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On the space of self homotopy equivalences of the projective plane

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1. Introduction.

Let X be a connected CW complex with base point, and let $G(X)$ and $G_0(X)$ be the spaces of self homotopy equivalences of X and self homotopy equivalences of X preserving the base point respectively.

It seems that little is known about the relation between $G(X)$ and X except an H-space X (see [**10**]). When X is not an H-space, we have Gottlieb's theorem on $G(K(\pi, 1))$ ([**4**]) and Hansen's theorem on $G(S^2)$ ([**7**]).

Of the projective plane P^2, it is known that the spaces $\mathrm{Top}(P^2)$ and $\mathrm{Diff}(P^2)$ both have the same homotopy type as $SO(3)$, where $\mathrm{Top}(P^2)$ and $\mathrm{Diff}(P^2)$ are the space of homeomorphisms of P^2 and the space of diffeomorphisms of P^2 respectively. This was proved for $\mathrm{Top}(P^2)$ by M.-E. Hamstrom [**5**] in 1965 and for $\mathrm{Diff}(P^2)$ by C. J. Earle and J. Eells [**3**] in 1969.

In this paper we shall prove the following

THEOREM. *There is a homeomorphism*

$$G(P^2) \cong SO(3) \times (G_0(P^2)/O(2)).$$

2. Compact Lie group actions.

Throughout this paper, all spaces will be Hausdorff spaces with base points when necessary and all spaces of maps will be equipped with the compact open topology.

For Lie group actions on manifolds, we have the following

PROPOSITION 1. *Let L be a compact Lie group which transitively acts on a connected closed manifold M. And let S be the isotropy subgroup at the base point. Then there is a homeomorphism*

$$G(M) \cong L \times_S G_0(M).$$

PROOF. We begin by defining an L-action on $G(M)$ by $\sigma \cdot f = \sigma \circ f$ for $\sigma \in L$,

$f \in G(M)$. Let ω be the evaluation map at the base point x_0 of $G(M)$ onto $L/S \cong M$. Then we see that ω is an L-map. Obviously, we have $\omega^{-1}(x_0)= G_0(M)$. By (4.4) Proposition in [**2**], there exists a homeomorphism

$$G(M) \cong L \times_S G_0(M) .$$

Let M be the n-sphere S^n. Then $O(n+1)$ transitively acts on S^n in the usual manner and its isotropy subgroup is $O(n)$. So, we have the following

EXAMPLE 1. There are homeomorphisms

$$G(S^n) \cong O(n+1) \times_{O(n)} G_0(S^n) ,$$

$$G^+(S^n) \cong SO(n+1) \times_{SO(n)} G_0^+(S^n) ,$$

where $G^+(S^n)$ and $G_0^+(S^n)$ denote the path components at the identity element of $G(S^n)$ and $G_0(S^n)$ respectively.

Similarly, $SO(n+1)$ transitively acts on the n dimensional real projective space P^n in the usual manner. Then there exists a natural homeomorphism

$$SO(n+1)/O'(n) \cong P^n ,$$

$$O'(n) = \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \sigma \end{pmatrix} \in SO(n+1) \right\} ,$$

where $O'(n)$ is isomorphic to $O(n)$. So, we have the following

EXAMPLE 2. There is a homeomorphism

$$G(P^n) \cong SO(n+1) \times_{O'(n)} G_0(P^n) .$$

Now, when a topological group L effectively acts on a connected closed manifold M, we easily see that L freely acts on $G(M)$. Also when a topological group L effectively acts on a connected closed manifold M leaving the base point fixed, we see that L freely acts on $G_0(M)$. Especially, for a compact Lie group L we have the following principal fibre bundles ([**1**], [**2**])

$$L \longrightarrow G_0(M) \longrightarrow G_0(M)/L .$$

In the sequel we need

PROPOSITION 2.2. *Let L be a compact Lie group and X be an* ANR *for the class of all metrizable spaces. If L freely acts on X, then the orbit space X/L is an* ANR.

PROOF. For each point x of X/L, there exists a neighbourhood V of x such that $V \times L$ is homeomorphic to an open subspace of X (see 5.4, Theorem in Chap. II ([**1**])). Since X is an ANR, $V \times L$ is an ANR (see [**6**], [**8**]). So,

V is a retract of ANR $V \times L$. Consequently, V is an ANR. That is, X/L is locally an ANR. This means that X/L is an ANR.

3. Hansen's theorem on $G(S^2)$.

Consider the usual $SO(2)$-action on the based 2-sphere S^2. This action induces the free $SO(2)$-action on $G_0^+(S^2)$. Therefore we have a principal fibre bundle

$$SO(2) \xrightarrow{i} G_0^+(S^2) \xrightarrow{p} G_0^+(S^2)/SO(2).$$

About this principal fibre bundle, we have the following

PROPOSITION 3.1. *The principal fibre bundle*

$$SO(2) \xrightarrow{i} G_0^+(S^2) \xrightarrow{p} G_0^+(S^2)/SO(2)$$

is trivial.

PROOF. First, we shall show that $G_0^+(S^2)/SO(2)$ has the same homotopy type as CW complex. Clearly $G_0^+(S^2)$ is an ANR for the class of all metrizable spaces (see Theorem 3.1 in Chap. VI [8]). By Proposition 2.2 the orbit space $G_0^+(S^2)/SO(2)$ is an ANR. Therefore, by the famous Milnor's theorem ([**9**]), $G_0^+(S^2)/SO(2)$ has the homotopy type of a CW complex.

Next, since $i_* : \pi_1(SO(2)) \to \pi_1(G_0^+(S^2))$ is an isomorphism, $p_* : \pi_n(G_0^+(S^2)) \to \pi_n(G_0^+(S^2)/SO(2))$ are isomorphisms for $n \geqq 2$. So, we can easily see that $G_0^+(S^2)/SO(2)$ has the same homotopy type as the universal covering space $\tilde{\Omega}$ of $G_0^+(S^2)$. Furthermore $\tilde{\Omega}$ has the same homotopy type as the universal covering space $\tilde{\Omega}^2(S^3)$ of the double loop space $\Omega^2(S^3)$ of S^3. Therefore we have

$$\pi_1(\tilde{\Omega}) = 0, \qquad \pi_2(\tilde{\Omega}) \cong \pi_4(S^3) \cong \boldsymbol{Z}_2.$$

Consequently, we have $H_2(\tilde{\Omega}, \boldsymbol{Z}) \cong \boldsymbol{Z}_2$. So, by the universal coefficient theorem we get $H^2(\tilde{\Omega}; \boldsymbol{Z}) = 0$. This means the classifying map of $G_0^+(S^2)/SO(2)$ into $BSO(2)$ for the principal fibre bundle must be trivial. Thus our bundle is trivial.

As a special case of Example 1, there is the principal fibre bundle

$$SO(3) \longrightarrow G^+(S^2) \longrightarrow G_0^+(S^2)/SO(2).$$

About this bundle we have immediately the following from Proposition 3.1

COROLLARY 3.2. *The principal fibre bundle*

$$SO(3) \longrightarrow G^+(S^2) \longrightarrow G_0^+(S^2)/SO(2)$$

is trivial.

PROOF. By Proposition 3.1 we see that the principal fibre bundle

$$SO(2) \longrightarrow G_0^+(S^2) \longrightarrow G_0^+(S^2)/SO(2)$$

has a cross-section σ. Since this bundle is the restriction of our principal fibre bundle on $G_0^+(S^2)$, our bundle has the cross-section σ.

Corollary 3.2 provides the following equivalences

$$G^+(S^2) \cong SO(3) \times (G_0^+(S^2)/SO(2)),$$

$$G^+(S^2) \simeq SO(3) \times \tilde{\Omega}^2(S^3).$$

The second homotopy equivalence was proved by V. L. Hansen ([7]).

4. A splitting of $G_0(P^2)$.

Consider the usual $SO(2)$-action on the based projective plane P^2. This action is obviously effective and induces the free action on $G_0^+(P^2)$. Thus we have the following principal fibre bundle

$$SO(2) \longrightarrow G_0^+(P^2) \longrightarrow G_0^+(P^2)/SO(2).$$

We shall show that this principal fibre bundle has a cross-section.

Let π be the covering map of S^2 onto P^2 and let $\mathrm{map}_0(S^2, P^2; \pi)$ denote the path component of π in the based map space $\mathrm{map}_0(S^2, P^2)$ from S^2 into P^2. Furthermore we define an $SO(2)$-action on $\mathrm{map}_0(S^2, P^2; \pi)$ by $\sigma \cdot f = \sigma \circ f$ for $\sigma \in SO(2)$ and $f \in \mathrm{map}_0(S^2, P^2; \pi)$. Then we see that this action is free, so we have the following principal fibre bundle

$$SO(2) \longrightarrow \mathrm{map}_0(S^2, P^2; \pi) \longrightarrow \mathrm{map}_0(S^2, P^2; \pi)/SO(2).$$

Denote $\pi^\# : G_0^+(P^2) \to \mathrm{map}_0(S^2, P^2; \pi)$ the map induced by π. We easily see that $\pi^\#$ is an $SO(2)$-map. Thus we have a bundle map between principal fibre bundles as follows

$$\begin{array}{ccccc} SO(2) & \longrightarrow & G_0^+(P^2) & \longrightarrow & G_0^+(P^2)/SO(2) \\ \| & & \downarrow \pi^\# & & \downarrow \bar{\pi} \\ SO(2) & \longrightarrow & \mathrm{map}_0(S^2, P^2; \pi) & \longrightarrow & \mathrm{map}_0(S^2, P^2; \pi)/SO(2), \end{array}$$

where $\bar{\pi}$ is the map between orbit spaces induced by $\pi^\#$. Then we have the following

LEMMA 4.1. *If the principal fibre bundle:*

$$SO(2) \longrightarrow \mathrm{map}_0(S^2, P^2; \pi) \longrightarrow \mathrm{map}_0(S^2, P^2; \pi)/SO(2)$$

has a cross-section, then our principal fibre bundle:

$$SO(2) \longrightarrow G_0^+(P^2) \longrightarrow G_0^+(P^2)/SO(2)$$

has also a cross-section.

Proof is straightforward and it is omitted.

Next, we consider the map $\pi_\#$ of $G_0^+(S^2)$ into $\mathrm{map}_0(S^2, P^2; \pi)$ induced by π. Then it is easily seen that $\pi_\#$ is an $SO(2)$-map. So, we have a bundle map between principal bundles as follows

$$\begin{array}{ccccc} SO(2) & \longrightarrow & G_0^+(S^2) & \longrightarrow & G_0^+(S^2)/SO(2) \\ \| & & \downarrow \pi_\# & & \downarrow \bar{\pi}_\# \\ SO(2) & \longrightarrow & \mathrm{map}_0(S^2, P^2; \pi) & \longrightarrow & \mathrm{map}_0(S^2, P^2; \pi)/SO(2), \end{array}$$

where $\bar{\pi}_\#$ is the map between orbit spaces induced by $\pi_\#$.

In the following we shall show that $\pi_\#$ is a homotopy equivalence.

Now, there is a fibration

$$S^2 \xrightarrow{i} P^2 \xrightarrow{j} P^\infty,$$

where P^∞ denotes the infinite real projective space and i coincides with the covering map π homotopically. Therefore we have the following fibration

$$G_0^+(S^2) \xrightarrow{i_\#} \mathrm{map}_0(S^2, P^2; \pi) \xrightarrow{j_\#} \mathrm{map}_0(S^2, P^\infty)$$

and we know that $\mathrm{map}_0(S^2, P^\infty)$ is contractible. This means that $i_\#$ is a homotopy equivalence. That is, $\pi_\#: G_0^+(S^2) \to \mathrm{map}_0(S^2, P^2; \pi)$ is a homotopy equivalence.

We are ready to show our main lemma.

LEMMA 4.2. *The principal fibre bundle*

$$SO(2) \longrightarrow G_0^+(P^2) \longrightarrow G_0^+(P^2)/SO(2)$$

has a cross-section.

PROOF. By the exactness of homotopy sequences for fibrations, we see that $\bar{\pi}_\#$ is a weak homotopy equivalence. On the other hand, both the spaces $G_0^+(S^2)/SO(2)$ and $\mathrm{map}_0(S^2, P^2; \pi)/SO(2)$ have homotopy types of CW complexes. Consequently we see that $\bar{\pi}_\#$ is actually a homotopy equivalence. Now, the proof of our lemma follows from Proposition 3.1 and Lemma 4.1.

Finally, from Lemma 4.2 it can be proved that the principal fibre bundle

$$SO(3) \longrightarrow G(P^2) \longrightarrow G_0(P^2)/O(2) \cong G_0^+(P^2)/SO(2)$$

has a cross-section (cf. Corollary 3.2). Immediately we get

THEOREM 4.3. *There is a homeomorphism*

$$G(P^2) \cong SO(3)\times(G_0(P^2)/O(2)) \cong SO(3)\times(G_0^+(P^2)/SO(2)).$$

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