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Structure theorems for positive radial solutions to $\operatorname{div}(|Du|^{m-2}Du)+K(|x|)u^q=0$ in $\boldsymbol{R}^n$

Dedicated to Professor Takaŝi Kusano on his 60th birthday

By Nichiro KAWANO[1], Eiji YANAGIDA[2]
and Shoji YOTSUTANI[3]

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§1. Introduction.

In this paper we investigate the structure of positive radial solutions to the following quasilinear elliptic equation

$$\text{(E)} \qquad \operatorname{div}(|Du|^{m-2}Du)+K(|x|)u^q=0\,, \qquad x\in\boldsymbol{R}^n\,,$$

where $q>m-1$, $n>m>1$, and $|x|=\{\sum_{i=1}^n x_i^2\}^{1/2}$. When $m=2$, the equation (E) reduces to the semilinear elliptic equation

$$\Delta u+K(|x|)u^q=0\,.$$

Recently, in Theorem 1 of [**KYY**], we have established a structure theorem for positive radial solutions of the latter equation. (See also [**Y1**] and [**Y2**].) The aim of this paper is to show that the result is extended to the equation (E).

Since we are only concerned with positive radial solutions (i.e., solutions with $u(x)=u(|x|)>0$ for all $x\in\boldsymbol{R}^n$), we will study the initial value problem

$$(\mathrm{K}_\alpha) \qquad \begin{cases} (r^{n-1}|u_r|^{m-2}u_r)_r+r^{n-1}K(r)(u^+)^q=0\,, & r>0\,, \\ u(0)=\alpha>0\,, \end{cases}$$

where $r=|x|$ and $u^+=\max\{u, 0\}$. We assume that

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$$
\text{(K)} \qquad \begin{cases} K(r) \in C^1((0, \infty))\,; \\ K(r) = O(r^{\nu}) \quad \text{at } r=0 \text{ for some } \nu > -m\,; \\ K(r) \geqq 0 \text{ and } K(r) \not\equiv 0 \quad \text{on } (0, \infty)\,; \\ \text{if } m > 2, \text{ then } \inf\{r^{-\nu}K(r)\,;\ 0 < r \leqq R\} > 0 \quad \text{for every } R > 0. \end{cases}
$$

Then, for each $\alpha>0$, the initial value problem (K_α) has a unique solution $u(r) \in C([0, \infty)) \cap C^2((0, \infty))$, which will be denoted by $u(r\,;\alpha)$.

We classify solutions of (K_α) according to the behavior as $r \to \infty$. For the sake of convenience, we introduce the following definitions. We say that

(i) $u(r\,;\alpha)$ is a *crossing solution* if $u(r\,;\alpha)$ has a finite zero,

(ii) $u(r\,;\alpha)$ is a *slowly decaying solution* if $u(r\,;\alpha)$ is positive on $[0, \infty)$ and $\lim_{r\to\infty} r^{(n-m)/(m-1)}u(r\,;\alpha)=\infty$,

(iii) $u(r\,;\alpha)$ is a *rapidly decaying solution* if $u(r\,;\alpha)$ is positive on $[0, \infty)$, $\lim_{r\to\infty} r^{(n-m)/(m-1)}u(r\,;\alpha)$ exists and is finite and positive.

It can be shown that every solution of (K_α) is classified into one of the above three types (see (d) of Proposition 2.1 below).

Let $G(r)$ be a function defined by

$$
G(r) := \frac{1}{q+1} r^n K(r) - \frac{n-m}{m} \int_0^r s^{n-1} K(s) ds\,.
$$

We note that, by (K), $K(r)$ satisfies

$$
\int_0^r s^{n-1} K(s) ds < \infty
$$

for any $r \in (0, \infty)$. We also note that

$$
G_r(r) = \frac{1}{q+1} r^{(n-m)(q+1)/m} \{ r^{\{(m-1)n+m-(n-m)q\}/m} K(r) \}_r\,,
$$

and

$$
G(r) = O(r^{n+\nu}) \qquad \text{at } r=0\,.
$$

For $G(r)$, we introduce the assumption

$$
\text{(G)} \qquad \begin{cases} G(r) \not\equiv 0 \quad \text{on } (0, \infty)\,; \\ \text{there exists } R_1 \in [0, \infty) \text{ such that} \\ \quad G(r) \geqq 0 \text{ for } r \in (0, R_1) \text{ and } G_r(r) \leqq 0 \text{ for } r \in (R_1, \infty)\,. \end{cases}
$$

The function $G(r)$ will play an important role for the investigation of the structure of solutions to (K_α).

Now we state our main theorem, which generalizes Theorem 1 of [**KYY**] to the quasilinear equation (E).

THEOREM 1. *Suppose that* (K) *and* (G) *hold. Then the structure of solutions to* (K_α) *is classified into one of the following three types:*

(i) Type C: *$u(r;\alpha)$ is a crossing solution for every $\alpha>0$.*

(ii) Type S: *$u(r;\alpha)$ is a slowly decaying solution for every $\alpha>0$.*

(iii) Type M: *There exists a unique positive number α^* such that $u(r;\alpha)$ is a crossing solution for every $\alpha\in(\alpha^*, \infty)$, $u(r;\alpha^*)$ is a rapidly decaying solution, and $u(r;\alpha)$ is a slowly decaying solution for every $\alpha\in(0, \alpha^*)$.*

If $G(r)\equiv 0$ on $(0, \infty)$, then $u(r;\alpha)$ is a rapidly decaying solution for every $\alpha>0$ (see Remark 2.2). Such a structure will be called of Type R.

The above theorem covers the semilinear case $m=2$, and the statement of the theorem is precisely the same as in the semilinear case. However, to generalize the result to the quasilinear case, we encounter some technical difficulties that we must overcome. For example, although Green's formula is a very useful tool to treat the semilinear case, such a formula does not work well in the quasilinear case. Also the Kelvin transformation does not work well in the quasilinear case.

In Theorem 1, the essential condition (G) is stated in terms of $G(r)$ and its derivative $G_r(r)$ and does not include the explicit restriction on the exponent q except $q>m-1$. Moreover, as we will see, we do not need the precise information about the asymptotic behavior of slowly decaying solutions for the proof of Theorem 1.

By virtue of Theorem 1, under the conditions (K) and (G), the type of the structure is one of the three types. Then a natural question is of what type the structure is. Concerning a sufficient condition so that the structure is either of Type C or Type S, the results by Kawano-Ni-Yotsutani [**KNY**] which are extensions of Ding-Ni [**DN**] and Kusano-Naito [**KN**] are very useful. The following Theorem A is given in Theorem 9.3 of [**KNY**], and Theorem B is obtained by combining Theorem 9.2 of [**KNY**] with a similar argument in the proof of Theorem 6 of [**NY**].

THEOREM A. *Suppose that* (K) *holds and that $G(r)\not\equiv 0$ and $G(r)\geqq 0$ on* $(0, \infty)$. *Then the structure of solutions to* (K_α) *is of* Type C.

THEOREM B. *Suppose that* (K) *holds and that $G(r)\not\equiv 0$ and $G(r)\leqq 0$ on* $(0, \infty)$. *Then the structure of solutions to* (K_α) *is of* Type S.

We see from Theorem 1 that, under the assumptions (K) and (G), the structure is of Type M if there exist both a slowly decaying solution and a crossing solution. The following theorems will be useful.

THEOREM 2. *Suppose that* (K) *holds. If $K(r)$ satisfies*

$$K(r)=k_{\infty}r^{\ell}+o(r^{\ell}) \qquad at\ r=\infty$$

for some $k_{\infty}>0$ *and*

$$\ell<\frac{(n-m)q-(m-1)n-m}{m},$$

then there exists $\alpha_s>0$ *such that* $u(r;\alpha)$ *is a slowly decaying solution of* (K_α) *for every* $\alpha\in(0, \alpha_s)$. *Moreover,*

(i) *if* $\ell\geqq-m$, *then* $u(r;\alpha)\to 0$ *as* $r\to\infty$, *and*

(ii) *if* $\ell<-m$, *then* $u(r;\alpha)\to u_\infty(\alpha)$ *as* $r\to\infty$, *where* $u_\infty(\alpha)$ *is a positive number depending on* α.

THEOREM 3. *Suppose that* (K) *holds. If* $K(r)$ *satisfies*

$$K(r)=k_0r^{\nu}+o(r^{\nu}) \qquad at\ r=0$$

for some $k_0>0$ *and*

$$\nu>\frac{(n-m)q-(m-1)n-m}{m},$$

then there exists $\alpha_z>0$ *such that* $u(r;\alpha)$ *is a crossing solution of* (K_α) *for every* $\alpha\in(\alpha_z, \infty)$.

This paper is organized as follows. In § 2, we describe some preliminary results which will be used throughout this paper. In § 3, we give simplified new proofs of Theorem A and B. In § 4, we give a proof of our main theorem (Theorem 1). In § 5, we give proofs of Theorems 2 and 3. In § 6, we give some applications to equations closely related with Matukuma-type equations and a generalized Batt-Faltenbacher-Horst equation [**BFH**].

§ 2. Preliminaries.

In this section, we collect some fundamental facts which will be frequently used throughout this paper. We also show some useful characterizations of rapidly decaying solutions and slowly decaying solutions in terms of the Pohozaev identity.

We consider the initial value problem

$$(F_\alpha)\qquad \begin{cases} (r^{n-1}|u_r|^{m-2}u_r)_r+r^{n-1}f(r, u^+)=0\,, & r>0\,,\\ u(0)=\alpha>0\,, \end{cases}$$

where $u^+=\max\{u, 0\}$. Needless to say, (K_α) is a special form of (F_α) with $f(r, u)=K(r)u^q$.

We now collect several hypotheses which will be assumed (but not simultaneously) under various circumstances in this section. We introduce

$$\text{(F.1)} \qquad \begin{cases} f(r, u),\ f_u(r, u) \in C((0, \infty)\times[0, \infty)); \\ \sup\{r^{-\nu}|f(r, u)|+r^{-\nu}|f_u(r, u)| : 0<r\leqq R,\ 0\leqq u\leqq M\} < \infty \\ \quad \text{for some } \nu>-m \text{ and for every } M,\ R>0; \\ \text{if } m>2, \text{ then } \inf\{r^{-\nu}|f(r, u)| : 0<r\leqq R,\ L\leqq u\leqq M\} > 0 \\ \quad \text{for every } L,\ M,\ R>0; \end{cases}$$

(F.2) $f(r, u) \geqq 0$ on $(0, \infty)\times[0, \infty)$,

(F.3) $f_r(r, u) \in C((0, \infty)\times[0, \infty))$,

(F.4) $F(r, u) \leqq cuf(r, u)$ for all $u>0$ and for sufficiently large $r>0$, where c is a positive constant with $0<c<1/m$,

(F.4)$'$ $F(r, u) \leqq c'f(r, u)$ for sufficiently small $u>0$ and sufficiently large $r>0$, where c' is a positive constant,

(F.5) $nF(r, u)-((n-m)/m)uf(r, u)+rF_r(r, u) \leqq 0$ for all $u\geqq 0$ and sufficiently large $r>0$.

Here $F(r, u)$ is defined by $F(r, u):=\int_0^u f(r, \xi^+)d\xi$. It is easy to check that $f=K(r)u^q$ satisfies (F.1), (F.2), (F.3) and (F.4) if (K), (G) and the inequality $q>m-1$ hold.

The following facts are fundamental.

PROPOSITION 2.1. *Suppose that* (F.1) *and* (F.2) *hold. Then there exists a unique solution* $u(r)\in C([0, \infty))\cap C^2((0, \infty))$ *of* (F_α). *Moreover,* $u(r)$ *has the following properties*:

(a) $\lim_{r\downarrow 0} ru_r(r)=0$,

(b) $u_r(r)=-\left\{\int_0^r (s/r)^{n-1}f(s, u^+(s))ds\right\}^{1/(m-1)} \leqq 0$ *for all* $r>0$,

(c) $u(r)$ *is non-increasing on* $[0, \infty)$,

(d) *if* $u(r)$ *is positive on* $[0, \infty)$, *then* $r^{1-(n-m)/(m-1)}\{r^{(n-m)/(m-1)}u(r)\}_r$ *is non-increasing on* $(0, \infty)$, *and* $r^{(n-m)/(m-1)}u(r)$ *is non-decreasing on* $[0, \infty)$,

(e) *if* $\lim_{r\to\infty} u(r)=0$, *then*

$$\lim_{r\to\infty} r^{(n-m)/(m-1)}u(r) = -\frac{m-1}{n-m}\lim_{r\to\infty} r^{(n-1)/(m-1)}u_r(r)$$
$$= \frac{m-1}{n-m}\left\{\int_0^\infty r^{n-1}f(r, u(r))dr\right\}^{1/(m-1)} \leqq \infty .$$

PROOF. By Propositions 6.1 and 6.2 of [**KNY**], we obtain the existence and uniqueness of the solution and the properties (a), (b) and (c). We have (d) in view of the fact that u satisfies

$$\frac{d}{dr}\{r^{1-(n-m)/(m-1)}\{r^{(n-m)/(m-1)}u(r)\}_r\}$$
$$=-\frac{1}{(m-1)|u_r|^{n-2}}rf(r, u^+)\leqq 0\,.$$

We get (e) by using the l'Hôspital's rule and (b). Q. E. D.

The following Pohozaev identity is useful for investigating the properties of solutions.

PROPOSITION 2.2. *Suppose that* (F.1), (F.2) *and* (F.3) *hold. If u is a unique solution of* (F_α), *then the following identity holds*:

$$\frac{d}{dr}P(r\,;u)=r^{n-1}\left\{nF(r, u(r))-\frac{n-m}{m}u(r)f(r, u^+(r))+rF_r(r, u(r))\right\}, \tag{2.1}$$

where

$$P(r\,;u):=\frac{n-m}{m}r^{n-1}u(r)|u_r(r)|^{m-2}u_r(r)+\frac{m-1}{m}r^n|u_r(r)|^m+r^nF(r, u(r)).$$

PROOF. See, e.g., Theorem 3.1 of [**KNY**]. Q. E. D.

The following lemma is fundamental and implicitly stated in Proposition 4.3 of [**NY**].

LEMMA 2.1. *Suppose that* (F.1), (F.2) *and* (F.3) *hold. If u is a solution of* (F_α), *then*

$$\lim_{r\downarrow 0}P(r\,;u)=0\,.$$

PROOF. Since u is a solution of (F_α), we have

$$\lim_{r\downarrow 0}r^{n-1}u|u_r|^{m-1}=\lim_{r\downarrow 0}r^{n-m}(r|u_r|)^{m-1}u=0$$

and

$$\lim_{r\downarrow 0}r^n|u_r|^m=\lim_{r\downarrow 0}r^{n-m}(r|u_r|)^m=0$$

by (a) of Proposition 2.1. On the other hand, by (F.1), we have

$$\begin{aligned} r^nF(r, u(r)) &\leqq r^nF(r, \alpha)\\ &\leqq r^n\alpha\sup\{f(r, u)\,;\,0\leqq u\leqq\alpha\}\longrightarrow 0 \qquad \text{as } r\downarrow 0. \end{aligned}$$

Thus we get the conclusion. Q. E. D.

In the following lemmas, we will give very useful characterizations of rapidly decaying solutions and slowly decaying solutions, respectively.

LEMMA 2.2. *Suppose that* (F.1), (F.2) *and* (F.4)′ *hold. If u is a rapidly decaying solution of* (F_α), *then there exists a sequence $\{\tilde{r}_i\}$ such that $\tilde{r}_i \to \infty$ and*

$$P(\tilde{r}_i\,;u) \longrightarrow 0$$

as $i \to \infty$.

PROOF. Since u is a rapidly decaying solution, we have

$$\lim_{r\to\infty} u(r)=0$$

and

$$\lim_{r\to\infty} r^{n-1}|u_r(r)|^{m-1}=\int_0^\infty r^{n-1}f(r,\,u(r))dr<\infty$$

by (e) of Proposition 2.1. Hence, in view of $n>m$, we get

$$\lim_{r\to\infty} r^{n-1}u|u_r|^{m-1}=\lim_{r\to\infty}(r^{n-1}|u_r|^{m-1})u=0$$

and

$$\lim_{r\to\infty} r^n|u_r|^m=\lim_{r\to\infty}(r^{n-1}|u_r|^{m-1})^{m/(m-1)}r^{(m-n)/(m-1)}=0\,.$$

On the other hand, it follows from (F.4)′ that

$$\int_0^\infty r^{n-1}F(r,\,u(r))dr \leqq c'\int_0^\infty r^{n-1}f(r,\,u(r))dr<\infty\,.$$

Hence there exists a sequence $\{\tilde{r}_i\}$ such that $\tilde{r}_i\to\infty$ and $\tilde{r}_i{}^n F(\tilde{r}_i,\,u(\tilde{r}_i))\to 0$ as $i\to\infty$. Thus we obtain the conclusion. Q. E. D.

REMARK 2.1. It is easily seen from the proof above that, if u_j ($j=1, 2, \cdots, J$) are rapidly decaying solutions, we can find a common subsequence $\{\tilde{r}_i\}$ such that $\tilde{r}_i\to\infty$ and $P(\tilde{r}_i\,;u_j)\to 0$ as $i\to\infty$ for every $j=1, 2, \cdots, J$. This fact follows from the inequalities

$$\int_0^\infty \sum_{j=1}^J r^{n-1}F(r,\,u_j(r))dr \leqq c'\sum_{j=1}^J\int_0^\infty r^{n-1}f(r,\,u_j(r))dr<\infty$$

and the non-negativity of $F(r,\,u)$.

LEMMA 2.3. *Suppose that* (F.1), (F.2) *and* (F.4) *hold. If u is a slowly decaying solution of* (F_α) *satisfying $u(r\,;\alpha)\not\equiv\alpha$ on $(0,\infty)$, then there exists a sequence $\{\hat{r}_i\}$ such that $\hat{r}_i\to\infty$ as $i\to\infty$ and*

$$P(\hat{r}_i\,;u)<0$$

for every i.

PROOF. Since $u(r;\alpha)\not\equiv\alpha$ on $(0, \infty)$, we have $f(r, u(r;\alpha))\not\equiv 0$ on $(0, \infty)$. Thus we see that $u_r(r;\alpha)<0$ for sufficiently large r. It follows from (F.4) and the equation in (F_α) that

$$\begin{aligned}
P(r;u) &\leqq \frac{n-m}{m} r^{n-1} u |u_r|^{n-2} u_r + \frac{m-1}{m} r^n |u_r|^n + c r^n u f \\
&= \frac{m-1}{m} r^{n-(n-m)/(m-1)} |u_r|^{m-2} u_r \Big\{ \frac{n-m}{m-1} r^{(n-m)/(m-1)-1} u \\
&\qquad + r^{(n-m)/(m-1)} u_r \Big\} + c r^n u f \\
&= \frac{m-1}{m} r^{n-(n-m)/(m-1)} |u_r|^{m-2} u_r \{ r^{(n-m)/(m-1)} u \}_r - c r u \{ r^{n-1} |u_r|^{m-2} u_r \}_r \\
&= -r^n u |u_r|^{m-2} u_r \Big\{ -\frac{m-1}{m} \frac{\{ r^{(n-m)/(m-1)} u \}_r}{r^{(n-m)/(m-1)} u} + c \frac{-\{ r^{n-1} |u_r|^{m-2} u_r \}_r}{-r^{n-1} |u_r|^{m-2} u_r} \Big\} \\
&= -r^n u |u_r|^{m-2} u_r \frac{d}{dr} \Big\{ -\frac{m-1}{m} \log (r^{(n-m)/(m-1)} u) \\
&\qquad + (m-1) c \log (-r^{(n-1)/(m-1)} u_r) \Big\} \\
&= -(m-1) r^n u |u_r|^{m-2} u_r \frac{d}{dr} \Big\{ \Big(c - \frac{1}{m} \Big) \log (r^{(n-m)/(m-1)} u) \\
&\qquad + c \log \Big(\frac{-r^{(n-1)/(m-1)} u_r}{r^{(n-m)/(m-1)} u} \Big) \Big\} .
\end{aligned}$$

Since u is a slowly decaying solution, we have

$$\lim_{r\to\infty} r^{(n-m)/(m-1)} u = \infty .$$

Moreover, it follows from (d) of Proposition 2.1 that $\{r^{(n-m)/(m-1)}u\}_r \geqq 0$ on $(0, \infty)$, which implies that

$$\frac{-r^{(n-m)/(m-1)} u_r}{r^{(n-m)/(m-1)-1} u} \leqq \frac{n-m}{m-1} .$$

Thus we see from $0<c<1/m$ that

$$\Big(c - \frac{1}{m} \Big) \log (r^{(n-m)/(m-1)} u) + c \log \Big(\frac{-r^{(n-1)/(m-1)} u_r}{r^{(n-m)/(m-1)} u} \Big) \longrightarrow -\infty$$

as $r\to\infty$. Therefore there exists a sequence $\{\hat{r}_i\}$ such that $\hat{r}_i\to\infty$ as $i\to\infty$ and

$$\frac{d}{dr} \Big\{ \Big(c - \frac{1}{m} \Big) \log (r^{(n-m)/(m-1)} u) + c \log \Big(\frac{-r^{(n-1)/(m-1)} u_r}{r^{(n-m)/(m-1)} u} \Big) \Big\} \Big|_{r=\hat{r}_i} < 0$$

for every i. Thus we complete the proof. Q.E.D.

As a direct consequence of Proposition 2.2 and Lemmas 2.1, 2.2 and 2.3, we get the following statement.

REMARK 2.2. Suppose that (F.1), (F.2), (F.3) and (F.4) hold. Suppose further that $F(r, u)\not\equiv 0$ in $r\in(0, \infty)$ for every fixed $u>0$ and that

$$nF(r, u)-\frac{n-m}{m}uf(r, u)+rF_r(r, u) \equiv 0$$

for all $(r, u)\in(0, \infty)\times[0, \infty)$. Then, $P(r; u)\equiv 0$ for all $r>0$ in view of Proposition 2.2. By virtue of Lemma 2.1, this implies that the solution $u(r; \alpha)$ of (F_α) is a rapidly decaying solution for every $\alpha>0$, i. e., the structure of solutions to (F_α) is of Type R.

We will give some sufficient conditions so that a solution of (F_α) may either be a crossing solution or a slowly decaying solution. The following lemmas can be proved in the same manner as Lemmas 2.4, 2.5, 2.6 and 2.7 of [**KYY**], respectively. So we omit the proofs.

LEMMA 2.4. *Suppose that* (F.1), (F.2) *and* (F.4) *hold, and let* $u=u(r; \alpha)$ *be a solution of* (F_α) *satisfying* $u(r; \alpha)\not\equiv\alpha$. *If there exist* $\delta=\delta(\alpha)>0$ *and* $r_0=r_0(\alpha)>0$ *such that*

$$P(r; u) \geqq \delta \qquad \textit{for all } r\in(r_0, \infty),$$

then u is a crossing solution.

LEMMA 2.5. *Suppose that* (F.1), (F.2) *and* (F.4) *hold, and let* $u=u(r; \alpha)$ *be a solution of* (F_α). *If there exist* $\delta=\delta(\alpha)>0$ *and* $r_0=r_0(\alpha)>0$ *such that*

$$P(r; u) \leqq 0 \qquad \textit{for all } r\in(0, \infty), \tag{2.2}$$

$$P(r; u) \leqq -\delta \qquad \textit{for all } r\in(r_0, \infty), \tag{2.3}$$

then u is a slowly decaying solution.

LEMMA 2.6. *Suppose that* (F.1), (F.2), (F.3), (F.4) *and* (F.5) *hold. Then the set*

$$\{\alpha>0;\ u(r; \alpha) \textit{ is a slowly decaying solution of } (\mathrm{F}_\alpha)$$
$$\textit{satisfying } u(r; \alpha)\not\equiv\alpha \textit{ on } (0, \infty)\}$$

is an open set.

LEMMA 2.7. *Suppose that* (F.1) *and* (F.2) *hold. Then the set*

$$\{\alpha>0;\ u(r; \alpha) \textit{ is a crossing solution of } (\mathrm{F}_\alpha)\}$$

is an open set.

§ 3. New proofs of Theorems A and B.

We will give new proofs of Theorems A and B by using the characterizations obtained in the previous section. The following identities are slight modifications of the Pohozaev identities.

LEMMA 3.1. *Suppose that* (K) *holds. Then there exists a unique solution* $u=u(r)\in C([0, \infty))\cap C^2((0, \infty))$ *of* (K_α) *satisfying the identities*

$$\frac{d}{dr}P(r\,;u)=G_r(r)u^+(r)^{q+1} \tag{3.1}$$

and

$$P(r\,;u)=G(r)u^+(r)^{q+1}-(q+1)\int_0^r G(s)u^+(s)^q u_r(s)ds\,, \tag{3.2}$$

where

$$P(r\,;u):=\frac{n-m}{m}r^{n-1}u(r)|u_r(r)|^{m-2}u_r(r)+\frac{m-1}{m}r^n|u_r(r)|^m+\frac{1}{q+1}r^nK(r)u^+(r)^{q+1}.$$

PROOF. We have (3.1) as a direct consequence of Proposition 2.2.
The identity (3.1) is equivalent to

$$\frac{d}{dr}P(r\,;u)=\frac{d}{dr}\{G(r)u^+(r)^{q+1}\}-(q+1)G(r)u^+(r)^q u_r(r).$$

Integrating this over $[\varepsilon, r]$, letting $\varepsilon\downarrow 0$, and using $P(\varepsilon\,;u)\to 0$ and $G(\varepsilon)\to 0$ as $\varepsilon\downarrow 0$, we obtain (3.2). Q. E. D.

REMARK 3.1. The rearrangement as in the right-hand side of (3.2) was employed in Lemma 1 of Kusano and Naito [**KN**] in case $m=2$.

Now let us complete the proofs of Theorems A and B.

PROOF OF THEOREM A. Let $u=u(r\,;\alpha)$ be a solution of (K_α). It follows from the assumption and (3.2) that there exists $\delta=\delta(\alpha)>0$ and $r_0=r_0(\alpha)$ such that

$$P(r\,;u)\geqq\delta>0 \qquad \text{for all } r\in(r_0, \infty).$$

Since $u(r\,;\alpha)\not\equiv\alpha$ on $(0, \infty)$ by (K), we get the conclusion by Lemma 2.4. Q.E.D.

PROOF OF THEOREM B. Let $u=u(r\,;\alpha)$ be a solution of (K_α). It follows from the assumption and (3.2) that there exist $\delta=\delta(\alpha)>0$ and $r_0=r_0(\alpha)$ such that

$$P(r\,;u)\leqq 0 \qquad \text{for all } r\in(0, \infty),$$

$$P(r\,;u)\leqq-\delta \qquad \text{for all } r\in(r_0, \infty).$$

Hence we get the conclusion by Lemma 2.5. Q. E. D.

§4. Proof of Theorem 1.

In this section we give a proof of Theorem 1. The following two propositions are important in the proof.

PROPOSITION 4.1. *Suppose that* (K) *and* (G) *hold. If* $\varphi(r)$ *is a rapidly decaying solution of* (K_α), *then it holds that*

$$P(r;\varphi) \geqq 0 \quad \textit{and} \quad P(r;\varphi) \not\equiv 0 \qquad \textit{on } (0, \infty).$$

PROPOSITION 4.2. *Suppose that* (K) *holds. If there exists a rapidly decaying solution* $\varphi(r)$ *of* (K_α) *satisfying*

$$P(r;\varphi) \geqq 0 \quad \textit{and} \quad P(r;\varphi) \not\equiv 0 \qquad \textit{on } (0, \infty),$$

then the structure of solution of (K_α) *is of* Type M *with* $\alpha^*=\varphi(0)$.

Before we prove the above propositions, we will prove Theorem 1.

PROOF OF THEOREM 1. If there exists a rapidly decaying solution, then the structure of solutions of (K_α) is of Type M by Propositions 4.1 and 4.2.

Consider the case where there is no rapidly decaying solution. We note that $u(r;\alpha)\not\equiv\alpha$ on $(0, \infty)$ for every $\alpha>0$ in view of $K(r)\not\equiv 0$ on $(0, \infty)$. By Lemmas 2.6 and 2.7, the sets of initial values of slowly decaying solutions and crossing solutions are open sets. Hence, if there exists a slowly decaying solution, then there is no crossing solution so that the structure of solutions is of Type S. Similarly, if there exists a crossing solution, then the structure of solutions is of Type C. Q. E. D.

Now we prove Proposition 4.1.

PROOF OF PROPOSITION 4.1. Since $G(r)\not\equiv 0$ by assumption and $G(r)\to 0$ as $r\downarrow 0$, we have $G_r(r)\not\equiv 0$. This implies $P(r;\varphi)\not\equiv 0$ on $(0, \infty)$ in view of (3.1). We will show that $P(r;\varphi)\geqq 0$ on $(0, \infty)$.

It follows from Lemma 3.1 that

$$P(r;\varphi) = G(r)\varphi(r)^{q+1}-(q+1)\int_0^r G(s)\varphi(s)^q\varphi_r(s)ds.$$

Since $G(r)\geqq 0$ on $[0, R_1)$ and $\varphi_r(r)\leqq 0$, this implies that

$$P(r;\varphi) \geqq 0 \qquad \text{on } (0, R_1).$$

Suppose that there exists $r_1\in(R_1, \infty)$ such that $P(r_1;\varphi)<0$. Then, since $G_r(r)\leqq 0$ on (R_1, ∞), it follows from (3.1) that

$$P(r\,;\varphi) \leqq P(r_1\,;\varphi) < 0 \qquad \text{on } [r_1, \infty).$$

By Lemma 2.2, this implies that $\varphi(r)$ cannot be a rapidly decaying solution. This is a contradiction.

We divide the proof of Proposition 4.2 into several steps. In what follows, we use the notation

$$r_z = \inf\{r \in (0, \infty)\,;\ K(r) > 0\}.$$

We note that $r_z \in [0, \infty)$ in view of (K) and that

$$u(r\,;\alpha) \equiv \alpha \qquad \text{for } r \in [0, r_z].$$

LEMMA 4.1. *Suppose that* (K) *holds. Let* $u=u(r\,;\alpha)$ *be a solution of* (K_α), *let* $\psi=u(r\,;\alpha')$ *be a positive solution of* $(K_{\alpha'})$ *with* $\alpha' \neq \alpha$, *and let* $R>r_z$.

(i) *If* $u>\psi$ *on* $[0, R)$, *then*

$$(u/\psi)_r = 0 \qquad \text{on } (0, r_z],$$

$$(u/\psi)_r < 0 \qquad \text{on } (r_z, R].$$

(ii) *If* $u<\psi$ *on* $[0, R)$, *then*

$$(u/\psi)_r = 0 \qquad \text{on } (0, r_z],$$

$$(u/\psi)_r > 0 \qquad \text{on } (r_z, R].$$

PROOF. We will prove (i). Firstly, since $K(r)\equiv 0$ on $(0, r_z]$, we have $u(r\,;\alpha)\equiv\alpha$ and $\psi(r\,;\alpha')\equiv\alpha'$ on $(0, r_z]$. Hence $(u/\psi)_r\equiv 0$ on $(0, r_z]$.

Next we prove that $(u/\psi)_r<0$ if $r-r_z>0$ is sufficiently small. From (K_α) and $(K_{\alpha'})$, we have

$$\frac{|u_r(r)|^{m-2}u_r(r)}{u(r)^{m-1}} = -r^{1-n}\int_{r_z}^{r} \frac{s^{n-1}K(s)u(s)^q}{u(r)^{m-1}}\,ds\,, \tag{4.1}$$

$$\frac{|\psi_r(r)|^{m-2}\psi_r(r)}{\psi(r)^{m-1}} = -r^{1-n}\int_{r_z}^{r} \frac{s^{n-1}K(s)\psi(s)^q}{\psi(r)^{m-1}}\,ds\,. \tag{4.2}$$

At $r=r_z$, the assumption $q>m-1$ implies that

$$\frac{u(r_z)^q}{u(r_z)^{m-1}} > \frac{\psi(r_z)^q}{\psi(r_z)^{m-1}}.$$

Hence, by continuity, if $r-r_z>0$ is sufficiently small, then

$$\frac{u(s)^q}{u(r)^{m-1}} > \frac{\psi(s)^q}{\psi(r)^{m-1}} \qquad \text{for } s \in (r_z, r).$$

This implies that the integrand of (4.1) is greater than that of (4.2). Hence we obtain

$$\frac{|u_r(r)|^{m-2}u_r(r)}{u(r)^{m-1}}<\frac{|\phi_r(r)|^{m-2}\phi_r(r)}{\phi(r)^{m-1}},$$

which is equivalent to $u_r/u<\phi_r/\phi$. Thus the inequality $(u/\phi)_r<0$ holds if $r-r_z>0$ is sufficiently small.

Finally we derive a contradiction by assuming that there exists an $R_0\in(r_z, R)$ such that

$$(u/\phi)_r<0 \qquad \text{for } r\in(r_z, R_0),$$

$$(u/\phi)_r=0 \qquad \text{for } r=R_0.$$

Then we have

$$\frac{u(s)}{\phi(s)}>\frac{u(R_0)}{\phi(R_0)} \qquad \text{for } s\in(r_z, R_0),$$

or equivalently

$$\frac{u(s)}{u(R_0)}>\frac{\phi(s)}{\phi(R_0)} \qquad \text{for } s\in(r_z, R_0).$$

Hence, if $s\in(r_z, R_0)$, then

$$\begin{aligned}\frac{u(s)^q}{u(R_0)^{m-1}}&=\left\{\frac{u(s)}{u(R_0)}\right\}^{m-1}u(s)^{q-(m-1)}\\&>\left\{\frac{\phi(s)}{\phi(R_0)}\right\}^{m-1}\phi(s)^{q-(m-1)}\\&=\frac{\phi(s)^q}{\phi(R_0)^{m-1}}.\end{aligned}$$

By (4.1) and (4.2), this implies

$$\frac{|u_r(R_0)|^{m-2}u_r(R_0)}{u(R_0)^{m-1}}<\frac{|\phi_r(R_0)|^{m-2}\phi_r(R_0)}{\phi(R_0)^{m-1}},$$

i.e., $(u/\phi)_r<0$ at $r=R_0$. This is a contradiction. Thus the proof of (i) is completed.

The proof of (ii) is obtained similarly. Q.E.D.

REMARK 4.1. In the semilinear case ($m=2$), Lemma 4.1 is an easy consequence of the Green's formula. In the quasilinear case ($m\neq 2$), the Green's formula does not work well.

LEMMA 4.2. *Suppose that* (K) *holds. Let* $u=u(r;\alpha)$ *be a solution of* (K_α) *and let* $\phi=u(r;\alpha')$ *be a positive solution of* $(\mathrm{K}_{\alpha'})$ *with* $\alpha'\neq\alpha$ *satisfying* $P(r;\phi)\geq 0$ *and* $P(r;\phi)\not\equiv 0$ *on* $[0, \infty)$.

(i) *If* $\alpha>\alpha'$ *and if* $u>0$ *on* $[0, R)$ *for some* $R>r_z$, *then*

$$(u/\phi)_r = 0 \qquad on\ (0, r_z],$$

$$(u/\phi)_r < 0 \qquad on\ (r_z, R).$$

(ii) *If* $\alpha<\alpha'$, *then* $u>0$ *on* $[0, \infty)$ *and*

$$(u/\phi)_r = 0 \qquad on\ (0, r_z],$$

$$(u/\phi)_r > 0 \qquad on\ (r_z, \infty).$$

PROOF. We will show (i). If u and ϕ do not intersect, then the conclusion follows from Lemma 4.1. Consider the case where u and ϕ intersect at some point in $(0, R)$, and put

$$\bar{R} := \inf\{r \in (0, R)\,;\ u(r) = \phi(r)\}.$$

We note that $r_z<\bar{R}<R$.

Suppose that there exists $a\in(\bar{R}, R)$ such that

$$(u/\phi)_r < 0 \qquad \text{on } (r_z, a),\ (u/\phi)_r|_{r=a}=0,$$

and put $b:= u(a)/\phi(a)$. Then we have $0<b<1$, because u/ϕ is strictly decreasing on $[\bar{R}, a]$ and $u(\bar{R})/\phi(\bar{R})=1$. Also we have

$$u(a) = b\phi(a), \qquad u_r(a) = b\phi_r(a), \tag{4.3}$$

by noting $(u/\phi)_r|_{r=a}=0$.

It follows from (3.1) that

$$\frac{d}{dr}\{P(r\,;u)-b^{q+1}P(r\,;\phi)\} = \{(u/\phi)^{q+1}-b^{q+1}\}\frac{d}{dr}P(r\,;\phi),$$

which implies that

$$\frac{d}{dr}\{P(r\,;u)-b^{q+1}P(r\,;\phi)\}$$

$$=\frac{d}{dr}\{\{(u/\phi)^{q+1}-b^{q+1}\}P(r\,;\phi)\}-P(r\,;\phi)\frac{d}{dr}(u/\phi)^{q+1}.$$

Integrating this over $[\varepsilon, a]$ and letting $\varepsilon\downarrow 0$, we see from (4.3) that

$$P(a\,;u)-b^{q+1}P(a\,;\phi) = -(q+1)\int_0^a P(s\,;\phi)(u/\phi)^q(u/\phi)_r ds.$$

Using (4.3) in the left-hand side, we obtain

$$\frac{m-1}{m}(b^m-b^{q+1})r^{n-(n-m)(m-1)}|\phi_r|^{m-2}\phi_r\{r^{(n-m)/(m-1)}\phi\}_r\Big|_{r=a}$$

$$=-(q+1)\int_0^a P(s;\phi)(u/\phi)^q(u/\phi)_r ds\,.$$

The left-hand side is nonpositive, because $0<b<1$, $q>m-1$, $\phi_r\leqq 0$ and $\{r^{(n-m)/(m-1)}\phi\}_r\geqq 0$ by (d) of Proposition 2.1. On the other hand, since $(u/\phi)_r<0$ on (r_z, a), it follows from the assumption on $P(r;\phi)$ that the right-hand side is nonnegative. Consequently we obtain

$$\{r^{(n-m)/(m-1)}\phi\}_r|_{r=a}=0\,, \tag{4.4}$$

$$P(r;\phi)\equiv 0 \quad \text{on } (0, a)\,. \tag{4.5}$$

It follows from (d) of Proposition 2.1 and (4.4) that

$$\{r^{(n-m)/(m-1)}\phi\}_r\equiv 0 \quad \text{on } [a,\infty)\,.$$

Thus we obtain

$$K(r)\equiv 0 \quad \text{on } [a,\infty)\,,$$

which implies that

$$G_r(r)\equiv 0 \quad \text{on } (a,\infty)\,.$$

Hence we get

$$P(r;\phi)\equiv 0 \quad \text{on } (0,\infty)$$

in view of (3.1) and (4.5). However this contradicts the assumption $P(r;\phi)\not\equiv 0$. Thus the proof of (i) is completed.

The proof of (ii) is obtained similarly. Q. E. D.

Now let us complete the proof of Proposition 4.2.

PROOF OF PROPOSITION 4.2. First we consider the case where $\alpha\in(\varphi(0),\infty)$. We will derive a contradiction by assuming that $u=u(r;\alpha)$ is positive on $(0,\infty)$. If u is positive on $(0,\infty)$, it follows from (i) of Lemma 4.2 that u/φ is non-increasing on $(0,\infty)$. Hence u must be a rapidly decaying solution so that $\lim_{r\to\infty}u/\varphi$ exists and is finite and positive. It follows from (3.1) that

$$\frac{d}{dr}P(r;u)=(u/\varphi)^{q+1}\frac{d}{dr}P(r;\varphi)$$

$$=\frac{d}{dr}\{(u/\varphi)^{q+1}P(r;\varphi)\}-P(r;\varphi)\frac{d}{dr}(u/\varphi)^{q+1}.$$

Integrating this over $[\varepsilon, r]$ and letting $\varepsilon\to 0$, we obtain

$$(u/\varphi)^{q+1}P(r;\varphi)-P(r;u)=(q+1)\int_0^r P(s;\varphi)(u/\varphi)^q(u/\varphi)_r ds\,. \tag{4.6}$$

By the way, we see from the assumption on $P(r;\varphi)$ and (i) of Lemma 4.2 that there exist $\delta>0$ and $r_1>0$ such that

$$(q+1)\int_0^r P(s;\varphi)(u/\varphi)^q(u/\varphi)_r ds \leqq -\delta \qquad \text{for } r\in(r_1, \infty).$$

On the other hand, by Lemma 2.2, there exists a sequence $\{\tilde{r}_i\}$ such that $\tilde{r}_i\to\infty$ and

$$(u/\varphi)^{q+1}P(\tilde{r}_i;\varphi)-P(\tilde{r}_i;u) \longrightarrow 0$$

as $i\to\infty$. This is a contradiction. Hence $u(r;\alpha)$ must be a crossing solution for every $\alpha\in(\varphi(0), \infty)$.

Next we consider the case where $\alpha\in(0, \varphi(0))$. In this case $u(r;\alpha)$ is positive on $(0, \infty)$ and there exist $\delta>0$ and $r_1>0$ such that

$$(q+1)\int_0^r P(s;\varphi)(u/\varphi)^q(u/\varphi)_r ds \geqq \delta \qquad \text{for } r\in(r_1, \infty)$$

by virtue of (ii) of Lemma 4.2 and the assumption on $P(r;\varphi)$. Suppose now that $u(r;\alpha)$ is a rapidly decaying solution. Then, again by (4.6), we can derive a contradiction. Hence $u(r;\alpha)$ must be a slowly decaying solution for every $\alpha\in(0, \varphi(0))$. Q. E. D.

§ 5. Existence of slow-decay solutions and zero-hit solutions.

In this section we give proofs of Theorems 2 and 3 and some related results.

The following theorem is a generalization of Theorems 6.2 and 7.1 of [**NY**] for the semilinear case ($m=2$). In their proofs, the Kelvin transformation was effectively used. In the quasilinear case ($m\neq 2$), however, the Kelvin transformation does not work well. A new ingredient of our proofs is to use the characterization of solutions stated in Lemmas 2.3 and 2.4. By using such a technique, the results are considerably improved even in the semilinear case.

THEOREM 5.1. *Let* $u(r;\alpha)$ *be a solution of* (K_α). *Suppose that* (K) *holds and that*

$$\limsup_{r\to\infty} G(r) < 0 . \tag{5.1}$$

Then there exists $\alpha_s>0$ *such that* $u(r;\alpha)$ *is a slowly decaying solution for every* $\alpha\in(0, \alpha_s)$.

PROOF. Put

$$w = w(r;\alpha) = \alpha^{-1}u(r;\alpha) .$$

Then w satisfies

(5.2) $$(r^{n-1}|w_r|^{m-2}w_r)_r+\alpha^{q-m+1}r^{n-1}K(r)(w^+)^q=0\,,\qquad r>0\,,$$
$$w(0)=1\,.$$

We see from (b) of Proposition 2.1 that

$$\begin{aligned} r|w_r(r)| &= \alpha^{(q-m+1)/(m-1)}r\left\{\int_0^r (s/r)^{n-1}K(s)w^+(s)^q ds\right\}^{1/(m-1)} \\ &= \alpha^{(q-m+1)/(m-1)}\left\{\int_0^r (s/r)^{n-m}s^{m-1}K(s)w^+(s)^q ds\right\}^{1/(m-1)} \\ &\leqq \alpha^{(q-m+1)/(m-1)}\left\{\int_0^r s^{m-1}K(s)w^+(s)^q ds\right\}^{1/(m-1)}. \end{aligned}$$

Hence, for any $R>0$, we have

(5.3) $$rw_r(r\,;\alpha)\longrightarrow 0\qquad \text{as } \alpha\downarrow 0 \text{ uniformly on } [0,\,R]\,.$$

Also it follows from (b) of Proposition 2.1 that

$$w(r)=1-\alpha^{(q-m+1)/(m-1)}\int_0^r\left\{\int_0^t (s/t)^{n-1}K(s)w^+(s)^q ds\right\}^{1/(m-1)}dt\,.$$

Hence w must satisfy

(5.4) $$w(r\,;\alpha)\longrightarrow 1\qquad \text{as } \alpha\downarrow 0 \text{ uniformly on } [0,\,R]\,.$$

Moreover w satisfies the Pohozaev identity

(5.5) $$c(\alpha)\frac{d}{dr}Q(r\,;w)=G_r(r)w^+(r)^{q+1}\,,$$

where

$$Q(r\,;w):=\frac{n-m}{m}r^{n-1}w|w_r|^{m-2}w_r+\frac{m-1}{m}r^n|w_r|^m+\frac{1}{q+1}\alpha^{q-m+1}r^nK(r)(w^+)^{q+1},$$

and

$$c(\alpha):=\alpha^{-(q-m+1)}\,.$$

It is easy to check that

(5.6) $$P(r\,;u)\equiv\alpha^m Q(r\,;w)\,.$$

It follows from the assumption (5.1) that there exists $r_0>0$ such that

$$G(r_0)<0$$

and

(5.7) $$G(r)<G(r_0)\qquad \text{for any } r>r_0\,.$$

Integrating (5.5) over $[0,\,r_0]$ by parts, we get

$$c(\alpha)Q(r_0\,;\,w) = G(r_0)w^+(r_0)^{q+1} - (q+1)\int_0^{r_0} \frac{G(s)}{s} w^+(s)^q s w_r(s) ds\,,$$

which implies

$$\lim_{\alpha\downarrow 0} c(\alpha)Q(r_0\,;\,w) = G(r_0) < 0 \tag{5.8}$$

by (5.3), (5.4) and the integrability of $|G(r)|/r$ on $[0, r_0]$. Thus we see from (5.4) that

$$u(r\,;\,\alpha) > 0 \qquad \text{on } [0, r_0]\,, \tag{5.9}$$

and from (5.6) and (5.8) that

$$P(r_0\,;\,u(\cdot\,;\,\alpha)) < 0 \tag{5.10}$$

for every sufficiently small $\alpha>0$.

Rearranging the right-hand side of (3.1), we have

$$\frac{d}{dr}P(r\,;\,u) = \{G(r)-G(r_0)\}_r u^+(r)^{q+1}\,.$$

Integrating this over $[r_0, r]$ by parts, we get

$$\begin{aligned} &P(r\,;\,u) - P(r_0\,;\,u) \\ &= \{G(r)-G(r_0)\}\, u^+(r)^{q+1} - (q+1)\int_{r_0}^{r} \{G(s)-G(r_0)\}\, u^+(s)^q u_r(s) ds < 0 \end{aligned}$$

by (5.7). Thus we see from (5.10) that

$$P(r\,;\,u) < P(r_0\,;\,u) < 0 \qquad \text{on } [r_0, \infty)$$

for sufficiently small fixed $\alpha>0$. Consequently, by (5.9), (5.10) and Lemma 2.3, $u(r\,;\,\alpha)$ is a slowly decaying solution for every sufficiently small $\alpha>0$. Q. E. D.

PROOF OF THEOREM 2. First we will show (i). By assumption, we have

$$r^n K(r) = k_\infty r^{n+\ell} + o(r^{n+\ell}) \qquad \text{at } r=\infty\,,$$

and

$$n+\ell > m+\ell \geqq 0\,.$$

Hence we obtain

$$\int_0^r s^{n-1}K(s)ds = \frac{k_\infty}{n+\ell} r^{n+\ell} + o(r^{n+\ell}) \qquad \text{at } r=\infty\,.$$

Therefore, by the assumption on ℓ,

$$\begin{aligned} G(r) &= \frac{1}{q+1} r^n K(r) - \frac{n-m}{m}\int_0^r s^{n-1}K(s)ds \\ &= k_\infty\left\{\frac{1}{q+1} - \frac{n-m}{m(n+\ell)}\right\} r^{n+\ell} + o(r^{n+\ell}) \longrightarrow -\infty \end{aligned}$$

as $r\to\infty$. Hence it follows from Theorem 5.1 that $u(r;\alpha)$ is a slowly decaying solution for every sufficiently small $\alpha>0$. Moreover we see from Theorem 5.1 of [**KNY**] that $u(r;\alpha)\to 0$ as $r\to\infty$.

Next we will show (ii). We see from (b) of Proposition 2.1 that

$$u(r;\alpha)\leqq\alpha \qquad \text{on } [0,\infty)$$

and

$$u(r)=\alpha-\int_0^r\left\{\int_0^t(s/t)^{n-1}K(s)u^+(s)^q ds\right\}^{1/(m-1)}dt\,.$$

Thus we obtain

$$u(r)\geqq\alpha-\alpha^{q/(m-1)}\int_0^\infty\left\{\int_0^t(s/t)^{n-1}K(s)ds\right\}^{1/(m-1)}dt\,. \tag{5.11}$$

Here we have

$$\begin{aligned}\left\{\int_0^t(s/t)^{n-1}K(s)ds\right\}^{1/(m-1)}&=\left\{t^{1-n}\int_0^t s^{n-1}K(s)ds\right\}^{1/(m-1)}\\&=\left\{\frac{k_\infty}{n+\ell}t^{\ell+1}\right\}^{1/(m-1)}+o(t^{(\ell+1)/(m-1)})\end{aligned}$$

at $t=\infty$, and

$$\frac{\ell+1}{m-1}<-1$$

by $\ell<-m$. Hence

$$\int_0^\infty\left\{\int_0^t(s/t)^{n-1}K(s)ds\right\}^{1/(m-1)}dt<\infty\,.$$

Thus, by (5.11) and $q/(m-1)>1$, we obtain

$$\lim_{r\to\infty}u(r;\alpha)>0$$

for every sufficiently small $\alpha>0$. Q. E. D.

The following theorem, which is a slight modification of Theorem 2.9 of [**LN**], characterizes the asymptotic behavior of slowly decaying solutions in the case where $K(r)$ decays sufficiently fast.

THEOREM 5.2. *Suppose that* (K) *holds and that* $K(r)$ *satisfies*

$$K(r)=O(r^\ell) \qquad \textit{at } r=\infty$$

for some $\ell<-m$. *If* u *is a slowly decaying solution of* (K_α), *then*

$$\lim_{r\to\infty}u(r)>0\,.$$

PROOF. Suppose that $\lim_{r\to\infty} u(r)=0$. It suffices for the proof to show that

$$u(r)=O(r^{(m-n)/(m-1)}) \quad \text{at } r=\infty\,. \tag{5.12}$$

Integrating (b) of Proposition 2.1 over $[r, \infty)$, we have

$$u(r)=\int_r^\infty\left\{t^{1-n}\int_0^t s^{n-1}K(s)u(s)^q ds\right\}^{1/(m-1)} dt\,. \tag{5.13}$$

To obtain the estimate (5.12), we divide our argument into three cases.

Case 1. $\ell<-n$. In this case, we have

$$\int_0^\infty s^{n-1}K(s)u(s)^q ds<\infty\,.$$

Then (5.12) follows from (5.13).

Case 2. $\ell=-n$. It follows from (5.13) that

$$\begin{aligned} u(r) &= O\Big(\int_r^\infty (t^{1-n}\log t)^{1/(m-1)}dt\Big) \\ &= O(r^{(m-n)/(m-1)}(\log r)^{1/(m-1)}) \quad \text{at } r=\infty\,. \end{aligned}$$

Hence we get

$$u(r)=O(r^{(m-n)/\{2(m-1)\}}) \quad \text{at } r=\infty\,.$$

Thus we have

$$K(r)u(r)^q=O(r^{-n+q(m-n)/\{2(m-1)\}}) \quad \text{at } r=\infty\,,$$

which implies (5.12) in view of (5.13).

Case 3. $-n<\ell<-m$. Define

$$\sigma(k):=\left\{1+\frac{q}{m-1}+\left(\frac{q}{m-1}\right)^2+\cdots+\left(\frac{q}{m-1}\right)^k\right\}\frac{m+\ell}{m-1}, \qquad k=0, 1, 2, \cdots.$$

By (5.13), we get

$$u(r)=O(r^{\sigma(0)}) \quad \text{at } r=\infty\,,$$

which implies

$$K(r)u(r)^q=O(r^{\ell+q\sigma(0)}) \quad \text{at } r=\infty\,.$$

If $\ell+q\sigma(0)\leqq -n$, then we can obtain the estimate (5.12) as in the previous case. Otherwise it follows from (5.13) that

$$u(r)=O(r^{\sigma(1)}) \quad \text{at } r=\infty\,.$$

Iterating this procedure k times, we have

$$u(r)=\begin{cases} O(r^{(m-n)/(m-1)}) & \text{if } \sigma(k)\leqq \dfrac{m-n}{m-1}, \\ O(r^{\sigma(k)}) & \text{if } \sigma(k)>\dfrac{m-n}{m-1}, \end{cases}$$

at $r=\infty$. Since $q/(m-1)>1$ and $m+\ell<0$, $\sigma(k)\to-\infty$ as $k\to\infty$. Consequently we obtain (5.12). Q. E. D.

Finally in this section, we complete the proof of Theorem 3.

PROOF OF THEOREM 3. Put

$$w=w(r\,;\alpha):=\alpha^{-1}u(\beta^{-1}r\,;\alpha)$$

and

$$K_\beta(r):=(\beta^{-1}r)^{-\nu}K(\beta^{-1}r),$$

where

$$\beta=\alpha^{(q-m+1)/(\nu+m)}.$$

Then w satisfies

$$(r^{n-1}|w_r|^{m-2}w_r)_r+r^{n-1}r^\nu K_\beta(r)u^q=0.$$

By the assumption on $K(r)$, $K_\beta(r)$ satisfies

$$K_\beta(r)\longrightarrow k_0 \qquad \text{as } \alpha\to\infty$$

uniformly in any bounded interval of r.

It follows from (b) of Proposition 2.1 that

$$w(r)=1-\int_0^r\left\{\int_0^t(s/t)^{n-1}s^\nu K_\beta(s)w^+(s)^q ds\right\}^{1/(m-1)}dt.$$

Hence

$$1-\int_0^r\left\{\int_0^t(s/t)^{n-1}s^\nu K_\beta(s)ds\right\}^{1/(m-1)}dt\leqq w(r)\leqq 1 \tag{5.14}$$

and

$$r|w_r(r)|\leqq\left\{\int_0^r s^{m-1}s^\nu K_\beta(s)ds\right\}^{1/(m-1)}. \tag{5.15}$$

Thus we see that, for any $R>0$, the family of continuous functions $\{w(r\,;\alpha)\,;\ \alpha>1\}$ is uniformly bounded and equicontinuous on $[0, R]$ in view of (5.14) and (5.15). By virtue of Ascoli-Arzelà's theorem, there exists a continuous function $v(r)$ defined on $[0,\infty)$ and a subsequence $\{\alpha_j\}$ such that

$$\alpha_j\longrightarrow\infty,$$

$$w(r\,;\alpha_j)\longrightarrow v(r) \text{ uniformly on } [0, R],$$

as $j\to\infty$, and

$$v(r)=1-k_0\int_0^r\left\{\int_0^t(s/t)^{n-1}s^\nu v^+(s)^q ds\right\}^{1/(m-1)}dt .$$

Thus $v\in C([0, \infty))\cap C^2([0, \infty))$ and

$$(r^{n-1}|v_r|^{m-2}v_r)_r+k_0 r^{n-1}r^\nu v^q=0 ,$$
$$v(0)=1 ,$$

in view of Propositions 6.1 and 6.2 of [**KNY**]. Therefore we see that

$$w(r ; \alpha) \longrightarrow v(r)$$

as $\alpha\to\infty$ uniformly on any bounded closed interval.

On the other hand, by virtue of Theorem B and the assumption on ν, v is a crossing solution. Consequently w is a crossing solution, which implies that u is a crossing solution for every sufficiently large $\alpha>0$. Q. E. D.

§ 6. Applications.

In this section we give some applications of Theorem 1, 2 and 3 to the Matukuma-type equation

$$\operatorname{div}(|Du|^{m-2}Du)+\frac{1}{1+|x|^\tau}u^q=0 , \qquad x\in \boldsymbol{R}^n ,$$

and a generalized Batt-Faltenbacher-Horst equation [**BFH**]:

$$\operatorname{div}(|Du|^{m-2}Du)+\frac{|x|^{\lambda-m}}{(1+|x|^m)^{\lambda/m}}u^q=0 , \qquad x\in \boldsymbol{R}^n .$$

THEOREM 6.1 (Matukuma-type equation). *Suppose that $K(r)$ is given by*

$$K(r)=\frac{1}{1+r^\tau} , \qquad \tau\geqq 0 .$$

Then the structure of positive radial solutions of (K_α) *is as follows.*

Case I. $\tau=0$ (*Lane-Emden equation*).

(i) *If* $m-1<q<\dfrac{(m-1)n+m}{n-m}$, *the structure is of* Type C.

(ii) *If* $q=\dfrac{(m-1)n+m}{n-m}$, *the structure is of* Type R.

(iii) *If* $q>\dfrac{(m-1)n+m}{n-m}$, *the structure is of* Type S.

Case II. $0<\tau<m$.

(i) *If* $m-1<q\leqq\dfrac{(m-1)n+m-m\tau}{n-m}$, *the structure is of* Type C.

(ii) *If* $\dfrac{(m-1)n+m-m\tau}{n-m}<q<\dfrac{(m-1)n+m}{n-m}$, *the structure is of* Type M.

(iii) *If* $q\geqq\dfrac{(m-1)n+m}{n-m}$, *the structure is of* Type S.

Case III. $\tau=m$ (*Matukuma equation*).

(i) *If* $m-1<q<\dfrac{(m-1)n+m}{n-m}$, *the structure is of* Type M.

(ii) *If* $q\geqq\dfrac{(m-1)n+m}{n-m}$, *the structure is of* Type S.

Case IV. $\tau>m$.

(i) *If* $m-1<q<\dfrac{(m-1)n+m}{n-m}$ *the structure is of* Type M.

(ii) *If* $q\geqq\dfrac{(m-1)n+m}{n-m}$, *the structure is of* Type S.

Moreover, if u *is a slowly decaying solution, then*

$$\lim_{r\to\infty} u(r)\begin{cases} =0 & \textit{for } \text{Cases I, II } \textit{and } \text{III,} \\ >0 & \textit{for } \text{Case IV.}\end{cases}$$

PROOF. We note that

$$\begin{aligned} G_r(r) &= \frac{1}{q+1} r^{(n-m)(q+1)/m}\{r^{\{(m-1)n+m-(n-m)q\}/m}K(r)\}_r \\ &= \frac{n-m}{m(q+1)} r^{n-1}(1+r^\tau)^{-2}\Big\{-\Big(q-\frac{(m-1)n+m-m\tau}{n-m}\Big)r^\tau-\Big(q-\frac{(m-1)n+m}{n-m}\Big)\Big\}. \end{aligned}$$

Then the conclusion follows from Theorems 1, 2, 3, A, B, 5.2 with $\ell=-\tau$, and Theorem 5.1 of [**KNY**]. Q.E.D.

THEOREM 6.2 (Batt-Faltenbacher-Horst equation). *Suppose that* $K(r)$ *is given by*

$$K(r)=\frac{r^{\lambda-m}}{(1+r^m)^{\lambda/m}}, \qquad \lambda>0\,.$$

Then the structure of solutions of (K_α) *is as follows.*

(i) *If* $m-1<q<\dfrac{(m-1)n+m(\lambda-m+1)}{n-m}$, *the structure is of* Type M.

(ii) *If* $q\geqq\dfrac{(m-1)n+m(\lambda-m+1)}{n-m}$, *the structure is of* Type S.

Moreover, if u is a slowly decaying solution, then

$$\lim_{r\to\infty} u(r)=0 .$$

PROOF. We note that

$$G_r(r)=\frac{1}{q+1}r^{(n-m)(q+1)/m}\{r^{\{(m-1)n+m-(n-m)q\}/m}K(r)\}_r$$

$$=\frac{n-m}{m(q+1)}r^{n+\lambda-m-1}(1+r^m)^{-\lambda/m-1}$$

$$\times\left\{-(q-m+1)r^m-\left(q-\frac{(m-1)n+m(\lambda-m+1)}{n-m}\right)\right\}.$$

Then the conclusion follows from Theorems 1, 2, 3, A, and Theorem 5.1 of [**KNY**]. Q. E. D.

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Nichiro KAWANO
Department of Mathematics
Faculty of Education
Miyazaki University
Miyazaki 889-21
Japan

Eiji YANAGIDA
Department of Information Science
Faculty of Science
Tokyo Institute of Technology
Meguro-ku, Tokyo 152
Japan

Shoji YOTSUTANI
Department of Applied Mathematics
and Informatics
Faculty of Science and Technology
Ryukoku University
Seta, Ohtsu 520-21
Japan