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On some improperly posed problem for degenerate quasilinear elliptic equations

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Introduction.

We consider the Cauchy problem for degenerate quasilinear elliptic equations, and we give an estimate for their solutions with a prescribed bound. For linear elliptic equations such an estimate is known (see e.g., [**1**], [3] and [**5**]). In general the Cauchy problem is not well-posed for elliptic equations, that is, it is improperly posed for these equations. We give the following example due to Lavrentiev's book [**5**, p. 19]:

Let Ω be a bounded domain in the plane. Let its boundary $\partial\Omega$ be smooth, and let $\boldsymbol{n}$ be the outer normal of $\partial\Omega$. Let Γ_1 be an open subset of $\partial\Omega$ and $\Gamma_2=\partial\Omega-\Gamma_1$. The part Γ_1 is said to be an initial surface. Let u be in $C^1(\bar{\Omega})$ and harmonic in Ω. We assume that

$$|u(x)|+\left|\frac{\partial}{\partial \boldsymbol{n}}u(x)\right| \leqq \varepsilon \qquad x\in\Gamma_1,$$

$$|u(x)|+\left|\frac{\partial}{\partial \boldsymbol{n}}u(x)\right| \leqq M \qquad x\in\Gamma_2.$$

Then the inequality

$$|u(x)| \leqq C(M, \varepsilon) \qquad x\in\Omega$$

holds. Here $C(M, \varepsilon)$ is a constant depending only on ε, M and Ω such that $C(M, \varepsilon)\to 0$ as $\varepsilon\to 0$. If $\varepsilon=0$ in particular, u vanishes identically in Ω. From this we see that the unique continuation property holds for harmonic functions.

In our previous paper [**2**], we have derived a similar estimate as above for a certain class of degenerate quasilinear elliptic equations. In [**2**] we have treated this problem in $\boldsymbol{R}^N$, and we have assumed the strict convexity of the initial surface. However there is a derivative loss in the required estimate. In this paper we extend our result in [**2**] to a larger class of degenerate quasilinear elliptic equations and we append another proof different from [**2**] (see Theorem 2). Further we see that there is no loss of derivative for the estimate in our Theorem 1, in comparison with [**2**].

Let D be a domain in $\boldsymbol{R}^N$ with coordinates $x=(x_1, \cdots, x_N)$. Let $F(x, v)$ be a given function, and let us consider the variational quantity

$$J(v)=\int_D\left(\frac{1}{p}\sum_{i=1}^N|\partial_{x_i}v|^p-F(x, v)\right)dx\,, \qquad p>1\,. \tag{0.1}$$

Then the critical point u of $J(v)$ satisfies

$$\sum_{i=1}^N\partial_{x_i}(|\partial_{x_i}u|^{p-2}\partial_{x_i}u)=-(\partial F/\partial v)(x, u) \qquad \text{in } D\,. \tag{0.2}$$

If the integrand $\sum_{i=1}^N|\partial_{x_i}v|^p$ is replaced by $|\nabla v|^p$ in (0.1), the equation (0.2) is changed into

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u)=-(\partial F/\partial v)(x, u) \qquad \text{in } D\,. \tag{0.3}$$

Thus it seems to us that the difference between (0.2) and (0.3) is small.

Each one of left-hand sides of (0.2) and (0.3) is written together with $\operatorname{div}(A(x, \nabla u)\cdot\nabla u)$, where $A(x, \xi)$ is a mapping from $D\times\boldsymbol{R}^N$ into $\boldsymbol{R}^N$. We set $L_A(u)=\operatorname{div}(A(x, \nabla u)\cdot\nabla u)$. The operator L_A is said to be degenerate quasilinear elliptic, if it is satisfied that for a. e. $x\in\boldsymbol{R}^N$ and for all $\xi\in\boldsymbol{R}^N$

$$|A(x, \xi)|\leqq C|\xi|^{p-1}, \qquad A(x, \xi)\cdot\xi\geqq c|\xi|^p,$$

where c, $C>0$.

If $p=2$ in particular, both principal parts of (0.2) and (0.3) are Laplacien, and it is known that the estimate with prescribed bound holds for the Cauchy problem under some assumptions on F (see [3]). Kazdan [4] discussed the strong unique continuation property for solutions of semilinear elliptic systems in Riemannian manifolds, whose principal parts are Laplacien. He proposed also the following question in the final section of [4]: Does the unique continuation property hold for solutions u of (0.3)? It does if $N=2$ and $F=0$, because the function $\partial_x u-i\partial_y u$ is a quasiconformal mapping due to Manfredi [7]. On the other hand Martio [8] gave a counterexample of $L_A u=0$ such that the unique continuation does not hold, where two functions $A(x, \xi)$ and $u(x)$ are constructed for $p=N\geqq 3$. Though we cannot answer comprehensively the above question, we give a partial affirmative answer for the equations of type (0.2) under the assumption that the initial surface is strictly convex. Our method is to yield an estimate based on those in [3] and [9].

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1. Statement of our theorems.

Throughout this paper we consider the nonlinear operator

$$L_p(u)=\sum_{i=1}^{N}\partial_{x_i}(|\partial_{x_i}u|^{p-2}\partial_{x_i}u), \qquad p>1.$$

We write $x=(x_1, \cdots, x_N)$, $x'=(x_1, \cdots, x_{N-1})$ and $y=x_N$. The origin in $\boldsymbol{R}^N$ is denoted by O. Let D be a domain in $\boldsymbol{R}^N$ such that D is in the half space $\{y>0\}$. Let Γ be an open subset of ∂D with $\Gamma\ni O$. Let Γ be of class C^1.

We assume that there is a positive number $a<1/2$ such that for any c with $0<c\leqq a$, $D\cap\{y<c\}$ is connected and

$$\partial(D\cap\{y<c\})=\{O\}\cup(\Gamma\cap\{0<y<c\})\cup(\bar{D}\cap\{y=c\}).$$

This means that D is strictly convex at O. We fix such a positive number a. From now on we write $D_c=D\cap\{y<c\}$ and $\Gamma_c=\Gamma\cap\{y<c\}$.

We denote by $(\ ,\ )$ the $L^2(D_a)$-inner product. For any v, $w\in C^1(\bar{D}_a)$ we define the inner product of $L_p(v)$ and w as follows:

$$(1.1)\qquad (L_p(v), w)=-\sum_{i=1}^{N}\int_{D_a}|\partial_{x_i}v|^{p-2}\partial_{x_i}v\cdot\partial_{x_i}w\,dx$$
$$+\sum_{i=1}^{N}\int_{\partial D_a}|\partial_{x_i}v|^{p-2}\partial_{x_i}v\cdot w\cos(\boldsymbol{n}, x_i)dS,$$

where $\boldsymbol{n}$ is the outer normal of ∂D_a and dS is the surface element of ∂D_a. Let u be in $C^1(\bar{D}_a)$ and f be in $L^1(D_a)$. In what follows we say that

$$|L_p(u)|\leqq|f| \qquad \text{in } D_a,$$

if and only if for any $\varphi\in C^1(\bar{D}_a)$

$$|(L_p(u), \varphi)|\leqq\int_{D_a}|f||\varphi|dx.$$

Let k be a non-negative integer and α be a real number with $0<\alpha<1$. The Hölder space $C^{k,\alpha}(\bar{D}_a)$ is defined as the subspace of $C^k(\bar{D}_a)$ consisting of functions whose k-th order partial derivatives are uniformly Hölder continuous in $\bar{D}_a$ with exponent α. It is known that solutions of the degenerate quasilinear elliptic equations are in $C^{1,\alpha}$ (see e.g., [**6**]).

First we have

THEOREM 1. *Let $p\geqq 2$ and let u belong to $C^{1,\alpha}(\bar{D}_a)$ for α with $1/2<\alpha<1$. Let*

$$|L_p(u)|\leqq K|u|^{p-1} \qquad \text{in } D_a,$$

for a constant K. Then, if

$$\int_{\Gamma_a}(|u|^p+|\nabla u|^p)dS\leqq\varepsilon,$$

$$\int_{\bar{D}\cap\{y=a\}} (|u|^p + |\nabla u|^p) dS \leqq M$$

and $\varepsilon \exp(2^{p-1}K)$, $\varepsilon \exp(p/2) \leqq a^{2p}M$, *it holds that*

$$\int_{D_{a/2}} (|u|^p + |\nabla u|^p) dx \leqq C(1+K)^2 \varepsilon^{a/2} M^{(2-a)/2},$$

where C *is a positive constant depending only on* p.

The assumption $p \geqq 2$ is essential in the above theorem (see Lemma 2). If $p > 1$, we have the following theorem. We denote $|\nabla^2 u|^2 = \sum_{i,j=1}^N (\partial_{x_i}\partial_{x_j} u)^2$.

THEOREM 2. *Let* $p>1$. *Assume that either* u *is in* $C^3(\bar{D}_a)$, *or* u *is in* $C^2(\bar{D}_a)$ *and* $|\partial_{x_i} u|^{p-2} \partial_{x_i}^2 u$ *are in* $L^1(D_a)$ *for* $i=1, \cdots, N$. *Let* $L_p(u)=0$ *in* D_a. *Then, if*

$$\int_{\Gamma_a} (|u|^p + |\nabla u|^p + |\nabla^2 u|^p) dS \leqq \varepsilon,$$

$$\int_{\bar{D}\cap\{y=a\}} (|u|^p + |\nabla u|^p + |\nabla^2 u|^p) dS \leqq M$$

and $\varepsilon \exp(p/2) \leqq M$, *it holds that*

$$\int_{D_{a/2}} (|u|^p + |\nabla u|^p) dx + a^2 \left[\sum_{i=1}^{N-1} \int_{D_{a/2}} |\partial_{x_i} u|^{p-2} (\partial_y \partial_{x_i} u)^2 dx \right.$$
$$\left. + \int_{D_{a/2}} |\partial_y u|^{p-2} (\partial_y^2 u)^2 dx \right] \leqq C \varepsilon^{a/2} M^{(2-a)/2},$$

where C *is a positive constant depending only on* p.

For any function v in class C^1, we see that if $v=0$ on E, $|\nabla v|=0$ a.e. on E, by using Lebesgue's density point theorem. Thus in Theorem 2, we read $|\partial_{x_i} u|^{p-2} (\partial_{x_i} \partial_{x_j} u)^2 = 0$, if $\partial_{x_i} u = 0$, $i, j = 1, \cdots, N$.

2. Preliminaries.

First we prepare

LEMMA 1. *Let* $p \geqq 2$. *Then for* $X, Y \in \boldsymbol{R}$

$$||X|^{p-2}X - |Y|^{p-2}Y| \leqq (p-1)|X-Y|(|X|+|Y|)^{p-2}.$$

PROOF. We set

$$F(t) = |Y + t(X-Y)|^{p-2}(Y + t(X-Y)), \qquad t \in \boldsymbol{R}.$$

Then

$$
\begin{aligned}
|X|^{p-2}X-|Y|^{p-2}Y &= \int_0^1 F'(t)dt \\
&= (p-1)(X-Y)\int_0^1 |Y+t(X-Y)|^{p-2}dt .
\end{aligned}
$$

Using the trivial inequality

$$|Y+t(X-Y)| \leqq |X|+|Y|, \qquad 0\leqq t\leqq 1,$$

we complete the proof. Q. E. D.

Next we have

LEMMA 2. *Let $p\geqq 2$. Then for $X, Y\in \boldsymbol{R}$*

$$X\cdot[|X+Y|^{p-2}(X+Y)-|Y|^{p-2}Y] \geqq 2^{2-p}|X|^{p} .$$

PROOF. We set

$$F(t)= |1+t|^{p-2}(1+t)-|t|^{p-2}t, \qquad t\in \boldsymbol{R} .$$

It is enough to prove that the minimum of $F(t)$ is 2^{2-p}. Since

$$F'(t)=(p-1)(|1+t|^{p-2}-|t|^{p-2}),$$

we see that $F(t)$ takes its minimum at $t=-1/2$. It completes the proof. Q. E. D.

The following lemma is essentially that of Poincaré. We repeat the proof known for its completeness.

LEMMA 3. *Let $q\geqq 1$. Then it holds that for $u\in C^1(\bar{D}_a)$*

$$\int_{D_a}|u|^q dx \leqq 2^{q-1}\Big(a\int_{\Gamma_a}|u|^q dS+a^q\int_{D_a}|\partial_y u|^q dx\Big).$$

PROOF. Let $(x', y(x'))\in\Gamma_a$, where $y(x')$ may be multivalued. First we assume that $(x', y)\in D_a$ for any y with $y(x')\leqq y\leqq a$. Since

$$u(x', y)-u(x', y(x'))=\int_{y(x')}^{y}(\partial_y u)(x', t)dt,$$

we have

$$|u(x', y)| \leqq |u(x', y(x'))|+\int_{y(x')}^{y}|(\partial_y u)(x', t)|dt .$$

Hence

$$|u(x', y)|^q \leqq 2^{q-1}\Big[|u(x', y(x'))|^q+\Big|\int_{y(x')}^{a}(\partial_y u)(x', t)dt\Big|^q\Big],$$

and

$$\int_{y(x')}^{a}|u(x',y)|^q dy \leqq 2^{q-1}\Big[(a-y(x'))|u(x',y(x'))|^q + (a-y(x'))^q\int_{y(x')}^{a}|(\partial_y u)(x',y)|^q dy\Big].$$

We repeat the same calculus also for the general case and we note the trivial inequalities

$$dS=\Big[1+\sum_{i=1}^{N-1}(\partial_{x_i}y(x'))^2\Big]^{1/2}dx' \geqq dx',$$

$$\sum_i a_i^q b_i \leqq (\sum_i a_i)^q(\sum_i b_i), \qquad a_i,\ b_i \geqq 0.$$

Thus the required inequality is obtained. Q. E. D.

For any function u we define

$$(D_y^h u)(x',y)=\frac{1}{h}[u(x',y+h)-u(x',y)], \qquad h\in\boldsymbol{R}.$$

Here we assume that $h\neq 0$ and $|h|$ is sufficiently small.

LEMMA 4. *Let u be continuous in a neighborhood of $\bar{D}_a$. Then*

$$\int_{D_a}(D_y^h u)(x)dx \longrightarrow \int_{\partial D_a}u(x',y)\cos(\boldsymbol{n},y)dS \qquad \text{as } h\to 0.$$

PROOF. We may assume that Γ_a is expressed by a function $y=y(x')$ $(x'\in I)$ and $(x',y)\in\bar{D}_a$ for y with $y(x')\leqq y\leqq a$. The general case is parallel to that of the above.

We can write

$$\int_{y(x')}^{a}[u(x',y+h)-u(x',y)]dy=\int_a^{a+h}u(x',y)dy-\int_{y(x')}^{y(x')+h}u(x',y)dy.$$

It becomes

$$\int_{D_a}(D_y^h u)(x',y)dx=\frac{1}{h}\Big[\int_I\int_a^{a+h}u(x',y)dydx' - \int_I\int_{y(x')}^{y(x')+h}u(x',y)dydx'\Big].$$

Letting $h\to 0$, we complete the proof. Q. E. D.

3. Proof of Theorem 1.

The function u in Theorem 1 may be assumed to be of class $C^{1,\alpha}$ $(1/2<\alpha<1)$ in a neighborhood of $\bar{D}_a$.

Let $\lambda \leqq -1$ for the time being. We set $v(x', y) = \exp(\lambda y) \cdot u(x', y)$. It follows from (1.1) that

(3.1) $$(L_p(u), \exp((p-1)\lambda y) \cdot D_y^h v) = -\sum_{i=1}^{N-1} (|\partial_{x_i} v|^{p-2} \partial_{x_i} v, D_y^h \partial_{x_i} v)$$

$$-(\exp((1-p)\lambda y) \cdot |\partial_y v - \lambda v|^{p-2} (\partial_y v - \lambda v), \partial_y (\exp((p-1)\lambda y) \cdot D_y^h v) + I_1^{(h)},$$

where

$$I_1^{(h)} = \sum_{i=1}^{N-1} \int_{\partial D_a} |\partial_{x_i} v|^{p-2} \partial_{x_i} v \cdot D_y^h v \cos(\boldsymbol{n}, x_i) dS$$

$$+ \int_{\partial D_a} |\partial_y v - \lambda v|^{p-2} (\partial_y v - \lambda v) D_y^h v \cos(\boldsymbol{n}, y) dS.$$

If we write

$$I_1 = \sum_{i=1}^{N-1} \int_{\partial D_a} |\partial_{x_i} v|^{p-2} \partial_{x_i} v \cdot \partial_y v \cos(\boldsymbol{n}, x_i) dS$$

$$+ \int_{\partial D_a} |\partial_y v - \lambda v|^{p-2} (\partial_y v - \lambda v) \partial_y v \cos(\boldsymbol{n}, y) dS,$$

then $I_1^{(h)} \to I_1$ as $h \to 0$.

We see generally that for any C^1 function F

$$(D_y^h F(u))(x', y)$$

$$= (D_y^h u)(x', y) \cdot \int_0^1 F'(u(x', y) + t(u(x', y+h) - u(x', y))) dt.$$

Hence

$$(D_y^h |\partial_{x_i} v|^p)(x', y) = p(D_y^h \partial_{x_i} v)(x', y) \cdot$$

$$\int_0^1 [|(\partial_{x_i} v)(x', y) + t((\partial_{x_i} v)(x', y+h) - (\partial_{x_i} v)(x', y))|^{p-2} \cdot$$

$$((\partial_{x_i} v)(x', y) + t((\partial_{x_i} v)(x', y+h) - (\partial_{x_i} v)(x', y)))] dt.$$

From now on we denote simply by the same C any constant depending only on p. From Lemma 1

$$|(|\partial_{x_i} v|^{p-2} \partial_{x_i} v, D_y^h \partial_{x_i} v) - p^{-1}(1, D_y^h |\partial_{x_i} v|^p)|$$

$$\leqq C \int_{D_a} |(D_y^h \partial_{x_i} v)(x', y)| \cdot$$

$$|(\partial_{x_i} v)(x', y+h) - (\partial_{x_i} v)(x', y)| \cdot$$

$$(|(\partial_{x_i} v)(x', y+h)| + |(\partial_{x_i} v)(x', y)|)^{p-2} dx.$$

Thus if we set

$$-\sum_{i=1}^{N-1}(|\partial_{x_i}v|^{p-2}\partial_{x_i}v,\ D_y^h\partial_{x_i}v) = -p^{-1}\sum_{i=1}^{N-1}(1,\ D_y^h|\partial_{x_i}v|^p)+J_1^{(h)},$$

then

$$|J_1^{(h)}| \leqq C\sum_{i=1}^{N-1}\int_{D_a}|(D_y^h\partial_{x_i}v)(x',\ y)|\cdot|(\partial_{x_i}v)(x',\ y+h)-(\partial_{x_i}v)(x',\ y)|\cdot (|(\partial_{x_i}v)(x',\ y+h)|+|(\partial_{x_i}v)(x',\ y)|)^{p-2}dx\,. \tag{3.2}$$

Since v is of class $C^{1,\alpha}$, $1/2<\alpha<1$, we have

$$|(D_y^h\partial_{x_i}v)(x',\ y)|\cdot|(\partial_{x_i}v)(x',\ y+h)-(\partial_{x_i}v)(x',\ y)| \leqq L|h|^{2\alpha-1} \longrightarrow 0 \quad \text{as } h\to 0\,.$$

That is,

$$J_1^{(h)} \longrightarrow 0 \quad \text{as } h\to 0\,.$$

Next setting

$$I_2^{(h)} = -p^{-1}\sum_{i=1}^{N-1}(1,\ D_y^h|\partial_{x_i}v|^p),$$

we have from Lemma 4

$$I_2^{(h)} \longrightarrow I_2 \quad \text{as } h\to 0\,,$$

where

$$I_2 = -p^{-1}\sum_{i=1}^{N-1}\int_{\partial D_a}|\partial_{x_i}v|^p\cos(\boldsymbol{n},\ y)dS\,.$$

From (3.1) and the above it follows that

$$(L_p(u),\ \exp((p-1)\lambda y)\cdot D_y^h v) = -(|\partial_y v-\lambda v|^{p-2}(\partial_y v-\lambda v),\ D_y^h\partial_y v +(p-1)\lambda D_y^h v)+I_1^{(h)}+I_2^{(h)}+J_1^{(h)}\,. \tag{3.3}$$

Now we write

$$\begin{aligned}-(|\partial_y v-\lambda v|^{p-2}(\partial_y v-\lambda v),\ D_y^h\partial_y v+(p-1)\lambda D_y^h v)\\ = -(|\partial_y v-\lambda v|^{p-2}(\partial_y v-\lambda v),\ D_y^h(\partial_y v-\lambda v))\\ -p\lambda(|\partial_y v-\lambda v|^{p-2}(\partial_y v-\lambda v),\ D_y^h v)\end{aligned}$$

and

$$\begin{aligned}-(|\partial_y v-\lambda v|^{p-2}(\partial_y v-\lambda v),\ D_y^h(\partial_y v-\lambda v))\\ = -p^{-1}(1,\ D_y^h|\partial_y v-\lambda v|^p)+J_2^{(h)}\,.\end{aligned}$$

Similarly to (3.2) it holds that

$$|J_2^{(h)}| \leqq C\int_{D_a} |(D_y^h(\partial_y v-\lambda v))(x', y)| \cdot$$
$$|(\partial_y v-\lambda v)(x', y+h)-(\partial_y v-\lambda v)(x', y)| \cdot$$
$$(|(\partial_y v-\lambda v)(x', y+h)|+|(\partial_y v-\lambda v)(x', y)|)^{p-2} dx .$$

And

$$|(D_y^h(\partial_y v-\lambda v))(x', y)| \cdot |(\partial_y v-\lambda v)(x', y+h)-(\partial_y v-\lambda v)(x', y)| \leqq L|h|^{2\alpha-1}.$$

Thus

$$J_2^{(h)} \longrightarrow 0 \quad \text{as } h \to 0 .$$

Setting

$$I_3^{(h)} = -p^{-1}(1, D_y^h|\partial_y v-\lambda v|^p),$$

we have as previously

$$I_3^{(h)} \longrightarrow I_3 \equiv -p^{-1}\int_{\partial D_a} |\partial_y v-\lambda v|^p \cos(\boldsymbol{n}, y) dS \quad \text{as } h \to 0 .$$

Further we write

$$\begin{aligned}
&(|\partial_y v-\lambda v|^{p-2}(\partial_y v-\lambda v), D_y^h v) \\
&\quad = (|\partial_y v-\lambda v|^{p-2}(\partial_y v-\lambda v), \partial_y v) \\
&\qquad +(|\partial_y v-\lambda v|^{p-2}(\partial_y v-\lambda v), D_y^h v-\partial_y v) \\
&\quad = (|\partial_y v-\lambda v|^{p-2}(\partial_y v-\lambda v), \partial_y v)-p^{-1}\lambda^{-1}J_3^{(h)} .
\end{aligned}$$

Then

$$J_3^{(h)} \longrightarrow 0 \quad \text{as } h \to 0 .$$

Lastly we can write

$$\begin{aligned}
-\lambda(|-\lambda v|^{p-2}(-\lambda v), \partial_y v) &= p^{-1}|\lambda|^p\int_{\partial D_a} |v|^p \cos(\boldsymbol{n}, y) dS \\
&\geqq p^{-1}|\lambda|^p\int_{\Gamma_a} |v|^p \cos(\boldsymbol{n}, y) dS .
\end{aligned}$$

It follows from Lemma 2 and the above inequality

$$\begin{aligned}
&-p\lambda(|\partial_y v-\lambda v|^{p-2}(\partial_y v-\lambda v), \partial_y v) \\
&\quad = -p\lambda(|\partial_y v-\lambda v|^{p-2}(\partial_y v-\lambda v)-|-\lambda v|^{p-2}(-\lambda v), \partial_y v) \\
&\qquad -p\lambda(|-\lambda v|^{p-2}(-\lambda v), \partial_y v) \\
&\quad \geqq p2^{2-p}|\lambda|(1, |\partial_y v|^p)+|\lambda|^p\int_{\Gamma_a} |v|^p \cos(\boldsymbol{n}, y) dS .
\end{aligned}$$

From (3.3) and the above we conclude that

$$(L_p(u),\ \exp((p-1)\lambda y)\cdot D_y^h v)$$
$$\geqq p2^{2-p}|\lambda|(1,\ |\partial_y v|^p)+|\lambda|^p\int_{\Gamma_a}|v|^p\cos(\boldsymbol{n},\ y)dS+\sum_{i=1}^3 I_i^{(h)}+\sum_{i=1}^3 J_i^{(h)}.$$

On the other hand it is seen from our assumption that

$$(L_p(u),\ \exp((p-1)\lambda y)\cdot D_y^h v)\leqq K\int_{D_a}|u|^{p-1}\exp((p-1)\lambda y)\cdot|D_y^h v|dx.$$

Combining these inequalities and letting $h\to 0$, we obtain

$$p2^{2-p}|\lambda|\int_{D_a}|\partial_y v|^p dx$$
$$\leqq K\int_{D_a}|v|^{p-1}|\partial_y v|dx+|\lambda|^p\int_{\Gamma_a}|v|^p dS+\sum_{i=1}^3|I_i|.$$

The first integral on the right-hand side is replaced by

$$p^{-1}(p-1)\int_{D_a}|v|^p dx+p^{-1}\int_{D_a}|\partial_y v|^p dx.$$

Hence

$$(1-p^{-2}2^{p-2}K|\lambda|^{-1})\int_{D_a}|\partial_y v|^p dx \tag{3.4}$$
$$\leqq p^{-2}(p-1)2^{p-2}K|\lambda|^{-1}\int_{D_a}|v|^p dx$$
$$+p^{-1}2^{p-2}|\lambda|^{-1}(|\lambda|^p\int_{\Gamma_a}|v|^p dS+\sum_{i=1}^3|I_i|).$$

Here we use the following inequalities:

$$\int_{D_a}|v|^p dx\leqq 2^{p-1}\Big(a\int_{\Gamma_a}|v|^p dS+a^p\int_{D_a}|\partial_y v|^p dS\Big)\quad \text{(by Lemma 3)}$$

and

$$\sum_{i=1}^3|I_i|\leqq C|\lambda|\int_{\partial D_a}\exp(p\lambda y)\cdot(|u|^p+|\nabla u|^p)dS. \tag{3.5}$$

Then (3.4) becomes

$$(1-p^{-2}2^{p-2}K|\lambda|^{-1}-p^{-2}(p-1)2^{2p-3}a^pK|\lambda|^{-1})\int_{D_a}|\partial_y v|^p dx$$
$$\leqq C\Big[K\int_{\Gamma_a}\exp(p\lambda y)\cdot|u|^p dS+|\lambda|^p\int_{\Gamma_a}\exp(p\lambda y)\cdot|u|^p dS$$
$$+\int_{\partial D_a}\exp(p\lambda y)\cdot(|u|^p+|\nabla u|^p)dS\Big].$$

We note that $p^{-2}2^{p-2}+p^{-2}(p-1)2^{2p-3}a^p \leqq p^{-1}2^{p-1}$. Therefore setting $\lambda_0=p^{-1}2^pK$, we obtain for $\lambda \leqq -\max(1, \lambda_0)$

$$\int_{D_a} |\partial_y v|^p dx \leqq C\Big[(1+K)|\lambda|^p \int_{\Gamma_a} (|u|^p+|\nabla u|^p)dS$$

$$+\exp(pa\lambda)\cdot \int_{\bar{D}\cap\{y=a\}} (|u|^p+|\nabla u|^p)dS\Big].$$

Using again Lemma 3, we see that

$$\int_{D_a} |v|^p dx \leqq C\Big[(1+K)|\lambda|^p \int_{\Gamma_a} (|u|^p+|\nabla u|^p)dS \tag{3.6}$$

$$+a^p \exp(pa\lambda)\cdot \int_{\bar{D}\cap\{y=a\}} (|u|^p+|\nabla u|^p)dS\Big].$$

Since $\exp(p\lambda y)\cdot|\partial_y u|^p \leqq C(|\partial_y v|^p+|\lambda|^p|v|^p)$, we conclude that

$$\int_{D_a} \exp(p\lambda y)\cdot|\partial_y u|^p dx \leqq C\Big[(1+K)|\lambda|^{2p}\cdot \int_{\Gamma_a} (|u|^p+|\nabla u|^p)dS \tag{3.7}$$

$$+(1+a^p|\lambda|^p)\exp(pa\lambda)\cdot \int_{\bar{D}\cap\{y=a\}} (|u|^p+|\nabla u|^p)dS\Big].$$

Next we estimate the integrals $\int_{D_a} \exp(p\lambda y)|\partial_{x_i} u|^p dx$, $i=1, \cdots, N-1$. By the definition (1.1) we have

$$\sum_{i=1}^{N-1}(\exp(p\lambda y),\ |\partial_{x_i}u|^p) = -(\exp(p\lambda y),\ |\partial_y u|^p)$$

$$-p\lambda(\exp(p\lambda y),\ |\partial_y u|^{p-2}\partial_y u\cdot u)-(L_p(u),\ \exp(p\lambda y)\cdot u)$$

$$+\sum_{i=1}^{N-1}\int_{\partial D_a} \exp(p\lambda y)\cdot|\partial_{x_i}u|^{p-2}\partial_{x_i}u\cdot u\cos(\boldsymbol{n},\ x_i)dS$$

$$+\int_{\partial D_a} \exp(p\lambda y)\cdot|\partial_y u|^{p-2}\partial_y u\cdot u\cos(\boldsymbol{n},\ y)dS.$$

Using the inequality

$$|\lambda(\exp(p\lambda y),\ |\partial_y u|^{p-2}\partial_y u\cdot u)|$$

$$\leqq p^{-1}(p-1)\int_{D_a} \exp(p\lambda y)\cdot|\partial_y u|^p dx$$

$$+p^{-1}|\lambda|^p\int_{D_a} \exp(p\lambda y)\cdot|u|^p dx\,,$$

we obtain

$$\sum_{i=1}^{N-1}\int_{D_a}\exp(p\lambda y)\cdot|\partial_{x_i}u|^p dx$$

$$\leqq (p-1)\int_{D_a}\exp(p\lambda y)\cdot|\partial_y u|^p dx+|\lambda|^p\int_{D_a}\exp(p\lambda y)\cdot|u|^p dx$$

$$+K\int_{D_a}\exp(p\lambda y)\cdot|u|^p dx$$

$$+\sum_{i=1}^{N-1}\int_{\partial D_a}\exp(p\lambda y)\cdot|\partial_{x_i}u|^{p-1}|u|\,dS+\int_{\partial D_a}\exp(p\lambda y)\cdot|\partial_y u|^{p-1}|u|\,dS.$$

Hence

$$\sum_{i=1}^{N-1}\int_{D_a}\exp(p\lambda y)\cdot|\partial_{x_i}u|^p dx \leqq C\Big[\int_{D_a}\exp(p\lambda y)\cdot|\partial_y u|^p dx$$

$$+(K+|\lambda|^p)\int_{D_a}\exp(p\lambda y)\cdot|u|^p dx$$

$$+\int_{\partial D_a}\exp(p\lambda y)\cdot(|u|^p+|\nabla u|^p)dS\Big].$$

Combining this with (3.6) and (3.7), we finally conclude that

$$\text{(3.8)}\qquad \int_{D_a}\exp(p\lambda y)\cdot(|u|^p+|\nabla u|^p)dx$$

$$\leqq C\Big[(1+K)^2|\lambda|^{2p}\int_{\Gamma_a}(|u|^p+|\nabla u|^p)dS$$

$$+(1+K)(1+a^p|\lambda|^p)\exp(pa\lambda)\cdot\int_{\bar{D}\cap\{y=a\}}(|u|^p+|\nabla u|^p)dS\Big].$$

From (3.8) it follows immediately that

$$\int_{D_{a/2}}(|u|^p+|\nabla u|^p)dx \leqq C\exp(pa|\lambda|/2)\cdot$$

$$[(1+K)^2\varepsilon|\lambda|^{2p}+(1+K)(1+a^p|\lambda|^p)M\exp(pa\lambda)]\,.$$

Here we use the trivial inequalities:

$$a^{2p}|\lambda|^{2p}, \qquad 1+a^p|\lambda|^p \leqq C\exp(pa|\lambda|/4)\,.$$

Then the above inequality becomes

$$\text{(3.9)}\qquad \int_{D_{a/2}}(|u|^p+|\nabla u|^p)dx \leqq C(1+K)[(1+K)a^{-2p}\varepsilon\cdot$$

$$\exp(3pa|\lambda|/4)+M\exp(pa\lambda/4)]\,.$$

Writing $\varepsilon'=\varepsilon a^{-2p}$ and setting $\lambda=-2p^{-1}\log(M/\varepsilon')$, we have

$$3pa|\lambda|/4=\frac{3}{2}a\log(M/\varepsilon'), \qquad pa\lambda/4=-\frac{a}{2}\log(M/\varepsilon')$$

and

$$\varepsilon'^{(2-3a)/2}M^{3a/2}\leqq \varepsilon'^{a/2}M^{(2-a)/2}.$$

The condition $\lambda\leqq-\max(1,\lambda_0)$ is equivalent to $\varepsilon'\exp(2^{p-1}K)$, $\varepsilon'\exp(p/2)\leqq M$. Accordingly it follows from (3.9) that if $\varepsilon\exp(2^{p-1}K)$, $\varepsilon\exp(p/2)\leqq a^{2p}M$,

$$\int_{D_{a/2}}(|u|^p+|\nabla u|^p)dx\leqq C(1+K)^2a^{-pa}\varepsilon^{a/2}M^{(2-a)/2}.$$

Since a^{-a} is bounded, we finally obtain the required estimate. Q. E. D.

4. Proof of Theorem 2.

We may assume that u is of class C^2 in a neighborhood of $\bar{D}_a$. We define the function v for $\lambda\leqq-1$ as in the previous section. We proceed along the different line from the proof of Theorem 1. We will not use Lemma 2. The proof below is entirely independent from that of [2]. The situation is delicate, if p is close to 1. So we proceed carefully.

We set for $\delta>0$

$$L_p^\delta(u)\equiv\sum_{i=1}^N\partial_{x_i}(((\partial_{x_i}u)^2+\delta)^{(p-2)/2}\partial_{x_i}u).$$

In place of (3.1) it holds that

$$\begin{aligned}(L_p^\delta(u),\ &\exp((p-1)\lambda y)\cdot\partial_yD_y^hv)\\ =&-\sum_{i=1}^{N-1}(((\partial_{x_i}u)^2+\delta)^{(p-2)/2}\partial_{x_i}u,\ \exp((p-1)\lambda y)\cdot\partial_{x_i}\partial_yD_y^hv)\\ &-(((\partial_yu)^2+\delta)^{(p-2)/2}\partial_yu,\ \partial_y(\exp((p-1)\lambda y)\cdot\partial_yD_y^hv))+I_4(h,\delta)\\ =&-\sum_{i=1}^{N-1}(((\partial_{x_i}v)^2+\delta\exp(2\lambda y))^{(p-2)/2}\partial_{x_i}v,\ \partial_{x_i}\partial_yD_y^hv)\\ &-(((\partial_yv-\lambda v)^2+\delta\exp(2\lambda y))^{(p-2)/2}(\partial_yv-\lambda v),\\ &\qquad\partial_y^2D_y^hv+(p-1)\lambda\partial_yD_y^hv)+I_4(h,\delta),\end{aligned}$$

where

$$\begin{aligned}I_4(h,\delta)=\sum_{i=1}^{N-1}\int_{\partial D_a}&((\partial_{x_i}v)^2+\delta\exp(2\lambda y))^{(p-2)/2}\partial_{x_i}v\cdot\\ &\partial_yD_y^hv\cos(\boldsymbol{n},x_i)dS\\ &+\int_{\partial D_a}((\partial_yv-\lambda v)^2+\delta\exp(2\lambda y))^{(p-2)/2}\cdot\\ &(\partial_yv-\lambda v)\partial_yD_y^hv\cos(\boldsymbol{n},y)dS.\end{aligned}$$

The limit of $I_4(h, \delta)$ as $h \to 0$ is written by $I_4(\delta)$.

From now on we set

$$f_i = ((\partial_{x_i} v)^2 + \delta \exp(2\lambda y))^{1/2}, \qquad 1 \leqq i \leqq N-1,$$

$$f_N = ((\partial_y v - \lambda v)^2 + \delta \exp(2\lambda y))^{1/2}.$$

By integration by parts

$$\begin{aligned}
&-(f_i^{p-2}\partial_{x_i} v,\ \partial_{x_i}\partial_y D_y^h v) = (f_i^{p-2}\partial_y \partial_{x_i} v,\ \partial_{x_i} D_y^h v) \\
&+(p-2)(f_i^{p-4}(\partial_{x_i} v \cdot \partial_y \partial_{x_i} v + \lambda\delta \exp(2\lambda y))\partial_{x_i} v,\ \partial_{x_i} D_y^h v) \\
&-\int_{\partial D_a} f_i^{p-2}\partial_{x_i} v \cdot \partial_{x_i} D_y^h v \cos(\boldsymbol{n}, y) dS.
\end{aligned}$$

And

$$\begin{aligned}
&-(f_N^{p-2}(\partial_y v - \lambda v),\ \partial_y^2 D_y^h v + (p-1)\lambda \partial_y D_y^h v) \\
&= -(f_N^{p-2}(\partial_y v - \lambda v),\ \partial_y D_y^h(\partial_y v - \lambda v)) \\
&\quad - p\lambda(f_N^{p-2}(\partial_y v - \lambda v),\ \partial_y D_y^h v) \\
&= (f_N^{p-2}\partial_y(\partial_y v - \lambda v),\ D_y^h(\partial_y v - \lambda v)) \\
&\quad +(p-2)(f_N^{p-4}((\partial_y v - \lambda v)\partial_y(\partial_y v - \lambda v) + \lambda\delta \exp(2\lambda y)) \cdot \\
&\qquad\qquad (\partial_y v - \lambda v),\ D_y^h(\partial_y v - \lambda v)) \\
&\quad - p\lambda(f_N^{p-2}(\partial_y v - \lambda v),\ \partial_y D_y^h v) \\
&\quad -\int_{\partial D_a} f_N^{p-2}(\partial_y v - \lambda v) D_y^h(\partial_y v - \lambda v) \cos(\boldsymbol{n}, y) dS.
\end{aligned}$$

Hence it follows that

$$\begin{aligned}
(4.1) \qquad &(L_p^\delta(u),\ \exp((p-1)\lambda y) \cdot \partial_y D_y^h v) \\
&= \sum_{i=1}^{N-1} (f_i^{p-4}((p-1)(\partial_{x_i} v)^2 + \delta \exp(2\lambda y))\partial_y \partial_{x_i} v,\ D_y^h \partial_{x_i} v) \\
&\quad +(p-2)\lambda\delta \sum_{i=1}^{N-1} (f_i^{p-4} \exp(2\lambda y) \cdot \partial_{x_i} v,\ D_y^h \partial_{x_i} v) \\
&\quad +(f_N^{p-4}((p-1)(\partial_y v - \lambda v)^2 + \delta \exp(2\lambda y)) \cdot \\
&\qquad\qquad \partial_y(\partial_y v - \lambda v),\ D_y^h(\partial_y v - \lambda v)) \\
&\quad +(p-2)\lambda\delta(f_N^{p-4} \exp(2\lambda y) \cdot (\partial_y v - \lambda v),\ D_y^h(\partial_y v - \lambda v)) \\
&\quad - p\lambda(f_N^{p-2}(\partial_y v - \lambda v),\ D_y^h \partial_y v) + \sum_{i=4}^{6} I_i(h, \delta),
\end{aligned}$$

where $I_5(h, \delta)$ and $I_6(h, \delta)$ are two surface integrals such that if $h \to 0$

$$I_5(h, \delta) \longrightarrow I_5(\delta) \equiv -\sum_{i=1}^{N-1}\int_{\partial D_a} f_i^{p-2}\partial_{x_i}v\cdot\partial_{x_i}\partial_y v \cos(\boldsymbol{n}, y)dS,$$

$$I_6(h, \delta) \longrightarrow I_6(\delta) \equiv -\int_{\partial D_a} f_N^{p-2}(\partial_y v-\lambda v)\cdot\partial_y(\partial_y v-\lambda v)\cdot\cos(\boldsymbol{n}, y)dS.$$

From (4.1) we see that

$$\begin{aligned}(4.2)\qquad &(L_p^\delta(u), \exp((p-1)\lambda y)\cdot\partial_y^2 v)\\ &=\sum_{i=1}^{N-1}(f_i^{p-4}((p-1)(\partial_{x_i}v)^2+\delta\exp(2\lambda y))\cdot\partial_y\partial_{x_i}v,\ \partial_y\partial_{x_i}v)\\ &+(p-2)\lambda\delta\sum_{i=1}^{N-1}(f_i^{p-4}\exp(2\lambda y)\cdot\partial_{x_i}v,\ \partial_y\partial_{x_i}v)\\ &+(f_N^{p-4}((p-1)(\partial_y v-\lambda v)^2+\delta\exp(2\lambda y))\cdot\\ &\qquad\qquad \partial_y(\partial_y v-\lambda v),\ \partial_y(\partial_y v-\lambda v))\\ &+(p-2)\lambda\delta(f_N^{p-4}\exp(2\lambda y)\cdot(\partial_y v-\lambda v),\ \partial_y(\partial_y v-\lambda v))\\ &-p\lambda(f_N^{p-2}(\partial_y v-\lambda v),\ \partial_y^2 v)+\sum_{i=4}^{6} I_i(\delta)\,.\end{aligned}$$

If $p\geqq 2$, we can take naturally $\delta\to+0$ in (4.2). Thus we may assume that $1<p<2$. By Cauchy's inequality

$$\begin{aligned}|f_i^{p-4}\delta\exp(2\lambda y)\cdot\partial_{x_i}v| &\leqq \frac{1}{2}\delta^{1/2}\exp(\lambda y)f_i^{p-2}\\ &\leqq C\delta^{(p-1)/2}\longrightarrow 0 \qquad \text{as } \delta\to+0\,.\end{aligned}$$

Similarly

$$|f_N^{p-4}\delta\exp(2\lambda y)\cdot(\partial_y v-\lambda v)| \longrightarrow 0 \qquad \text{uniformly as } \delta\to+0\,.$$

Further it is seen that

$$(4.3)\qquad (L_p^\delta(u), \exp((p-1)\lambda y)\cdot\partial_y^2 v)\longrightarrow 0 \qquad \text{as } \delta\to+0\,.$$

In fact, if u is in C^3, this is clear by integration by parts. If u is in C^2 and $|\partial_{x_i}u|^{p-2}\partial_{x_i}^2 u\in L^1(D_a)$, $i=1, \cdots, N$, (4.3) is correct by the following inequalities and Lebesgue's convergence theorem:

$$|\partial_{x_i}((\partial_{x_i}u)^2+\delta)^{(p-2)/2}\partial_{x_i}u| \leqq C|\partial_{x_i}u|^{p-2}|\partial_{y_i}^2 u|, \qquad i=1, \cdots, N.$$

Therefore, taking $\delta\to+0$ in (4.2) and using Fatou's lemma, we obtain from (4.3)

$$0 \geqq (p-1)\sum_{i=1}^{N-1}(|\partial_{x_i}v|^{p-2}\partial_y\partial_{x_i}v,\ \partial_y\partial_{x_i}v) \tag{4.4}$$
$$+(p-1)(|\partial_y v-\lambda v|^{p-2}\partial_y(\partial_y v-\lambda v),\ \partial_y(\partial_y v-\lambda v))$$
$$-p\lambda(|\partial_y v-\lambda v|^{p-2}(\partial_y v-\lambda v),\ \partial_y^2 v)+\sum_{i=1}^{6} I_i\,,$$

where $I_i=\lim_{\delta\to+0} I_i(\delta)$, $i=4, 5, 6$. And we remember the final part of the first section.

We write

$$-p\lambda(|\partial_y v-\lambda v|^{p-2}(\partial_y v-\lambda v),\ \partial_y^2 v)$$
$$=-p\lambda(|\partial_y v-\lambda v|^{p-2}(\partial_y v-\lambda v),\ \partial_y(\partial_y v-\lambda v))$$
$$-p\lambda^2(|\partial_y v-\lambda v|^{p-2}(\partial_y v-\lambda v),\ \partial_y v)$$

and

$$-p\lambda(|\partial_y v-\lambda v|^{p-2}(\partial_y v-\lambda v),\ \partial_y(\partial_y v-\lambda v))$$
$$=-\lambda\int_{\partial D_a}|\partial_y v-\lambda v|^p\cos(\boldsymbol{n},\ y)dS\equiv I_7\,,\qquad\text{say.}$$

Here $I_7=p\lambda I_3$. Then (4.4) becomes

$$0 \geqq (p-1)\Big[\sum_{i=1}^{N-1}(|\partial_{x_i}v|^{p-2}\partial_y\partial_{x_i}v,\ \partial_y\partial_{x_i}v) \tag{4.5}$$
$$+(|\partial_y v-\lambda v|^{p-2}\partial_y(\partial_y v-\lambda v),\ \partial_y(\partial_y v-\lambda v))\Big]$$
$$-p\lambda^2(|\partial_y v-\lambda v|^{p-2}(\partial_y v-\lambda v),\ \partial_y v)+\sum_{i=4}^{7} I_i\,.$$

On the other hand it follows from the proof of Theorem 1 that

$$(L_p(u),\ \exp((p-1)\lambda y)\cdot\partial_y v)$$
$$=-p\lambda(|\partial_y v-\lambda v|^{p-2}(\partial_y v-\lambda v),\ \partial_y v)+\sum_{i=1}^{3} I_i\,.$$

Combining this with (4.5), we obtain

$$(p-1)\Big[\sum_{i=1}^{N-1}(|\partial_{x_i}v|^{p-2}\partial_y\partial_{x_i}v,\ \partial_y\partial_{x_i}v) \tag{4.6}$$
$$+(|\partial_y v-\lambda v|^{p-2}\partial_y(\partial_y v-\lambda v),\ \partial_y(\partial_y v-\lambda v))\Big]$$
$$\leqq \lambda\sum_{i=1}^{3} I_i-\sum_{i=1}^{7} I_i\,.$$

Note that

$$(\partial_y f_i^{p/2})^2 \leqq (p^2/2) f_i^{p-4}((\partial_{x_i} v)^2 (\partial_y \partial_{x_i} v)^2 + \lambda^2 \delta^2 \exp(4\lambda y)),$$

$$(\partial_y f_N^{p/2})^2 \leqq (p^2/2) f_N^{p-4}((\partial_y v - \lambda v)^2 \cdot (\partial_y (\partial_y v - \lambda v)^2 + \lambda^2 \delta^2 \exp(4\lambda y))).$$

Using Lemma 3 and taking $\delta \to +0$, we have

$$\begin{aligned}
&\sum_{i=1}^{N-1} \int_{D_a} |\partial_{x_i} v|^p dx + \int_{D_a} |\partial_y v - \lambda v|^p dx \\
&\quad \leqq C\Big[a \sum_{i=1}^{N-1} \int_{\Gamma_a} |\partial_{x_i} v|^p dS + a \int_{\Gamma_a} |\partial_y v - \lambda v|^p dS \\
&\qquad + a^2 \sum_{i=1}^{N-1} \int_{D_a} |\partial_{x_i} v|^{p-2} (\partial_y \partial_{x_i} v)^2 dx \\
&\qquad + a^2 \int_{D_a} |\partial_y v - \lambda v|^{p-2} (\partial_y (\partial_y v - \lambda v))^2 dx \Big]
\end{aligned}$$

(from (4.6))

$$\begin{aligned}
&\quad \leqq Ca\Big[\sum_{i=1}^{N-1} \int_{\Gamma_a} |\partial_{x_i} v|^p dS + \int_{\Gamma_a} |\partial_y v - \lambda v|^p dS \\
&\qquad + a|\lambda| \sum_{i=1}^{3} |I_i| + a \sum_{i=4}^{7} |I_i| \Big].
\end{aligned}$$

Obviously

$$\sum_{i=4}^{7} |I_i| \leqq C\lambda^2 \int_{\partial D_a} \exp(p\lambda y) \cdot (|u|^p + |\nabla u|^p + |\nabla^2 u|^p) dS.$$

Therefore from the above and (3.5) it follows that

$$\begin{aligned}
(4.7) \qquad &\int_{D_a} \exp(p\lambda y) \cdot |\nabla u|^p dx \leqq C\Big[\int_{\Gamma_a} \exp(p\lambda y) \cdot |\nabla u|^p dS \\
&\qquad + a^2 \lambda^2 \int_{\partial D_a} \exp(p\lambda y) \cdot (|u|^p + |\nabla u|^p + |\nabla^2 u|^p) dS\Big].
\end{aligned}$$

Using the inequalities

$$\exp(2\lambda y) \cdot (\partial_y \partial_{x_i} u)^2 \leqq 2((\partial_y \partial_{x_i} v)^2 + \lambda^2 (\partial_{x_i} v)^2),$$

$$\exp(2\lambda y) \cdot (\partial_y^2 u)^2 \leqq 2((\partial_y (\partial_y v - \lambda v))^2 + \lambda^2 (\partial_y v - \lambda v)^2),$$

we have from (4.6)

$$\begin{aligned}
&\sum_{i=1}^{N-1} \int_{D_a} \exp(p\lambda y) \cdot |\partial_{x_i} u|^{p-2} (\partial_y \partial_{x_i} u)^2 dx \\
&\qquad + \int_{D_a} \exp(p\lambda y) \cdot |\partial_y u|^{p-2} (\partial_y^2 u)^2 dx
\end{aligned}$$

$$\leqq 2(p-1)^{-1}\Big(\lambda\sum_{i=1}^{3} I_i - \sum_{i=4}^{7} I_i\Big)$$

$$+2\lambda^2\Big(\sum_{i=1}^{N-1}\int_{D_a}|\partial_{x_i}v|^p dx+\int_{D_a}|\partial_y v-\lambda v|^p dx\Big).$$

Combining this with (4.7) we conclude that

$$\int_{D_a}\exp(p\lambda y)\cdot|\nabla u|^p dx$$

$$+a^2\Big[\sum_{i=1}^{N-1}\int_{D_a}\exp(p\lambda y)\cdot|\partial_{x_i}u|^{p-2}(\partial_y\partial_{x_i}u)^2 dx$$

$$+\int_{D_a}\exp(p\lambda y)\cdot|\partial_y u|^{p-2}(\partial_y^2 u)^2 dx\Big]$$

$$\leqq C(1+a^2\lambda^2)^2\int_{\partial D_a}\exp(p\lambda y)\cdot(|u|^p+|\nabla u|^p+|\nabla^2 u|^p)dS$$

$$\leqq C(1+a^2\lambda^2)^2(\varepsilon+M\exp(pa\lambda)).$$

Here we note that $(1+a^2\lambda^2)\leqq C\exp(pa|\lambda|/4)$. Therefore

$$\int_{D_{a/2}}|\nabla u|^p dx+a^2\Big[\sum_{i=1}^{N-1}\int_{D_{a/2}}|\partial_{x_i}u|^{p-2}(\partial_y\partial_{x_i}u)^2 dx$$

$$+\int_{D_{a/2}}|\partial_y u|^{p-2}(\partial_y^2 u)^2 dx\Big]$$

$$\leqq C\exp(3pa|\lambda|/4)(\varepsilon+M\exp(pa\lambda)).$$

Setting $\lambda=-2p^{-1}\log(M/\varepsilon)$, we complete the essential part of the proof of Theorem 2. The remaining part is immediately obtained by the above inequality and Lemma 3. Q. E. D.

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