrightarrow[\ F\ ]{} & N. \end{array}$$

Let $\tilde{F}_*: H\times G\to\Gamma$ be a homomorphism such that

$$\tilde{F}\circ(h,\, g)=\tilde{F}_*(h,\, g)\circ\tilde{F}$$

for all $(h,\, g)\in H\times G$. We put

$$\hat{\Gamma}=\tilde{F}_*(H\times G) \quad\text{and}\quad \hat{N}=\tilde{N}/\hat{\Gamma}.$$

Since $\hat{\Gamma}$ is a subgroup of Γ, the quotient space $\hat{N}$ is a complex manifold. Note that $\hat{N}$ is not necessarily compact.

Let $\hat{\Pi}: \tilde{N}\to\hat{N}$ be the canonical projection, and let $\hat{F}: X\times M\to\hat{N}$ be the holomorphic map satisfying

$$\hat{F}\circ(\rho,\, \pi)=\hat{\Pi}\circ\tilde{F}.$$

We have the following commutative diagram:

$$\begin{array}{ccc} \tilde{X}\times\tilde{M} & \xrightarrow{\ \tilde{F}\ } & \tilde{N} \\ {\scriptstyle(\rho,\,\pi)}\big\downarrow {\scriptstyle H\times G} & & {\scriptstyle\hat{\Pi}}\big\downarrow{\scriptstyle\hat{\Gamma}} \\ X\times M & \xrightarrow{\ \hat{F}\ } & \hat{N}. \end{array}$$

For any $f\in X$, we have a holomorphic map $\hat{f}: M\to\hat{N}$ given by

$$\hat{f}(\cdot)=\hat{F}(f,\, \cdot).$$

For every point $p\in M$, we define the holomorphic map $\hat{p}: X\to\hat{N}$ by

$$\hat{p}(f)=\hat{f}(p)=\hat{F}(f,\, p)$$

for all $f\in X$.

We fix these notations throughout this section.

Note that the Proper Mapping Theorem implies that both $\hat{f}(M)$ and $\hat{p}(X)$ are compact analytic subsets of $\hat{N}$, and hence they are compact complex spaces.

Now we have the following lemma.

LEMMA 4. *If $f_1,\, f_2\in X$ satisfy $\hat{f}_1(M)\cap\hat{f}_2(M)\neq\emptyset$, then $\hat{f}_1=\hat{f}_2$. In particular, $f_1=f_2$.*

PROOF. Take two poins $(\tilde{f}_1,\, \tilde{p}_1)$ and $(\tilde{f}_2,\, \tilde{p}_2)$ of $\tilde{X}\times\tilde{M}$ such that $(\rho,\, \pi)(\tilde{f}_1,\, \tilde{p}_1)=(f_1,\, p_1)$ and $(\rho,\, \pi)(\tilde{f}_2,\, \tilde{p}_2)=(f_2,\, p_2)$. The assumption $\hat{F}(f_1,\, p_1)=\hat{F}(f_2,\, p_2)$ implies that there exists an element $(h_0,\, g_0)\in H\times G$ satisfying $\tilde{F}(\tilde{f}_2,\, \tilde{p}_2)=\tilde{F}_*(h_0,\, g_0)\circ\tilde{F}(\tilde{f}_1,\, \tilde{p}_1)=\tilde{F}(h_0(\tilde{f}_1),\, g_0(\tilde{p}_1))$. Define $\varphi_1,\, \varphi_2: \tilde{M}\to\tilde{N}$ by

$$\varphi_1(\tilde{p})=\tilde{F}(h_0(\tilde{f}_1),\, \tilde{p}),$$

$$\varphi_2(\tilde{p})=\tilde{F}(\tilde{f}_2,\, \tilde{p})$$

for any $\tilde{p}\in\tilde{M}$. Then $(\varphi_1)_*=(\varphi_2)_*: G\to\hat{\Gamma}$ and $\varphi_2(\tilde{p}_2)=\varphi_1(g_0(\tilde{p}_1))$. Hence, Lemma

3 implies that $\varphi_1=\varphi_2$ on $\tilde{M}$. Therefore, $\hat{F}(f_1, \cdot)=\hat{F}(f_2, \cdot)$ on M. ■

LEMMA 5. *For any point $p\in M$, the holomorphic map $\hat{p}: X\to\hat{N}$ is injective. If $p_1, p_2\in X$ satisfy $\hat{p}_1(X)\cap\hat{p}_2(X)\neq\emptyset$, then $\hat{p}_1=\hat{p}_2$.*

PROOF. The first assertion of this Lemma is clear from Lemma 4.

The second assertion is proved as follows: Take two elements $f_1, f_2\in X$ with $\hat{p}_1(f_1)=\hat{p}_2(f_2)$. Then we have $\hat{f}_1(p_1)=\hat{f}_2(p_2)$. By Lemma 4, we get $\hat{f}_1=\hat{f}_2$, and hence $\hat{p}_1(f_1)=\hat{p}_2(f_1)$. The rigidity theorem due to Borel and Narasimhan ([**1**], Theorem 3.6 and its remark) shows $\hat{p}_1=\hat{p}_2$. ■

Since X may have singular points, we extend the notion of holomorphic maps on complex spaces after Whitney. Let X and Y be complex spaces. We say that a map $f: X\to Y$ is continuous weakly holomorphic, or c-holomorphic for short, if it is continuous on X and is holomorphic at every regular point of X (see Whitney [**21**], p. 149). A map $f: X\to Y$ is said to be c-biholomorphic if it is homeomorphic, and both f and f^{-1} are c-holomorphic. Two complex spaces X and Y are c-biholomorphically equivalent if there exists a c-biholomorphic map between them. We give a typical example of c-biholomorphically equivalent complex spaces which are not biholomorphically equivalent: Let X be the complex plane, and let Y be the complex space $\{(z, w)\in \boldsymbol{C}^2 \mid w^2=z^3\}$. Define the map $f: X\to Y$ by $f(t)=(t^2, t^3)$. Then f is homeomorphic and holomorphic. The inverse map f^{-1} is given by $f^{-1}(z, w)=w/z$, and hence it is not holomorphic at the singular point (0, 0) of Y. Clearly f^{-1} is c-holomorphic, and X, Y are c-biholomorphically equivalent.

Now we have the following assersion.

PROPOSITION 1. *For any point $p\in M$, the holomorphic map $\hat{p}: X\to\hat{p}(M)$ is c-biholomorphic.*

PROOF. From Lemma 5 we see that $\hat{p}: X\to\hat{p}(M)$ is homeomorhic. Since $\hat{p}$ is holomorphic, it is clear that the graph of the inverse map $\hat{p}^{-1}$ is an analytic subset of $\hat{p}(M)\times X$. Hence $\hat{p}^{-1}$ is c-holomorphic (see Whitney [**21**], p. 149). ■

COROLLARY. *If N is a projective algebraic C-hyperbolic manifold, then any component X of* Hol(M, N) *is c-biholomorphically equivalent to a projective algebraic variety.*

PROOF. Let $\hat{\Pi}_0: \hat{N}=\tilde{N}/\hat{\Gamma}\to N=\tilde{N}/\Gamma$ be the canonical projection. Then $\hat{\Pi}_0\circ\hat{p}(X)$ is a projective algebraic variety and $\hat{\Pi}_0|\hat{p}(X): \hat{p}(X)\to\Pi_0\circ\hat{p}(X)$ is a finite holomorphic map. Thus $\hat{p}(X)$ is also a projective algebraic variety. Hence Proposition 1 implies that X is c-biholomorphically equivalent to a projective algebraic variety. ■

PROPOSITION 2. *For any two points $f_1, f_2 \in X$, complex spaces $f_1(M)$, $f_2(M)$ are c-biholomorphically equivalent.*

PROOF. First, we define the map $\Phi: \hat{f}_1(M) \to \hat{f}_2(M)$ by

$$\Phi(x) = \hat{f}_2(p), \qquad p \in \hat{f}_1^{-1}(x).$$

This map Φ is well-defined. In fact, for any $p, q \in \hat{f}_1^{-1}(x)$, Lemma 5 implies that $\hat{p} = \hat{q}$, and so $\hat{f}_2(p) = \hat{f}_2(q)$.

Second, we show that Φ is continuous on $\hat{f}_1(M)$. Assume that Φ is not continuous at $x_0 \in \hat{f}_1(M)$. Then there exists an infinite sequence $\{x_n\}$ converging to x_0 in $\hat{f}_1(M)$ such that $\{\Phi(x_n)\}$ does not converge to $\Phi(x_0)$. We find a neighborhood U of $\Phi(x_0)$ in $\hat{f}_2(M)$ and a subsequence $\{\Phi(x_{n_j})\}$ such that $\Phi(x_{n_j}) \notin U$ for all n_j. For each n_j, take a point $p_{n_j} \in M$ with $\hat{f}_1(p_{n_j}) = x_{n_j}$. Since M is compact, we may assume that $\{p_{n_j}\}$ converges to a point $p_0 \in M$. Because $\hat{f}_1$ is continuous and $\{x_{n_j}\}$ converges to x_0, we have $\hat{f}_1(p_0) = x_0$, and so $\Phi(x_0) = \hat{f}_2(p_0)$. By the continuity of $\hat{f}_2$, we obtain $\Phi(x_{n_j}) = \hat{f}_2(p_{n_j}) \to \hat{f}_2(p_0) = \Phi(x_0)$. This is a contradiction.

Third, we prove that Φ is c-holomorphic. It is sufficient to see that the graph G_Φ of Φ is an analytic subset of $\hat{f}_1(M) \times \hat{f}_2(M)$. By the definition of Φ, we get

$$G_\Phi = \{(\hat{f}_1(p), \hat{f}_2(p)) \in \hat{f}_1(M) \times \hat{f}_2(M) \mid p \in M\}.$$

Since $\hat{f}_1 \times \hat{f}_2: M \to \hat{f}_1(M) \times \hat{f}_2(M)$ is holomorphic, and M is a compact complex space, the Proper Mapping Theorem shows that G_Φ is an analytic subset of $\hat{f}_1(M) \times \hat{f}_2(M)$.

Finally, we define the map $\Psi: \hat{f}_2(M) \to \hat{f}_1(M)$ by

$$\Psi(y) = \hat{f}_1(p), \qquad p \in \hat{f}_2^{-1}(y).$$

By the same reasoning as above, we see that Ψ is a c-holomorphic map. It is clear that Ψ is the inverse map of Φ. ■

THEOREM 2. *For any fixed point $f_0 \in X$, complex spaces $\hat{F}(X \times M)$ and $X \times \hat{f}_0(M)$ are c-biholomorphically equivalent.*

PROOF. Define the map $\Phi: X \times \hat{f}_0(M) \to \hat{F}(X \times M)$ by

$$\Phi(f, x) = \hat{f}(p), \qquad p \in \hat{f}_0^{-1}(x).$$

This map Φ is well-defined. In fact, for any $p, q \in \hat{f}_0^{-1}(x)$, Lemma 5 implies that $\hat{p} = \hat{q}$, and hence $\hat{f}(p) = \hat{f}(q)$.

Next, we show that Φ is continuous on $X \times \hat{f}_0(M)$. Assume that Φ is not continuous at $(f, x) \in X \times \hat{f}_0(M)$. Then there exists an infinite sequence $\{(f_n, x_n)\}$ converging to (f, x) in $X \times \hat{f}_0(M)$ such that $\{\Phi(f_n, x_n)\}$ does not converge to

$\Phi(f, x)$. We find a neighborhood U of $\Phi(f, x)$ in $\hat{F}(X\times M)$ and a subsequence $\{\Phi(f_{n_j}, x_{n_j})\}$ such that $\Phi(f_{n_j}, x_{n_j})\notin U$ for all n_j. For each n_j, take a point $p_{n_j}\in M$ with $\hat{f}_0(p_{n_j})=x_{n_j}$. Since M is compact, we may assume that $\{p_{n_j}\}$ converges to a point $q\in M$. Because $\hat{f}_0$ is continuous and $\{x_{n_j}\}$ converges to x, we have $\hat{f}_0(q)=x$, and so $\Phi(f, x)=\hat{f}(q)$. By the continuity of $\hat{F}$, we obtain $\Phi(f_{n_j}, x_{n_j})=\hat{F}(f_{n_j}, p_{n_j})\to\hat{F}(f, q)=\Phi(f, x)$. This is a contradiction.

In order to prove that Φ is c-holomorphic, it is sufficient to see that the graph G_Φ of Φ is an analytic subset of $X\times\hat{f}_0(M)\times\hat{F}(X\times M)$. By the definition of Φ, we get

$$G_\Phi=\{(f, \hat{f}_0(p), \hat{F}(f, p))\in X\times\hat{f}_0(M)\times\hat{F}(X\times M)\mid f\in X,\ p\in M\}.$$

Since $\mathrm{id}\times\hat{f}_0\times\hat{F}: X\times M\to X\times\hat{f}_0(M)\times\hat{F}(X\times M)$ is holomorphic, and $X\times M$ is a compact complex space, the Proper Mapping Theorem shows that G_Φ is an analytic subset of $X\times\hat{f}_0(M)\times\hat{F}(X\times M)$.

Finally, the inverse map Ψ of Φ is given by

$$\Psi(y)=(f, \hat{f}_0(p)),\qquad (f, p)\in\hat{F}^{-1}(y).$$

By the similar reasoning as above, we see that Ψ is a c-holomorphic map and it is the inverse map of Φ. ∎

From Theorem 2, $\dim\hat{f}(M)$, the dimension of the complex space $\hat{f}(M)$, is independent of $f\in X$. We call $\dim\hat{f}(M)$ the rank of X and denote it by $\mathrm{rank}X$.

THEOREM 3. *For any component X of* $\mathrm{Hol}(M, N)$,

$$\dim X+\mathrm{rank}X\leqq\dim N.$$

If $\dim X+\mathrm{rank}X=\dim N$, *then $X\times\hat{f}_0(M)$ is biholomorphically equivalent to $\hat{N}$, where $f_0\in X$. In particular, X and $\hat{f}_0(M)$ are nonsingular, and $\hat{\Gamma}$ is a finite index subgroup of Γ so that the canonical projection $\hat{\Pi}_0$: $\hat{N}=\tilde{N}/\hat{\Gamma}\to N=\tilde{N}/\Gamma$ is a surjective finite holomorphic map.*

PROOF. Theorem 2 implies that $X\times\hat{f}_0(M)$ is c-biholomorphically equivalent to the analytic subset $\hat{F}(X\times M)$. Thus we have $\dim X+\dim\hat{f}_0(X)=\dim\hat{F}(X\times M)\leqq\dim\hat{N}=\dim N$, and hence $\dim X+\mathrm{rank}X\leqq\dim N$.

If $\dim X+\mathrm{rank}X=\dim N$, then we get $\dim\hat{F}(X\times M)=\dim\hat{N}$. Thus $\hat{F}(X\times M)$ is a non-empty, open and closed subset of $\hat{N}$. Since $\hat{N}$ is a connected complex manifold, we see that $\hat{F}(X\times M)=\hat{N}$ and $X\times\hat{f}_0(M)$ is biholomorphically equivalent to $\hat{N}$. Hence both X and $\hat{f}_0(M)$ are nonsingular. Since $\hat{N}$ is a compact complex manifold and $\dim\hat{N}=\dim N$, we see that the canonical projection $\hat{\Pi}_0$: $\hat{N}=\tilde{N}/\hat{\Gamma}\to N=\tilde{N}/\Gamma$ is a surjective finite holomorphic map. In particular, $\hat{\Gamma}$ is a finite index subgroup of Γ. ∎

This theorem also proves Corollary 2 to Theorem 1.

THEOREM 4. *Let N be a compact complex manifold represented by a quotient space Ω/Γ such that Ω is a bounded domain in $\boldsymbol{C}^n$ and Γ is a fixed-point-free discrete subgroup of* Aut(Ω). *Let $\ell(\Omega)$ be the maximum dimension of all complex spaces included in the boundary $\partial\Omega$ of Ω. If a holomorphic map $f: M\to N$ is of* rank $>\ell(\Omega)$, *then f is rigid. Moreover,* $\dim X\leqq \ell(\Omega)$ *for any component X of* Hol(M, N) *with $X\neq Const$.*

PROOF. For the first assertion it is sufficient to show that $\dim X=0$ for any component X of Hol(M, N) which contains f. Assume that $\dim X>0$. Since N is projective algebraic, Corollary to Proposition 1 implies that X is c-biholomorphically equivalent to a projective algebraic variety. Thus we have a 1-dimensional irreducible analytic subset R of X. Let $\sigma: R_0\to R$ be the normalization of R. Then R_0 is a compact Riemann surface of genus >1. In fact, if R_0 is of genus $\leqq 1$, then for a point $p_0\in M$, the holomorphic map $\hat{p}_0$: $X\to\hat{N}$ is constant on R, which contradicts Lemma 5. Hence the universal covering surface of R_0 is the unit disc Δ. Denote by H_0 the universal covering transformation group of R_0.

Define the holomorphic map $\Phi: R_0\times M\to N$ by $\Phi(\varphi, p)=F(\sigma(\varphi), p)=\sigma(\varphi)(p)$ for any $(\varphi, p)\in R_0\times M$. Let $\tilde{\Phi}=(\tilde{\Phi}_1, \cdots, \tilde{\Phi}_n): \Delta\times\tilde{M}\to\Omega$ be a lift of Φ. From Lemma 5 we can take a point $t_0\in\Delta$ so that $\tilde{\Phi}(\cdot, \tilde{p})$ is injective in a neighborhood of t_0 for any $\tilde{p}\in\tilde{M}$.

Let r be the rank of f. Since $\tilde{\Phi}(t, \cdot)$ is of rank r for any $t\in\Delta$, we may assume that there exist holomorphic local coordinates $z_1, \cdots, z_m$ around a point $\tilde{p}_0\in\tilde{M}$ such that

$$\det\Big(\frac{\partial\tilde{\Phi}_j}{\partial z_k}(t_0, \tilde{p}_0)\Big)_{1\leq j, k\leq r}\neq 0 .$$

For any $t\in\Delta$ we set

$$d(t)=\det\Big(\frac{\partial\tilde{\Phi}_j}{\partial z_k}(t, p_0)\Big)_{1\leq j, k\leq r} .$$

Then $d(t)$ is a bounded holomorphic function on Δ.

Take a neighborhood K of t_0 which is relatively compact in Δ. For almost every boundary point $\tau\in\partial\Delta$, there exists a sequence $\{h_j\}_{j=1}^{\infty}$ of H_0 satisfying the following conditions:

(1) $h_j(t)\to\tau$ through a Stolz domain with vertex at τ as $j\to\infty$ for any $t\in K$.

(2) $\tilde{\Phi}(h_j(t), \tilde{p}_0)\to\zeta\in\bar{\Omega}$ as $j\to\infty$ for any $t\in K$.

(3) $d(h_j(t))\to d_0\in\boldsymbol{C}$ as $j\to\infty$ for any $t\in K$.

(4) the sequence $\{\tilde{\Phi}_*((h_j, \mathrm{id}))\}_{j=1}^{\infty}$ of Γ converges uniformly on compact sets of Ω to a holomorphic map $T: \Omega\to\bar{\Omega}$.

It is proved that the range of T is contained in $\partial\Omega$ as follows: From conditions (2) and (4), we have

$$\zeta=\lim_{j\to\infty}\tilde{\Phi}(h_j(t),\ \tilde{p}_0)=T\circ\tilde{\Phi}(t,\ \tilde{p}_0),$$

where $\zeta\in\bar{\Omega}$ is independent of $t\in K$. If $T(\Omega)\not\subset\partial\Omega$, H. Cartan's theorem (Narasimhan [**17**], Theorem 4, p. 78) implies that T is an analytic automorphism of Ω. Thus $T\circ\tilde{\Phi}(t,\ \tilde{p}_0)$ depends on t because $\Phi(t,\ \tilde{p}_0)$ is injective in a neighborhood of t_0. We have a contradiction.

Now we define a holomorphic map $A: \tilde{M}\to\partial\Omega$ by

$$A(\tilde{p})=\lim_{j\to\infty}\tilde{\Phi}(h_j(t_0),\ \tilde{p})=T\circ\tilde{\Phi}(t_0,\ \tilde{p}).$$

Then A is of rank$\leq\ell(\Omega)$. Hence, by the assumption that $\ell(\Omega)<r$ and condition (3), we obtain $d_0=0$. This means that the bounded holomorphic function d on the unit disk Δ has boundary value 0 for almost all $\tau\in\partial\Delta$. Thus $d=0$ on Δ, which contradicts the condition $d(t_0)\neq 0$. Therefore, we have $\dim X=0$.

In order to prove the second assertion, assume that some component X of Hol(M, N) with $X\neq Const$ satisfies $\dim X>\ell(\Omega)$. Consider the Douady space Hol(X, N). Take a point $f_0\in X$. For any $p\in M$ we have a holomorphic map $\hat{p}: X\to\hat{N}$. By Theorem 2 these maps $\hat{p}$'s are parametrized by an analytic subset $\hat{f}_0(M)$ of $\hat{N}$. Hence there exists a component Y of Hol(X, N) such that Y includes all $\Pi_0\circ\hat{p}$, where Π_0 is the canonical projection of $\hat{N}=\Omega/\hat{\Gamma}$ to $N=\Omega/\Gamma$. Note that $\dim Y>0$ because $X\neq Const$. On the other hand, from Lemma 5 the holomorphic $\hat{p}: X\to\hat{N}$ is injective. Thus, $\Pi_0\circ\hat{p}$ is of rank $\dim X>\ell(\Omega)$. By the first assertion of this theorem, $\Pi_0\circ\hat{p}$ is rigid. This is a contradiction. ■

§3. An example.

In Imayoshi [**8**], we saw some typical examples of 2-dimensional compact C-hyperbolic manifolds N and Douady spaces Hol(M, N). In this section, using a Kodaira surface in Kodaira [**14**], we construct a 3-dimensional C-hyperbolic projective algebraic manifold N such that N is not biholomorphically equivalent to a product of complex manifolds, and such that for a certain compact Riemann surface C the Douady space Hol(C, N) has a 1-dimensional component with singular points.

Fix a complex torus T, i.e., a compact Riemann surface of genus 1. Note that a torus has the canonical additive group structure. Construct a 2-sheeted ramified covering $\pi_{R_0}: R_0\to T$ so that π_{R_0} is ramified over two points $t_1, t_2\in T$. By Riemann-Hurwitz relation, R_0 is a compact Riemann surface of genus 2. After Kas [**10**], Example 1, let us construct a Kodaira surface M as follows: Let R be a compact Riemann surface of genus 3 such that R is a 2-sheeted unramified covering of R_0 with covering projection π_R. Taking a certain compact Riemann surface S of genus 9, which is a 4-sheeted unramified covering surface of R, we can construct a Kodaira surface M. This 2-dimensional compact complex manifold M has a holomorphic map $\Phi: M \to S \times R$ which makes M an r-sheeted cyclic branched covering of $S \times R$ such that $P_S = P_1 \circ \Phi: M \to S$ and $P_R = P_2 \circ \Phi: M \to S$ are both non-trivial regularly fibered surfaces, where P_1 and P_2 are the projections of $S \times R$ onto the first and second factors, respectively. We have the following commutative diagram:

$$\begin{array}{ccccc} & \overset{P_S}{\swarrow} & M & \overset{P_R}{\searrow} & \\ & & \downarrow \Phi & & \\ S & \xleftarrow{P_1} & S \times R & \xrightarrow{P_2} & R. \end{array}$$

Hence, the universal covering space $\tilde{M}$ is biholomorphically equivalent to a bounded domain in $\boldsymbol{C}^2$, and so M is a C-hyperbolic projective algebraic manifold. Moreover, $\tilde{M}$ is biholomorphically equivalent to neither a 2-dimensional polydisc nor a 2-dimensional strongly pseudoconvex domain (see Imayoshi [**7**], Corollary 1 to Theorem 1, and Theorems 2, 3).

Now we take a 2-sheeted ramified covering $\pi_A: A \to T$ which ramified over $t'_j \in T$, $j=1, \cdots, 2m$. We also construct a 3-sheeted ramified covering $\pi_B: B \to T$ such that π_B is ramified over $t''_k \in T$, $k=1, \cdots, 2n$ and these branch numbers are all 2. We may assume that $\{t'_j - t''_k \mid 1 \leqq j \leqq 2m,\ 1 \leqq k \leqq 2n\}$ does not meet $\{t_1, t_2\}$. We set

$$\rho = \pi_{R_0} \circ \pi_R \circ P_R : M \longrightarrow T,$$

$$\Pi = \pi_A - \pi_B : A \times B \longrightarrow T,$$

$$N = \{(a, b, p) \in A \times B \times M \mid \pi_A(a) - \pi_B(b) = \rho(p)\}.$$

We shall see that N satisfies the following assertions:

(1) N is a 3-dimensional C-hyperbolic projective algebraic manifold.

(2) N is not biholomorphically equivalent to a product of complex manifolds.

(3) N contains a submanifold which is biholomorphically equivalent to a product of Riemann surfaces.

(4) There exists a compact Riemann surface C such that $\mathrm{Hol}(C, N)$ has a 1-dimensional irreducible component with singular points.

In order to prove assertion (1), we consider a compact analytic subset of $A \times B \times R$ defined by

$$Z = \{(a, b, x) \in A \times B \times R \mid \pi_A(a) - \pi_B(b) = \pi_{R_0} \circ \pi_R(x)\}.$$

Since $\pi_{R_0} \circ \pi_R$, π_A, and π_B are ramified over $\{t_1, t_2\}$, $\{t'_j\}_{j=1}^{2m}$, and $\{t''_k\}_{k=1}^{2n}$, respectively, the analytic subset Z is non-singular. We see that Z is connected as follows: Take any points $(a_0, b_0, x_0), (a_1, b_1, x_1) \in Z$. Since $A \times B$ is connected, we have a continuous curve $C_0: [0, 1] \to A \times B$ with $C_0(0)=(a_0, b_0)$, $C_0(1)=(a_1, b_1)$. Then $D_0 = \Pi \circ C_0$ is a continuous curve on T. Thus we find a continuous curve E_0 on R with $D_0 = \pi_{R_0} \circ \pi_R \circ E_0$. Consequently, we obtain a continuous curve F_0 on Z defined by $F_0(t) = (C_0(t), E_0(t))$. Therefore, Z is connected, and hence it is a 2-dimensional compact complex manifold.

By the same reasoning as the case of Z, the compact analytic subset N of $A \times B \times M$ is non-singular. We define a holomorphic map $\Psi: N \to Z$ by

$$\Psi(a, b, p) = (a, b, P_R(p))$$

for all $(a, b, p) \in N$. It is easy to see that Ψ is of rank 2 at every point of N because P_R is of rank 1 at an arbitrary point of M, and $\pi_{R_0} \circ \pi_R$, π_A, π_B are ramified over $\{t_1, t_2\}$, $\{t'_j\}_{j=1}^{2m}$, $\{t''_k\}_{k=1}^{2n}$, respectively. For every $(a, b, x) \in Z$ the fiber $\Psi^{-1}(a, b, x)$ of Ψ over (a, b, x) is biholomorphically equivalent to the fiber $P_R^{-1}(x)$ of P_R over $x \in R$. In particular, every fiber $\Psi^{-1}(a, b, x)$ is non-singular and connected. Hence N is connected and is a complex manifold. Since $A \times B \times M$ is a C-hyperbolic projective algebraic manifold, its submanifold N is also C-hyperbolic projective algebraic.

Let us prove assertion (2). Assume that there exists a biholomorphic map $F: N_1 \times N_2 \to N$, where N_1 is a compact Riemann surface and N_2 is a 2-dimensional compact complex manifold. Let P_A, P_B, P_M be projections of $N \subset A \times B \times M$ to A, B, M, respectively. We put $F_A = P_A \circ F$, $F_B = P_B \circ F$, $F_M = P_M \circ F$. It is seen that $F_M(\cdot, q_2): N_1 \to M$ is constant for every $q_2 \in N_2$. In fact, suppose that $F_M(\cdot, q_2): N_1 \to M$ is non-constant for some $q'_2 \in N_2$. Then $F_M(\cdot, q_2): N_1 \to M$ is non-constant for every $q_2 \in N_2$. From Imayoshi [**6**], Theorem 9, the set $\{F_M(\cdot, q_2) | q_2 \in N_2\}$ is finite. Thus we get $\dim F_M(N_1 \times N_2) = 1$, which contradicts F_M is surjective. We see that $F_A(\cdot, q_2): N_1 \to A$ is constant for every $q_2 \in N_2$ or $F_B(\cdot, q_2): N_1 \to B$ is constant for every $q_2 \in N_2$. In fact, assume that $F_A(\cdot, q_2)$: $N_1 \to A$ is non-constant for some $q'_2 \in N_2$ and $F_B(\cdot, q_2): N_1 \to B$ is nonconstant for some $q''_2 \in N_2$. Then both $F_A(\cdot, q_2): N_1 \to A$ and $F_B(\cdot, q_2): N_1 \to B$ are non-constant for every $q_2 \in N_2$. By de Franchis' theorem (see, for example Imayoshi [**6**], Theorem 2), the sets $\{F_A(\cdot, q_2) | q_2 \in N_2\}$ and $\{F_B(\cdot, q_2) | q_2 \in N_2\}$ are finite. Hence we obtain $\dim F_A \times F_B(N_1 \times N_2) = 1$, which contradicts $F_A \times F_B$ is surjective. Thus we may assume that $F_A(\cdot, q_2): N_1 \to A$ is constant for every $q_2 \in N_2$. Then by the relation $\pi_A \circ F_A(\cdot, q_2) - \pi_B \circ F_B(\cdot, q_2) = \rho \circ F_M(\cdot, q_2)$ on N_1, we conclude that $F_B(\cdot, q_2)$ is also constant. This is a contradiction.

Now we see assertion (3). By Sard's theorem, we can find a point $t_0 \in T$ such that the analytic subset $\Pi^{-1}(t_0) = \{(a, b) | \pi_A(a) - \pi_B(b) = t_0\}$ of $A \times B$ is non-singular. Let D be a connected component of $\Pi^{-1}(t_0)$. Take a point $x_0 \in R$ with $t_0 = \pi_{R_0} \circ \pi_R(x_0)$. We set $C = P_R^{-1}(x_0)$. It is easy to show that $D \times C$ is a

2-dimensional complex submanifold of N.

Finally we show assertion (4). Take a point $x_0 \in R$. We set $C=P_R^{-1}(x_0)$ and $t_0=\pi_{R_0}\circ\pi_R(x_0)\in T$. Let D be an irreducible component of the analytic subset $\Pi^{-1}(t_0)$ of $A\times B$. Then we see that the Douady space $\mathrm{Hol}(C, N)$ has an irreducible component X given by

$$X=\{f_{(a,b)} \mid f_{(a,b)}: C\to N, (a, b)\in D\},$$

where $f_{(a,b)}$ defined by $f_{(a,b)}(p)=(a, b, p)$ for any $p\in C$. In fact, let Y be an irreducible component of $\mathrm{Hol}(C, N)$ with $X\subset Y$. Take any element $f\in Y$. For any $(a, b)\in A\times B$ the holomorphic map $P_M\circ f_{(a,b)}=\mathrm{id}$ on C. Thus $P_M\circ f$ is non-constant, and so $P_M\circ f=P_M\circ f_{(a,b)}=\mathrm{id}$ on C by de Franchis' theorem. Since the holomorphic map $P_A\circ f_{(a,b)}: C\to A$ is constant map with value a, the holomorphic map $P_A\circ f: C\to A$ is also constant. Similarly, $P_B\circ f: C\to B$ is also constant. Hence f is contained in X, and $Y\subset X$. Thus X is an irreducible component of $\mathrm{Hol}(C, N)$. It is seen that X is biholomorphically equivalent to D as follows: By the universality property of $\mathrm{Hol}(C, N)$, the holomorphic map of $D\times C$ into N sending (a, b, p) to (a, b, p) induces the bijective holomorphic map $G: D\to X\subset \mathrm{Hol}(C, N)$ given by $G(a, b)=f_{(a,b)}$. The inverse map $G^{-1}: X\to D$ of G with $G^{-1}(f_{(a,b)})=(a, b)$ is also holomorphic because for a fixed $p_0\in C$ the map of X into N sending $f_{(a,b)}$ into $f_{(a,b)}(p_0)=(a, b, p_0)$ is holomorphic, and so is the map of X to D sending $f_{(a,b)}$ into (a, b). Thus G gives a biholomorphic map between X and D.

If we choose $t_1', t_2''\in T$ with $t_1'=t_2''$ and $t_0=0\in T$, the D has a singular point as same as the singular point $(0, 0)$ of the analytic subset $\{(z, w)\in \boldsymbol{C}^2 \mid w^2=z^3\}$. Therefore, X has a singular point.

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