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On a class of hypoelliptic differential operators with double characteristics

Dedicated to Professor Mutsuhide Matsumura on his 60th birthday in 1991

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Introduction and results.

This paper is devoted to the study of *hypoellipticity* for second-order degenerate *elliptic* differential operators $P(x, D)$ with real coefficients on $\boldsymbol{R}^n$ of the form:

$$P(x, D)=\frac{\partial^2}{\partial x_1^2}+\sum_{i,j=2}^{n}\frac{\partial}{\partial x_i}\left(a^{ij}(x)\frac{\partial}{\partial x_j}\right)+\sum_{i=1}^{n}b^i(x)\frac{\partial}{\partial x_i}+c(x),$$

where:

1) The a^{ij} are the components of a C^∞ symmetric contravariant tensor of type $\binom{2}{0}$ on $\boldsymbol{R}^n$, and

$$\sum_{i,j=2}^{n}a^{ij}(x)\xi_i\xi_j\geqq 0 \qquad \text{on } T^*(\boldsymbol{R}^n).$$

Here $T^*(\boldsymbol{R}^n)$ is the cotangent bundle of $\boldsymbol{R}^n$.

2) $b^i\in C^\infty(\boldsymbol{R}^n)$.

3) $c\in C^\infty(\boldsymbol{R}^n)$.

Let u be a distribution on an open subset Ω of $\boldsymbol{R}^n$. The singular support of u, denoted by $\operatorname{sing\,supp} u$, is the complement of the largest open subset of Ω on which u is of class C^∞. A differential operator $P(x, D)$ is said to be *hypoelliptic* in Ω if it satisfies the condition:

$$\operatorname{sing\,supp} u=\operatorname{sing\,supp} Pu \qquad \text{for all} \quad u\in\mathcal{D}'(\Omega).$$

This condition is equivalent to the following:

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$$\begin{cases} \text{For any open subset } \Omega' \text{ of } \Omega, \text{ we have} \\ u \in \mathscr{D}'(\Omega),\ Pu \in C^\infty(\Omega') \Longrightarrow u \in C^\infty(\Omega'). \end{cases}$$

We say that $P(x, D)$ is *globally hypoelliptic* in Ω if it satisfies the weaker condition:

$$u \in \mathscr{D}'(\Omega),\ Pu \in C^\infty(\Omega) \Longrightarrow u \in C^\infty(\Omega).$$

To state our fundamental hypothesis for the operator $P(x, D)$, we let

$$\Phi = \frac{\partial}{\partial x_1} \otimes \frac{\partial}{\partial x_1} + \sum_{i,j=2}^{n} a^{ij}(x) \frac{\partial}{\partial x_i} \otimes_S \frac{\partial}{\partial x_j},$$

which lies in the space $\Gamma(\boldsymbol{R}^n, T(\boldsymbol{R}^n) \otimes_S T(\boldsymbol{R}^n))$ of C^∞ symmetric contravariant tensor fields of type $\binom{2}{0}$ on $\boldsymbol{R}^n$. Here the notation $\otimes_S$ stands for the symmetric tensor product:

$$\frac{\partial}{\partial x_i} \otimes_S \frac{\partial}{\partial x_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_j} \otimes \frac{\partial}{\partial x_i}\right).$$

Denote by $\Gamma(\boldsymbol{R}^n, T^*(\boldsymbol{R}^n))$ (resp. $\Gamma(\boldsymbol{R}^n, T(\boldsymbol{R}^n))$) the space of C^∞ covariant (resp. contravariant) vector fields on $\boldsymbol{R}^n$. Then, making use of Φ, we can define a mapping

$$\begin{aligned} \Psi : \Gamma(\boldsymbol{R}^n, T^*(\boldsymbol{R}^n)) &\longrightarrow \Gamma(\boldsymbol{R}^n, T(\boldsymbol{R}^n)) \\ \zeta &\longmapsto \Phi(\zeta, \cdot). \end{aligned}$$

In terms of local coordinates $x=(x_1, x_2, \cdots, x_n)$, we have for $\zeta=\sum_{i=1}^n \zeta_i dx_i$

$$\Psi(\zeta) = \zeta_1 \frac{\partial}{\partial x_1} + \sum_{i,j=2}^{n} a^{ij}(x) \zeta_i \frac{\partial}{\partial x_j}.$$

We let

$$\begin{aligned} X_1 &= \text{the image of } \Psi \\ &= \{\Psi(\zeta)\,;\ \zeta \in \Gamma(\boldsymbol{R}^n, T^*(\boldsymbol{R}^n))\}. \end{aligned}$$

Further we define the *drift vector field* X_0 by

$$X_0 = \sum_{i=1}^{n} b^i(x) \frac{\partial}{\partial x_i}.$$

The fundamental hypothesis for the operator $P(x, D)$ is the following:

(H) *The Lie algebra $\mathcal{L}(X)$ over $\boldsymbol{R}$ generated by $X=X_1 \cup X_0$ has rank n outside a closed subset S of the hypersurface $\{x=(x_1, x_2, \cdots, x_n) \in \boldsymbol{R}^n\,;\ x_1=0\}$.*

By the celebrated theorem of Hörmander ([**Hr1**, Theorem 1.1]), we know that the operator $P(x, D)$ is hypoelliptic outside the set S. Furthermore, Oleĭnik and Radkevič proved (cf. [**OR**, Theorem 2.6.3]; [**A**, Theorem 1]) that:

If the set S is compact in Ω, then the operator $P(x, D)$ is globally hypoelliptic *in Ω.*

The purpose of this paper is to give sufficient conditions for *hypoellipticity* for the operator $P(x, D)$ under condition (H). Some previous results in this direction are due to Fedii [**F**], Kusuoka-Stroock [**KS**], Morimoto [**Mo**], Hoshiro [**Ho**] and also Morioka [**Ma**]. The results here extend and improve substantially those results in a *unified* theory.

To state hypotheses for the a^{ij}, we let

$$\alpha(x, \xi') = \alpha(x_1, x', \xi') = \sum_{i,j=2}^{n} a^{ij}(x_1, x')\xi_i\xi_j,$$

where

$$x' = (x_2, \cdots, x_n), \quad \xi' = (\xi_2, \cdots, \xi_n),$$

and the variable x_1 is considered as a *parameter*.

For the a^{ij}, we assume that:

(A.1) There exists a constant $a_0 > 0$ such that

$$\sum_{i,j=2}^{n} \left| \frac{\partial^2 \alpha}{\partial x_i \partial \xi_j}(x_1, x', \xi') \right|^2 \leqq a_0 \alpha(x_1, x', \xi') \quad \text{on } T^*(\boldsymbol{R}^{n-1}).$$

This condition is satisfied if $\alpha(x, \xi')$ is diagonal, that is, if $a^{ij}(x)=0$ for $i \neq j$.

(A.2) The function

$$\mu(x) = \mu(x_1, x') = \min_{|\xi'|=1} \alpha(x_1, x', \xi')$$

is Lipschitz continuous in the variable x_1 and is of class C^∞ in the variables x', and satisfies the condition:

$$\mu(x_1, x') > 0 \qquad \text{outside the set } S.$$

We remark that condition (A.2) implies that the operator $P(x, D)$ is elliptic outside the set S, so condition (H) is satisfied.

For the b^i, we assume that:

(B) There exists a constant $b_0 > 0$ such that

$$\sum_{i=2}^{n} |b^i(x)| \leqq b_0 \sqrt{\mu(x)} \qquad \text{on } \boldsymbol{R}^n.$$

Now we can state our main result (cf. [**WS**, Theorem 4.9]):

THEOREM 1. *Assume that conditions* (A.1), (A.2) *and* (B) *are satisfied and that*

$$\lim_{x_1 \to 0} \frac{\tilde{\lambda}(x_1, x') \log \mu(x_1, x')}{\sqrt{\lambda(x_1, x')}} = 0 \tag{0.1}$$

uniformly in the variables $x' = (x_2, \cdots, x_n)$ over compact subsets of $\boldsymbol{R}^{n-1}$ which *intersect the set S, where*

$$\lambda(x_1, x') = \sum_{i=2}^{n} a^{ii}(x_1, x'),$$

$$\tilde{\lambda}(x_1, x') = \int_0^{x_1} \lambda(t, x')dt.$$

Then the operator $P(x, D)$ is hypoelliptic *in $\boldsymbol{R}^n$, that is,*

$$\operatorname{sing\,supp} Pu = \operatorname{sing\,supp} u \qquad \text{for all } u \in \mathcal{D}'(\boldsymbol{R}^n).$$

REMARK. If the function $\lambda(x_1, x')$ is monotone increasing for $x_1>0$ and is monotone decreasing for $x_1<0$, that is, if we have

$$x_1 \lambda_{x_1}(x_1, x') \geqq 0 \qquad \text{on } \boldsymbol{R}^n,$$

then the above condition (0.1) may be replaced by the following simpler one:

$$\lim_{x_1 \to 0} \sqrt{\lambda(x_1, x')}\, x_1 \log \mu(x_1, x') = 0. \tag{0.1'}$$

In fact, it suffices to note that we have

$$|\tilde{\lambda}(x_1, x')| = \left| \int_0^{x_1} \lambda(t, x')dt \right| \leqq \lambda(x_1, x')|x_1| \qquad \text{on } \boldsymbol{R}^n.$$

Thus Theorem 1 is a generalization of Theorem 4 of Hoshiro [**Ho2**].

EXAMPLE 1. Consider the following operator $P(x, D)$ on $\boldsymbol{R}^3$:

$$P(x, D) = \frac{\partial^2}{\partial x_1^2} + \frac{\partial}{\partial x_2}\left(f(x)\frac{\partial}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(g(x)\frac{\partial}{\partial x_3}\right),$$

where f and g are non-negative functions on $\boldsymbol{R}^3$ such that

$$f(x_1, x_2, x_3) > 0 \qquad \text{for } x_1 \neq 0,$$

$$g(x_1, x_2, x_3) > 0 \qquad \text{for } x_1 \neq 0.$$

Then the operator $P(x, D)$ is hypoelliptic in $\boldsymbol{R}^3$ if the following two conditions are satisfied:

$$\lim_{x_1 \to 0} \frac{\int_0^{x_1} f(t, x')dt \log g(x_1, x')}{\sqrt{f(x_1, x')}} = 0.$$

$$\lim_{x_1 \to 0} \frac{\int_0^{x_1} g(t, x')dt \log f(x_1, x')}{\sqrt{g(x_1, x')}} = 0.$$

Here the convergence is uniform in the variables $x' = (x_2, x_3)$ over compact subsets of $\boldsymbol{R}^2$.

Our method can be applied to the study of hypoellipticity for second-order degenerate *parabolic* differential operators $Q(x, D)$ with real coefficients on $\boldsymbol{R}^n$ of the form:

$$Q(x, D)=\frac{\partial^2}{\partial x_1^2}+\sum_{i,j=2}^{n-1}\frac{\partial}{\partial x_i}\left(a^{ij}(x)\frac{\partial}{\partial x_j}\right)+b^n(x)\frac{\partial}{\partial x_n}+c(x),$$

where:

1) The a^{ij} are the components of a C^∞ symmetric contravariant tensor of type $\binom{2}{0}$ on $\boldsymbol{R}^n$, and

$$\sum_{i,j=2}^{n-1} a^{ij}(x)\xi_i\xi_j \geqq 0 \qquad \text{on } T^*(\boldsymbol{R}^n).$$

2) $b^n\in C^\infty(\boldsymbol{R}^n)$.

3) $c\in C^\infty(\boldsymbol{R}^n)$.

The next result is due to Oleĭnik and Radkevič (cf. [**OR**, Theorem 2.6.3]; [**A**, Theorem 2]):

If condition (H) *is satisfied for* $Q(x, D)$ *and the set* S *is compact in* Ω, *and if* $b^n(x)\neq 0$ *on* S, *then the operator* $Q(x, D)$ *is* globally hypoelliptic *in* Ω.

Now we let

$$\alpha(x, \xi'')=\alpha(x_1, x', \xi'')=\sum_{i,j=2}^{n-1} a^{ij}(x_1, x')\xi_i\xi_j,$$

where

$$\xi''=(\xi_2, \cdots, \xi_{n-1}).$$

For the a^{ij}, we assume that:

(A.1′) There exists a constant $a_0>0$ such that

$$\sum_{\substack{2\leq i\leq n\\ 2\leq j\leq n-1}}\left|\frac{\partial^2\alpha}{\partial x_i\partial\xi_j}(x_1, x', \xi'')\right|^2\leqq a_0\alpha(x_1, x', \xi'') \qquad \text{on } T^*(\boldsymbol{R}^{n-1}).$$

This condition is satisfied if $\alpha(x, \xi'')$ is diagonal, that is, if $a^{ij}(x)=0$ for $i\neq j$.

(A.2′) The function

$$\mu(x)=\mu(x_1, x')=\min_{|\xi''|=1}\alpha(x_1, x', \xi'')$$

is Lipschitz continuous in the variable x_1 and is of class C^∞ in the variables x', and satisfies the condition:

$$\mu(x_1, x')>0 \qquad \text{outside the set } S.$$

For the function b^n, we assume that:

(B′) $b^n(x_1, x')\neq 0$ outside the set S, and either $b^n(x)\geqq 0$ on $\boldsymbol{R}^n$ or $b^n(x)\leqq 0$ on $\boldsymbol{R}^n$.

We remark that conditions (A.2′) and (B′) imply that condition (H) is satisfied.

Then we have the following:

THEOREM 2. *Assume that conditions* (A.1′), (A.2′) *and* (B′) *are satisfied and that*

$$\lim_{x_1\to 0}\frac{\tilde{\lambda}(x_1, x')\log|b^n(x_1, x')|}{\sqrt{\lambda(x_1, x')}}=0, \tag{0.2a}$$

$$\lim_{x_1\to 0}\frac{\tilde{b}^n(x_1, x')^2\log\mu(x_1, x')}{b^n(x_1, x')}=0, \tag{0.2b}$$

$$\lim_{x_1\to 0}\frac{\lambda(x_1, x')\log\mu(x_1, x')}{b^n(x_1, x')}=0, \tag{0.2c}$$

where

$$\lambda(x_1, x')=\sum_{i=2}^{n-1}a^{ii}(x_1, x'),$$

$$\tilde{\lambda}(x_1, x')=\int_0^{x_1}\lambda(t, x')dt,$$

$$\tilde{b}^n(x_1, x')=\int_0^{x_1}b^n(t, x')dt.$$

Here the convergence is uniform in the variables $x'=(x_2, \cdots, x_n)$ *over compact subsets of* $\boldsymbol{R}^{n-1}$ *which intersect the set* S.

Then the operator $Q(x, D)$ *is* hypoelliptic *in* $\boldsymbol{R}^n$.

The next example is a generalization of Theorem 4 of Hoshiro [**Ho1**].

EXAMPLE 2. Consider the following operator $Q(x, D)$ on $\boldsymbol{R}^3$:

$$Q(x, D)=\frac{\partial^2}{\partial x_1^2}+\frac{\partial}{\partial x_2}\left(f(x)\frac{\partial}{\partial x_2}\right)+g(x)\frac{\partial}{\partial x_3},$$

where f and g are non-negative functions on $\boldsymbol{R}^3$ such that

$$f(x_1, x_2, x_3)>0 \quad \text{for} \quad x_1\neq 0,$$

$$g(x_1, x_2, x_3)>0 \quad \text{for} \quad x_1\neq 0.$$

Then the operator $Q(x, D)$ is hypoelliptic in $\boldsymbol{R}^3$ if the following two conditions are satisfied:

$$\lim_{x_1\to 0}\frac{\int_0^{x_1}f(t, x')dt\log g(x_1, x')}{\sqrt{f(x_1, x')}}=0.$$

$$\lim_{x_1\to 0}\frac{\left(\int_0^{x_1}g(t, x')dt\right)^2\log f(x_1, x')}{g(x_1, x')}=0.$$

Here the convergence is uniform in the variables $x'=(x_2, x_3)$ over compact subsets of $\boldsymbol{R}^2$. (We remark that condition (0.2c) is superfluous for Example 2, since one may take $\mu=\lambda$ in inequality (3.9) in the proof of Theorem 2.)

The rest of this paper is organized as follows. In Section 1, we consider a family of modifications $P_{\Lambda_\delta}(x, D)$ of the operator $P(x, D)$ which is adapted to the study of hypoellipticity. The operators $P_{\Lambda_\delta}(x, D)$ are introduced in the study of propagation of singularities for hyperbolic pseudodifferential operators with double characteristics by Kajitani-Wakabayashi [**KW**]. We give a general criterion for hypoellipticity for the operator $P(x, D)$ under a weaker condition (H′) in terms of the operators $P_{\Lambda_\delta}(x, D)$ (Theorem 1.1). This criterion is more useful if it is combined with the well-known Poincaré inequality (Corollary 1.2).

Sections 2 and 3 are devoted to the proof of Theorems 1 and 2, respectively, indicating applications of such a criterion to the study of hypoellipticity for the operators $P(x, D)$ and $Q(x, D)$. The proof follows the pattern given in Section 5 of Kajitani-Wakabayashi [**KW**]. That is, we calculate the symbol of the operators $P_{\Lambda_\delta}(x, D)$ in question, and then apply a sharpened form of Gårding's inequality due to Fefferman-Phong [**FP**] (Theorem 2.1 and Corollary 2.2) to the operators $P_{\Lambda_\delta}(x, D)$. It is Lemma 2.4 that allows us to make good use of the Fefferman-Phong inequality.

This paper is inspired by the work of Wakabayashi and Suzuki [**WS**]. It is a genuine pleasure to acknowledge the great debt which I owe to S. Wakabayashi, with whom I had extensive and fruitful conversations while working on this paper. I am also grateful to T. Hoshiro, Y. Morimoto and N. Iwasaki for some useful comments.

1. A criterion for hypoellipticity.

In this section we give a general criterion for hypoellipticity for the operator $P(x, D)$ which is a variant of Theorem 1.2 of Kajitani-Wakabayashi [**KW**]. For the sake of completeness, we reproduce here its proof due to Wakabayashi (cf. [**WS**, Theorem 1.1]).

First we recall the definition of the symbol class $S^m_{1,0}(\boldsymbol{R}^n\times\boldsymbol{R}^n)$ for $m\in\boldsymbol{R}$. We say that a C^∞ function $p(x, \xi)$ on the cotangent bundle $T^*(\boldsymbol{R}^n)$ belongs to the class $S^m_{1,0}(\boldsymbol{R}^n\times\boldsymbol{R}^n)$ if, for any multi-indices α and β, there exists a constant $C_{\alpha,\beta}>0$ such that

$$|\partial_\xi^\alpha\partial_x^\beta p(x, \xi)| \leqq C_{\alpha,\beta}(1+|\xi|^2)^{(m-|\alpha|)/2} \qquad \text{for all} \quad (x, \xi)\in T^*(\boldsymbol{R}^n).$$

Here we have identified the cotangent bundle $T^*(\boldsymbol{R}^n)$ with the space $\boldsymbol{R}^n\times\boldsymbol{R}^n$.

Let $\lambda(\xi)$ be a real-valued symbol in the class $S^1_{1,0}(\boldsymbol{R}^n\times\boldsymbol{R}^n)$ such that

$$\lambda(\xi)=\begin{cases}\langle\xi'\rangle & \text{if } |\xi'|\geqq\frac{1}{2}|\xi| \text{ and } |\xi|\geqq 4,\\ \frac{1}{4}\langle\xi\rangle & \text{if } |\xi'|\leqq\frac{1}{4}|\xi| \text{ and } |\xi|\geqq 4,\end{cases}$$

and that

$$\frac{1}{4}\langle\xi\rangle\leqq\lambda(\xi)\leqq\langle\xi\rangle,\qquad \lambda(\xi)\geqq 1,$$

where

$$\xi=(\xi_1,\xi'),\qquad \xi'=(\xi_2,\cdots,\xi_n),$$
$$\langle\xi'\rangle=(1+|\xi'|^2)^{1/2},$$
$$\langle\xi\rangle=(1+|\xi|^2)^{1/2}.$$

Let $x^0=(x_1^0, x_2^0, \cdots, x_n^0)$ be a point of a subset T of the $(n-k)$ dimensional surface $\{x=(x_1, x_2, \cdots, x_n)\in \boldsymbol{R}^n;\ x_1=x_2=\cdots=x_k=0\}$. If $0\leqq\delta\leqq 1$, $a\geqq 0$, $N\geqq 0$ and $s\in\boldsymbol{R}$, we let

$$\begin{aligned}\Lambda_\delta(x''',\xi)&=\Lambda_\delta(x''',\xi;\ a, N, s)\\&=(-s+a|x'''-x^{0'''}|^2)\log\lambda(\xi)+N\log(1+\delta\lambda(\xi)),\end{aligned}$$

where

$$x'''=(x_{k+1},\cdots,x_n),$$
$$x^{0'''}=(x_{k+1}^0,\cdots,x_n^0).$$

We remark that

$$e^{\Lambda_\delta(x''',\xi)}=\lambda(\xi)^{(-s+a|x'''-x^{0'''}|^2)}(1+\delta\lambda(\xi))^N,$$
$$e^{-\Lambda_\delta(x''',\xi)}=\lambda(\xi)^{(s-a|x'''-x^{0'''}|^2)}(1+\delta\lambda(\xi))^{-N},$$

and that

$$|\partial_\xi^\alpha\partial_{x'''}^\beta(e^{\pm\Lambda_\delta(x''',\xi)})|\leqq C_{\alpha,\beta}\langle\xi\rangle^{-|\alpha|}(1+\log\langle\xi\rangle)^{|\beta|}e^{\pm\Lambda_\delta(x''',\xi)},$$

where the constant $C_{\alpha,\beta}$ is independent of δ.

Furthermore we introduce a family of second-order pseudodifferential operators $P_{\Lambda_\delta}(x, D)$ defined by the formula

$$P_{\Lambda_\delta}(x,D)=e^{-\Lambda_\delta}(x''',D)P(x,D)e^{\Lambda_\delta}(x''',D),$$

where $e^{\pm\Lambda_\delta}(x''', D)$ are properly supported pseudodifferential operators with symbols $e^{\pm\Lambda_\delta(x''',\xi)}$, respectively.

Now we can state a criterion for hypoellipticity for the operator $P(x, D)$:

THEOREM 1.1. *Assume that:*
(H′) *The operator* $P(x, D)$ *is* hypoelliptic *outside a closed subset* T *of the*

$(n-k)$ dimensional surface $\{x=(x'', x''')\in \boldsymbol{R}^n;\ x''=0\}$, where $x''=(x_1, \cdots, x_k)$ and $x'''=(x_{k+1}, \cdots, x_n)$ for $1\leqq k\leqq n-1$.

Furthermore, assume that, for each point x^0 of the set T, there exist an open neighborhood $U(x^0)$ of x^0 and numbers $a_0\geqq 0$, $N_0\geqq 0$ and $s_0\in \boldsymbol{R}$ such that:

For any $a\geqq a_0$, any $N\geqq N_0$, and any $s\geqq s_0$, there exist functions $\theta(x''')\in C^\infty(\boldsymbol{R}^{n-k})$ and $\phi(x)\in C^\infty(\boldsymbol{R}^n)$ with $\operatorname{supp}(1-\theta)\cap\{x^{0'''}\}=\emptyset$ and $\operatorname{supp}\phi\cap T=\emptyset$ and constants $0<\delta_0\leqq 1$ and $C>0$ such that the estimate

$$\|v\| \leqq C(\|P_{\Lambda_\delta}(x, D)v\|+\|(1-\theta(x'''))v\|+\|\phi(x)v\|) \tag{1.1}$$

holds for all $v\in C_0^\infty(U(x^0))$ and all $0<\delta\leqq\delta_0$. Here $\|\cdot\|$ is the norm of the space $L^2(\boldsymbol{R}^n)$.

Then the operator $P(x, D)$ is hypoelliptic in $\boldsymbol{R}^n$.

PROOF. Let x^0 be an arbitrary point of the set T. Assume that $u\in\mathscr{D}'(\boldsymbol{R}^n)$ and the function

$$f = P(x, D)u$$

is of class C^∞ in a neighborhood of x_0.

Without loss of generality, one may assume that

$$x^0 = (0, 0),$$
$$U(x^0) = \{x = (x'', x''') \in \boldsymbol{R}^n;\ |x''| < 1,\ |x'''| < 1\}.$$

We take three open neighborhoods U_1, U_2, U_3 of $x^0=(0, 0)$ such that

$$U_1 = \left\{x = (x'', x''') \in \boldsymbol{R}^n;\ |x''| < \frac{3}{4},\ |x'''| < \frac{3}{4}\right\},$$

$$U_2 = \left\{x = (x'', x''') \in \boldsymbol{R}^n;\ |x''| < \frac{1}{2},\ |x'''| < \frac{1}{2}\right\},$$

$$U_3 = \left\{x = (x'', x''') \in \boldsymbol{R}^n;\ |x''| < \frac{1}{4},\ |x'''| < \frac{1}{4}\right\}.$$

One may assume that for some $s'\in\boldsymbol{R}$

$$u \in \mathcal{E}'(\boldsymbol{R}^n)\cap H^{s'}(\boldsymbol{R}^n),$$

and that the function f is of class C^∞ near the set U_1. For each $\sigma>s'$, we can choose numbers $a\geqq a_0$ and $s\geqq s_0$ such that

$$\begin{cases} s-\dfrac{1}{16}a > \sigma, \\ s-\dfrac{1}{4}a < s'-1, \end{cases}$$

and also choose a number $N\geqq N_0$ such that

$$N > s - s' + 2.$$

Now, by the calculus of pseudodifferential operators, one can find an *elliptic* symbol $q_\delta(x''', \xi) = q_\delta(x''', \xi; a, N, s)$ in the class $S^0_{1,0}(\boldsymbol{R}^n \times \boldsymbol{R}^n)$ such that

$$e^{\Lambda_\delta}(x''', D)e^{-\Lambda_\delta}(x''', D)q_\delta(x''', D) \equiv I \quad \text{modulo an operator of order } -\infty.$$

If χ and χ_1 are functions in $C_0^\infty(U_1)$ such that

$$\begin{cases} \chi(x) = 1 & \text{on } U_2, \\ \chi_1(x) = 1 & \text{near } \operatorname{supp} \chi, \end{cases}$$

we let

$$v_\delta = \chi_1(x)e^{-\Lambda_\delta}(x''', D)q_\delta(x''', D)(\chi u).$$

Then we have

$$\begin{aligned} &\|P_{\Lambda_\delta}(x, D)v_\delta - e^{-\Lambda_\delta}(x''', D)(\chi f) - e^{-\Lambda_\delta}(x''', D)[P, \chi]u\| \\ (1.2) \qquad &= \|e^{-\Lambda_\delta}(x''', D)P(x, D)(e^{\Lambda_\delta}(x''', D)\chi_1(x)e^{-\Lambda_\delta}(x''', D)q_\delta(x''', D) - I)(\chi u)\| \\ &\leqq C\|u\|_{s'}. \end{aligned}$$

Here and in the following the letter C denotes a generic positive constant *independent* of $\delta (0 < \delta \leqq \delta_0)$, and $\|\cdot\|_s$ is the norm of the Sobolev space $H^s(\boldsymbol{R}^n)$ of order s.

Furthermore, since the operator $e^{-\Lambda_\delta}(x''', D)$ is of order at most s, it follows that

$$(1.3) \qquad \|e^{-\Lambda_\delta}(x''', D)(\chi f)\| \leqq C\|\chi f\|_s.$$

We also have

$$(1.4) \qquad \|e^{-\Lambda_\delta}(x''', D)[P, \chi]u\| \leqq C\|u\|_{s'}.$$

In fact, if $\tilde{\chi}$ is a function in $C_0^\infty(U_1)$ such that $\tilde{\chi}(x) = 1$ on U_2 and $\tilde{\chi}\chi = \tilde{\chi}$ and if η is a function in $C^\infty(\boldsymbol{R}^n)$ such that $\eta(x) = 1$ near supp $(1 - \tilde{\chi})$ and $\operatorname{supp} \eta \cap U_2 = \emptyset$, then it follows that

$$\begin{aligned} e^{-\Lambda_\delta}(x''', D)[P, \chi]u = e^{-\Lambda_\delta}(x''', D)\tilde{\chi}[P, \chi]u + \eta e^{-\Lambda_\delta}(x''', D)(1 - \tilde{\chi})[P, \chi]u \\ + (1 - \eta)e^{-\Lambda_\delta}(x''', D)(1 - \tilde{\chi})[P, \chi]u. \end{aligned}$$

But we remark that the operators $\tilde{\chi}[P, \chi]$ and $(1 - \eta)e^{-\Lambda_\delta}(x''', D)(1 - \tilde{\chi})$ are of order $-\infty$, and the operator $\eta e^{-\Lambda_\delta}(x''', D)(1 - \tilde{\chi})$ is of order at most $s' - 1$, since $s - a|x'''|^2 < s' - 1$ for $|x'''| \geqq 1/2$. Hence we find that

$$\|e^{-\Lambda_\delta}(x''', D)[P, \chi]u\| \leqq C\|u\|_{s'}.$$

Therefore, we obtain from inequalities (1.2), (1.3) and (1.4) that

$$(1.5) \qquad \|P_{\Lambda_\delta}(x, D)v_\delta\| \leqq C(\|\chi f\|_s + \|u\|_{s'}).$$

For the term $\|(1-\theta(x'''))v_\delta\|$, without loss of generality, one may assume that

$$\operatorname{supp}(1-\theta) \subset \left\{x''' \in \boldsymbol{R}^{n-k}\,;\ |x'''| \geqq \frac{1}{2}\right\}.$$

Then we have

$$\|(1-\theta(x'''))v_\delta\| \leqq C\|u\|_{s'-1}, \tag{1.6}$$

since the operator $(1-\theta)e^{-\Lambda_\delta}(x''', D)$ is of order at most $s'-1$.

On the other hand, if $\tilde{\phi}$ is a function in $C^\infty(\boldsymbol{R}^n)$ such that $\tilde{\phi}\phi=\phi$ and $\operatorname{supp}\tilde{\phi}\cap T=\emptyset$, then it follows from condition (H′) that

$$\begin{aligned}\|\phi(x)v_\delta\| &\leqq \|\phi(x)\chi_1(x)e^{-\Lambda_\delta}(x''', D)q_\delta(x''', D)((1-\tilde{\phi})\chi u)\| \\ &\quad +\|\phi(x)\chi_1(x)e^{-\Lambda_\delta}(x''', D)q_\delta(x''', D)(\tilde{\phi}(\chi u))\| \\ &\leqq C(\|u\|_{s'}+\|\tilde{\phi}(\chi u)\|_s),\end{aligned} \tag{1.7}$$

since the operator $\phi e^{-\Lambda_\delta}(x''', D)q_\delta(x''', D)(1-\tilde{\phi})$ is of order $-\infty$ and the function $\tilde{\phi}(\chi u)$ is of class C^∞. Further we find that the function ϕv_δ is of class C^∞.

If we take another function χ_2 in $C_0^\infty(U_1)$ such that

$$\chi_2(x)=1 \quad \text{near } \operatorname{supp}\chi_1,$$

then we have for all $\delta>0$

$$v_\delta=\chi_1\phi v_\delta+(1-\chi_1\phi)\chi_2 v_\delta \in H^2(\boldsymbol{R}^n)\cap\mathcal{E}'(U_1),$$

since the operator $e^{-\Lambda_\delta}(x''', D)$ is of order $s-N$ and $s'-(s-N)>2$. But, if $\{w_j\}$ is a sequence in $C_0^\infty(U_1)$ such that

$$w_j \longrightarrow \chi_2 v_\delta \quad \text{in } H^2(\boldsymbol{R}^n),$$

then it is easy to verify the following:

(a) $\tilde{w}_j=\chi_1\phi v_\delta+(1-\chi_1\phi)w_j\in C_0^\infty(U_1)\subset C_0^\infty(U(x^0))$.
(b) $\tilde{w}_j\to v_\delta$ in $H^2(\boldsymbol{R}^n)$.
(c) $P_{\Lambda_\delta}(x, D)\tilde{w}_j\to P_{\Lambda_\delta}(x, D)v_\delta$ in $L^2(\boldsymbol{R}^n)$.
(d) $(1-\theta(x'''))\tilde{w}_j\to(1-\theta(x'''))v_\delta$ in $L^2(\boldsymbol{R}^n)$.
(e) $\phi(x)\tilde{w}_j\to\phi(x)v_\delta$ in $L^2(\boldsymbol{R}^n)$.

This proves that estimate (1.1) remains valid for the functions v_δ.

Therefore, it follows from inequalities (1.5), (1.6) and (1.7) that we have for all $0<\delta\leqq\delta_0$

$$\|v_\delta\| \leqq C(\|\chi f\|_s+\|u\|_{s'}+\|\tilde{\phi}(\chi u)\|_s).$$

Hence, letting $\delta\downarrow 0$, we find that

$$v_\delta \longrightarrow v_0 \quad \text{weakly in } L^2(\boldsymbol{R}^n),$$

where

$$v_0 = \chi_1(x)\lambda(D)^{(s-a|x'''|^2)}q_0(x''', D)(\chi u) \in L^2(\boldsymbol{R}^n).$$

But we remark that

$$\begin{cases} \lambda(\xi) \geqq 1, \\ q_0(x''', \xi) = 1 + \cdots \quad \text{near } x''' = 0, \\ \chi_1(x) = 1 \quad \text{near } x^0 = 0, \end{cases}$$

and also that we have for $|x'''| \leqq 1/4$

$$s - a|x'''|^2 > \sigma.$$

Thus, taking a function $\chi_3 \in C_0^\infty(U_3)$ such that

$$\chi_3(x) = 1 \quad \text{near } x^0 = 0,$$

we find that

$$\chi_3 u \in H^\sigma(\boldsymbol{R}^n).$$

This proves that u is of class C^∞ at $x^0=0$, since σ is arbitrary.

The proof of Theorem 1.1 is now complete.

If we combine Theorem 1.1 with the well-known Poincaré inequality, we obtain the following useful criterion for hypoellipticity (cf. [**WS**], [**Mo**]):

COROLLARY 1.2. *Assume that condition* (H′) *is satisfied and that, for each point x^0 of the set T, there exist an open neighborhood $U(x^0)$ of x^0 and numbers $a_0 \geqq 0$, $N_0 \geqq 0$ and $s_0 \in \boldsymbol{R}$ such that:*

For any $a \geqq a_0$, any $N \geqq N_0$ and any $s \geqq s_0$, there exist constants $0 < \delta_0 \leqq 1$, $C_1 > 0$ and $C_2 > 0$ such that we have for all $v \in C_0^\infty(U(x^0))$ and all $0 < \delta \leqq \delta_0$

$$|(P_{\Lambda_\delta}(x, D)v, v)| \geqq C_1\|D_{x_1}v\|^2 - C_2\|v\|^2. \tag{1.8}$$

Here $(\cdot, \cdot)$ is the inner product of the space $L^2(\boldsymbol{R}^n)$ and $D_{x_1} = 1/\sqrt{-1}\,\partial/\partial x_1$.

Then the operator $P(x, D)$ is hypoelliptic in $\boldsymbol{R}^n$.

PROOF. First we recall the Poincaré inequality:

LEMMA 1.3. *Let Ω be an open subset of $\boldsymbol{R}^n$ such that each line parallel to some line meets Ω in a set of width at most L. Then we have for all $u \in H_0^1(\Omega)$*

$$\|u\| \leqq L\Big(\sum_{j=1}^n \|D_{x_j}u\|^2\Big)^{1/2}.$$

Here $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the Sobolev space $H^1(\Omega)$.

Now, without loss of generality, one may assume that

$$x^0 = (0, 0),$$
$$U(x^0) = \{x = (x'', x''') \in \boldsymbol{R}^n;\ |x''| < 1,\ |x'''| < 1\}.$$

We choose a function $\psi(t)$ in $C_0^\infty(\boldsymbol{R})$ such that

$$\begin{cases} 0 \leqq \psi(t) \leqq 1 \quad \text{on } \boldsymbol{R}, \\ \operatorname{supp}\psi \subset \{|t| \leqq 1\}, \\ \psi(t) = 1 \quad \text{if} \quad |t| \leqq \dfrac{1}{2}. \end{cases}$$

If v is a function in $C_0^\infty(U(x^0))$, then it can be decomposed as follows:

$$v = v_1 + v_2 + v_3,$$

where

$$v_1 = \psi\Big(\frac{|x''|}{d}\Big)\psi(|x'''|)v,$$

$$v_2 = \Big(1-\psi\Big(\frac{|x''|}{d}\Big)\Big)\psi(|x'''|)v,$$

$$v_3 = (1-\psi(|x'''|))v,$$

and $d>0$ is a small parameter and will be chosen later on.

Then, applying Poincaré's inequality to the function v_1, we have

$$\|v_1\| \leqq \sqrt{2}\,d\|D_{x_1}v_1\|. \tag{1.9}$$

But we remark that

$$D_{x_1}v_1(x) = \psi\Big(\frac{|x''|}{d}\Big)\psi(|x'''|)D_{x_1}v(x) + \frac{1}{d}D_{x_1}\psi\Big(\frac{|x''|}{d}\Big)\frac{x_1}{|x''|}\psi(|x'''|)v(x),$$

and

$$D_{x_1}\psi\Big(\frac{|x''|}{d}\Big) = 0 \qquad \text{for } |x''| \leqq \frac{d}{2}.$$

Thus, if we let

$$\psi_d(x) = \Big(1-\psi\Big(\frac{3|x''|}{d}\Big)\Big)\psi\Big(\frac{|x'''|}{3}\Big),$$

we obtain that

$$\psi_d(x) = 1 \quad \text{on } \operatorname{supp}\Big[D_{x_1}\psi\Big(\frac{|x''|}{d}\Big)\psi(|x'''|)\Big],$$

so that

$$D_{x_1}\psi\Big(\frac{|x''|}{d}\Big)\frac{x_1}{|x''|}\psi(|x'''|)v(x) = D_{x_1}\psi\Big(\frac{|x''|}{d}\Big)\frac{x_1}{|x''|}\psi(|x'''|)\psi_d(x)v(x).$$

Hence we have

$$\|D_{x_1}v_1\| \leqq \|D_{x_1}v\| + \frac{1}{d}C_d\|\psi_d(x)v\|, \tag{1.10}$$

with

$$C_d = \max\left|D_{x_1}\phi\left(\frac{|x''|}{d}\right)\right|.$$

Therefore, combining inequalities (1.9) and (1.10), we obtain that

$$\|v_1\| \leqq \sqrt{2}\, d\|D_{x_1}v\| + \sqrt{2}\, C_d\|\phi_d(x)v\|\,.$$

Similarly, we have for the function v_2

$$\|v_2\| \leqq \|\phi_d(x)v\|\,.$$

In fact, it suffices to note that

$$\phi_d(x) = 1 \quad \text{on supp}\left[\left(1-\phi\left(\frac{|x''|}{d}\right)\right)\phi(|x'''|)\right].$$

Hence we have for all $v \in C_0^\infty(U(x^0))$

$$\begin{aligned} \|v\|^2 &= \|v_1+v_2+v_3\|^2 \\ &\leqq 3(\|v_1\|^2+\|v_2\|^2+\|v_3\|^2) \\ &\leqq 12d^2\|D_{x_1}v\|^2 + C'_d(\|(1-\phi(|x'''|))v\|^2+\|\phi_d(x)v\|^2), \end{aligned} \tag{1.11}$$

with a constant $C'_d > 0$.

On the other hand, we have by the Schwarz inequality

$$|(P_{\Lambda_\delta}(x, D)v, v)| \leqq 4\|P_{\Lambda_\delta}(x, D)v\|^2 + \|v\|^2. \tag{1.12}$$

Therefore, combining inequalities (1.8), (1.11) and (1.12), we have for all $v \in C_0^\infty(U(x^0))$ and all $0 < \delta \leqq \delta_0$

$$\|v\| \leqq C''_d(\|P_{\Lambda_\delta}(x, D)v\| + \|(1-\phi(|x'''|))v\| + \|\phi_d(x)v\|),$$

if we take

$$0 < d < \frac{\sqrt{C_1}}{2\sqrt{3}\sqrt{C_2+1}}.$$

Thus Corollary 1.2 follows from an application of Theorem 1.1.

2. Proof of Theorem 1.

Our proof of Theorem 1 is based on Corollary 1.2.

1) First we give a version of the criterion in Corollary 1.2 adapted to the present context.

Let $x^0 = (x_1^0, x_2^0, \cdots, x_n^0)$ be a point of a closed subset S of the hypersurface $\{x = (x_1, x') \in \boldsymbol{R}^n;\ x_1 = 0\}$, where $x' = (x_2, \cdots, x_n)$. Without loss of generality, one may assume that

$$x^0=(0,\ 0).$$

Let $\lambda(\xi)$ be a real-valued symbol in the class $S^1_{1,0}(\boldsymbol{R}^n\times\boldsymbol{R}^n)$ such that

$$\lambda(\xi)=\begin{cases}\langle\xi'\rangle & \text{if } |\xi'|\geqq\frac{1}{2}|\xi| \text{ and } |\xi|\geqq 4,\\ \frac{1}{4}\langle\xi\rangle & \text{if } |\xi'|\leqq\frac{1}{4}|\xi| \text{ and } |\xi|\geqq 4,\end{cases}$$

and that

$$\frac{1}{4}\langle\xi\rangle\leqq\lambda(\xi)\leqq\langle\xi\rangle,\quad \lambda(\xi)\geqq 1.$$

If $0\leqq\delta\leqq 1$, $a\geqq 0$, $N\geqq 0$ and $s\in\boldsymbol{R}$, we let

$$\begin{aligned}\Lambda_\delta(x',\xi)&=\Lambda_\delta(x',\xi;\ a,\ N,\ s)\\ &=(-s+a|x'|^2)\log\lambda(\xi)+N\log(1+\delta\lambda(\xi)),\end{aligned}$$

and

$$P_{\Lambda_\delta}(x,\ D)=e^{-\Lambda_\delta}(x',\ D)P(x,\ D)e^{\Lambda_\delta}(x',\ D),$$

where $e^{\pm\Lambda_\delta}(x',\ D)$ are properly supported pseudodifferential operators with symbols $e^{\pm\Lambda_\delta(x',\xi)}$, respectively:

$$\begin{aligned}e^{\Lambda_\delta(x',\xi)}&=\lambda(\xi)^{(-s+a|x'|^2)}(1+\delta\lambda(\xi))^N,\\ e^{-\Lambda_\delta(x',\xi)}&=\lambda(\xi)^{(s-a|x'|^2)}(1+\delta\lambda(\xi))^{-N}.\end{aligned}$$

By virtue of Corollary 1.2, in order to prove the hypoellipticity for the operator $P(x,\ D)$, it suffices to show that there exists an open neighborhood $U_{\varepsilon_0}=\{x=(x_1,\ x')\in\boldsymbol{R}^n;\ |x_1|<\varepsilon_0,\ |x'|<1\}$ of $x^0=(0,\ 0)$ such that we have for all $v\in C_0^\infty(U_{\varepsilon_0})$ and all $0<\delta\leqq 1$

$$\text{(2.1)}\qquad \mathrm{Re}(P_{\Lambda_\delta}(x,\ D)v,\ v)\geqq C_1\|D_{x_1}v\|^2-C_2\|v\|^2,$$

with constants $C_1>0$ and $C_2>0$ independent of δ.

2) In the proof of inequality (2.1), we make good use of the following Fefferman-Phong inequality (cf. [**FP**, Theorem]; [**Hr2**, Corollary 18.6.11]):

THEOREM 2.1. *If $p(x',\ \xi')$ is a symbol in the class $S^2_{1,0}(\boldsymbol{R}^{n-1}\times\boldsymbol{R}^{n-1})$ such that $p(x',\ \xi')\geqq 0$ on $\boldsymbol{R}^{n-1}\times\boldsymbol{R}^{n-1}$, then we have for all $v\in C_0^\infty(\boldsymbol{R}^{n-1})$*

$$\mathrm{Re}(p(x',\ D')v,\ v)\geqq -C\|v\|^2.$$

Here the constant C may be chosen uniformly in the $p(x',\ \xi')$ in a bounded subset of $S^2_{1,0}(\boldsymbol{R}^{n-1}\times\boldsymbol{R}^{n-1})$.

COROLLARY 2.2. *Let $p(x_1,\ x',\ \xi')$ be a symbol in the class $S^2_{1,0}(\boldsymbol{R}^{n-1}\times\boldsymbol{R}^{n-1})$ such that $p(x_1,\ x',\ \xi')\geqq 0$ on $\boldsymbol{R}^{n-1}\times\boldsymbol{R}^{n-1}$, where the variable x_1 is considered as a* *parameter. If the family* $\{p(x_1, x', \xi')\}_{x_1\in \boldsymbol{R}}$ *forms a bounded subset of* $S^2_{1,0}(\boldsymbol{R}^{n-1}\times\boldsymbol{R}^{n-1})$*, then we have for all* $u\in C_0^\infty(\boldsymbol{R}^n)$

$$\mathrm{Re}(p(x_1, x', D')u, u) \geqq -C\|u\|^2. \tag{2.2}$$

Here the constant C *may be chosen uniformly in the* $p(x_1, x', \xi')$.

PROOF. If we apply Theorem 2.1 to the functions $u(x_1, \cdot)\in C_0^\infty(\boldsymbol{R}^{n-1})(x_1\in \boldsymbol{R})$, we obtain that

$$\mathrm{Re}\int_{\boldsymbol{R}^{n-1}} p(x_1, x', D')u(x_1, x')\cdot \overline{u(x_1, x')}dx' \geqq -C\int_{\boldsymbol{R}^{n-1}}|u(x_1, x')|^2 dx'.$$

Hence inequality (2.2) follows by integrating the both sides with respect to x_1.

3) In order to calculate the symbol of the operator $P_{\Lambda_\delta}(x, D)$, we remark that the operator $P(x, D)$ is *micro-elliptic* outside a conic neighborhood of a point $(x^0, \xi^0)=(x^0, 0, \xi_2^0, \cdots, \xi_n^0)$ in the bundle $T^*(\boldsymbol{R}^n)\backslash 0$ of non-zero cotangent vectors. Here a *conic* subset C of $T^*(\boldsymbol{R}^n)$ is such a set that $(x, \xi)\in C$ implies $(x, r\xi)\in C$ for all $r>0$. Hence, without loss of generality, one may assume that

$$4 \leqq |\xi| \leqq 2|\xi'|, \quad \xi=(\xi_1, \xi'),$$

and that

$$\begin{cases} \lambda(\xi) = (1+|\xi'|^2)^{1/2} = \langle\xi'\rangle, \\ \Lambda_\delta(x', \xi) = \Lambda_\delta(x', \xi') = (-s+a|x'|^2)\log\langle\xi'\rangle + N\log(1+\delta\langle\xi'\rangle). \end{cases}$$

Then, for the derivatives of the symbol $\Lambda_\delta(x', \xi')$, we have the following:

$$\Lambda_{\delta x_j}(x', \xi') = 2ax_j\log\langle\xi'\rangle, \qquad 2\leqq j\leqq n,$$

$$\Lambda_{\delta x_j x_k}(x', \xi') = 2a\delta_{jk}\log\langle\xi'\rangle, \qquad 2\leqq j,\, k\leqq n,$$

$$\Lambda_{\delta\xi_j}(x', \xi') = \left\{\left[(-s+a|x'|^2)+N\frac{\delta\langle\xi'\rangle}{1+\delta\langle\xi'\rangle}\right]\frac{\xi_j}{\langle\xi'\rangle}\right\}\frac{1}{\langle\xi'\rangle}, \quad 2\leqq j\leqq n,$$

$$\begin{aligned}\Lambda_{\delta\xi_j\xi_k}(x', \xi') = \Bigg\{&\left[(-s+a|x'|^2)+N\frac{\delta\langle\xi'\rangle}{1+\delta\langle\xi'\rangle}\right]\left(\delta_{jk}-\frac{\xi_j}{\langle\xi'\rangle}\frac{\xi_k}{\langle\xi'\rangle}\right)\\ &-N\frac{\delta\langle\xi'\rangle}{1+\delta\langle\xi'\rangle}\frac{\delta\langle\xi'\rangle}{1+\delta\langle\xi'\rangle}\frac{\xi_j}{\langle\xi'\rangle}\frac{\xi_k}{\langle\xi'\rangle}\Bigg\}\frac{1}{\langle\xi'\rangle^2}, \quad 2\leqq j,\, k\leqq n.\end{aligned}$$

Here and in the following, for the derivatives of a symbol $p(x, \xi)$, we use the shorthand

$$p_{x_i} = p_{x_i}(x, \xi) = \frac{\partial p}{\partial x_i}(x, \xi),$$

$$p_{\xi_i} = p_{\xi_i}(x, \xi) = \frac{\partial p}{\partial \xi_i}(x, \xi).$$

But, since $|\xi'|\leqq|\xi|\leqq 2|\xi'|$ in a conic neighborhood of $\xi^0=(0, \xi_2^0, \cdots, \xi_n^0)$, it follows that

$$\Lambda_{\delta x_j}(x', \xi') \in \cap_{\rho>0} S^{\rho}_{1,0}(\boldsymbol{R}^n\times\boldsymbol{R}^n), \qquad 2\leqq j\leqq n,$$
$$\Lambda_{\delta x_j x_k}(x', \xi') \in \cap_{\rho>0} S^{\rho}_{1,0}(\boldsymbol{R}^n\times\boldsymbol{R}^n), \qquad 2\leqq j, k\leqq n,$$
$$\Lambda_{\delta \xi_j}(x', \xi') \in S^{-1}_{1,0}(\boldsymbol{R}^n\times\boldsymbol{R}^n), \qquad 2\leqq j\leqq n,$$
$$\Lambda_{\delta \xi_j \xi_k}(x', \xi') \in S^{-2}_{1,0}(\boldsymbol{R}^n\times\boldsymbol{R}^n), \qquad 2\leqq j, k\leqq n.$$

Therefore, we find that the symbol $P_{\Lambda_\delta}(x, \xi)$ of $P_{\Lambda_\delta}(x, D)$ is given by the following (cf. [**KW**, Section 5]):

$$\begin{aligned} P_{\Lambda_\delta}(x, \xi) \equiv (1+q_\delta(x', \xi))\Big[& p(x, \xi)+\sqrt{-1}\sum_{j=2}^n (p_{\xi_j}\Lambda_{\delta x_j}-p_{x_j}\Lambda_{\delta\xi_j}) \\ & -\frac{1}{2}\sum_{j,k=2}^n p_{\xi_j\xi_k}\Lambda_{\delta x_j}\Lambda_{\delta x_k}+\sum_{j,k=2}^n p_{\xi_j x_k}\Lambda_{\delta x_j}\Lambda_{\delta\xi_k} \\ & +\frac{1}{2}\sum_{j,k=2}^n p_{\xi_j\xi_k}\Lambda_{\delta x_j x_k} \\ & +\sum_{j,k=2}^n p_{\xi_j}\Lambda_{\delta\xi_k}\Lambda_{\delta x_j x_k}+\sum_{j,k=2}^n p_{x_j}\Lambda_{\delta x_k}\Lambda_{\delta\xi_j\xi_k}\Big] \\ & \operatorname{mod} S^0_{1,0}(\boldsymbol{R}^n\times\boldsymbol{R}^n). \end{aligned}$$

Here $p(x, \xi)$ is a symbol in the class $S^2_{1,0}(\boldsymbol{R}^n\times\boldsymbol{R}^n)$ given by

$$p(x, \xi) = \xi_1^2+\sum_{i,j=2}^n a^{ij}(x)\xi_i\xi_j-\sqrt{-1}\sum_{i=1}^n\Big(b^i(x)+\sum_{j=1}^n\frac{\partial a^{ij}}{\partial x_j}(x)\Big)\xi_i-c(x),$$

and $q_\delta(x', \xi)=q_\delta(x', \xi; a, N, s)$ is a symbol in the class $\cap_{\rho>0}S^{-1+\rho}_{1,0}(\boldsymbol{R}^n\times\boldsymbol{R}^n)$ given by

$$q_\delta(x', \xi) = \sqrt{-1}\sum_{j=2}^n\Lambda_{\delta\xi_j}\Lambda_{\delta x_j}+\frac{1}{2}\sum_{j,k=2}^n(\Lambda_{\delta\xi_j\xi_k}+\Lambda_{\delta\xi_j}\Lambda_{\delta\xi_k})(\Lambda_{\delta x_j x_k}-\Lambda_{\delta x_j}\Lambda_{\delta x_k}).$$

But, since we have for $|\xi|$ sufficiently large (uniformly in $\delta>0$)

$$\frac{1}{2}\leqq|1+q_\delta(x', \xi)|\leqq 2,$$

one can find an elliptic symbol $r_\delta(x', \xi)=r_\delta(x', \xi; a, N, s)$ in the class $S^0_{1,0}(\boldsymbol{R}^n\times\boldsymbol{R}^n)$ such that we have for $|\xi|$ sufficiently large (uniformly in $\delta>0$)

$$r_\delta(x', \xi)(1+q_\delta(x', \xi)) = 1.$$

We let

$$\tilde{P}_{\Lambda_\delta}(x, D) = r_\delta(x', D)P_{\Lambda_\delta}(x, D), \tag{2.3}$$

where $r_\delta(x', D)$ is a properly supported, *elliptic* pseudodifferential operator with symbol $r_\delta(x', \xi)$ such that we have for $|\xi|$ sufficiently large (uniformly in $\delta>0$)

$$\frac{1}{2} \leqq |r_\delta(x', \xi)| \leqq 2.$$

Then we have by a direct calculation

$$\begin{aligned}(2.4) \qquad \tilde{P}_{\Lambda_\delta}(x, \xi) \equiv\ & \xi_1^2 + \alpha(x, \xi') + \sum_{j=2}^n b^j(x)\Lambda_{\delta x_j} - \frac{1}{2}\sum_{j,k=2}^n \alpha_{\xi_j\xi_k}\Lambda_{\delta x_j}\Lambda_{\delta x_k} \\ & + \sum_{j,k=2}^n \alpha_{\xi_j x_k}\Lambda_{\delta x_j}\Lambda_{\delta \xi_k} + \frac{1}{2}\sum_{j,k=2}^n \alpha_{\xi_j\xi_k}\Lambda_{\delta x_j x_k} \\ & + \sum_{j,k=2}^n \alpha_{\xi_j}\Lambda_{\delta\xi_k}\Lambda_{\delta x_j x_k} + \sum_{j,k=2}^n \alpha_{x_j}\Lambda_{\delta x_k}\Lambda_{\delta \xi_j\xi_k} \\ & + \sqrt{-1}\Big[-\sum_{k=1}^n b^k(x)\xi_k + \sum_{j=2}^n \alpha_{\xi_j}\Lambda_{\delta x_j} - \sum_{j=2}^n \alpha_{x_j}\Lambda_{\delta\xi_j}\Big] \\ & \bmod S^0_{1,0}(\boldsymbol{R}^n \times \boldsymbol{R}^n),\end{aligned}$$

where

$$\alpha(x, \xi') = \sum_{i,j=2}^n a^{ij}(x)\xi_i\xi_j.$$

In order to estimate the terms $\alpha_{\xi_j}\Lambda_{\delta\xi_k}\Lambda_{\delta x_j x_k}$ and $\alpha_{x_j}\Lambda_{\delta x_k}\Lambda_{\delta\xi_j\xi_k}$ in formula (2.4), we need the following:

LEMMA 2.3. *Let $d(x, \xi)$ and $e(x, \xi)$ be symbols in the classes $S^{-1+\rho}_{1,0}(\boldsymbol{R}^n \times \boldsymbol{R}^n)$ and $S^{-2+\rho}_{1,0}(\boldsymbol{R}^n \times \boldsymbol{R}^n)$ for some $0<\rho<1$, respectively. Then, for every $\varepsilon>0$, one can find constants $C_\varepsilon>0$ and $C'_\varepsilon>0$ such that*

$$(2.5) \qquad |\alpha_{\xi_j}(x, \xi')d(x, \xi)| \leqq \varepsilon\alpha(x, \xi') + C_\varepsilon \quad \text{on } T^*(\boldsymbol{R}^n),$$

$$(2.6) \qquad |\alpha_{x_j}(x, \xi')e(x, \xi)| \leqq \varepsilon\alpha(x, \xi') + C'_\varepsilon \quad \text{on } T^*(\boldsymbol{R}^n).$$

PROOF. Since $\alpha(x, \xi') \geqq 0$ on $T^*(\boldsymbol{R}^n)$, it follows from an application of Lemma 1.7.1 of Oleĭnik-Radkevič [**OR**] that

$$(2.7) \qquad |\alpha_{\xi_j}(x, \xi')|^2 \leqq a^{jj}(x)\alpha(x, \xi') \quad \text{on } T^*(\boldsymbol{R}^n),$$

$$(2.8) \qquad |\alpha_{x_j}(x, \xi')|^2 \leqq 2\Big(\sup_{\substack{x \in \boldsymbol{R}^n \\ 2 \leq l, m \leq n}} |\alpha_{x_l x_m}(x, \xi')|\Big)\alpha(x, \xi') \quad \text{on } T^*(\boldsymbol{R}^n).$$

Thus, using the Schwarz inequality, we obtain from inequality (2.7) that for every $\varepsilon>0$

$$|\alpha_{\xi_j}(x, \xi')d(x, \xi)| \leqq \varepsilon\alpha(x, \xi') + \frac{1}{4\varepsilon}a^{jj}(x)d(x, \xi)^2 \quad \text{on } T^*(\boldsymbol{R}^n).$$

This proves estimate (2.5), since $d(x, \xi)^2$ belongs to the class $S^{-2+2\rho}_{1,0}(\boldsymbol{R}^n \times \boldsymbol{R}^n)$ for some $0<\rho<1$.

Similarly, estimate (2.6) can be proved by using inequality (2.8).

Now we recall that for all $\xi=(\xi_1, \xi')$ in a conic neighborhood of $\xi^0=(0, \xi^{0\prime})$

$$|\xi'| \leqq |\xi| \leqq 2|\xi'|,$$

and hence that

$$\Lambda_{\delta\xi_k}(x', \xi')\Lambda_{\delta x_j x_k}(x', \xi') \in \cap_{\rho>0} S^{-1+\rho}_{1,0}(\boldsymbol{R}^n \times \boldsymbol{R}^n), \qquad 2 \leqq j,\ k \leqq n,$$

$$\Lambda_{\delta x_k}(x', \xi')\Lambda_{\delta\xi_j\xi_k}(x', \xi') \in \cap_{\rho>0} S^{-2+\rho}_{1,0}(\boldsymbol{R}^n \times \boldsymbol{R}^n), \qquad 2 \leqq j,\ k \leqq n.$$

Therefore, applying Lemma 2.3 to the terms $\alpha_{\xi_j}\Lambda_{\delta\xi_k}\Lambda_{\delta x_j x_k}$ and $\alpha_{x_j}\Lambda_{\delta x_k}\Lambda_{\delta\xi_j\xi_k}$, we have for every $\varepsilon>0$

$$\alpha_{\xi_j}(x, \xi')\Lambda_{\delta\xi_k}(x', \xi')\Lambda_{\delta x_j x_k}(x', \xi') \geqq -\varepsilon\alpha(x, \xi') \mod S^0_{1,0}(\boldsymbol{R}^n \times \boldsymbol{R}^n),$$

$$\alpha_{x_j}(x, \xi')\Lambda_{\delta x_k}(x', \xi')\Lambda_{\delta\xi_j\xi_k}(x', \xi') \geqq -\varepsilon\alpha(x, \xi') \mod S^0_{1,0}(\boldsymbol{R}^n \times \boldsymbol{R}^n).$$

On the other hand, by virtue of conditions (B) and (A.1), we can estimate the terms $b^j\Lambda_{\delta x_j}$ and $\alpha_{\xi_j x_k}\Lambda_{\delta x_j}\Lambda_{\delta\xi_k}$ in formula (2.4) as follows:

$$b^j(x)\Lambda_{\delta x_j}(x', \xi') \geqq -\varepsilon\alpha(x, \xi') \mod S^0_{1,0}(\boldsymbol{R}^n \times \boldsymbol{R}^n).$$

$$\alpha_{\xi_j x_k}(x, \xi')\Lambda_{\delta x_j}(x', \xi')\Lambda_{\delta\xi_k}(x', \xi') \geqq -\varepsilon\alpha(x, \xi') \mod S^0_{1,0}(\boldsymbol{R}^n \times \boldsymbol{R}^n).$$

Summing up, we obtain from formula (2.4) that in a conic neighborhood of (x^0, ξ^0)

$$\operatorname{Re}\tilde{P}_{\Lambda_\delta}(x, \xi) \geqq \xi_1^2 + \frac{1}{2}\alpha(x, \xi') - C\Big(\sum_{j,k=2}^{n} |a^{jk}(x)|\Big)(\log\langle\xi'\rangle)^2 \mod S^0_{1,0}(\boldsymbol{R}^n \times \boldsymbol{R}^n),$$

where $C>0$ is a constant independent of δ. But we remark that

$$|a^{jk}(x)| \leqq \sqrt{a^{jj}(x)a^{kk}(x)} \leqq \frac{1}{2}(a^{jj}(x)+a^{kk}(x)).$$

Hence we have in a conic neighborhood of (x^0, ξ^0)

$$\begin{aligned}\text{(2.9)}\qquad \operatorname{Re}\tilde{P}_{\Lambda_\delta}(x, \xi) &\geqq \xi_1^2 + \frac{1}{2}\alpha(x, \xi') - 2C\Big(\sum_{j=2}^{n} a^{jj}(x)\Big)(\log\langle\xi'\rangle)^2 \\ &= \frac{1}{2}\xi_1^2 + \Big[\frac{1}{2}\xi_1^2 + \frac{1}{2}\alpha(x, \xi') - 2C\lambda(x)(\log\langle\xi'\rangle)^2\Big] \\ &\mod S^0_{1,0}(\boldsymbol{R}^n \times \boldsymbol{R}^n),\end{aligned}$$

where

$$\lambda(x) = \sum_{j=2}^{n} a^{jj}(x).$$

4) The next lemma allows us to replace the symbol $(1/2)\xi_1^2$ in the bracket in formula (2.9) by a symbol of a pseudodifferential operator on $\boldsymbol{R}^{n-1}$:

LEMMA 2.4. *Let $F(x)$ be a non-negative C^∞ function on $\boldsymbol{R}^n$ and l a positive integer. If $a(x_1, x', D')$ is a properly supported, pseudodifferential operator on $\boldsymbol{R}^{n-1}$ with symbol*

$$a(x_1, x', \xi') = F(x_1, x')(\log\langle\xi'\rangle)^l,$$

where the variable x_1 *is considered as a parameter, we define a formally self-adjoint operator* $\mathcal{A}(x_1, x', D')$ *by the formula*

$$\mathcal{A}(x_1, x', D') = \frac{1}{2}[a(x_1, x', D') + a(x_1, x', D')^*].$$

Then we have for all $u \in C_0^\infty(\boldsymbol{R}^n)$

$$(D_{x_1}^2 u, u) \geqq ((\mathcal{A}_{x_1}(x, D') - \mathcal{A}(x, D')^2)u, u).$$

Here $\mathcal{A}_{x_1}(x, D') = \partial\mathcal{A}(x, D')/\partial x_1$.

PROOF. Since $\mathcal{A}^* = \mathcal{A}$, it follows that

$$\begin{aligned}(D_{x_1}^2 u, u) &= ((D_{x_1} + \sqrt{-1}\mathcal{A}(x, D'))(D_{x_1} - \sqrt{-1}\mathcal{A}(x, D'))u, u) \\ &\quad + ((\mathcal{A}_{x_1}(x, D') - \mathcal{A}(x, D')^2)u, u) \\ &= \|(D_{x_1} - \sqrt{-1}\mathcal{A}(x, D'))u\|^2 + ((\mathcal{A}_{x_1}(x, D') - \mathcal{A}(x, D')^2)u, u) \\ &\geqq ((\mathcal{A}_{x_1}(x, D') - \mathcal{A}(x, D')^2)u, u).\end{aligned}$$

This proves the lemma.

Lemma 2.4 tells us that the differential operator $D_{x_1}^2$ can be estimated from below by the pseudodifferential operator $\mathcal{A}_{x_1}(x, D') - \mathcal{A}(x, D')^2$ on $\boldsymbol{R}^{n-1}$ in the sense of the inner product of $L^2(\boldsymbol{R}^n)$. In terms of symbols, one may estimate the symbol ξ_1^2 as follows:

$$\begin{aligned}\xi_1^2 \geqq F_{x_1}(x_1, x')(\log\langle\xi'\rangle)^l - F(x_1, x')^2(\log\langle\xi'\rangle)^{2l} \\ \mod S_{1,0}^0(\boldsymbol{R}^{n-1} \times \boldsymbol{R}^{n-1}).\end{aligned}$$

This trick is due to Wakabayashi.

5) Now, applying Lemma 2.4 with

$$a(x_1, x', \xi') = 2(2C+1)\left(\int_0^{x_1} \lambda(t, x')dt\right)(\log\langle\xi'\rangle)^2,$$

we find that the symbol $(1/2)\xi_1^2$ may be replaced by the following:

$$(2C+1)\lambda(x_1, x')(\log\langle\xi'\rangle)^2 - 2(2C+1)^2\tilde{\lambda}(x_1, x')^2(\log\langle\xi'\rangle)^4,$$

where

$$\tilde{\lambda}(x_1, x') = \int_0^{x_1} \lambda(t, x')dt.$$

In view of formula (2.9), this proves that in a conic neighborhood of (x^0, ξ^0)

$$\operatorname{Re} \tilde{P}_{\Lambda_\delta}(x, \xi) \geqq \frac{1}{2}\xi_1^2+\left[\frac{1}{2}\xi_1^2+\frac{1}{2}\alpha(x_1, x', \xi')-2C\lambda(x_1, x')(\log\langle\xi'\rangle)^2\right]$$

$$\geqq \frac{1}{2}\xi_1^2+\pi(x_1, x', \xi') \mod S^0_{1,0}(\boldsymbol{R}^{n-1}\times\boldsymbol{R}^{n-1}),$$

where $\pi(x_1, x', \xi')$ is a symbol in the class $S^2_{1,0}(\boldsymbol{R}^{n-1}\times\boldsymbol{R}^{n-1})$ given by the following formula:

$$\pi(x_1, x', \xi') = \frac{1}{2}\alpha(x_1, x', \xi')+\lambda(x_1, x')(\log\langle\xi'\rangle)^2$$
$$-2(2C+1)^2\tilde{\lambda}(x_1, x')^2(\log\langle\xi'\rangle)^4.$$

Thus we are reduced to the *positivity* of the symbol $\pi(x_1, x', \xi')$.

(a) First, if we have

$$\lambda(x_1, x')(\log\langle\xi'\rangle)^2-2(2C+1)^2\tilde{\lambda}(x_1, x')^2(\log\langle\xi'\rangle)^4 \geqq 0,$$

then it follows that

$$\pi(x_1, x', \xi') \geqq 0.$$

(b) Next we assume that

$$\lambda(x, x')(\log\langle\xi'\rangle)^2-2(2C+1)^2\tilde{\lambda}(x_1, x')^2(\log\langle\xi'\rangle)^4 \leqq 0,$$

that is,

$$\log\langle\xi'\rangle \geqq \frac{\sqrt{\lambda(x_1, x')}}{\sqrt{2}\,(2C+1)|\tilde{\lambda}(x_1, x')|}. \tag{2.10}$$

Then we shall show that condition (0.1) implies that in a conic neighborhood of (x^0, ξ^0)

$$\frac{1}{2}\alpha(x_1, x', \xi') \geqq 2(2C+1)^2\tilde{\lambda}(x_1, x')^2(\log\langle\xi'\rangle)^4, \tag{2.11}$$

which proves that

$$\pi(x_1, x', \xi') \geqq 0.$$

By condition (A.2), it follows that

$$\alpha(x_1, x', \xi') \geqq \mu(x_1, x')|\xi'|^2 \quad \text{on } T^*(\boldsymbol{R}^{n-1}).$$

Thus it suffices to show that

$$\mu(x_1, x')|\xi'|^2 \geqq 4(2C+1)^2\tilde{\lambda}(x_1, x')^2(\log\langle\xi'\rangle)^4. \tag{2.12}$$

If we take the logarithm of the both sides, we obtain that

$$\log\mu(x_1, x')+2\log|\xi'| \geqq \log[4(2C+1)^2]+2\log|\tilde{\lambda}(x_1, x')|$$
$$+4\log(\log\langle\xi'\rangle).$$

This condition is satisfied if we have for $|\xi'|$ sufficiently large

$$\log \mu(x_1, x') + \log\langle\xi'\rangle \geqq 2\log|\tilde{\lambda}(x_1, x')|. \tag{2.12'}$$

Therefore, combining inequalities (2.10) and (2.12′), we obtain that condition (2.11) is satisfied if we have for $|x_1|$ sufficiently small

$$\log \mu(x_1, x') + \frac{\sqrt{\lambda(x_1, x')}}{\sqrt{2}\,(2C+1)|\tilde{\lambda}(x_1, x')|} \geqq 0,$$

since $\log|\tilde{\lambda}(x_1, x')|<0$ for $|x_1|$ sufficiently small.

Summing up, we have proved that if the condition

$$\lim_{x_1\to 0} \frac{\tilde{\lambda}(x_1, x')\log \mu(x_1, x')}{\sqrt{\lambda(x_1, x')}} = 0 \tag{0.1}$$

is satisfied, then we have

$$\operatorname{Re} \tilde{P}_{\Lambda_\delta}(x, \xi) \geqq \frac{1}{2}\xi_1^2 + \pi(x_1, x', \xi') \mod S^0_{1,0}(\boldsymbol{R}^{n-1}\times\boldsymbol{R}^{n-1}),$$

and further the symbol $\pi(x_1, x', \xi')$ is *non-negative* and forms a bounded subset of the class $S^2_{1,0}(\boldsymbol{R}^{n-1}\times\boldsymbol{R}^{n-1})$ for $|x_1|\leqq\varepsilon_0$ if $\varepsilon_0>0$ is sufficiently small.

6) Therefore, applying Corollary 2.2 to the operator $\pi(x_1, x', D')$, we obtain that if $\varepsilon_0>0$ is sufficiently small, then we have for all $v\in C_0^\infty(U_{\varepsilon_0})$ and all $0<\delta\leqq 1$

$$\operatorname{Re}(\tilde{P}_{\Lambda_\delta}(x, D)v, v) \geqq \frac{1}{2}\|D_{x_1}v\|^2 - \tilde{C}\|v\|^2,$$

with a constant $\tilde{C}>0$ independent of δ. Hence, in view of formula (2.3), this proves inequality (2.1).

The proof of Theorem 1 is now complete.

3. Proof of Theorem 2.

The proof of Theorem 2 is essentially the same as that of Theorem 1.

1) Let $x^0=(x_1^0, x_2^0, \cdots, x_n^0)$ be a point of a closed subset S of the hypersurface $\{x=(x_1, x')\in\boldsymbol{R}^n;\ x_1=0\}$, where $x'=(x_2, \cdots, x_n)$. Without loss of generality, one may assume that

$$x^0=(0, 0).$$

If $0\leqq\delta\leqq 1$, $a\geqq 0$, $N\geqq 0$ and $s\in\boldsymbol{R}$, we let

$$\begin{aligned}\Lambda_\delta(x', \xi) &= \Lambda_\delta(x', \xi;\ a, N, s)\\ &= (-s+a|x'|^2)\log\lambda(\xi) + N\log(1+\delta\lambda(\xi)),\end{aligned}$$

and

$$Q_{\Lambda_\delta}(x, D) = e^{-\Lambda_\delta}(x', D)Q(x, D)e^{\Lambda_\delta}(x', D),$$

where $e^{\pm \Lambda_\delta}(x', D)$ are properly supported pseudodifferential operators with symbols $e^{\pm \Lambda_\delta(x', \xi)}$, respectively:

$$e^{\Lambda_\delta(x',\xi)} = \lambda(\xi)^{(-s+a|x'|^2)}(1+\delta\lambda(\xi))^N,$$
$$e^{-\Lambda_\delta(x',\xi)} = \lambda(\xi)^{(s-a|x'|^2)}(1+\delta\lambda(\xi))^{-N}.$$

By virtue of Corollary 1.2, it suffices to show that there exists an open neighborhood $U_{\varepsilon_0}=\{x=(x_1, x')\in \boldsymbol{R}^n;\ |x_1|<\varepsilon_0,\ |x'|<1\}$ of $x^0=(0, 0)$ such that we have for all $v\in C_0^\infty(U_{\varepsilon_0})$ and all $0<\delta\leqq 1$

$$|(Q_{\Lambda_\delta}(x, D)v, v)| \geqq C_1\|D_{x_1}v\|^2 - C_2\|v\|^2, \tag{3.1}$$

with constants $C_1>0$ and $C_2>0$ independent of δ.

2) Since the operator $Q(x, D)$ is *micro-elliptic* outside a conic neighborhood of a point $(x^0, \xi^0)=(x^0, 0, \xi_2^0, \cdots, \xi_n^0)$ in the bundle $T^*(\boldsymbol{R}^n)\backslash 0$ of non-zero cotangent vectors, one may assume that

$$4 \leqq |\xi| \leqq 2|\xi'|, \quad \xi=(\xi_1, \xi'),$$

and that

$$\begin{cases} \lambda(\xi) = (1+|\xi'|^2)^{1/2} = \langle \xi' \rangle, \\ \Lambda_\delta(x', \xi) = \Lambda_\delta(x', \xi') = (-s+a|x'|^2)\log\langle\xi'\rangle + N\log(1+\delta\langle\xi'\rangle). \end{cases}$$

Then, arguing as in the proof of Theorem 1 (cf. formula (2.4)), one can find an elliptic symbol $s_\delta(x', \xi)=s_\delta(x', \xi;\ a, N, s)$ in the class $S^0_{1,0}(\boldsymbol{R}^n\times\boldsymbol{R}^n)$ such that we have for $|\xi|$ sufficiently large (uniformly in $\delta>0$)

$$\begin{aligned} s_\delta(x', \xi)Q_{\Lambda_\delta}(x, \xi) \equiv\ & \xi_1^2+\alpha(x, \xi'')+b^n(x)\Lambda_{\delta x_n} - \frac{1}{2}\sum_{j,k=2}^{n-1}\alpha_{\xi_j\xi_k}\Lambda_{\delta x_j}\Lambda_{\delta x_k} \\ & + \sum_{\substack{2\leqq j\leqq n-1\\ 2\leqq k\leqq n}} \alpha_{\xi_j x_k}\Lambda_{\delta x_j}\Lambda_{\delta\xi_k} + \frac{1}{2}\sum_{j,k=2}^{n-1}\alpha_{\xi_j\xi_k}\Lambda_{\delta x_j x_k} \\ & + \sum_{\substack{2\leqq j\leqq n-1\\ 2\leqq k\leqq n}} \alpha_{\xi_j}\Lambda_{\delta\xi_k}\Lambda_{\delta x_j x_k} + \sum_{j,k=2}^{n}\alpha_{x_j}\Lambda_{\delta x_k}\Lambda_{\delta\xi_j\xi_k} \\ & + \sqrt{-1}\Big[-b^n(x)\xi_n + \sum_{j=2}^{n-1}\alpha_{\xi_j}\Lambda_{\delta x_j} - \sum_{j=2}^{n}\alpha_{x_j}\Lambda_{\delta\xi_j}\Big] \\ & \mod S^0_{1,0}(\boldsymbol{R}^n\times\boldsymbol{R}^n), \end{aligned}$$

where

$$\alpha(x, \xi'') = \sum_{i,j=2}^{n-1} a^{ij}(x)\xi_i\xi_j, \quad \xi''=(\xi_2, \cdots, \xi_{n-1}).$$

We let

$$\tilde{Q}_{\Lambda_\delta}(x, D) = s_\delta(x', D)Q_{\Lambda_\delta}(x, D), \tag{3.2}$$

where $s_\delta(x', D)$ is a properly supported, *elliptic* pseudodifferential operator with symbol $s_\delta(x', \xi)$ such that we have for $|\xi|$ sufficiently large (uniformly in $\delta>0$)

$$\frac{1}{2} \leqq |s_\delta(x', \xi)| \leqq 2.$$

Now we remark that

$$|(\tilde{Q}_{\Lambda_\delta}(x, D)v, v)| = [(\mathrm{Re}(\tilde{Q}_{\Lambda_\delta}(x, D)v, v))^2 + (\mathrm{Im}(\tilde{Q}_{\Lambda_\delta}(x, D)v, v))^2]^{1/2} \tag{3.3}$$
$$\geqq \frac{\sqrt{2}}{2}(\mathrm{Re}(\tilde{Q}_{\Lambda_\delta}(x, D)v, v) + |\mathrm{Im}(\tilde{Q}_{\Lambda_\delta}(x, D)v, v)|).$$

First we estimate the term $|\mathrm{Im}(\tilde{Q}_{\Lambda_\delta}(x, D)v, v)|$. To do so, arguing as in the proof of Lemma 2.3, we have for every $\varepsilon>0$

$$|\alpha_{\xi_j}(x, \xi'')\Lambda_{\delta x_j}(x', \xi')| \leqq \varepsilon\alpha(x, \xi'') + \frac{1}{\varepsilon}a^2 x_j^2 a^{jj}(x)(\log\langle\xi'\rangle)^2$$
$$\mod S^0_{1,0}(\boldsymbol{R}^n \times \boldsymbol{R}^n),$$
$$|\alpha_{x_j}(x, \xi'')\Lambda_{\delta\xi_j}(x', \xi')| \leqq \varepsilon\alpha(x, \xi'') \mod S^0_{1,0}(\boldsymbol{R}^n \times \boldsymbol{R}^n).$$

Here we recall that for all $\xi=(\xi_1, \xi')$ in a conic neighborhood of $\xi^0=(0, \xi^{0\prime})$

$$|\xi'| \leqq |\xi| \leqq 2|\xi'|.$$

Furthermore, condition (B′) implies that the function b^n does not change sign. Hence, for every $\varepsilon>0$, one can find a constant $C_\varepsilon>0$ such that

$$|\mathrm{Im}(\tilde{Q}_{\Lambda_\delta}(x, D)v, v)| \geqq \mathrm{Re}(|b_n(x)|\langle D_{x_n}\rangle v, v) - \varepsilon\,\mathrm{Re}(\alpha(x, D'')v, v)$$
$$- C_\varepsilon \sum_{j=2}^{n-1} \mathrm{Re}(a^{jj}(x)(\log\langle D'\rangle)^2 v, v),$$

where $\langle D_{x_n}\rangle$ and $\log\langle D'\rangle$ are pseudodifferential operators with symbols $\langle\xi_n\rangle = (1+\xi_n^2)^{1/2}$ and $\log((1+|\xi'|^2)^{1/2})$, respectively.

Next we estimate the term $\mathrm{Re}(\tilde{Q}_{\Lambda_\delta}(x, D)v, v)$. Similarly, applying Lemma 2.3 to the terms $\alpha_{\xi_j}\Lambda_{\delta\xi_k}\Lambda_{\delta x_j x_k}$ and $\alpha_{x_j}\Lambda_{\delta x_k}\Lambda_{\delta\xi_j\xi_k}$, we have for every $\varepsilon>0$

$$\alpha_{\xi_j}(x, \xi'')\Lambda_{\delta\xi_k}(x', \xi')\Lambda_{\delta x_j x_k}(x', \xi') \geqq -\varepsilon\alpha(x, \xi'') \mod S^0_{1,0}(\boldsymbol{R}^n \times \boldsymbol{R}^n),$$
$$\alpha_{x_j}(x, \xi'')\Lambda_{\delta x_k}(x', \xi')\Lambda_{\delta\xi_j\xi_k}(x', \xi') \geqq -\varepsilon\alpha(x, \xi'') \mod S^0_{1,0}(\boldsymbol{R}^n \times \boldsymbol{R}^n).$$

Moreover, by virtue of condition (A.1′), we can estimate the terms $\alpha_{\xi_j x_k}\Lambda_{\delta x_j}\Lambda_{\delta\xi_k}$ as follows:

$$\alpha_{\xi_j x_k}(x, \xi'')\Lambda_{\delta x_j}(x', \xi')\Lambda_{\delta\xi_k}(x', \xi') \geqq -\varepsilon\alpha(x, \xi'') \mod S^0_{1,0}(\boldsymbol{R}^n \times \boldsymbol{R}^n).$$

We also have

$$b^n(x)\Lambda_{\delta x_n}(x', \xi') \geqq -2a|x_n||b^n(x)|\log\langle\xi'\rangle \mod S^0_{1,0}(\boldsymbol{R}^n \times \boldsymbol{R}^n).$$

Hence, arguing as in the proof of formula (2.9), we obtain that for some constants $C_1>0$ and $C_2>0$ independent of δ

$$\mathrm{Re}(\tilde{Q}_{\Lambda_\delta}(x, D)v, v) \geqq (D_{x_1}^2 v, v) + \frac{3}{4}\mathrm{Re}(\alpha(x, D'')v, v)$$
$$-C_1 \mathrm{Re}(|b_n(x)| \log\langle D'\rangle v, v)$$
$$-C_2 \sum_{j=2}^{n-1} \mathrm{Re}(a^{jj}(x)(\log\langle D'\rangle)^2 v, v).$$

Therefore, we can find a second-order pseudodifferential operator $\tilde{R}_{\Lambda_\delta}(x, D)$ with symbol $\tilde{R}_{\Lambda_\delta}(x, \xi)$ such that

$$|(\tilde{Q}_{\Lambda_\delta}(x, D)v, v)| \geqq \frac{\sqrt{2}}{2}\mathrm{Re}(\tilde{R}_{\Lambda_\delta}(x, D)v, v), \tag{3.3'}$$

and that in a conic neighborhood of (x^0, ξ^0)

$$\tilde{R}_{\Lambda_\delta}(x, \xi) \geqq \xi_1^2 + \frac{1}{2}\alpha(x, \xi'') + |b^n(x)|\langle \xi_n\rangle \tag{3.4}$$
$$-A|b^n(x)|\log\langle\xi'\rangle - B\lambda(x)(\log\langle\xi'\rangle)^2$$
$$\mathrm{mod}\ S_{1,0}^0(\boldsymbol{R}^n \times \boldsymbol{R}^n).$$

Here $A>0$ and $B>0$ are constants independent of δ, and

$$\lambda(x) = \sum_{j=2}^{n-1} a^{jj}(x).$$

Thus we are reduced to the study of the symbol $\tilde{R}_{\Lambda_\delta}(x, \xi)$.

3-i) Assume that

$$\xi^0 = (0, \xi_2^0, \cdots, \xi_{n-1}^0, \xi_n^0) \quad \text{with } \xi_n^0 \neq 0.$$

Then we remark that, for all ξ in a conic neighborhood of ξ^0, there exists a constant $c_1>0$ such that

$$c_1|\xi'| \leqq |\xi_n| \leqq |\xi'|, \quad \xi' = (\xi_2, \cdots, \xi_{n-1}, \xi_n).$$

Hence, by formula (3.4), we have for $|\xi_n|$ sufficiently large (uniformly in $\delta>0$)

$$\tilde{R}_{\Lambda_\delta}(x, \xi) \geqq \xi_1^2 + \frac{1}{2}\alpha(x, \xi'') + |b^n(x)|\langle\xi_n\rangle$$
$$-A|b^n(x)|\log\langle\xi_n\rangle - B\lambda(x)(\log\langle\xi_n\rangle)^2$$
$$\geqq \frac{1}{2}\xi_1^2 + \left[\frac{1}{2}\xi_1^2 + \frac{1}{2}|b^n(x_1, x')||\xi_n| - B\lambda(x_1, x')(\log\langle\xi_n\rangle)^2\right]$$
$$\mathrm{mod}\ S_{1,0}^0(\boldsymbol{R}^{n-1} \times \boldsymbol{R}^{n-1}).$$

Therefore, arguing as in step 5) of the proof of Theorem 1, we find that if the condition

$$\lim_{x_1 \to 0} \frac{\tilde{\lambda}(x_1, x')\log|b^n(x_1, x')|}{\sqrt{\lambda(x_1, x')}} = 0 \tag{0.2a}$$

is satisfied, then we have in a conic neighborhood of (x^0, ξ^0)

$$\tilde{R}_{\Lambda_\delta}(x, \xi) \geqq \frac{1}{2}\xi_1^2+\rho_1(x_1, x', \xi') \quad \mathrm{mod}\, S^0_{1,0}(\boldsymbol{R}^{n-1}\times\boldsymbol{R}^{n-1}),$$

and the symbol $\rho_1(x_1, x', \xi')$ is *non-negative* and forms a bounded subset of the class $S^2_{1,0}(\boldsymbol{R}^{n-1}\times\boldsymbol{R}^{n-1})$ for $|x_1|\leqq\varepsilon_0$ if $\varepsilon_0>0$ is sufficiently small.

3-ii) Assume that

$$\xi^0=(0, \xi^{0\prime\prime}, \xi^0_n) \quad \text{with } \xi^{0\prime\prime}=(\xi^0_2, \cdots, \xi^0_{n-1})\neq 0.$$

Then we remark that, for all ξ in a conic neighborhood of ξ^0, there exists a constant $c_2>0$ such that

$$c_2|\xi'| \leqq |\xi''| \leqq |\xi'|, \quad \xi'=(\xi'', \xi_n).$$

Hence, by formula (3.4), we have in a conic neighborhood of (x^0, ξ^0)

$$\begin{aligned}(3.5)\quad \tilde{R}_{\Lambda_\delta}(x, \xi) &\geqq \xi_1^2+\frac{1}{2}\alpha(x, \xi'')+|b^n(x)|\langle\xi_n\rangle-A|b^n(x)|\log\langle\xi''\rangle \\ &\quad -B\lambda(x)(\log\langle\xi''\rangle)^2 \\ &\geqq \frac{1}{2}\xi_1^2+\left[\frac{1}{2}\xi_1^2+\frac{1}{2}\alpha(x, \xi'')-A|b^n(x)|\log\langle\xi''\rangle-B\lambda(x)(\log\langle\xi''\rangle)^2\right] \\ &\quad \mathrm{mod}\, S^0_{1,0}(\boldsymbol{R}^n\times\boldsymbol{R}^n).\end{aligned}$$

Now, applying Lemma 2.4 with

$$a(x_1, x', \xi'')=2(A+1)\left(\int_0^{x_1}|b^n(t, x')|\,dt\right)\log\langle\xi''\rangle,$$

we find that the symbol $(1/2)\xi_1^2$ in the bracket in formula (3.5) may be replaced by the following:

$$(A+1)|b^n(x_1, x')|\log\langle\xi''\rangle-2(A+1)^2\tilde{b}^n(x_1, x')^2(\log\langle\xi''\rangle)^2,$$

where

$$\tilde{b}^n(x_1, x')=\int_0^{x_1}|b^n(t, x')|\,dt.$$

This proves that in a conic neighborhood of (x^0, ξ^0)

$$\begin{aligned}\tilde{R}_{\Lambda_\delta}(x, \xi) &\geqq \frac{1}{2}\xi_1^2+\left[\frac{1}{2}\xi_1^2+\frac{1}{2}\alpha(x_1, x', \xi'')-A|b^n(x_1, x')|\log\langle\xi''\rangle\right. \\ &\quad \left. -B\lambda(x_1, x')(\log\langle\xi''\rangle)^2\right] \\ &\geqq \frac{1}{2}\xi_1^2+\rho_2(x_1, x', \xi'') \quad \mathrm{mod}\, S^0_{1,0}(\boldsymbol{R}^{n-1}\times\boldsymbol{R}^{n-1}),\end{aligned}$$

where $\rho_2(x_1, x', \xi'')$ is a symbol in the class $S^2_{1,0}(\boldsymbol{R}^{n-1}\times\boldsymbol{R}^{n-1})$ given by the following formula:

$$\rho_2(x_1, x', \xi'') = \frac{1}{2}\alpha(x_1, x', \xi'') + |b^n(x_1, x')| \log\langle\xi''\rangle - C(\tilde{b}^n(x_1, x')^2 + \lambda(x_1, x'))(\log\langle\xi''\rangle)^2,$$

with

$$C = \max(2(A+1)^2, B).$$

Thus we are reduced to the *positivity* of the symbol $\rho_2(x_1, x', \xi'')$.

(a) First, if we have

$$|b^n(x_1, x')| \log\langle\xi''\rangle - C(\tilde{b}^n(x_1, x')^2 + \lambda(x_1, x'))(\log\langle\xi''\rangle)^2 \geqq 0,$$

then it follows that

$$\rho_2(x_1, x', \xi'') \geqq 0.$$

(b) Next we assume that

$$|b^n(x_1, x')| \log\langle\xi''\rangle - C(\tilde{b}^n(x_1, x')^2 + \lambda(x_1, x'))(\log\langle\xi''\rangle)^2 \leqq 0,$$

that is,

$$\log\langle\xi''\rangle \geqq \frac{|b^n(x_1, x')|}{C(\tilde{b}^n(x_1, x')^2 + \lambda(x_1, x'))}. \tag{3.6}$$

Then we shall show that conditions (0.2b) and (0.2c) imply that in a conic neighborhood of (x^0, ξ^0)

$$\frac{1}{2}\alpha(x_1, x', \xi'') \geqq C(\tilde{b}^n(x_1, x')^2 + \lambda(x_1, x'))(\log\langle\xi''\rangle)^2, \tag{3.7}$$

which proves that

$$\rho_2(x_1, x', \xi'') \geqq 0. \tag{3.8}$$

By condition (A.2′), it follows that

$$\alpha(x_1, x', \xi'') \geqq \mu(x_1, x')|\xi''|^2 \quad \text{on } T^*(\boldsymbol{R}^{n-1}).$$

Thus it suffices to show that

$$\mu(x_1, x')|\xi''|^2 \geqq 2C(\tilde{b}^n(x_1, x')^2 + \lambda(x_1, x'))(\log\langle\xi''\rangle)^2. \tag{3.9}$$

If we take the logarithm of the both sides, we obtain that

$$\log \mu(x_1, x') + 2\log|\xi''| \geqq \log 2C + \log(\tilde{b}^n(x_1, x')^2 + \lambda(x_1, x')) + 2\log(\log\langle\xi''\rangle).$$

This condition is satisfied if we have for $|\xi''|$ sufficiently large

$$\log \mu(x_1, x') + \log\langle\xi''\rangle \geqq \log(\tilde{b}^n(x_1, x')^2 + \lambda(x_1, x')). \tag{3.9′}$$

Thus, combining inequalities (3.6) and (3.9′), we obtain that condition (3.7) is satisfied if we have for $|x_1|$ sufficiently small

$$\log \mu(x_1, x')+\frac{|b^n(x_1, x')|}{C(\tilde{b}^n(x_1, x')^2+\lambda(x_1, x'))} \geqq 0,$$

since $\log(\tilde{b}^n(x_1, x')^2+\lambda(x_1, x'))<0$ for $|x_1|$ sufficiently small.

Therefore, we find that the conditions

$$\lim_{x_1\to 0}\frac{\tilde{b}^n(x_1, x')^2 \log \mu(x_1, x')}{b^n(x_1, x')}=0, \tag{0.2b}$$

$$\lim_{x_1\to 0}\frac{\lambda(x_1, x') \log \mu(x_1, x')}{b^n(x_1, x')}=0 \tag{0.2c}$$

imply the desired condition (3.7) and hence condition (3.8).

Summing up, we have proved that if conditions (0.2a), (0.2b) and (0.2c) are satisfied, then we have

$$\tilde{R}_{\Lambda_\delta}(x, \xi) \geqq \frac{1}{2}\xi_1^2+\rho(x_1, x', \xi') \mod S^0_{1,0}(\boldsymbol{R}^{n-1}\times\boldsymbol{R}^{n-1}),$$

and further the symbol $\rho(x_1, x', \xi')$ is *non-negative* and forms a bounded subset of the class $S^2_{1,0}(\boldsymbol{R}^{n-1}\times\boldsymbol{R}^{n-1})$ for $|x_1|\leqq\varepsilon_0$ if $\varepsilon_0>0$ is sufficiently small.

4) Therefore, applying Corollary 2.2 to the operator $\rho(x_1, x', D')$, we obtain that if $\varepsilon_0>0$ is sufficiently small, then we have for all $v\in C_0^\infty(U_{\varepsilon_0})$ and all $0<\delta\leqq 1$

$$\mathrm{Re}(\tilde{R}_{\Lambda_\delta}(x, D)v, v) \geqq \frac{1}{2}\|D_{x_1}v\|^2-\tilde{C}\|v\|^2,$$

with a constant $\tilde{C}>0$ independent of δ. In view of inequality (3.3′) and formula (3.2), this proves inequality (3.1).

The proof of Theorem 2 is now complete.

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