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Hodge-Witt cohomology of complete intersections

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1. Statement of the theorem.

In this note, we prove the following assertions.

THEOREM. *Let k be a perfect field of characteristic $p>0$ and X a smooth complete intersection of dimension n in a projective space over k.*

(a) *If $i\neq j$ and $i+j\neq n$, $n+1$, $H^{j}(X, W\Omega_{X}^{i})=0$.*

(b) *If $2i\neq n$, $n+1$ and $0\leqq i\leqq n$, $H^{i}(X, W\Omega_{X}^{i})=W$ and F is bijective on $H^{i}(X, W\Omega_{X}^{i})$.*

(c) *$H^{n-i}(X, W\Omega_{X}^{i})$ is a Cartier module (in the sense of* [**5**], Ch. I, Def. 2.4).

(d) *If $2i\neq n+1$, $H^{n-i+1}(X, W\Omega_{X}^{i})/F^{\infty}B=0$.*

(e) *If $2i=n+1$, $H^{n-i+1}(X, W\Omega_{X}^{i})/F^{\infty}B=W$ and F is bijective on $H^{n-i+1}(X, W\Omega_{X}^{i})/F^{\infty}B$.*

We follow the notation of [1], [4] and [5]. In particular, $W=W(k)$ (resp. K) is the ring of Witt vectors with coefficients in k (resp. the fraction field of W). $H^{\cdot}(X/W)$ (resp. $H^{j}(X, W\Omega_{X}^{i})$) denotes the crystalline cohomology group (resp. the Hodge-Witt cohomology group) of X. F (resp. V) stands for the Frobenius morphism (resp. the Verschiebung morphism). For a commutative group A and an endomorphism m of A, ${}_{m}A$ (resp. A/m) denotes $\mathrm{Ker}[m: A\to A]$ (resp. $\mathrm{Coker}\,[m: A\to A]$).

2. Proof of the theorem.

Throughout this section, k denotes a perfect field of characteristic $p>0$ and X a smooth complete intersection of dimension n in a projective space over k.

We first recall known facts on the Hodge cohomology and the crystalline cohomology of a smooth complete intersection in a projective space:

(I) $H^{j}(X, \Omega_{X}^{i})=0$ if $i\neq j$ and $i+j\neq n$;

(II) $H^{i}(X, \Omega_{X}^{i})=k$ if $2i\neq n$ and $0\leqq i\leqq n$;

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(III) $H^i(X/W)=0$ if i is odd and $i\neq n$;

(IV) $H^i(X/W)=W$ if $i=2r\neq n$ and $0\leqq i\leqq 2n$. In this case, $H^{2r}(X/W)_K$ is generated by the classes of algebraic cycles, and therefore $H^{2r}(X/W)_K=H^{2r}(X/W)_K^{[r]}=H^r(X, W\Omega_X^r)_K$ (cf. [2], Th. 1.5, [1], Ch. VII, Remarque 1.1.11, [4], Ch. II, Cor. 3.5).

We shall prove the theorem step by step.

STEP 1. (a) *If* $i\neq j$ *and* $i+j<n$, $H^j(X, W\Omega_X^i)=0$.

(b) *If* $0\leqq 2i<n$, $H^i(X, W\Omega_X^i)=W$ *and* F *is bijective on* $H^i(X, W\Omega_X^i)$.

PROOF. We shall prove the assertions by induction on i.

First note that the assertion (a) holds true for $i=-1$ since $W\Omega_X^{-1}=0$ and that the assertion (b) holds true for $i=0$ (cf. [4], Ch. II, Cor. 2.17).

Assume now that:

(1) $H^j(X, W\Omega_X^{i-1})=0$ if $j\neq i-1$ and $i-1+j<n$;

(2) $H^{i-1}(X, W\Omega_X^{i-1})=W$ and F is bijective on $H^{i-1}(X, W\Omega_X^{i-1})$ if $0\leqq i-1<n/2$.

The commutative diagram of pro-sheaves on X with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & W.\Omega_X^{i-1} & \xrightarrow{F} & W.\Omega_X^{i-1} & \longrightarrow & W.\Omega_X^{i-1}/F & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle Fd} & & \downarrow{\scriptstyle dV} & & \downarrow{\scriptstyle dV} & & \\
0 & \longrightarrow & W.\Omega_X^i & \xrightarrow{V} & W.\Omega_X^i & \longrightarrow & W.\Omega_X^i/V & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & \Omega_X^i & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

([4], Ch. I, Cor. 3.5, Cor. 3.19) defines a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & H^{j-1}(X, \Omega_X^i) & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & H^j(X, W\Omega_X^{i-1})/F & \longrightarrow & H^j(X, W\Omega_X^{i-1}/F) & \longrightarrow & {}_FH^{j+1}(X, W\Omega_X^{i-1}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle dV} & & \downarrow{\scriptstyle dV} & & \downarrow{\scriptstyle Fd} & & \\
0 & \longrightarrow & H^j(X, W\Omega_X^i)/V & \longrightarrow & H^j(X, W\Omega_X^i/V) & \longrightarrow & {}_VH^{j+1}(X, W\Omega_X^i) & \longrightarrow & 0 \\
 & & & & \downarrow & & & & \\
 & & & & H^j(X, \Omega_X^i)\,. & & & &
\end{array}$$

By the hypothesis of induction, we have

$$H^j(X, W\Omega_X^{i-1})/F=0 \quad \text{and} \quad {}_FH^{j+1}(X, W\Omega_X^{i-1})=0 \qquad \text{for } j<n-i.$$

Then we obtain

$$H^j(X, W\Omega_X^{i-1})/F=0 \qquad \text{for } j<n-i.$$

By (I) and (II), we have

$$\dim H^j(X, \Omega_X^i)=\begin{cases} 0 & \text{if } j\neq i \text{ and } i+j<n \\ 1 & \text{if } j=i \text{ and } i+j<n. \end{cases}$$

This implies that

$$\dim H^j(X, W\Omega_X^i/V)=\begin{cases} 0 & \text{if } j\neq i \text{ and } i+j<n \\ 0 \text{ or } 1 & \text{if } j=i \text{ and } i+j<n, \end{cases}$$

and hence

$$\dim H^j(X, W\Omega_X^i)/V=\begin{cases} 0 & \text{if } j\neq i \text{ and } i+j<n \\ 0 \text{ or } 1 & \text{if } j=i \text{ and } i+j<n. \end{cases}$$

Since $H^j(X, W\Omega_X^i)$ is V-adically separated ([**4**], Ch. II, Cor. 2.5), we obtain

$$H^j(X, W\Omega_X^i)=0 \qquad \text{if } j\neq i \text{ and } j<n-1.$$

By (IV) we have $H^i(X, W\Omega_X^i)_K=H^{2i}(X/W)_K=K$ if $2i<n$. Then we get $H^i(X, W\Omega_X^i)/V\neq 0$ and therefore $\dim H^i(X, W\Omega_X^i)/V=1$. It follows that $H^i(X, W\Omega_X^i)=W$ and that F is bijective on $H^i(X, W\Omega_X^i)$.

While proving Step 1, we have shown the following assertion.

STEP 2. *$H^{n-i}(X, W\Omega_X^i)$ is V-torsion-free. Hence $H^{n-i}(X, W\Omega_X^i)$ is a Cartier module.*

STEP 3. (a) *The differential $d: H^j(X, W\Omega_X^i)\to H^j(X, W\Omega_X^{i+1})$ is zero if $i+j\neq n$.*

(b) *$H^j(X, W\Omega_X^i)$ is of finite type over W if $i+j>n+1$.*

PROOF. First note that the differential $d: H^j(X, W\Omega_X^i)\to H^j(X, W\Omega_X^{i+1})$ is zero if and only if $\dim \operatorname{Domino} H^j(X, W\Omega_X^\cdot)^i=0$ (cf. [**5**], Ch. I, Prop. 2.18.).

By Step 1, $\dim \operatorname{Domino} H^j(X, W\Omega_X^\cdot)^i=0$ if $i+j<n$. Hence, by Ekedhal's duality ([**3**], Ch. IV, Cor. 3.5.1), $\dim \operatorname{Domino} H^j(X, W\Omega_X^\cdot)^i=0$, and therefore the differential $d: H^j(X, W\Omega_X^i)\to H^j(X, W\Omega_X^{i+1})$ is zero, if $i+j>n$. It follows that $H^{j+1}(X, W\Omega_X^i)$ is of finite type over W if $i+j>n$.

STEP 4. (a) *If $i\neq j$ and $i+j>n+1$, $H^j(X, W\Omega_X^i)=0$.*

(b) *If $n+1<2i\leqq 2n$, $H^i(X, W\Omega_X^i)=W$ and F is bijective on $H^i(X, W\Omega_X^i)$.*

PROOF. By Step 3, X is of Hodge-Witt type in degree r for $r>n+1$, that is, $H^j(X, W\Omega_X^i)$ is of finite type over W for each (i, j) with $i+j=r>n+1$. Hence we have a decomposition of W-module

$$H^r(X/W) = \bigoplus_{i+j=r} H^j(X, W\Omega_X^i)$$

([5], Ch. IV, Th. 4.5).

Case 1. r is odd.

By (III) we have $H^r(X/W)=0$, and therefore $H^j(X, W\Omega_X^i)=0$ for each (i, j) with $i+j=r$.

Case 2. r is even and $n+1<r\leqq 2n$.

By (IV) we have $H^r(X/W)=W$, and therefore, $H^j(X, W\Omega_X^i)$ is torsion-free for each (i, j) with $i+j=r$, and $\sum_{i+j=r} \mathrm{rk}_W H^j(X, W\Omega_X^i)=1$. However, by (IV) we have $H^i(X, W\Omega_X^i)_K=H^r(X/W)_K=K$ if $n/2<i\leqq n$. Hence we obtain

$$\mathrm{rk}_W H^j(X, W\Omega_X^i) = \begin{cases} 1 & \text{if } i=j=r/2 \\ 0 & \text{if } i\neq j,\ i+j=r. \end{cases}$$

STEP 5. (a) *If* $2i\neq n+1$, $H^{n-i+1}(X, W\Omega_X^i)/F^\infty B=0$.

(b) *If* $2i=n+1$, $H^{n-i+1}(X, W\Omega_X^i)/F^\infty B=W$ *and* F *is bijective on* $H^{n-i+1}(X, W\Omega_X^i)/F^\infty B$.

PROOF. Consider now the commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
0\to H^{n-i+1}(X, W\Omega_X^{i-1})/F & \longrightarrow & H^{n-i+1}(X, W\Omega_X^{i-1}/F) & \longrightarrow & {}_F H^{n-i+2}(X, W\Omega_X^{i-1}) & \to 0 \\
\downarrow dV & & \downarrow dV & & \downarrow Fd & \\
0\to H^{n-i+1}(X, W\Omega_X^i)/V & \longrightarrow & H^{n-i+1}(X, W\Omega_X^i/V) & \longrightarrow & {}_V H^{n-i+2}(X, W\Omega_X^i) & \to 0 \\
 & & \downarrow & & & \\
 & & H^j(X, \Omega_X^i). & & &
\end{array}$$

Put

$$\begin{bmatrix} M^0 \\ \downarrow dV \\ M^1 \end{bmatrix} = \begin{bmatrix} H^{n-i+1}(X, W\Omega_X^{i-1})/F \\ \downarrow dV \\ H^{n-i+1}(X, W\Omega_X^i)/V \end{bmatrix}$$

and

$$\begin{bmatrix} L^0 \\ \downarrow dV \\ L^1 \end{bmatrix} = \begin{bmatrix} H^{n-i+1}(X, W\Omega_X^{i-1}/F) \\ \downarrow dV \\ H^{n-i+1}(X, W\Omega_X^i/V) \end{bmatrix}.$$

Case 1. $n\neq 2i-1$.

By (I) we have $H^{n-i+1}(X, \Omega_X^i)=0$. This implies that $dV: L^0\to L^1$ is surjective, and therefore that $L^1=M^1=F^\infty BM^1$ ([5], Ch. I, 1.4). Then we have

$$[H^{n-i+1}(X, W\Omega_X^i)/F^\infty B]/V = H^{n-i+1}(X, W\Omega_X^i)/(F^\infty B+V)=0.$$

Since $H^{n-i+1}(X, W\Omega_X^i)/F^\infty B$ is V-adically separated (loc. cit. Ch. I, Th. 2.9), we obtain

$$[H^{n-i+1}(X, W\Omega_X^i)/F^\infty B]/V = 0.$$

Case 2. $n=2i-1$.

By (II) we have $\dim H^i(X, \Omega_X^i) = 1$. This implies that $\dim M^1/F^\infty B \leqq \dim L^1/F^\infty B = 0$ or 1. Further, we have $H^i(X, W\Omega_X^i)_K = H^{2i}(X/W)_K = K$ by (IV). Then we get $[H^i(X, W\Omega_X^i)/F^\infty B]/V \neq 0$ and therefore $\dim[H^i(X, W\Omega_X^i)/F^\infty B]/V = 1$. It follows that $H^i(X, W\Omega_X^i)/F^\infty B = W$ and that F is bijective on $H^i(X, W\Omega_X^i)/F^\infty B$.

The proof of the theorem is now completed.

COROLLARY. *Let X be a smooth complete intersection of dimension n in a projective space over a perfect field k of characteristic $p>0$.*

(a) *If $i \neq j$ and $i+j \neq n, n+1$, $\underline{H}^j(X, W\Omega_{X,\log}^i)=0$.*

(b) *If $2i \neq n, n+1$ and $0 \leqq i \leqq n$, $\underline{H}^i(X, W\Omega_{X,\log}^i) = \boldsymbol{Z}_p$.*

(c) *$H^{n-i}(X, W\Omega_{X,\log}^i)$ is a free $\boldsymbol{Z}_p$-module and* $\mathrm{rk}_{\boldsymbol{Z}_p} \underline{H}^{n-i}(X, W\Omega_{X,\log}^i) = \dim_K H^n(X/W)_K^{[i]}$.

(d) *If $2i \neq n+1$, $\underline{H}^{n-i+1}(X, W\Omega_{X,\log}^i) = \underline{U}^{n-i+1}(X, W\Omega_{X,\log}^i)$.*

(e) *If $2i = n+1$, $\underline{H}^{n-i+1}(X, W\Omega_{X,\log}^i)/\underline{U}^{n-i+1}(X, W\Omega_{X,\log}^i) = \boldsymbol{Z}_p$.*

PROOF. By Illusie-Raynaud [**5**], Ch. IV, Th. 3.3, we see that

(1) $\underline{H}^j(X, W\Omega_{X,\log}^i)$ is an extension of a pro-étale quasi-algebraic group $\underline{D}^j(X, W\Omega_{X,\log}^i)$ by a connected unipotent quasi-algebraic group $\underline{U}^j(X, W\Omega_{X,\log}^i)$;

(2) $\dim \underline{U}^j(X, W\Omega_{X,\log}^i) = \dim \text{Domino } H^j(X, W\Omega_X^\cdot)^{i-1}$;

(3) $\underline{D}^j(X, W\Omega_{X,\log}^i)(\bar{k})$ is isomorphic to ${}_{F-1}(\text{Heart } H^j(X_{\bar{k}}, W\Omega_X^\cdot)^i)_{ss}$.

Now we can deduce the assertions from the theorem as follows.

Case 1. $i+j \neq n, n+1$.

By Step 3, the differentials $d: H^j(X, W\Omega_X^{i-1}) \to H^j(X, W\Omega_X^i)$ and $d: H^j(X, W\Omega_X^i) \to H^j(X, W\Omega_X^{i+1})$ are zero. Hence

$$\text{Heart } H^j(X, W\Omega_X^\cdot)^i = H^j(X, W\Omega_X^i)$$

(cf. [**5**], Ch. I. Prop. 2.18), and therefore

$$\text{Heart } H^j(X, W\Omega_X^\cdot)^i = \begin{cases} W & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

This implies (a) and (b).

Case 2. $i+j=n$.

By Step 3, the differential $d: H^j(X, W\Omega_X^{i-1}) \to H^j(X, W\Omega_X^i)$ is zero. Hence

$$\text{Heart } H^j(X, W\Omega_X^\cdot)^i = V^{-\infty}Z \subset H^j(X, W\Omega_X^i),$$

and therefore Heart $H^j(X, W\Omega_X^\cdot)^i$ is torsion-free. This implies (c).

Case 3. $i+j=n+1$.
By Step 3, the differential $d: H^j(X, W\Omega_X^i) \to H^j(X, W\Omega_X^{i+1})$ is zero. Hence

$$\text{Heart}\, H^j(X, W\Omega_X^{\cdot})^i = H^j(X, W\Omega_X^i)/F^\infty B\,,$$

and therefore

$$\text{Heart}\, H^j(X, W\Omega_X^{\cdot})^i = \begin{cases} W & \text{if } i=j \\ 0 & \text{if } i \neq j\,. \end{cases}$$

This implies (d) and (e).

REMARK. By Deligne (cf. [6]), *general* smooth complete intersections of dimension n and of multidegree $(d_1, \cdots, d_m)$ in a projective space are ordinary. In this case, $H^j(X, W\Omega_X^i)$ is a free W-module of rank $h^{ij}(X)=\dim_k H^j(X, \Omega_X^i)$ and F is bijective on $H^j(X, W\Omega_X^i)$ for each (i, j).

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