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An inverse problem in bifurcation theory, II

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1. The main result.

We consider the nonlinear boundary value problem

$$(1.1) \qquad \begin{cases} \dfrac{d^2u}{dx^2}+\lambda u = uf(u), & 0<x<\pi, \\ u(0)=u(\pi)=0, \end{cases}$$

where λ is a real parameter and f is a Lipschitz continuous, real function defined on $\boldsymbol{R}$. Without loss of generality, we assume that $f(0)=0$. By a solution of (1.1) we mean a pair $(\lambda, u)\in \boldsymbol{R}\times C^2[0, \pi]$ satisfying (1.1). Let $\Gamma_n(f)$, $n=1, 2, \cdots$, denote the set of $(\lambda, h)\in \boldsymbol{R}^2$ for which there exists a solution (λ, u) of (1.1) satisfying the following conditions:

(i) $u(x)$ has exactly $n-1$ zeros in $(0, \pi)$;

(ii) The first stationary value of $u(x)$ is equal to h.

The set $\Gamma_n(f)$ is considered to be a representation in $\boldsymbol{R}^2$ of a set of nontrivial solutions of (1.1) bifurcating from the trivial solution $(n^2, 0)$ (note that n^2 is the n-th eigenvalue of the linearized problem of (1.1)).

In the previous paper [2] the author established a result that a nonlinear term f is determined uniquely from its solution set $\Gamma_1(f)$ and, in particular, that

$$\Gamma_1(f)=\{(1, h)\in \boldsymbol{R}^2 : h\neq 0\}$$

implies $f\equiv 0$. The purpose of the present paper is to show that a nonlinear term f is not determined uniquely by the condition

$$(1.2) \qquad \Gamma_2(f)=\{(4, h)\in \boldsymbol{R}^2 : h\neq 0\}$$

and to find nonlinear terms f satisfying the condition (1.2).

To state our result precisely we need some terminology. Let $0\leqq\alpha\leqq 1/2$ and let X_+ be the function space

$$(1.3) \qquad X_+ := \{g(h)\in C^1[0, \infty) : g(0)=0;\ |g|_0+|g'(0)|+|g'|_\alpha =: \|g\|_{X_+}<\infty\},$$

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where

$$|g|_0 := \sup_{0 \le h < \infty} |g(h)|,$$

and

$$|g'|_\alpha := \sup_{0 \le h,\, k < \infty,\, h \ne k} \frac{|(1+k^{\alpha+1})g'(k)-(1+h^{\alpha+1})g'(h)|}{|k-h|^\alpha}.$$

Also we let

$$X_- := \{g(h) \in C^1(-\infty, 0] : g(-h) \in X_+\} \tag{1.4}$$

and let $\|g\|_{X_-} := \|g(-h)\|_{X_+}$. It is routine to verify that $X_\pm$ become Banach spaces when furnished with the norms $\|g\|_{X_\pm}$, respectively. Moreover we define the sets $U_\pm$ by

$$U_\pm := \{f_\pm \in X_\pm : \|f_\pm\|_{X_\pm} < \varepsilon\}. \tag{1.5}$$

With this notation, the main result of this paper can be stated as follows:

THEOREM 1.1. *Let $0<\alpha<1/2$, let ε be sufficiently small and let $U_\pm$ be the sets defined by* (1.5). *Then, for any $f_- \in U_-$, there exists a unique function $f_+ \in U_+$ such that the function*

$$f(h) := \begin{cases} f_+(h), & \text{if } h \geqq 0, \\ f_-(h), & \text{if } h \leqq 0, \end{cases} \tag{1.6}$$

satisfies the condition (1.2).

Theorem 1.1 implies that a nonlinear term f of (1.1) is determined uniquely if, not only the condition (1.2), but also the section of f on the half interval $(-\infty, 0]$ (or $[0, \infty)$) is prescribed. However, in the case f is an even function, the following uniqueness result can be proved:

THEOREM 1.2. *Suppose f is a Lipschitz continuous, real, even function defined on $\boldsymbol{R}$. If f satisfies the condition* (1.2), *then $f \equiv 0$.*

For the nonlinear boundary value problem

$$\begin{cases} \dfrac{d^2u}{dx^2} + \lambda g(u) = 0, & 0 < x < \pi, \\ u(0) = u(\pi) = 0, \end{cases}$$

where $g(u)$ is a continuous function on $\boldsymbol{R}$ satisfying the condition $ug(u)>0$ for all $u \ne 0$, a solution set $\Gamma_2(g)$ can be defined by the same way as for (1.1) and the problem of finding nonlinear terms g satisfying the condition $\Gamma_2(g) = \{(4, h) \in \boldsymbol{R}^2 : h \ne 0\}$ can be posed. But, as is easily seen by the substitution $\lambda^{1/2}x = t$, this problem is reduced to the problem of finding g such that every solution of $\ddot{u} + g(u) = 0$ ($\dot{} = d/dt$) have the same period 2π, which has been studied by Urabe [8, **9**, **10**], Levin and Schatz [**3**] and Obi [**5**, **6**].

The present paper is organized as follows: In Section 3 we derive a necessary and sufficient condition for f to satisfy the condition (1.2), which is described as a nonlinear integral equation of f_+. In Section 4 we solve the integral equation. To this end we need some estimates, which are picked out in Section 2. Theorem 1.1 is proved at the end of Section 4. In Section 5 we treat the case f is an even function. Theorem 1.2 is an immediate consequence of Theorem 5.4.

2. Function spaces.

In this section we shall present some estimates of functions in the function spaces $X_\pm$ defined in (1.3), (1.4) and a function space Y_+, which will be defined in (2.5). The following is elementary.

LEMMA 2.1. *Let g be in X_+. Then*:

(i) *For any $h \in [0, \infty)$,*

$$|(1+h)g'(h)| \leqq 3\|g\|_{X_+}. \tag{2.1}$$

In particular, for any $h \in [0, \infty)$,

$$|hg'(h)| \leqq 3\|g\|_{X_+}. \tag{2.2}$$

(ii) *For any $h, k \in [0, \infty)$,*

$$|g'(k)-g'(h)| \leqq M_1|k-h|^\alpha \|g\|_{X_+}, \tag{2.3}$$

where M_1 is independent of h, k, g.

(iii) *For any $h, k \in [0, \infty)$,*

$$|k^{\alpha+1}g'(k)-h^{\alpha+1}g'(h)| \leqq M_1'|k-h|^\alpha \|g\|_{X_+}, \tag{2.4}$$

where M_1' is independent of h, k, g.

PROOF. From (1.3) we have, for any $h \in [0, \infty)$,

$$\begin{aligned}|(1+h^{\alpha+1})g'(h)| &\leqq |(1+h^{\alpha+1})g'(h)-g'(0)|+|g'(0)| \\ &\leqq (1+h^\alpha)\|g\|_{X_+}.\end{aligned}$$

This proves (i). It follows from (1.3) and (2.1) that, for $0 \leqq h < k$,

$$\begin{aligned}|g'(k)-g'(h)| &= \left|\frac{1}{1+k^{\alpha+1}}((1+k^{\alpha+1})g'(k)-(1+h^{\alpha+1})g'(h)) - \frac{k^{\alpha+1}-h^{\alpha+1}}{1+k^{\alpha+1}}g'(h)\right| \\ &\leqq \frac{(k-h)^\alpha}{1+k^{\alpha+1}}\|g\|_{X_+} + \frac{(k^{\alpha+1}-h^{\alpha+1})}{(1+h)(1+k^{\alpha+1})}3\|g\|_{X_+} \\ &\leqq M_1|k-h|^\alpha \|g\|_{X_+}.\end{aligned}$$

This proves (ii). The assertion (iii) is immediate from (1.3) and (2.3).

We now define a Banach space Y_+ by

(2.5) $Y_+ := \{\phi(h)\in C^1[0, \infty) : \phi(0)=0;\ |\phi|_0+|\phi'(0)|+|\phi'|_{\alpha+1/2} =: \|\phi\|_{Y_+}<\infty\}$,

where

$$|\phi'|_{\alpha+1/2} := \sup_{0\leq h,\ k<\infty,\ h\neq k} \frac{|k^{1/2}[(1+k^{\alpha+1})\phi'(k)-\phi'(0)]-h^{1/2}[(1+h^{\alpha+1})\phi'(h)-\phi'(0)]|}{|k-h|^{\alpha+1/2}}.$$

The following estimate will be used later.

LEMMA 2.2. *Let ϕ be in Y_+. Then, for $0\leqq h<k$,*

$$|(1+k^{\alpha+1})\phi'(k)-(1+h^{\alpha+1})\phi'(h)| \leqq M_2\left(1-\frac{h}{k}\right)^{1/2}(k-h)^{\alpha}\|\phi\|_{Y_+}, \tag{2.6}$$

where M_2 is independent of h, k.

PROOF. Let $\phi\in Y_+$ and $0\leqq h<k$. By the definition (2.5) we have

$$\begin{aligned}
&|(1+k^{\alpha+1})\phi'(k)-(1+h^{\alpha+1})\phi'(h)| \\
&= \Big|\frac{1}{k^{1/2}}\{k^{1/2}[(1+k^{\alpha+1})\phi'(k)-\phi'(0)]-h^{1/2}[(1+h^{\alpha+1})\phi'(h)-\phi'(0)]\} \\
&\quad -\left(1-\frac{h^{1/2}}{k^{1/2}}\right)[(1+h^{\alpha+1})\phi'(h)-\phi'(0)]\Big| \\
&\leqq \left\{\frac{1}{k^{1/2}}(k-h)^{\alpha+1/2}+\left(1-\frac{h^{1/2}}{k^{1/2}}\right)h^{\alpha}\right\}\|\phi\|_{Y_+} \\
&\leqq M_2\left(1-\frac{h}{k}\right)^{1/2}(k-h)^{\alpha}\|\phi\|_{Y_+}.
\end{aligned}$$

The following is an immediate consequence of Lemma 2.2.

LEMMA 2.3. *Let $0\leqq\alpha\leqq 1/2$ and let X_+ and Y_+ be function spaces defined by* (1.3) *and* (2.5) *respectively. Then $Y_+\subset X_+$.*

3. Boundary value problem.

In this section we shall give a necessary and sufficient condition for f to satisfy the condition (1.2). We start with the following

LEMMA 3.1. *A point $(\lambda, h)\in \boldsymbol{R}^2$ belongs to the set $\Gamma_2(f)$ if and only if the point (λ, h) satisfies the following conditions:*

(a) $h\neq 0$;

(b) *For any $t\in[0, 1)$,*

$$\lambda(1-t^2)-\int_t^1 2sf(hs)ds > 0; \tag{3.1}$$

(c) *There exists a number* $H\in\boldsymbol{R}$ *such that*

(c)$_1$ *for any* $t\in[0, 1)$,

$$\lambda(1-t^2)-\int_t^1 2sf(Hs)ds > 0; \tag{3.2}$$

(c)$_2$ *the equality*

$$h\Big(\lambda-\int_0^1 2sf(hs)ds\Big)^{1/2}+H\Big(\lambda-\int_0^1 2sf(Hs)ds\Big)^{1/2}=0 \tag{3.3}$$

holds;

(c)$_3$ *the equality*

$$\int_0^1\Big(\lambda(1-t^2)-\int_t^1 2sf(hs)ds\Big)^{-1/2}dt+\int_0^1\Big(\lambda(1-t^2)-\int_t^1 2sf(Hs)ds\Big)^{-1/2}dt=\frac{\pi}{2} \tag{3.4}$$

holds.

PROOF. Suppose that $(\lambda, h)\in\Gamma_2(f)$ and let (λ, u) be a solution of (1.1). Let ω denote the zero in $(0, \pi)$ of u. By an argument similar to that in the proof of [2, Lemma 2.1], it follows that $u'(\omega/2)=0$ and $u'(x)\neq 0$ for any $x\in(0, \omega/2)\cup(\omega/2, (\omega+\pi)/2)$. By multiplying both sides of the differential equation in (1.1) by $2u'(x)$ and integrating from $\omega/2$ to x, we have

$$u'(x)^2=\lambda(h^2-u(x)^2)+\int_h^{u(x)}2\xi f(\xi)d\xi\,. \tag{3.5}$$

The fact that $u'(x)\neq 0$ for any $x\in(0, \omega/2)$ implies (3.1).

Let H be the second stationary value of $u(x)$. Since the above argument remains true if h is replaced by H, we obtain

$$u'(x)^2=\lambda(H^2-u(x)^2)+\int_H^{u(x)}2\xi f(\xi)d\xi\,, \tag{3.6}$$

and (3.2). Substituting $x=\omega$ for (3.5) and (3.6) and noticing that $hH<0$ lead to (3.3). Furthermore, from (3.5) and (3.6), it follows that

$$\int_0^1\Big(\lambda(1-t^2)-\int_t^1 2sf(hs)ds\Big)^{-1/2}dt=\frac{\omega}{2},\quad \int_0^1\Big(\lambda(1-t^2)-\int_t^1 2sf(Hs)ds\Big)^{-1/2}dt=\frac{\pi-\omega}{2}.$$

This proves (3.4).

Conversely, suppose that (λ, h) satisfies the conditions (a), (b), (c). Let $h>0$. By (3.1), the function

$$x_1(u):=\int_0^{u/h}\left(\lambda(1-t^2)-\int_t^1 2sf(hs)ds\right)^{-1/2}dt$$

is a monotonically increasing function of u, defined on the interval $[0, h)$. From (3.4), we have $\omega:=2x_1(h)<\infty$. Let $u_1(x)$ be the inverse function of $x_1(u)$. Then the function $u_1(x)$ satisfies $u_1(0)=0$, $u_1(\omega/2)=h$ and $u_1'(\omega/2)=0$. In view of $u_1'(\omega/2)=0$ we extend $u_1(x)$ as a function on the interval $[0, \omega]$, by letting $u_1(x)=u_1(\omega-x)$. Similarly, using H which satisfies the condition (c), we define a function u_2 as the inverse of the function

$$x_2(u):=\pi-\int_0^{u/H}\left(\lambda(1-t^2)-\int_t^1 2sf(Hs)ds\right)^{-1/2}dt.$$

Then, from (3.4), we have $x_2(H)=(\omega+\pi)/2$ and therefore $u_2(x)$ can be defined as a function on the interval $[\omega, \pi]$. By the assumption (3.3), it follows that the function

$$u(x):=\begin{cases} u_1(x) & 0\leqq x\leqq\omega \\ u_2(x) & \omega\leqq x\leqq\pi \end{cases}$$

satisfies (1.1) and has exactly one zero in $(0, \pi)$. This proves $(\lambda, h)\in\Gamma_2(f)$. The case $h<0$ can be treated in a similar fashion.

We turn to the equation (3.3) with $\lambda=4$. Let f_+ and f_- denote the restrictions of f on the intervals $[0, \infty)$ and $(-\infty, 0]$, respectively and let $U_\pm$ be sets defined by (1.5).

LEMMA 3.2. *Let* $0\leqq\alpha\leqq 1/2$, *let* $0<\varepsilon<4$ *and let* $f_\pm\in U_\pm$ *respectively. Then there exists a unique function* $H(h)$, *defined on the interval* $[0, \infty)$, *satisfying, for any* $h\geqq 0$,

$$h\left(4-\int_0^1 2sf_+(hs)ds\right)^{1/2}+H(h)\left(4-\int_0^1 2sf_-(H(h)s)ds\right)^{1/2}=0\,. \tag{3.7}$$

The function $H(h)$ *possesses the following properties*:

(i) $H(0)=0$.

(ii) $H'(0)=-1$.

(iii) *There exist positive numbers* C_1 *and* C_2 *independent of* h *such that, for any* $h\geqq 0$,

$$-C_1h\leqq H(h)\leqq -C_2h\,. \tag{3.8}$$

In particular

$$\lim_{h\to\infty}H(h)=-\infty \tag{3.9}$$

(iv) *The function $H(h)$ is of class C^1 with the derivative*

$$H'(h)=\frac{h(4-f_+(h))}{H(h)(4-f_-(H(h)))}. \tag{3.10}$$

PROOF. Put

$$\Phi(h, H):=\left(4h^2-\int_0^h 2\xi f_+(\xi)d\xi\right)-\left(4H^2-\int_0^H 2\xi f_-(\xi)d\xi\right). \tag{3.11}$$

For any $H<0$, in view of the assumption $f_\pm\in U_\pm$,

$$\Phi_H(h, H)=-2H(4-f_-(H))>0.$$

Hence, for each $h\geqq 0$, $\Phi(h, H)$ is a monotonically increasing function of H on the interval $(-\infty, 0]$. Since, in view of the assumption $f_\pm\in U_\pm$,

$$\Phi(h, H)<\left(4h^2-\int_0^h 2\xi f_+(\xi)d\xi\right)-(4-\varepsilon)H^2,$$

$\Phi(h, H)\to-\infty$ as $H\to-\infty$. Moreover $\Phi(h, 0)\geqq 0$. Therefore, by virtue of the intermediate-value theorem, we conclude that, for each $h\geqq 0$, there exists exactly one number $H(h)\in(-\infty, 0]$ satisfying $\Phi(h, H(h))=0$, this is,

$$h^2\left(4-\int_0^1 2sf_+(hs)ds\right)=H(h)^2\left(4-\int_0^1 2sf_-(H(h)s)ds\right). \tag{3.12}$$

Since $H(h)\leqq 0$, the relation (3.12) may be rewritten as (3.7).

Obviously $H(0)=0$. Also, by (3.12) and the assumption $f_\pm(0)=0$, we have $H'(0)=-1$. It follows from (3.12) and the assumption $f_\pm\in U_\pm$ that

$$\frac{4-\varepsilon}{4+\varepsilon}\leqq\frac{H(h)^2}{h^2}\leqq\frac{4+\varepsilon}{4-\varepsilon}$$

This proves (iii). Applying the implicit function theorem to the mapping $\Phi(h, H)$ defined in (3.11) proves (iv).

The following result is basic for our work.

LEMMA 3.3. *Let $0\leqq\alpha\leqq 1/2$, let $0<\varepsilon<1$, let $f_\pm\in U_\pm$ respectively and let $H(h)$ be the function defined in Lemma 3.2. Then a function f, defined by (1.6), satisfies the condition (1.2) if and only if $f_\pm$ satisfies, for any $h\geqq 0$,*

$$\int_0^1\left(4(1-t^2)-\int_t^1 2sf_+(hs)ds\right)^{-1/2}dt+\int_0^1\left(4(1-t^2)-\int_t^1 2sf_-(H(h)s)ds\right)^{-1/2}dt=\frac{\pi}{2}. \tag{3.13}$$

PROOF. Suppose that $f_\pm$ satisfies (3.13) for any $h\geqq 0$. Then, from Lemma 3.1, $(4, h)\in\Gamma_2(f)$ for any $h>0$. An elementary observation shows that if $u(x)$ satisfies (1.1) then $\tilde{u}(x):=u(\pi-x)$ satisfies (1.1). It follows from this fact and the definition of $\Gamma_2(f)$ that, for each $h>0$, if $(4, h)\in\Gamma_2(f)$ then $(4, H(h))\in\Gamma_2(f)$. Therefore $(4, H(h))\in\Gamma_2(f)$ for any $h>0$. This and the property (iii) of Lemma 3.2 prove that $\{(4, h)\in\boldsymbol{R}^2: h\neq 0\}\subset\Gamma_2(f)$.

To prove that $\{(4, h)\in\boldsymbol{R}^2: h\neq 0\}\supset\Gamma_2(f)$, suppose that $(\lambda, h)\in\Gamma_2(f)$. Then, from Lemma 3.1, (λ, h) satisfies (3.4) for some H. But the assumption $f_\pm\in U_\pm$ yields

$$\frac{\pi}{(\lambda+\varepsilon)^{1/2}}\leqq\int_0^1\Big(\lambda(1-t^2)-\int_t^1 2sf_+(hs)ds\Big)^{-1/2}dt+\int_0^1\Big(\lambda(1-t^2)-\int_t^1 2sf_-(Hs)ds\Big)^{-1/2}dt.$$

Hence we have $(\lambda+\varepsilon)^{-1/2}\pi\leqq\pi/2$. This proves that if $(\lambda, h)\in\Gamma_2(f)$ then $\lambda\geqq 4-\varepsilon(>3)$. Therefore, to prove that $\{(4, h)\in\boldsymbol{R}^2: h\neq 0\}\supset\Gamma_2(f)$, it suffices to show, for each fixed $h\in\boldsymbol{R}$, the uniqueness of λ such that $(\lambda, h)\in\Gamma_2(f)$ in $\lambda\geqq 3$. It follows from an argument used in the proof of Lemma 3.2 that there exists a unique function $H(\lambda, h)$ satisfying

$$\lambda H(\lambda, h)^2-\int_0^{H(\lambda,h)}2\xi f_-(\xi)d\xi=\lambda h^2-\int_0^h 2\xi f_+(\xi)d\xi,$$

that the function $H(\lambda, h)$ can be estimated as

$$\frac{\lambda-\varepsilon}{\lambda+\varepsilon}\leqq\frac{h^2}{H(\lambda, h)^2}\leqq\frac{\lambda+\varepsilon}{\lambda-\varepsilon} \tag{3.14}$$

and that the derivative $H_\lambda(\lambda, h)$ of $H(\lambda, h)$ with respect to λ is written as

$$H_\lambda(\lambda, h)=\frac{h^2-H(\lambda, h)^2}{2H(\lambda, h)}\frac{1}{\lambda-f_-(H(\lambda, h))}. \tag{3.15}$$

From (3.14) and (3.15) we have

$$\left|\frac{H_\lambda(\lambda, h)}{H(\lambda, h)}\right|\leqq\frac{\varepsilon}{(\lambda-\varepsilon)^2}. \tag{3.16}$$

We now put

$$\begin{aligned}U(\lambda):=&\int_0^1\Big(\lambda(1-t^2)-\int_t^1 2sf_+(hs)ds\Big)^{-1/2}dt\\&+\int_0^1\Big(\lambda(1-t^2)-\int_t^1 2sf_-(H(\lambda, h)s)ds\Big)^{-1/2}dt.\end{aligned}$$

A calculation shows that

$$U_\lambda(\lambda)=-\frac{1}{2}\int_0^1(1-t^2)\Big(\lambda(1-t^2)-\int_t^1 2sf_+(hs)ds\Big)^{-3/2}dt$$

$$-\frac{1}{2}\int_0^1\Big((1-t^2)-\int_t^1 2s^2f_-(H(\lambda,\ h)s)dsH_\lambda(\lambda,\ h)\Big)\Big(\lambda(1-t^2)-\int_t^1 2sf_-(H(\lambda,\ h)s)ds\Big)^{-3/2}dt.$$

But, from (2.2) and (3.16), we obtain, for $\lambda\geqq 3$,

$$\begin{aligned}(1-t^2)&-\int_t^1 2s^2f_-(H(\lambda,\ h)s)dsH_\lambda(\lambda,\ h)\\ &=(1-t^2)-\int_t^1 2s(H(\lambda,\ h)sf_-(H(\lambda,\ h)s))ds\frac{H_\lambda(\lambda,\ h)}{H(\lambda,\ h)}\\ &\geqq(1-t^2)-\int_t^1 2s3\varepsilon ds\frac{\varepsilon}{(\lambda-\varepsilon)^2}\\ &=\Big(1-\frac{3\varepsilon^2}{(\lambda-\varepsilon)^2}\Big)(1-t^2)\\ &\geqq 0.\end{aligned}$$

Therefore, for $\lambda\geqq 3$, $U_\lambda(\lambda)<0$, which leads to the uniqueness of λ such that $(\lambda,\ h)\in\Gamma_2(f)$ in $\lambda\geqq 3$. This proves that f satisfies the condition (1.2). The converse is a direct consequence of Lemma 2.1.

We conclude this section with the following estimates of the function $H(h)$.

LEMMA 3.4. *Under the same assumption as in Lemma* 3.2, *the function* $H(h)$ *possesses the following properties*:

(i) *There exist constants* C_3, C_4 *independent of* h *such that, for any* $h\geqq 0$,

$$|H'(h)|\leqq C_3;\qquad |hH'(h)|\leqq C_4|H(h)|. \tag{3.17}$$

(ii) *There exists a constant* C_5 *independent of* h, k *such that, for* $0\leqq h\leqq k$,

$$|(1+k^{\alpha+1})H'(k)-(1+h^{\alpha+1})H'(h)|\leqq C_5(1+k^{1/2})(k-h)^{\alpha+1/2}. \tag{3.18}$$

(iii) *There exists a constant* C_6 *independent of* h *such that, for* $h\geqq 0$,

$$|(1+h^{\alpha+1})H'(h)f'_-(H(h))+f'_-(0)|\leqq C_6h^\alpha. \tag{3.19}$$

(iv) *There exists a constant* C_7 *independent of* t, h *such that, for* $h\geqq 0$, $t\leqq 0$,

$$|t(1+h^{\alpha+1})H'(h)(f'_-(t)-f'_-(H(h)))|\leqq C_7(|t|+|H(h)|)|H(h)-t|^\alpha. \tag{3.20}$$

PROOF. Using (3.10) and (3.8), we obtain

$$|H'(h)| \leqq \frac{4+\|f_+\|_{X_+}}{C_2(4-\varepsilon)} =: C_3 .$$

Hence $|hH'(h)| \leqq C_3 h \leqq C_3 C_2^{-1} |H(h)|$. This proves (i).

From (3.10) we have

$$(1+k^{\alpha+1})H'(k)-(1+h^{\alpha+1})H'(h) = \int_h^k \frac{d}{d\xi}\left\{\frac{(1+\xi^{\alpha+1})\xi(4-f_+(\xi))}{H(\xi)(4-f_-(H(\xi)))}\right\} d\xi .$$

But, using (3.12), (3.8), (2.1), (2.2), (3.17) and a tedious calculation, it follows that, for $\xi \geqq 0$,

$$\begin{aligned} &\left|\frac{d}{d\xi}\left\{\frac{(1+\xi^{\alpha+1})\xi(4-f_+(\xi))}{H(\xi)(4-f_-(H(\xi)))}\right\}\right| \\ &= \left|\frac{d}{d\xi}\left\{(1+\xi^{\alpha+1})\frac{\left(4-\int_0^1 2sf_-(H(\xi)s)ds\right)^{1/2}}{\left(4-\int_0^1 2sf_+(\xi s)ds\right)^{1/2}}\frac{(4-f_+(\xi))}{(4-f_-(H(\xi)))}\right\}\right| \\ &\leqq C_8(1+\xi^{\alpha}) , \end{aligned}$$

where C_8 is independent of ξ. Hence

$$\begin{aligned} |(1+k^{\alpha+1})H'(k)-(1+h^{\alpha+1})H'(h)| &\leqq C_8\left|\int_h^k (1+\xi^{\alpha})d\xi\right| \\ &\leqq C_5(1+k^{1/2})(k-h)^{\alpha+1/2} , \end{aligned}$$

where C_5 is independent of h, k. This proves (ii).

From (ii) of Lemma 3.2, (3.18), (2.1), (2.3) and (3.8) we have

$$\begin{aligned} &|(1+h^{\alpha+1})H'(h)f'_-(H(h))+f'_-(0)| \\ &= |\{(1+h^{\alpha+1})H'(h)-H'(0)\} f'_-(H(h))-(f'_-(H(h))-f'_-(0))| \\ &\leqq C_5(1+h^{1/2})h^{\alpha+1/2}(1+|H(h)|)^{-1}3\|f_-\|_{X_-}+M_1|H(h)|^{\alpha}\|f_-\|_{X_-} \\ &\leqq C_5(1+h^{1/2})h^{\alpha+1/2}(1+C_2 h)^{-1}3\|f_-\|_{X_-}+M_1 C_1^{\alpha} h^{\alpha}\|f_-\|_{X_-} \\ &\leqq C_6 h^{\alpha} . \end{aligned}$$

This proves (iii).

To prove (iv), let $g(h) := f_-(-h)$. It follows from the assumption $f_- \in X_-$, (3.8), (3.17) and (2.2) that

$$\begin{aligned} &|t(1+h^{\alpha+1})H'(h)(f'_-(t)-f'_-(H(h)))| \\ &= |H'(h)|\,|t(1+h^{\alpha+1})(f'(-H(h))-g'(-t))| \\ &= |H'(h)|\left|\frac{1+h^{\alpha+1}}{1+|H(h)|^{\alpha+1}}\right|\Big||t|\{(1+|H(h)|^{\alpha+1})g'(|H(h)|)-(1+|-t|^{\alpha+1})g'(-t)\} \\ &\qquad -(|H(h)|^{\alpha+1}-|t|^{\alpha+1})|t|g'(|t|)\Big| \end{aligned}$$

$$\leqq C_3\left(\frac{1+h^{\alpha+1}}{1+C_2^{\alpha+1}h^{\alpha+1}}\right)\left\{|t|\,|t-H(h)|^{\alpha}\|g\|_{X_+}+\Big|\,|H(h)|^{\alpha+1}-|t|^{\alpha+1}\Big|3\|g\|_{X_+}\right\}$$

$$\leqq C_7(|t|+|H(h)|)|H(h)-t|^{\alpha},$$

where C_7 is independent of h, t. This proves (iv).

4. Integral equation.

Let $\varepsilon<4$, $f_{\pm}\in U_{\pm}$ respectively and set

$$\Omega(f_+, f_-)=\int_0^1\left(4(1-t^2)-\int_t^1 2sf_+(hs)ds\right)^{-1/2}dt \tag{4.1}$$
$$+\int_0^1\left(4(1-t^2)-\int_t^1 2sf_-(H(h)s)ds\right)^{-1/2}dt-\frac{\pi}{2},$$

where $H(h)$ is the function defined in Lemma 3.2. Note that (3.13) is rewritten as $\Omega(f_+, f_-)=0$. In this section we shall solve the integral equation $\Omega(f_+, f_-)=0$, where f_- is given in U_-.

To facilitate matters, we let

$$D(h, t):=4(1-t^2)-\int_t^1 2sf_-(H(h)s)ds\,. \tag{4.2}$$

Then, by putting $\delta:=4-\varepsilon>0$, we have

$$D(h, t)>(1-t^2)\delta\,. \tag{4.3}$$

The following estimate is useful.

LEMMA 4.1. *Let* $0\leqq\alpha\leqq 1/2$, *let* $f_{\pm}\in U_{\pm}$ *respectively and let* $D(h, t)$ *be defined in* (4.2). *Then*:

(i) *For* $0\leqq h<k$ *and* $0\leqq t<1$,

$$\left|\frac{1}{D(k, t)^{3/2}}-\frac{1}{D(h, t)^{3/2}}\right|\leqq\frac{M_3}{(1-t^2)^{3/2}}\frac{(k-h)^{\alpha+1/2}}{1+h^{\alpha+1/2}}, \tag{4.4}$$

where M_3 *is independent of* h, k, t.

(ii) *For* $0\leqq h<k$ *and* $0\leqq t<1$,

$$(k^{1/2}-h^{1/2})\left|\frac{1}{D(k, t)^{3/2}}-\frac{1}{D(0, t)^{3/2}}\right|\leqq\frac{M_3'}{(1-t^2)^{3/2}}(k-h)^{\alpha+1/2}, \tag{4.5}$$

where M_3' *is independent of* h, k, t.

PROOF. It follows from (4.3), (2.1), (3.17), (3.8) that

$$\left|\frac{1}{D(k,\,t)^{3/2}}-\frac{1}{D(h,\,t)^{3/2}}\right|$$

$$=\left|3\int_h^k \frac{d\xi}{D(\xi,\,t)^{5/2}}\int_t^1 s^2 f'_-(H(\xi)s)H'(\xi)ds\right|$$

$$\leqq 3\delta^{-5/2}(1-t^2)^{-5/2}\int_t^1 s^2 ds\int_h^k 3(1+|H(\xi)|s)^{-1}d\xi C_3\|f_-\|_{X_-}$$

$$\leqq 3\delta^{-5/2}(1-t^2)^{-5/2}\int_t^1 s ds\int_h^k 3(1+C_2\xi)^{-1}d\xi\, C_3\|f_-\|_{X_-}$$

$$\leqq 9C_3\delta^{-5/2}(1-t^2)^{-3/2}\log\frac{1+C_2k}{1+C_2h}\|f_-\|_{X_-}$$

$$\leqq \frac{M_4}{(1-t^2)^{3/2}}\frac{(k-h)^{\alpha+1/2}}{1+h^{\alpha+1/2}},$$

where M_4 is independent of h, k. This proves (i).

In the case of $k\leqq 1$, it follows from (4.4) that

$$(k^{1/2}-h^{1/2})\left|\frac{1}{D(k,\,t)^{3/2}}-\frac{1}{D(0,\,t)^{3/2}}\right|\leqq\frac{M_3}{(1-t^2)^{3/2}}(k^{1/2}-h^{1/2})k^{\alpha+1/2}$$

$$\leqq\frac{M_5}{(1-t^2)^{3/2}}(k-h)^{\alpha+1/2}.$$

In the case of $k\geqq 1$, it follows from (4.3) that

$$(k^{1/2}-h^{1/2})\left|\frac{1}{D(k,\,t)^{3/2}}-\frac{1}{D(0,\,t)^{3/2}}\right|\leqq(k^{1/2}-h^{1/2})\left\{\left|\frac{1}{D(k,\,t)^{3/2}}\right|+\left|\frac{1}{D(0,\,t)^{3/2}}\right|\right\}$$

$$\leqq 2\delta^{-3/2}\frac{1}{(1-t^2)^{3/2}}(k^{1/2}-h^{1/2})$$

$$\leqq\frac{M'_5}{(1-t^2)^{3/2}}(k-h)^{\alpha+1/2}.$$

Since the constants M_5, M'_5 are independent of h, k, the proof of (ii) is complete.

We are now in a position to prove the following

LEMMA 4.2. *Suppose that* $0\leqq\alpha<1/2$, $\varepsilon<4$. *Let* $U_\pm$ *be the sets defined by* (1.5), *let* Y_+ *be the function space defined by* (2.5) *and let* $\Omega(f_+, f_-)$ *be the mapping defined by* (4.1). *Then:*

(i) $\Omega(f_+, f_-)$ *is a continuous mapping of* $U_+\times U_-$ *to* Y_+.

(ii) $\Omega(f_+, f_-)$ *is Fréchet differentiable with respect to* f_+ *in* $U_+\times U_-$. *The Fréchet derivative* $\Omega_{f_+}(0, 0)$ *of* $\Omega(f_+, f_-)$ *at* $(0, 0)$ *is written in the form*

$$(\Omega_{f_+}(0,\,0)f)(h)=\frac{1}{8}\int_0^1\frac{dt}{(1-t^2)^{3/2}}\int_t^1 sf(hs)ds. \tag{4.6}$$

(iii) *The Fréchet derivative* $\Omega_{f_+}(f_+, f_-)$ *is continuous in* $U_+\times U_-$.

PROOF. We shall show only that if $(f_+, f_-)\in U_+\times U_-$ then $\Omega(f_+, f_-)\in Y_+$, because other assertions can be shown similarly.

Let (f_+, f_-) be fixed in $U_+\times U_-$ and put

$$\phi(h):=\int_0^1\Big(4(1-t^2)-\int_t^1 2sf_-(H(h)s)ds\Big)^{-1/2}dt.$$

By (i), (ii) of Lemma 3.2, a calculation shows that

$$(1+h^{\alpha+1})\phi'(h)-\phi'(0)=p_1(h)+p_2(h),$$

where

$$p_1(h):=\int_0^1\frac{dt}{D(h,\ t)^{3/2}}\int_t^1 s^2\{(1+h^{\alpha+1})H'(h)f'_-(H(h)s)+f'_-(0)\}\,ds\,;$$

$$p_2(h):=-\frac{1}{2}\int_0^1\Big(\frac{1}{D(h,\ t)^{3/2}}-\frac{1}{D(0,\ t)^{3/2}}\Big)D_h(0,\ t)dt\,.$$

This yields

$$\begin{aligned}&k^{1/2}[(1+k^{\alpha+1})\phi'(k)-\phi'(0)]-h^{1/2}[(1+h^{\alpha+1})\phi'(h)-\phi'(0)]\\ &\qquad=\sum_{i=1}^{2}\{k^{1/2}p_i(k)-h^{1/2}p_i(h)\}\,.\end{aligned}\tag{4.7}$$

We assume that $0\leqq h<k$ and shall estimate $k^{1/2}p_i(k)-h^{1/2}p_i(h)$, $i=1, 2$, separately.

From (4.4) and (4.5), we have

$$\begin{aligned}&|k^{1/2}p_2(k)-h^{1/2}p_2(h)|\\ &=\Big|-\frac{1}{2}(k^{1/2}-h^{1/2})\int_0^1\Big(\frac{1}{D(h,\ t)^{3/2}}-\frac{1}{D(0,\ t)^{3/2}}\Big)D_h(0,\ t)dt\\ &\quad-\frac{1}{2}h^{1/2}\int_0^1\Big(\frac{1}{D(k,\ t)^{3/2}}-\frac{1}{D(h,\ t)^{3/2}}\Big)D_h(0,\ t)dt\Big|\\ &\leqq(\pi/4)M_3'\|f_-\|_{X_-}(k-h)^{\alpha+1/2}\\ &\quad+(\pi/4)M_3\|f_-\|_{X_-}\frac{h^{1/2}}{1+h^{\alpha+1/2}}(k-h)^{\alpha+1/2},\end{aligned}$$

from which it follows that there exists a constant M_6 independent of h, k such that

$$|k^{1/2}p_2(k)-h^{1/2}p_2(h)|\leqq M_6(k-h)^{\alpha+1/2}\,.\tag{4.8}$$

Changing the order of integration shows that

$$
\begin{aligned}
&k^{1/2}p_1(k)-h^{1/2}p_1(h)\\
&=k^{1/2}\int_0^1 t^2\{(1+k^{\alpha+1})H'(k)f'_-(H(k)t)+f'_-(0)\}\,dt\int_0^t\frac{ds}{D(k,s)^{3/2}}\\
&\quad-h^{1/2}\int_0^1 t^2\{(1+h^{\alpha+1})H'(h)f'_-(H(h)t)+f'_-(0)\}\,dt\int_0^t\frac{ds}{D(h,s)^{3/2}}\\
&=h^{1/2}\int_0^1 t^2\{(1+h^{\alpha+1})H'(h)f'_-(H(h)t)+f'_-(0)\}\,dt\int_0^t\Big(\frac{1}{D(k,s)^{3/2}}-\frac{1}{D(h,s)^{3/2}}\Big)ds\\
&\quad+k^{1/2}\int_0^1 t^2\{(1+k^{\alpha+1})H'(k)f'_-(H(k)t)+f'_-(0)\}\,dt\int_0^t\frac{ds}{D(k,s)^{3/2}}\\
&\quad-h^{1/2}\int_0^1 t^2\{(1+h^{\alpha+1})H'(h)f'_-(H(h)t)+f'_-(0)\}\,dt\int_0^t\frac{ds}{D(k,s)^{3/2}}\\
&=h^{1/2}\int_0^1 t^2\{(1+h^{\alpha+1})H'(h)f'_-(H(h)t)+f'_-(0)\}\,dt\int_0^t\Big(\frac{1}{D(k,s)^{3/2}}-\frac{1}{D(h,s)^{3/2}}\Big)ds\\
&\quad+k^{1/2}\int_0^{H(k)}\frac{t^2}{H(k)^3}\{(1+k^{\alpha+1})H'(k)f'_-(t)+f'_-(0)\}\,dt\int_0^{t/H(k)}\frac{ds}{D(k,s)^{3/2}}\\
&\quad-h^{1/2}\int_0^{H(h)}\frac{t^2}{H(h)^3}\{(1+h^{\alpha+1})H'(h)f'_-(t)+f'_-(0)\}\,dt\int_0^{t/H(h)}\frac{ds}{D(k,s)^{3/2}}\\
&=J_1+J_2+J_3+J_4+J_5\,,
\end{aligned}
$$

where

$$
\begin{aligned}
J_1&:=h^{1/2}\int_0^1 t^2\{(1+h^{\alpha+1})H'(h)f'_-(H(h)t)+f'_-(0)\}\,dt\int_0^t\Big(\frac{1}{D(k,s)^{3/2}}-\frac{1}{D(h,s)^{3/2}}\Big)ds\,;\\
J_2&:=\int_0^{H(k)}\frac{k^{1/2}t^2}{H(k)^3}f'_-(t)dt\int_0^{t/H(k)}\frac{ds}{D(k,s)^{3/2}}\{(1+k^{\alpha+1})H'(k)-(1+h^{\alpha+1})H'(h)\}\,;\\
J_3&:=\int_{H(h)}^{H(k)}\frac{k^{1/2}t^2}{H(k)^3}(1+h^{\alpha+1})H'(h)(f'_-(t)-f'_-(H(h)))\,dt\int_0^{t/H(k)}\frac{ds}{D(k,s)^{3/2}}\,;\\
J_4&:=\int_0^{H(h)}\Big\{\frac{k^{1/2}}{H(k)^3}\int_0^{t/H(k)}\frac{ds}{D(k,s)^{3/2}}-\frac{h^{1/2}}{H(h)^3}\int_0^{t/H(h)}\frac{ds}{D(k,s)^{3/2}}\Big\}\\
&\qquad\qquad\times t^2(1+h^{\alpha+1})H'(h)(f'_-(t)-f'_-(H(h)))dt\,;\\
J_5&:=\Big\{\int_0^{H(k)}\frac{k^{1/2}t^2}{H(k)^3}dt\int_0^{t/H(k)}\frac{ds}{D(k,s)^{3/2}}-\int_0^{H(h)}\frac{h^{1/2}t^2}{H(h)^3}dt\int_0^{t/H(h)}\frac{ds}{D(k,s)^{3/2}}\Big\}\\
&\qquad\qquad\times\{(1+h^{\alpha+1})H'(h)f'_-(H(h))+f'_-(0)\}\,.
\end{aligned}
$$

Since, by (2.1), (3.17), (3.8),

$$\begin{aligned}
&|t\{(1+h^{\alpha+1})H'(h)f'_-(H(h)t)+f'(0)\}| \\
&\quad \leqq t\{(1+h^{\alpha+1})|H'(h)|3(1+|H(h)|t)^{-1}\|f_-\|_{X_-}+\|f_-\|_{X_-}\} \\
&\quad \leqq t\{(1+h^{\alpha+1})C_3 3(1+C_2ht)^{-1}\|f_-\|_{X_-}+\|f_-\|_{X_-}\} \\
&\quad \leqq 3C_3(1+h^{\alpha+1})(1+C_2h)^{-1}\|f_-\|_{X_-}+\|f_-\|_{X_-}
\end{aligned}$$

for $0\leqq t\leqq 1$, it follows from (4.4) that there exists a constant M_7 independent of h, k such that $|J_1|\leqq M_7(k-h)^{\alpha+1/2}$.

Next, from (3.18), (2.1), (4.3), (3.8), we have

$$\begin{aligned}
|J_2| &\leqq \int_0^1 k^{1/2}t^2|f'_-(H(k)t)|dt\int_0^t \frac{ds}{D(k,s)^{3/2}}C_5(1+k^{1/2})(k-h)^{\alpha+1/2} \\
&\leqq \int_0^1 k^{1/2}t^3 3(1+|H(k)t|)^{-1}\|f_-\|_{X_-}\delta^{-3/2}(1-t^2)^{-1/2}dtC_5(1+k^{1/2})(k-h)^{\alpha+1/2} \\
&\leqq 3\delta^{-3/2}C_5\int_0^1 t^3(1+C_2kt)^{-1}(1-t^2)^{-1/2}dt\|f_-\|_{X_-}k^{1/2}(1+k^{1/2})(k-h)^{\alpha+1/2} \\
&\leqq 3\delta^{-3/2}C_5\int_0^1 t^2(1-t^2)^{-1/2}dt\|f_-\|_{X_-}(1+C_2k)^{-1}k^{1/2}(1+k^{1/2})(k-h)^{\alpha+1/2} \\
&\leqq M_8(k-h)^{\alpha+1/2},
\end{aligned}$$

where M_8 is a constant independent of h, k. Moreover, by virtue of (3.20), (4.3), (3.8) and (3.17), the term J_3 may be estimated as

$$\begin{aligned}
|J_3| &\leqq \left|\int_{H(h)}^{H(k)} \frac{k^{1/2}|t|}{|H(k)|^3}2C_7|t|\,|H(h)-t|^{\alpha}\delta^{-3/2}\frac{|t|}{(H(k)^2-t^2)^{1/2}}dt\right| \\
&\leqq 2C_7\delta^{-3/2}\int_{H(k)}^{H(h)} \frac{k^{1/2}}{|H(k)|^{1/2}}(H(h)-t)^{\alpha}(t-H(k))^{-1/2}dt \\
&\leqq 2C_7C_2^{-1/2}\delta^{-3/2}\int_{H(k)}^{H(h)}(H(h)-t)^{\alpha}(t-H(k))^{-1/2}dt \\
&= 2C_7C_2^{-1/2}\delta^{-3/2}B(1/2,\ \alpha+1)(H(h)-H(k))^{\alpha+1/2}, \\
&\leqq M_9(k-h)^{\alpha+1/2},
\end{aligned}$$

where $B(\cdot,\ \cdot)$ denotes the beta function.

The estimate (3.20) yields

$$\begin{aligned}
|J_4| \leqq \int_{H(h)}^0 &\left|\frac{k^{1/2}}{H(k)^3}\int_0^{t/H(k)}\frac{ds}{D(k,s)^{3/2}}-\frac{h^{1/2}}{H(h)^3}\int_0^{t/H(h)}\frac{ds}{D(k,s)^{3/2}}\right| \\
&\times 2C_7|t|\,|H(h)|(t-H(h))^{\alpha}dt
\end{aligned}$$

$$\leqq \int_{H(h)}^{0}\left|\frac{k^{1/2}-h^{1/2}}{H(k)^3}\int_0^{t/H(k)}\frac{ds}{D(k,\,s)^{3/2}}\right|2C_7|t|\,|H(h)|(t-H(h))^{\alpha}dt$$

$$+\int_{H(h)}^{0}h^{1/2}\left|\frac{1}{H(k)^3}\int_0^{t/H(k)}\frac{ds}{D(k,\,s)^{3/2}}-\frac{1}{H(h)^3}\int_0^{t/H(h)}\frac{ds}{D(k,\,s)^{3/2}}\right|$$

$$\times 2C_7|t|\,|H(h)|(t-H(h))^{\alpha}dt\,.$$

From (4.3) and (3.8) we have

$$\int_{H(h)}^{0}\left|\frac{k^{1/2}-h^{1/2}}{H(k)^3}\int_0^{t/H(k)}\frac{ds}{D(k,\,s)^{3/2}}\right||t|\,|H(h)|(t-H(h))^{\alpha}dt$$

$$\leqq \delta^{-3/2}\frac{k^{1/2}-h^{1/2}}{|H(k)|^3}|H(h)|\int_{H(h)}^{0}\frac{t^2(t-H(h))^{\alpha}}{(H(k)^2-t^2)^{1/2}}dt$$

$$\leqq \delta^{-3/2}\frac{k^{1/2}-h^{1/2}}{|H(k)|^{7/2}}|H(h)|^3\int_{H(h)}^{0}(t-H(h))^{\alpha-1/2}dt$$

$$\leqq \delta^{-3/2}(\alpha+1/2)^{-1}(k^{1/2}-h^{1/2})(C_2k)^{-7/2}(C_1h)^{\alpha+7/2}$$

$$\leqq M_{10}(k-h)^{\alpha+1/2},$$

where M_{10} is independent of h, k. Change of variables shows that

$$\int_{H(h)}^{0}h^{1/2}\left|\frac{1}{H(k)^3}\int_0^{t/H(k)}\frac{ds}{D(k,\,s)^{3/2}}-\frac{1}{H(h)^3}\int_0^{t/H(h)}\frac{ds}{D(k,\,s)^{3/2}}\right||t|\,|H(h)|(t-H(h))^{\alpha}dt$$

$$=h^{1/2}\int_{H(h)}^{0}\left\{\frac{1}{|H(h)|^3}\int_0^{t/H(h)}\frac{ds}{D(k,\,s)^{3/2}}-\frac{1}{|H(k)|^3}\int_0^{t/H(k)}\frac{ds}{D(k,\,s)^{3/2}}\right\}$$

$$\times|t|\,|H(h)|(t-H(h))^{\alpha}dt$$

$$=h^{1/2}(-H(h))^{\alpha}\left\{\int_0^1 t(1-t)^{\alpha}dt\int_0^t\frac{ds}{D(k,\,s)^{3/2}}\right.$$

$$\left.-\int_0^{H(h)/H(k)}\frac{H(h)}{H(k)}t(1-(H(k)/H(h))t)^{\alpha}dt\int_0^t\frac{ds}{D(k,\,s)^{3/2}}\right\}$$

$$=h^{1/2}(-H(h))^{\alpha}\left\{\int_{H(h)/H(k)}^{1}t(1-t)^{\alpha}dt\int_0^t\frac{ds}{D(k,\,s)^{3/2}}\right.$$

$$\left.+\int_0^{H(h)/H(k)}t\left[(1-t)^{\alpha}-\frac{H(h)}{H(k)}(1-(H(k)/H(h))t)^{\alpha}\right]dt\int_0^t\frac{ds}{D(k,\,s)^{3/2}}\right\},$$

which, by a similar argument to that in the end of the proof of [**2**, Lemma 3.1], may be bounded from above by the quantity constant$\times|H(k)-H(h)|^{\alpha+1/2}$ and therefore, in view of (3.17), by the quantity constant$\times(k-h)^{\alpha+1/2}$. Thus the term J_4 may be estimated as $|J_4|\leqq M_{11}(k-h)^{\alpha+1/2}$, where M_{11} is independent of h, k.

By changing the order of integration and using (3.19), (4.3), it follows that

$$|J_5| \leqq \left|\int_0^1 \frac{ds}{D(k,\ s)^{3/2}}\left(\int_{H(k)s}^{H(k)} \frac{k^{1/2}t^2}{H(k)^3}dt - \int_{H(h)s}^{H(h)} \frac{h^{1/2}t^2}{H(h)^3}dt\right)\right| C_6 h^{\alpha}$$
$$= (1/3)\int_0^1 \frac{1-s^3}{D(k,\ s)^{3/2}} ds C_6(k^{1/2}-h^{1/2})h^{\alpha}$$
$$\leqq M_{12}(k-h)^{\alpha+1/2},$$

where M_{12} is independent of h, k. Thus we conclude that

$$|k^{1/2}p_1(k)-h^{1/2}p_1(h)| \leqq M_{13}(k-h)^{\alpha+1/2}. \tag{4.9}$$

Combining (4.7), (4.8), (4.9) shows that $\phi \in Y_+$. Similarly, the first term in the right side of (4.1) belongs to the space Y_+. Therefore $\Omega(f_+,\ f_-) \in Y_+$.

The following lemma is an analogue of [**2**, Theorem 4.1].

LEMMA 4.3. *Suppose that* $0<\alpha \leqq 1/2$, $\varepsilon<4$. *Let* $U_\pm$ *be the sets defined by* (1.5), *let* Y_+ *be the function space defined by* (2.5) *and let* $\Omega(f_+,\ f_-)$ *be the mapping defined by* (4.1). *Then the Fréchet derivative* $\Omega_{f_+}(0,\ 0)$ *of* $\Omega(f_+,\ f_-)$ *at* (0, 0) *is a linear homeomorphism of* X_+ *onto* Y_+.

PROOF. In view of (4.6) the mapping $\Omega_{f_+}(0,\ 0)$ may be written in the form

$$8(\Omega_{f_+}(0,\ 0)f)(h) = \int_0^1 \frac{dt}{(1-t^2)^{3/2}}\int_t^1 sf(hs)ds = \int_0^1 \frac{t^2 f(ht)}{(1-t^2)^{1/2}}dt. \tag{4.10}$$

By the open mapping theorem, to prove the lemma, it suffices to show that the equation

$$\int_0^1 \frac{t^2 f(ht)}{(1-t^2)^{1/2}}\,dt = \phi(h), \tag{4.11}$$

where ϕ is given in Y_+, has a unique solution f in X_+. But, as was shown in [**2**, § 4], the equation (4.11) may be solved as

$$\begin{aligned} f(h) &= (2/\pi)h^{-2}\frac{d}{dh}\left\{h^3\int_0^1 \frac{t^3\phi(ht)}{(1-t^2)^{1/2}}\,dt\right\} \\ &= (6/\pi)\int_0^1 \frac{t^3\phi(ht)}{(1-t^2)^{1/2}}\,dt + (2/\pi)h\int_0^1 \frac{t^4\phi'(ht)}{(1-t^2)^{1/2}}\,dt. \end{aligned} \tag{4.12}$$

Hence, to prove the lemma, it suffices to show that, for each $\phi \in Y_+$, the function f defined by (4.12) belongs to the space X_+. To facilitate matters we let

$$q(h) := \int_0^1 \frac{t^3\phi(ht)}{(1-t^2)^{1/2}}dt; \qquad r(h) := h\int_0^1 \frac{t^4\phi'(ht)}{(1-t^2)^{1/2}}dt.$$

Obviously, $\sup_{0\leq h<\infty}|q(h)| \leqq (2/3)\|\phi\|_{Y_+}$. By Lemma 2.3, the assumption $\phi \in Y_+$ implies $\phi \in X_+$. Therefore, using (2.3), we obtain

$$
\begin{aligned}
&|t^{\alpha+1}\{(1+k^{\alpha+1})\phi'(kt)-(1+h^{\alpha+1})\phi'(ht)\}| \\
&\quad = |(t^{\alpha+1}-1)(\phi'(kt)-\phi'(ht))+\{(1+(kt)^{\alpha+1})\phi'(kt)-(1+(ht)^{\alpha+1})\phi'(ht)\}| \\
&\quad \leqq (M_1+1)\|\phi\|_{X_+}(k-h)^{\alpha},
\end{aligned}
$$

from which it follows that

$$
(4.13)\qquad |(1+k^{\alpha+1})q'(k)-(1+h^{\alpha+1})q'(h)| \leqq (\pi/2)(M_1+1)\|\phi\|_{X_+}(k-h)^{\alpha}.
$$

We now turn to the function $r(h)$. By virtue of (2.2), $\sup_{0\leqq h<\infty}|r(h)|\leqq 2\|\phi\|_{X_+}$. By integrating by parts, the function $r(h)$ may be rewritten as

$$
r(h)=\int_0^1\left\{\frac{d}{dt}\frac{t^4}{(1-t^2)^{1/2}}\right\}(\phi(h)-\phi(ht))dt.
$$

Differentiating this with respect to h leads to

$$
r'(h)=(3\pi/16)\phi'(h)+\int_0^1\left\{t\frac{d}{dt}\frac{t^4}{(1-t^2)^{1/2}}\right\}(\phi'(h)-\phi'(ht))dt.
$$

Hence, for $0\leqq h<k$, we have

$$
\begin{aligned}
&(1+k^{\alpha+1})r'(k)-(1+h^{\alpha+1})r'(h) \\
&\quad =(3\pi/16)\{(1+k^{\alpha+1})\phi'(k)-(1+h^{\alpha+1})\phi'(h)\} \\
(4.14)\qquad &\quad +\int_0^{h/k}\left\{t\frac{d}{dt}\frac{t^4}{(1-t^2)^{1/2}}\right\}\{((1+k^{\alpha+1})\phi'(k)-(1+h^{\alpha+1})\phi'(h)) \\
&\qquad\qquad -((1+k^{\alpha+1})\phi'(kt)-(1+h^{\alpha+1})\phi'(ht))\}\,dt \\
&\quad +\int_{h/k}^1\left\{t\frac{d}{dt}\frac{t^4}{(1-t^2)^{1/2}}\right\}\{(1+k^{\alpha+1})(\phi'(k)-\phi'(kt)) \\
&\qquad\qquad -(1+h^{\alpha+1})(\phi'(h)-\phi'(ht))\}\,dt.
\end{aligned}
$$

By (2.6) the first term in the right side of (4.14) may be estimated as

$$
|(3\pi/16)\{(1+k^{\alpha+1})\phi'(k)-(1+h^{\alpha+1})\phi'(h)\}| \leqq (3\pi/16)M_2\|\phi\|_{Y_+}(k-h)^{\alpha}.
$$

It follows from (2.4) and (2.6) that

$$
\begin{aligned}
&|((1+k^{\alpha+1})\phi'(k)-(1+h^{\alpha+1})\phi'(h))-((1+k^{\alpha+1})\phi'(kt)-(1+h^{\alpha+1})\phi'(ht))| \\
&\quad \leqq |(1+k^{\alpha+1})\phi'(k)-(1+h^{\alpha+1})\phi'(h)| \\
&\qquad +|(1+(kt)^{\alpha+1})\phi'(kt)-(1+(ht)^{\alpha+1})\phi'(ht)| \\
&\qquad +|t^{-\alpha-1}(1-t^{\alpha+1})((kt)^{\alpha+1}\phi'(kt)-(ht)^{\alpha+1}\phi'(ht))| \\
&\quad \leqq 2M_2\left(1-\frac{h}{k}\right)^{1/2}(k-h)^{\alpha}\|\phi\|_{Y_+}+M_1't^{-1}(1-t^{\alpha+1})(k-h)^{\alpha}\|\phi\|_{X_+}.
\end{aligned}
$$

Hence the second term in the right side of (4.14) may be estimated as

$$\left|\int_0^{h/k}\left\{t\frac{d}{dt}\frac{t^4}{(1-t^2)^{1/2}}\right\}\{((1+k^{\alpha+1})\phi'(k)-(1+h^{\alpha+1})\phi'(h))\right.$$

$$\left.-((1+k^{\alpha+1})\phi'(kt)-(1+h^{\alpha+1})\phi'(ht))\}\,dt\right|$$

$$\leqq 2M_2\left(1-\frac{h}{k}\right)^{1/2}\int_0^{h/k}\frac{5}{(1-t)^{3/2}}dt\|\phi\|_{X_+}(k-h)^{\alpha}$$

$$+M_1'\int_0^1(1-t^{\alpha+1})\left\{\frac{d}{dt}\frac{t^4}{(1-t^2)^{1/2}}\right\}dt\|\phi\|_{X_+}(k-h)^{\alpha}$$

$$\leqq M_{14}(k-h)^{\alpha},$$

where M_{14} is independent of h, k. It follows from (2.2), (2.6) that there exists a constant M_{15} independent of k such that

$$|(1+k^{\alpha+1})(\phi'(k)-\phi'(kt))|$$

$$=|\{(1+k^{\alpha+1})\phi'(k)-(1+(kt)^{\alpha+1})\phi'(kt)\}-(1-t^{\alpha+1})k^{\alpha+1}\phi'(kt)|$$

$$\leqq M_2(1-t)^{\alpha+1/2}\|\phi\|_{X_+}k^{\alpha}+3t^{-1}(1-t^{\alpha+1})\|\phi\|_{X_+}k^{\alpha}$$

$$\leqq M_{15}t^{-1}(1-t)^{\alpha+1/2}k^{\alpha}.$$

Hence the third term in the right side of (4.14) may be estimated as

$$\left|\int_{h/k}^1\left\{t\frac{d}{dt}\frac{t^4}{(1-t^2)^{1/2}}\right\}\{(1+k^{\alpha+1})(\phi'(k)-\phi'(kt))-(1+h^{\alpha+1})(\phi'(h)-\phi'(ht))\}\,dt\right|$$

$$\leqq 4M_{15}(k^{\alpha}+h^{\alpha})\int_{h/k}^1(1-t)^{\alpha-1}dt$$

$$\leqq 8M_{15}(k-h)^{\alpha}.$$

Thus we have proved that

$$|(1+k^{\alpha+1})r'(k)-(1+h^{\alpha+1})r'(h)|\leqq M_{16}(k-h)^{\alpha}, \tag{4.14}$$

where M_{16} is independent of h, k.

Combining (4.12), (4.13), (4.14) shows that $f\in X_+$.

We now present

PROOF OF THEOREM 1.1. Lemma 4.2 and Lemma 4.3 enable us to apply the implicit function theorem (see e.g., [**4**, Theorem 2.7.2]) to the operator $\Omega(f_+, f_-)$ and to conclude that there exists a neighborhood V_- and a unique continuous mapping $S: f_-\to f_+$, defined in V_-, such that $\Omega(f_+, f_-)=0$ with $f_-\in V_-$ is uniquely solvable as $f_+=S(f_-)$. Combining this observation and Lemma 3.3 proves Theorem 1.1.

5. Even nonlinear term.

In this section we shall treat the case the nonlinear term f of (1.1) is even. We start with the following elementary lemma.

LEMMA 5.1. *Suppose that f is a Lipschitz continuous, even, real function defined on $\boldsymbol{R}$. Let $u(x)$ be a solution of* (1.1) *with exactly one zero in* $(0, \pi)$ *and let h be the first stationary value of $u(x)$. Then the second stationary value H is equal to $-h$.*

PROOF. Let ω denote the zero of $u(x)$ in $(0, \pi)$. As is easily seen from the assumption that f is even, the function $v(x):=-u(2\omega-x)$ is a solution of $u''+\lambda u=uf(u)$ satisfying $v(\omega)=u(\omega)$, $v'(\omega)=u'(\omega)$. Hence, by the uniqueness theorem for an initial-value problem, $v(x)\equiv u(x)$, that is,

$$u(x)\equiv -u(2\omega-x)\,. \tag{5.1}$$

By substituting $x=0$ for (5.1), we have $\omega=\pi/2$. Therefore, from (5.1), it follows that $u(x)\equiv -u(\pi-x)$, which proves the lemma.

The following is an analogue of the corresponding result [**2**, Lemma 2.2] for $\Gamma_1(f)$.

LEMMA 5.2. *Under the same assumption on f as in Lemma* 5.1, *a point* $(\lambda, h)\in\boldsymbol{R}$ *belongs to* $\Gamma_2(f)$ *if and only if the point* (λ, h) *satisfies the following condition*:

(a) $h\neq 0$;

(b)

$$\lambda(1-t^2)-\int_t^1 2sf(hs)ds>0 \qquad \textit{for any } t\in[0, 1)\,;$$

(c)

$$\int_0^1\left(\lambda(1-t^2)-\int_t^1 2sf(hs)ds\right)^{-1/2}dt=\frac{\pi}{4}\,. \tag{5.2}$$

PROOF. Let $(\lambda, h)\in\Gamma_2(f)$ and let u be a solution of (1.1) associated with (λ, h). The condition (3.3) implies that H is the second stationary value of $u(x)$. Therefore, by Lemma 5.1, the equation (3.3) of H has a unique solution $H=-h$. In view of this observation, Lemma 5.2 is direct from Lemma 3.1.

The methods in the proof of [**2**, Theorem 2.3] apply to the equation (5.2), giving the following result.

THEOREM 5.3. *Suppose that f is a Lipschitz continuous, even, real function defined on $\boldsymbol{R}$. Then there exists a continuous, even function $\lambda(h)$, defined on $\boldsymbol{R}$,* *such that* $\Gamma_2(f)=\{(\lambda(h),\ h) : h\in\boldsymbol{R}\}$.

The main result of this section can be stated as follows:

THEOREM 5.4. *Suppose that* f_1, f_2 *are Lipschitz continuous, even, real functions defined on* $\boldsymbol{R}$. *If* $\Gamma_2(f_1)=\Gamma_2(f_2)$ *then* $f_1\equiv f_2$.

PROOF. By Theorem 5.3, there exists an even function $\lambda(h)$ defined on $\boldsymbol{R}$ such that, for any $h\geqq 0$,

$$\int_0^1\left(\lambda(h)(1-t^2)-\int_t^1 2sf_i(hs)ds\right)^{-1/2}dt=\frac{\pi}{4}\qquad(i=1,\ 2). \tag{5.3}$$

Put

$$D_i(h,\ t):=\lambda(h)(1-t^2)-\int_t^1 2sf_i(hs)ds\qquad(i=1,\ 2).$$

Using (5.3) and an interchange of the order of integration, we obtain, for any $h\geqq 0$,

$$\begin{aligned}0&=\int_0^1\frac{dt}{D_2(h,\ t)^{1/2}}-\int_0^1\frac{dt}{D_1(h,\ t)^{1/2}}\\&=\int_0^1\frac{dt}{D_2(h,\ t)^{1/2}D_1(h,\ t)^{1/2}\{D_2(h,\ t)^{1/2}+D_1(h,\ t)^{1/2}\}}\int_t^1 2s(f_2(hs)-f_1(hs))ds\\&=\int_0^1 2s(f_2(hs)-f_1(hs))ds\int_0^s\frac{dt}{D_2(h,\ t)^{1/2}D_1(h,\ t)^{1/2}\{D_2(h,\ t)^{1/2}+D_1(h,\ t)^{1/2}\}}.\end{aligned}$$

Therefore, for any $h\geqq 0$,

$$\int_0^h 2\xi(f_2(\xi)-f_1(\xi))d\xi\int_0^{\xi/h}\frac{dt}{D_2(h,\ t)^{1/2}D_1(h,\ t)^{1/2}\{D_2(h,\ t)^{1/2}+D_1(h,\ t)^{1/2}\}}=0. \tag{5.4}$$

If the assertion is not true, then there exist numbers a, b $(0\leqq a<b)$ such that (i) $f_2(\xi)-f_1(\xi)=0$ for $0\leqq\xi\leqq a$, (ii) $f_2(\xi)-f_1(\xi)>0$ (or <0) for $a<\xi\leqq b$. Substituting $h=b$ to (5.4) yields a contradiction and proves Theorem.

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