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# On Moishezon manifolds homeomorphic to $P_C^n$

Dedicated to Professor Kunihiko Kodaira

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## § 0. Introduction.

There are in general many different complex manifolds having the same underlying topological or differentiable structure. However there are a few exceptional cases where we can expect that homeomorphy to a given compact complex manifold implies analytic isomorphism to it, for instance, compact Hermitian symmetric spaces. Among compact Hermitian symmetric spaces, the complex projective space $P_C^n$ and a smooth hyperquadric $Q_C^n$ in $P_C^{n+1}$ seem to be nice exceptions which we can handle with algebraic methods.

The following conjecture is the problem we study in the present article.

CONJECTURE $\mathrm{MP}_n$. *Any Moishezon complex manifold homeomorphic to $P_C^n$ is isomorphic to $P_C^n$.*

There are some related conjectures, or rather, more accessible forms of Conjecture $\mathrm{MP}_n$ which are interesting themselves.

CONJECTURE $\mathrm{LM}_n$. *Let $X$ be a Moishezon manifold of dimension $n$, and $L$ a line bundle on $X$. Assume that* $\mathrm{Pic}\, X = \boldsymbol{Z}L$, $c_1(X) = dc_1(L)$ $(d \geqq n+1)$ *and* $h^0(X, O_X(L)) \geqq n+1$. *Then $X$ is isomorphic to $P_C^n$.*

CONJECTURE $\mathrm{LMP}_n$. *Let $X$ be a Moishezon manifold homeomorphic to $P_C^n$, and $L$ a line bundle on $X$ with $L^n=1$. Assume $h^0(X, O_X(L)) \geqq n+1$. Then $X$ is isomorphic to $P_C^n$.*

CONJECTURE $\mathrm{DP}_n$. *Any complex (global) deformation of $P_C^n$ is isomorphic to $P_C^n$.*

In the above conjectures a Moishezon (complex) manifold of dimension $n$ is by definition a compact complex manifold with $n$ algebraically independent meromorphic functions. This is equivalent to saying that it is bimeromorphic to an algebraic variety.

Conjecture $\mathrm{MP}_n$ (resp. Conjecture $\mathrm{LM}_n$) has been settled by Hirzebruch-Kodaira [3], and Yau [21] (resp. by Fujita [1], Kobayashi and Ochiai [6]), when the manifold under consideration is *projective or Kählerian*. See Siu [17] [18] and Tsuji [20] for Conjecture $DP_n$. I heard from Mabuchi in the summer of 1990 that Siu seemed to have completed a correction of [17], while I completed the present article in 1991 January. I was unable to look at the article of Siu until very recently it appeared as [18]. I cannot spend enough time for understanding [18] before submitting this article, but I hear from Mabuchi that [18] is correct.

Meanwhile Kollár [8] and the author [10] solved Conjecture $MP_3$ without extra assumptions, each supplementing the other. Peternell [15][16] also asserts $(MP_3)$. See also [8, 5.3.6].

(0.1) THEOREM [8][10]. *Any Moishezon threefold homeomorphic to $\boldsymbol{P}_{\boldsymbol{C}}^3$ is isomorphic to $\boldsymbol{P}_{\boldsymbol{C}}^3$.*

The purpose of the present paper is to give some partial solutions to the above conjectures, in particular, a complete solution to $(LM_4)$ and $(LMP_4)$, which implies $(DP_4)$.

For the proof of $(LM_4)$ or $(LMP_4)$, we study dualizing sheaves of reduced curves and surfaces in the present article, although the idea of the proof is essentialy the same as our previous papers [10][11]. Our new ingredient here is a subadjunction formula (2.A) for curves and surfaces.

(0.2) THEOREM. *Let $X$ be a Moishezon manifold of dimension $n$ with $b_2=1$, $L$ a line bundle on $X$. Assume that $c_1(X)=dc_1(L)$ $(d\geqq n+1)$, and $h^0(X, O_X(L))\geqq n$. If a complete intersection of general $(n-1)$-members of the complete linear system $|L|$ is nonempty outside the base locus $\mathrm{Bs}|L|$, then $X$ is isomorphic to $\boldsymbol{P}_{\boldsymbol{C}}^n$.*

The following theorems are proved by applying (0.2) or the idea of the proof of (0.2).

(0.3) THEOREM. *Let $X$ be a Moishezon fourfold, and $L$ a line bundle on $X$. Assume that $\mathrm{Pic}\, X=\boldsymbol{Z}L$, $c_1(X)=dc_1(L)$ $(d\geqq 5)$ and $h^0(X, O_X(L))\geqq 4$. Then $X$ is isomorphic to $\boldsymbol{P}_{\boldsymbol{C}}^4$.*

(0.4) THEOREM. *Let $X$ be a Moishezon fourfold homeomorphic to $\boldsymbol{P}_{\boldsymbol{C}}^4$, and $L$ a line bundle on $X$ with $L^4=1$. Assume $h^0(X, O_X(L))\geqq 3$. Then $X$ is isomorphic to $\boldsymbol{P}_{\boldsymbol{C}}^4$.*

(0.5) COROLLARY. *Any complex (global) deformation of $\boldsymbol{P}_{\boldsymbol{C}}^4$ is isomorphic to $\boldsymbol{P}_{\boldsymbol{C}}^4$.*

See also [17][18][20]. Now we shall explain an outline of our proof of (0.2). By Bertini's theorem, we choose a general $(n-1)$-dimensional subspace $V$ of $H^0(X, O_X(L))$ such that $l_V := \cap_{s\in V}(\text{zeroes of } s)$, the scheme-theoretic complete intersection associated to $V$, is pure one dimensional and nonsingular outside $\mathrm{Bs}|L|$. Then we show in section one that $l_V$ is a union of nonsingular rational curves $C$ with $LC=1$ and $N_{C/X}\cong O_C(1)^{\oplus(n-1)}$, of nonsingular elliptic curves $E$ with $LE=0$ and $N_{E/X}\cong O_E^{\oplus(n-1)}$ and of the base locus $\mathrm{Bs}|L|$, each of the curves being a connected component of $l_V$. This is proved by using the subadjunction formula (1.8) or (2.A) for curves, which generalizes an argument in [**10**]. The existence of a rational curve among the irreducible components of $l$ outside $\mathrm{Bs}|L|$ follows from the fact that $X$ is Moishezon.

In section 2 we prove an inequality which is a key to the proofs in section one.

Then in section 3, by using the results proved in section one, we show that $\dim|L|=n$ and that $X$ is rationally mapped onto $P_C^n$ by the rational map $\rho_{|L|}$ associated with $|L|$. Therefore $X$ is finite over $P_C^n$ outside proper subvarieties $B_X$ and $B_{P^n}$.

If a line on $P_C^n$ is not contained in $B_{P^n}$, its inverse image by $\rho_{|L|}$ is a complete intersection of $(n-1)$ members of $|L|$ and it is generically reduced and pure one-dimensional outside $B_X$. Then we can show as before that the inverse image $l$ is a union of a nonsingular rational curve $C$ and $\mathrm{Bs}|L|$ and that $C$ is a connected component of $l$.

Now $LC=1$ implies that $\rho_{|L|}$ is birational. Moreover those lines which are not contained in $B_{P^n}$ sweep out $P_C^n$, so that inverse images of the lines sweep out $X$. This implies that $\mathrm{Bs}|L|$ is empty. We also see that $\rho_{|L|}$ is unramified, so that $X$ is isomorphic to $P_C^n$. See also (1.6).

In section 3, applying (0.2) and the subadjunction formula (2.A) for surfaces, we also prove (0.3) and (3.3), the latter of which strengthens our earlier consequence on $P_C^3$ [**10**].

In section 4 (resp. section 5), we apply the results in section one to study ($\mathrm{LMP}_n$) (resp. to prove (0.4)). In the proof of (0.3) (resp. (0.4)) the complete intersection of two members of $|L|$ is proved to be isomorphic to $P_C^2$, from which (0.3) (resp. (0.4)) follows immediately. This also implies ($\mathrm{LM}_4$), ($\mathrm{LMP}_4$) and ($\mathrm{DP}_4$).

The main consequences of the present article were announced in [**13**], where the proof of (0.4) is sketched.

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## §1. A complete intersection $l_V$.

(1.1) Let $X$ be a nonsingular complete algebraic variety of dimension $n$ defined over $C$ (or a compact complex manifold of dimension $n$). We assume that there exists a line bundle $L$ on $X$ such that

$$(1.1.1) \qquad c_1(X) = dc_1(L) \quad \text{for some} \quad d \geqq n+1$$

$$(1.1.2) \qquad \dim H^0(X, L) \geqq n\,.$$

Let $B = \mathrm{Bs}|L|$ be the base locus of $|L|$. Let $V$ be a linear subspace of $H^0(X, L)$ of dimension $n-1$, $l_V$ a scheme-theoretic complete intersection $\bigcap_{s\in V\setminus\{0\}} D_s$ associated with $V$, where $D_s$ is the divisor defined by $s=0$. More precisely, the ideal sheaf of $O_X$ defining $l$ is given by $I_l = \sum_{s\in V} I_{D_s} = \sum_{s\in V} sO_X$. Let $C_V$ be *the sum of all the irreducible components of* $l$ *which are not totally contained in* $B$. We express it as $l_V = C_V + B$ for simplicity.

We call an irreducible component $C$ of $l$ (or of $C_V$) of dimension one *a reduced curve component* if $l$ is reduced generically along $C$. We assume that

$$(1.1.3) \qquad l_V \text{ has } \textit{a reduced curve component } C \quad \text{for some } V\,.$$

In the present section, we always assume (1.1.1)-(1.1.3). For the use in § 3, we also define

(1.2) DEFINITION. We say that $D_s$ $(s\in V)$ *intersect outside* $\mathrm{Bs}|L|$ if $C_V$ is nonempty. We say that $D_s$ $(s \in V)$ *intersect rationally outside* $\mathrm{Bs}|L|$ if $C_V$ is nonempty and moreover if at least one of the irreducible components of $C_V$ is a (possibly singular) rational curve.

(1.3) Let $l = l_V$, and let $C$ *a reduced curve component* of $l$, $I_C$ the ideal sheaf of $O_X$ defining $C$ with $\sqrt{I_C} = I_C$. We have nontrivial $O_C$-homomorphisms $\phi_C^0$ and $\phi_C$ which are isomorphisms on a Zariski open dense subset of $C$,

$$\begin{array}{ccc} \phi_C^0:\ (I_l/I_l^2)\otimes O_C & \longrightarrow & I_C/I_C^2 \\ \| & & \downarrow \\ \phi_C:\ (I_l/I_l^2)\otimes O_C & \longrightarrow & [I_C/I_C^2] \end{array}$$

where $[F] = F/\{O_C\text{-torsions in } F\}$ for an $O_C$-module $F$.

(1.4) LEMMA. *Let $C$ be an irreducible reduced curve component of $l := l_V$. Then*

$$(I_l/I_l^2)\otimes O_C \cong O_C(-L)^{\oplus(n-1)}$$

$$-(n-1)LC \leqq c_1([I_C/I_C^2])$$

*where $c_1([I_C/I_C^2]) := c_1([I_C/I_C^2]\otimes O_{\tilde{C}}/O_{\tilde{C}}\text{-torsions})$ for the normalization $\tilde{C}$ of $C$.*

PROOF. We have a commutative diagram of natural homomorphisms;

$$\begin{array}{ccc} O_X(-L)^{\oplus(n-1)} & \longrightarrow & I_l/I_l^2 \\ \downarrow & & \downarrow \\ O_C(-L)^{\oplus(n-1)} & \xrightarrow{\ \beta\ } & (I_l/I_l^2)\otimes O_C\,. \end{array}$$

where all the arrows are surjective. Moreover $(n-1)$ generators of $I_l$ are regular parameters on $C\backslash B$. Hence $\beta$ is injective on $C\backslash B$, and it is surjective anywhere on $C$. Since $O_C(-L)$ is $O_C$-torsion free, $\beta$ is an isomorphism. It follows that the composite homomorphism $\phi_C\cdot\beta$ is injective. Hence we have $-(n-1)LC\leqq c_1([I_C/I_C^2])$. q.e.d.

(1.5) LEMMA. *The following sequence is exact everywhere on* $C$;

$$0 \longrightarrow [I_C/I_C^2] \longrightarrow \Omega_X^1\otimes O_C \longrightarrow \Omega_C^1 \longrightarrow 0.$$

*where* $\Omega_C^1:=\Omega_X^1/I_C\Omega_X^1+O_X\{d\varphi;\ \varphi\in I_C\}$.

PROOF. We have a natural exact sequence

$$I_C/I_C^2 \xrightarrow{\eta} \Omega_X^1\otimes O_C \longrightarrow \Omega_C^1 \longrightarrow 0.$$

If $C$ is nonsingular at $p$, then $\eta$ is injective at $p$. Since $\Omega_X^1$ is locally free, the sheaf $\Omega_X^1\otimes O_C$ is locally $O_C$-free, in particular, it is $O_C$-torsion free. q.e.d.

In order to illustrate how our arguments in sections 1 and 3 proceed, we first prove the following easy Proposition.

(1.6) PROPOSITION. *Assume* $K_X=-dL(d\geqq n+1)$, $h^0(X, L)\geqq n+1$. *Let* $C$ *be a reduced curve component of* $C_V$ *with* $LC\geqq 1$ *which is not contained in* $B:=\mathrm{Bs}|L|$. *Assume that* $l_V$ *is connected and that* $C$ *is nonsingular everywhere. Then* $l_V=C_V=C\cong\boldsymbol{P}^1$, $L^n=LC=1$, $N_{C/X}\cong O_C(1)^{\oplus(n-1)}$, $d=n+1$ *and* $B$ *consists of at most a single point. Moreover if* $B$ *is empty, then* $X\cong\boldsymbol{P}^n$.

PROOF. Let $l=l_V$. Since $C$ is nonsingular, we have $[I_C/I_C^2]=I_C/I_C^2$. By (1.5) we have

$$c_1(I_C/I_C^2)=K_XC-c_1(\Omega_C^1)=-dLC-c_1(\Omega_C^1).$$

From (1.4) we infer,

$$-(n-1)LC\leqq c_1(I_C/I_C^2)=-dLC-c_1(\Omega_C^1)$$

$$2\leqq d-n+1\leqq(d-n+1)LC\leqq -c_1(\Omega_C^1)\leqq 2.$$

This implies that $C\cong\boldsymbol{P}^1$, $c_1(\Omega_C^1)=-2$, $d=n+1$ and $LC=1$. The homomorphism $\phi_C=\phi_C^0$ is an isomorphism, $I_C/I_C^2\cong O_C(-L)^{\oplus(n-1)}\cong O_C(-1)^{\oplus(n-1)}$. Since $\phi_C$ is surjective, we have $I_l+I_C^2=I_C$ along $C$. By applying Nakayama's lemma to the $O_X$-module $I_C/I_l$ we see that $I_l=I_C$ along $C$. Consequently $C$ is a connected component of $l$. By the assumption that $l$ is connected, we see $l=C_V=C$, $N_{C/X}=(I_C/I_C^2)^\vee\cong O_C(1)^{\oplus(n-1)}$, $L^n=LC=1$. Since $C$ is not contained in $B$, $B$ is empty or a single point in view of $LC=1$. If $B$ is empty, we have a morphism $f$ of $X$ into $\boldsymbol{P}^N$ associated with the linear system $|L|$ where $N=$

$h^0(X, L)-1$. Since $L^n=1$, $f(X)$ is a linear subspace of $\boldsymbol{P}^N$ with $\dim f(X)=n$, whence $N=n$ and $f$ is surjective and birational. Let $\omega_P$ be a meromorphic $n$ form on $\boldsymbol{P}^n$ with poles $(n+1)H$, $H$ a hyperplane of $\boldsymbol{P}^n$. Then by using local coordinates $z_P$ on $\boldsymbol{P}^n$ and $z$ on $X$ we write symbolically

$$f^*\omega_P = f^*dz_P/f^*H^{n+1} = f^*dz_P/D^{n+1}$$

$$f^*dz_P = \det(\text{Jacobian of } f)\cdot dz$$

for a member $D=f^*H\in|L|$. Since $f^*\omega_P$ is a meromorphic $n$ form on $X$, the divisor $(f^*\omega_P)$ is equal to $K_X=-(n+1)D$, whence we have $(f^*dz_P)=0$. Hence the birational morphism $f$ is unramified so that $X$ is isomorphic to $\boldsymbol{P}^n$.

q.e.d.

This is a prototype of our subsequent argument. However in general $l_V$ may be disconnected, and some component $C$ of $C_V$ may be singular at the intersection $C\cap B$.

(1.7) Now we come back to the situation in (1.1). Under the same notation as in (1.1), let $l=l_V$, and let $C$ be a reduced curve component of $l$.

Let $\nu\colon \tilde{C}\to C$ be the normalization of $C$. Then we obtain exact sequences,

$$0\longrightarrow \mathrm{Tor}_1^{O_C}(\Omega_C^1, O_{\tilde{C}})\longrightarrow [I_C/I_C^2]\otimes O_{\tilde{C}}\longrightarrow \Omega_X^1\otimes O_{\tilde{C}}\longrightarrow \Omega_C^1\otimes O_{\tilde{C}}\longrightarrow 0 \tag{1.7.1}$$

$$0\longrightarrow [[I_C/I_C^2]\otimes O_{\tilde{C}}]\longrightarrow \Omega_X^1\otimes O_{\tilde{C}}\longrightarrow \Omega_C^1\otimes O_{\tilde{C}}\longrightarrow 0 \tag{1.7.2}$$

because $\mathrm{Tor}_1^{O_C}(\Omega_X^1\otimes O_C, O_{\tilde{C}})=0$. We recall an injective $O_C$-homomorphism $\phi_C$ in (1.3),

$$\phi_C\colon (I_l/I_l^2)\otimes O_C\quad (\cong O_C(-L)^{\oplus(n-1)})\longrightarrow [I_C/I_C^2]. \tag{1.7.3}$$

Let $Q_C^0$ be $\mathrm{Coker}\,\phi_C$. By tensoring (1.7.3) with $O_{\tilde{C}}$, we obtain an exact sequence

$$\begin{aligned}\cdots\longrightarrow \mathrm{Tor}_1^{O_C}(Q_C^0, O_{\tilde{C}})\longrightarrow O_{\tilde{C}}(-\nu^*L)^{\oplus(n-1)}&\\ \longrightarrow [I_C/I_C^2]\otimes O_{\tilde{C}}\longrightarrow Q_C^0\otimes O_{\tilde{C}}\longrightarrow 0.&\end{aligned} \tag{1.7.4}$$

Since $\mathrm{supp}\,Q_C^0$ is contained in $\mathrm{Sing}\,C$, $\mathrm{Tor}_1^{O_C}(Q_C^0, O_{\tilde{C}})$ is also an $O_{\tilde{C}}$-torsion sheaf. Hence we have an exact sequence

$$0\longrightarrow O_{\tilde{C}}(-\nu^*L)^{\oplus(n-1)}\longrightarrow [I_C/I_C^2]\otimes O_{\tilde{C}}\longrightarrow Q_C^0\otimes O_{\tilde{C}}\longrightarrow 0. \tag{1.7.5}$$

Composed with a natural homomorphism

$$[I_C/I_C^2]\otimes O_{\tilde{C}}\longrightarrow [[I_C/I_C^2]\otimes O_{\tilde{C}}] := [I_C/I_C^2]\otimes O_{\tilde{C}}/O_{\tilde{C}}\text{-torsions},$$

we infer an exact sequence

$$0\longrightarrow O_{\tilde{C}}(-\nu^*L)^{\oplus(n-1)}\longrightarrow [[I_C/I_C^2]\otimes O_{\tilde{C}}]\longrightarrow Q_C\longrightarrow 0 \tag{1.7.6}$$

with $Q_C$ cokernel.

Finally we consider a natural homomorphism

$$\Omega_C^1 \otimes O_{\tilde{C}} \xrightarrow{\eta} \Omega_{\tilde{C}}^1 .$$

Letting $Q'_C = \mathrm{Coker}\,\eta$ and $Q''_C = \mathrm{Ker}\,\eta$, we have an exact sequence

$$(1.7.7) \qquad 0 \longrightarrow Q''_C \longrightarrow \Omega_C^1 \otimes O_{\tilde{C}} \longrightarrow \Omega_{\tilde{C}}^1 \longrightarrow Q'_C \longrightarrow 0 .$$

For a torsion sheaf $F$ we define the length $l(F)$ of $F$ to be the rank of $F$ as a $\boldsymbol{C}$-module.

(1.8) LEMMA. *Let $C$ be a reduced curve component of $l$. Assume $c_1(X) = dc_1(L)$. Then we have,*

$$(d-n+1)LC + c_1(\Omega_{\tilde{C}}^1) + l(Q_C) + l(Q''_C) - l(Q'_C) = 0 .$$

PROOF. From the above exact sequences we infer,

$$\begin{aligned}
\chi(\Omega_{\tilde{C}}^1) + l(Q''_C) - l(Q'_C) &= \chi(\Omega_C^1 \otimes O_{\tilde{C}}) \quad \text{by } (1.7.7) \\
&= \chi(\Omega_X^1 \otimes O_{\tilde{C}}) - \chi([[I_C/I_C^2] \otimes O_{\tilde{C}}]) \quad \text{by } (1.7.2) \\
&= \chi(\Omega_X^1 \otimes O_{\tilde{C}}) - (n-1)\chi(O_{\tilde{C}}(-\nu^* L)) - l(Q_C) \\
&= \chi(O_{\tilde{C}}) + K_X C + (n-1)LC - l(Q_C) \\
&= \chi(O_{\tilde{C}}) - (d-n+1)LC - l(Q_C) \quad \text{by } (1.1.1).
\end{aligned}$$

q.e.d.

Moreover we see

(1.9) THEOREM. $l(Q''_C) \geqq l(Q'_C)$. *Equality holds if and only if $C$ is nonsingular.*

This is proved in §2. See (2.5).

As a corollary to (1.8) and (1.9), we infer

(1.10) LEMMA. *Assume $c_1(X) = dc_1(L)$. Let $C$ be a reduced curve component of $l = l_V$. If $d \geqq n+1$, $LC \geqq 1$, then $d = n+1$, $LC = 1$, $\tilde{C} \cong C \cong \boldsymbol{P}^1$, $N_{C/X} \cong O_C(1)^{\oplus(n-1)}$ and $C$ is a connected component of $l_V$. Moreover if $C$ is not contained in $B = \mathrm{Bs}|L|$, then $C \cap B$ consists of at most one point.*

PROOF. Note that $c_1(\Omega_{\tilde{C}}^1) \geqq -2$, $(d-n+1)LC \geqq 2LC \geqq 2$, $l(Q_C) \geqq 0$. By (1.9), $l(Q''_C) \geqq l(Q'_C)$. Hence all the above inequalities are equalities by (1.8). Therefore $\tilde{C} \cong \boldsymbol{P}^1$, $LC = 1$, $d = n+1$, $l(Q_C) = 0$, $l(Q''_C) = l(Q'_C)$. Moreover $C$ is nonsingular by (1.9). Therefore the sequence (1.7.6) is the same as those in (1.3) and (1.7.3) where $\phi_C = \phi_C^0$ is an isomorphism. It follows that $N_{C/X} = (I_C/I_C^2)^\vee \cong$

$O_C(1)^{\oplus(n-1)}$, $I_l+I_C^2=I_C$ along $C$. Consequently $I_l=I_C$ along $C$ by Nakayama's lemma. This implies that $C$ is a connected component of $l$. In view of $LC=1$, $C\cap B$ consists of at most a single point if $C\subset C_V$. q.e.d.

(1.11) LEMMA. *Assume $c_1(X)=dc_1(L)$, $d$ arbitrary. Let $C$ be a reduced curve component of $C_V$. If $LC=0$ and if $C_V$ is nonsingular outside $B$, then $C$ is a smooth elliptic curve with $N_{C/X}\cong O_C^{\oplus(n-1)}$ and $C$ is a connected component of $l_V$ disjoint from $B$.*

PROOF. Let $l=l_V$. Any member $D$ of $|L|$ contains $B\cap C$. Hence if $B\cap C\neq\emptyset$, then $D$ contains $C$ because $LC=0$. Hence $C$ is contained in $B$, which contradicts $C\subset C_V$. Therefore $B\cap C=\emptyset$. By the assumption, any singular point of $C$ is contained in $B$. Therefore $C$ is nonsingular, $l(Q_C'')=l(Q_C')=0$ and $C$ passes through no singular points of $l_{\text{red}}$. This implies that $C$ is a connected component of $l$ and $I_C=I_l$ along $C$. Hence $l(Q_C)=0$ and $\phi_C$ is an isomorphism. In view of (1.8) we have $c_1(\Omega_{\tilde{C}}^1)=c_1(\Omega_C^1)=0$. Consequently $C$ is a smooth elliptic curve disjoint from $B$. Meanwhile there is a member $D$ of $|L|$ which does not contain $C$. Since $LC=0$, $D$ does not intersect $C$, which shows $L\otimes O_C\cong O_C$. It follows that $N_{C/X}\cong O_C^{\oplus(n-1)}$. q.e.d.

## §2. The inequality $l(Q_C'')\geqq l(Q_C')$ — Proof of (1.9).

(2.1) Let $C$ be an irreducible curve, $\nu\colon \tilde{C}\to C$ the normalization, $F$ a torsion $O_{\tilde{C}}$-module, $p$ (resp. $q$) a point of $C$ (resp. $\tilde{C}$). Then we define $e(F, q)$, $l(F, p)$ and $l(F)$ as follows,

$$e(F, q)=l(F_q)=\dim_{\mathbf{C}} F_q,$$

$$l(F, p)=\sum_{q\text{ above }p} l(F_q), \qquad l(F)=\sum_{p\in C} l(F, p).$$

It is clear that if $C$ is locally irreducible at $p$, then we have $e(F, q)=l(F, p)$ for the unique point $q$ of $\tilde{C}$ lying above $p$.

Let $\operatorname{Sing} C$ be the set of all singular points of $C$. Then consider the exact sequence

$$\text{(2.1.1)} \qquad 0\longrightarrow Q_C''\longrightarrow \Omega_C^1\otimes O_{\tilde{C}}\longrightarrow \Omega_{\tilde{C}}^1\longrightarrow Q_C'\longrightarrow 0.$$

Hence we have

$$l(Q_C')=\sum_{p\in\operatorname{Sing} C} l(Q_C', p), \qquad l(Q_C'')=\sum_{p\in\operatorname{Sing} C} l(Q_C'', p).$$

Now we consider the germ of $C$ at $p\in\operatorname{Sing} C$ locally. Let $C=C_1\cup\cdots\cup C_r$ be locally irreducible components of $C$ at $p$. Then we have an exact sequence

$$\text{(2.1.2)} \qquad 0\longrightarrow Q_\lambda''\longrightarrow \Omega_{C_\lambda}^1\otimes O_{\tilde{C}_\lambda}\longrightarrow \Omega_{\tilde{C}_\lambda}^1\longrightarrow Q_\lambda'\longrightarrow 0$$

where $Q'_\lambda := Q'_{C_\lambda}$, and $Q''_\lambda := Q''_{C_\lambda}$ for an irreducible component $C_\lambda$ at $p$. The local curve $C_\lambda$ is irreducible at $p$, and the normalization $\tilde{C}_\lambda$ of $C_\lambda$ has a unique point $q_\lambda$ above $p$. Then we have at $p$

$$\Omega^1_{\tilde{C}} \cong \oplus_\lambda \Omega^1_{\tilde{C}_\lambda} \cong \oplus_\lambda \Omega^1_{\tilde{C}_\lambda, q_\lambda}, \qquad O_{\tilde{C}} \cong \oplus_\lambda O_{\tilde{C}_\lambda} \cong \oplus_\lambda O_{\tilde{C}_\lambda, q_\lambda}.$$

Hence

$$\begin{aligned} Q'_C &\cong \oplus_\lambda \Omega^1_{\tilde{C}_\lambda} / \oplus_\lambda \Omega^1_{C_\lambda} \otimes O_{\tilde{C}_\lambda} \\ &\cong \oplus_\lambda (\Omega^1_{\tilde{C}_\lambda} / \Omega^1_{C_\lambda} \otimes O_{\tilde{C}_\lambda}) \\ &\cong \oplus_\lambda Q'_\lambda \end{aligned}$$

whence $l(Q'_C, p) = \sum_\lambda l(Q'_\lambda)$.

Next we consider $l(Q''_C, p)$. We have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Q''_C \arrow[r] & \Omega^1_C \otimes O_{\tilde{C}} \arrow[r, "\xi"] \arrow[d, "j"'] \arrow[dr, phantom, "\oplus\xi_\lambda"] & \Omega^1_{\tilde{C}} \arrow[d, equal] \\ 0 \arrow[r] & \oplus_\lambda Q''_\lambda \arrow[r] & \oplus_\lambda \Omega^1_{C_\lambda} \otimes O_{\tilde{C}_\lambda} \arrow[r] & \oplus_\lambda \Omega^1_{\tilde{C}_\lambda} \end{tikzcd}$$

with $j$ surjective. Hence $\operatorname{Ker}\xi$ is mapped onto $\oplus \operatorname{Ker}\xi_\lambda$. This shows

$$l(Q''_C, p) = l(\operatorname{Ker}\xi) \geqq \sum_{\lambda \in \Lambda} l(\operatorname{Ker}\xi_\lambda) = \sum_{\lambda \in \Lambda} l(Q''_\lambda).$$

Thus we obtain

(2.2) LEMMA. *Let $C_\lambda (\lambda \in \Lambda)$ be all the locally irreducible components of $C$ at $p$. Then*

$$l(Q'_C, p) = \sum_{\lambda \in \Lambda} l(Q'_\lambda)$$

$$l(Q''_C, p) \geqq \sum_{\lambda \in \Lambda} l(Q''_\lambda).$$

Next we prove

(2.3) LEMMA. *Assume that $C$ is locally irreducible at $p$. Then $l(Q''_C, p) \geqq l(Q'_C, p)$. Equality holds if and only if $C$ is nonsingular at $p$. If $C$ is singular at $p$, then $l(Q''_C, p) \geqq l(Q'_C, p) + 2$.*

PROOF. Let $x_1, \cdots, x_n$ be a local coordinate system of $X$ at $p$. Then we may assume that the normalization $\nu: \tilde{C} \to C$ $(\subset X)$ is locally given by

$$\begin{aligned} &x_1 = t^m \\ &x_j = f_j(t) = t^{m_j} g_j(t), \qquad g_j(0) \neq 0, \qquad (2 \leqq j \leqq s) \\ &x_j = 0 \qquad (s+1 \leqq j \leqq n) \end{aligned}$$

where $m < m_2 < m_3 < \cdots < m_s$, none of $m_j$ and $m_j - m_k$ is an integral multiple of

$m$, $s$ is the embedding dimension of $(C, p)$. By the choice of $m_2$, there is a positive integer $q$ such that $m \leqq qm < m_2 < (q+1)m$.

In terms of the parameter $t$, (by taking completions) we have

$$\begin{aligned} \Omega^1_{\tilde{C},q} &\cong \boldsymbol{C}[[t]]dt \\ \mathrm{Image}(\Omega^1_{C,p} \otimes O_{\tilde{C},q}) &\cong \boldsymbol{C}[[t]]t^{m-1}dt + \cdots + \boldsymbol{C}[[t]]\nu^* dx_s \\ &\cong \boldsymbol{C}[[t]]t^{m-1}dt + \cdots + \boldsymbol{C}[[t]](m_s t^{m_s-1} g_s + t^{m_s} g_s')dt \\ &\cong \boldsymbol{C}[[t]]t^{m-1}dt \end{aligned}$$

because $m_j > m$ $(j \geqq 2)$. Consequently

$$l(Q'_C, p) = l(\Omega^1_{\tilde{C},q} / \Omega^1_{C,q} \otimes O_{\tilde{C},q}) = m-1 . \tag{2.3.1}$$

Next consider $l(Q''_C, p)$. First we see that $J := I_C \cap \boldsymbol{C}[[x_1, \cdots, x_s]]$ is contained in $m_p^2$, $m_p$ being the maximal ideal of $O_{X,p}$. In fact, if there is an element $F \in J \cap (m_p \backslash m_p^2)$, then $F$ is part of a local coordinate system. Replacing one of the local parameters $x_1, \cdots, x_s$, say $x_s$, by $F$ then $C$ is contained in $x_s = x_{s+1} = \cdots = x_n = 0$ locally. This is absurd because we choose $s$ minimal, $s$ being equal to the embedding dimension of $(C, p)$.

When $m=1$, $C$ is nonsingular at $p$ and $\Omega^1_C \otimes O_{\tilde{C}} \cong \Omega^1_{\tilde{C}}$, $l(Q''_C, p) = l(Q'_C, p) = 0$.

So we may assume $m \geqq 2$. Let $e_j = dx_j \otimes 1 \in \Omega^1_X \otimes O_{\tilde{C}}$, $\bar{e}_j = dx_j \otimes 1 \in \Omega^1_C \otimes O_{\tilde{C}}$ for $1 \leqq j \leqq s$. Then the element $\sigma_j = (f_j'(t)/mt^{m-1})\bar{e}_1 - \bar{e}_j$ is contained in $Q''_C$. In fact, $\xi(\sigma_j) = (f_j'(t)/mt^{m-1})\nu^* dx_1 - \nu^* dx_j = 0$. Now we choose the minimal integer $N \geqq 0$ such that $t^N \sigma_2 = 0$. We note that

$$l(Q''_C, p) \geqq N . \tag{2.3.2}$$

Recall that

$$\Omega^1_C \otimes O_{\tilde{C}} \cong \sum_{j=1}^{s} \boldsymbol{C}[[t]]e_j / \boldsymbol{C}[[t]] \left\{ \sum_{j=1}^{s} \nu^*(\partial\varphi/\partial x_j) e_j ,\ \varphi \in I_C \right\} .$$

Hence $t^N \sigma_2 = 0$ means that there exist some $F_i \in \boldsymbol{C}[[t]]$ and $\varphi_i \in I_C$ $(1 \leqq i \leqq l)$ such that

$$t^N((f_2'(t)/mt^{m-1})e_1 - e_2) = \sum_{j=1}^{s} \left( \sum_{i=1}^{l} F_i(t) \nu^*(\partial\varphi_i/\partial x_j) \right) e_j . \tag{2.3.3}$$

The coefficient of $e_1$ in the right hand side is equal to $\sum_{i=1}^{l} F_i(t)\nu^*(\partial\varphi_i/\partial x_1)$. Take any element $\varphi \in I_C$ $(\subset m_p)$. We want to estimate a lower bound of $\deg \nu^*(\partial\varphi/\partial x_1)$. For this purpose, we may assume $\varphi \in I_C \cap \boldsymbol{C}[[x_1, \cdots, x_s]]$ $(\subset m_p^2)$. Expand $\varphi$ as

$$\varphi = \sum_{i_1 + \cdots + i_s \geqq 2} a_{i_1 \cdots i_s} x_1^{i_1} \cdots x_s^{i_s} .$$

Since $\varphi \in I_C$ is equivalent to $\nu^* \varphi = 0$, we have $a_{20\cdots 0} = 0$ because $x_1^2$ is the unique monomial in $x_j$'s with $\deg \nu^* x_1^2 = 2m$. We put $a_{10\cdots 0} = 0$.

(2.3.4) CLAIM. $a_{j0\cdots0}=0\ (1\leqq j\leqq 2q)$, $a_{j10\cdots0}=0\ (1\leqq j\leqq q)$.

PROOF OF (2.3.4). First we prove $a_{j0\cdots0}=0\ (1\leqq j\leqq 2q)$. Assume the contrary. We choose the minimal $j_0$ such that $a_{j_00\cdots0}\neq 0$. Since $\nu^*\varphi=0$, there is at least another monomial term $\gamma$ in $\varphi$ with degree $\leqq j_0m$. We choose $\gamma$ to be the monomial in $\varphi$ with minimum degree. We note that $\deg\nu^*(x_ix_j)\geqq 2m_2>2qm\geqq j_0m$ for any $i$, $j\geqq 2$. Therefore $\gamma=x_1^ix_j$ for some $i\geqq 1$, $j\geqq 2$. Since $\deg\gamma=\deg\nu^*(x_1^ix_j)=im+m_j$ and $m_j$ is not divisible by $m$, we see that there is another term $\delta=x_1^kx_l$ in $\varphi$ whose degree $km+m_l$ is equal to $im+m_j$. However this is impossible because $m_j-m_l(j\neq l)$ is not divisible by $m$. Hence $a_{j0\cdots0}=0\ (1\leqq j\leqq 2q)$. Similarly we can prove $a_{j10\cdots0}=0\ (1\leqq j\leqq q)$. q.e.d.

In view of (2.3.4), the expansion of $\varphi$ is

$$\varphi=\sum_{j\geqq 2q+1}a_jx_1^j+\sum_{i\geqq q+1}b_ix_1^ix_2+\sum_{j\geqq 2}c_jx_1x_2^j+\sum_{i\geqq 1,\,j\geqq 3}d_{ij}x_1^ix_j+\sum_{i,\,j\geqq 2}e_{ij}x_ix_j+\cdots$$

so that

$$\partial\varphi/\partial x_1=(2q+1)a_{2q+1}x_1^{2q}+(q+1)b_{q+1}x_1^qx_2+c_2x_2^2+d_{13}x_3+\cdots.$$

Hence we have,

$$\deg\nu^*(\partial\varphi/\partial x_1)\geqq\min(2qm,\ qm+m_2,\ 2m_2,\ m_3)=\min(2qm,\ m_3)$$

$$\deg\nu^*(\partial\varphi_i/\partial x_1)\geqq\min(2qm,\ m_3)\qquad\text{for any } i \text{ in } (2.3.3)$$

$$\deg t^{N-m+1}f_2'(t)\geqq\min(2qm,\ m_3)\qquad\text{by}\quad(2.3.3).$$

It follows from (2.3.1) and (2.3.2) that

$$N-m+1+m_2-1=N-m+m_2\geqq\min(2qm,\ m_3),$$

$$l(Q_C'',\ p)-l(Q_C',\ p)\geqq N-m+1\geqq 2qm-m_2+1\geqq(q-1)m+2\geqq 2$$

or

$$l(Q_C'',\ p)-l(Q_C',\ p)\geqq N-m+1\geqq m_3-m_2+1\geqq 2.$$

In either case $l(Q_C'',\ p)\geqq l(Q_C',\ p)+2$ as desired, which completes a proof of (2.3). q.e.d.

(2.4) LEMMA. *Let $(C_\lambda,\ p)$ be a germ of a locally irreducible component of $C$ $(\lambda\in\Lambda)$, $C=\cup_{\lambda\in\Lambda}C_\lambda$. Let $\Lambda_{\mathrm{ns}}$ (resp. $\Lambda_{\mathrm{s}}$) be the subset of $\Lambda$ consisting of all $\lambda\in\Lambda$ with $(C_\lambda,\ p)$ nonsingular (resp. singular). Assume $\#(\Lambda)\geqq 2$. Then*

$$l(Q_C'',\ p)\geqq\sum_{\lambda\in\Lambda}l(Q_\lambda'')+\#(\Lambda_{\mathrm{ns}})$$

$$l(Q_C'',\ p)\geqq l(Q_C',\ p)+2\#(\Lambda_{\mathrm{s}})+\#(\Lambda_{\mathrm{ns}})$$

PROOF. By (2.1.1) and (2.1.2), we see

$$l(Q''_C, p) = \sum_\lambda l(Q''_\lambda) + \sum_\lambda l(\mathrm{Ker}(\Omega^1_C \otimes O_{\tilde{C}_\lambda} \longrightarrow \Omega^1_{C_\lambda} \otimes O_{\tilde{C}_\lambda})),$$

where $\Omega^1_C \otimes O_{\tilde{C}_\lambda} \cong \Omega^1_C \otimes O_{C_\lambda}$, $\Omega^1_{C_\lambda} \otimes O_{\tilde{C}_\lambda} \cong \Omega^1_{C_\lambda}$ for $(C_\lambda, p)$ nonsingular. Hence it suffices to prove $l(\mathrm{Ker}(\Omega^1_C \otimes O_{C_\lambda} \to \Omega^1_{C_\lambda})) \geqq 1$ for $\lambda \in \Lambda_{\mathrm{ns}}$. Let $I_C$ (resp. $I_{C_\lambda}$) be the defining ideal of $C$ (resp. $C_\lambda$) in $O_X$. Then by definition,

(2.4.1) $$\Omega^1_C \otimes O_{C_\lambda} \cong \Omega^1_X / I_{C_\lambda} \Omega^1_X + O_X\{d\phi;\ \phi \in I_C\}$$

(2.4.2) $$\Omega^1_{C_\lambda} \cong \Omega^1_X / I_{C_\lambda} \Omega^1_X + O_X\{d\varphi;\ \varphi \in I_{C_\lambda}\}.$$

We assume $l(\mathrm{Ker}(\Omega^1_C \otimes O_{C_\lambda} \to \Omega^1_{C_\lambda}))=0$ for some $\lambda \in \Lambda_{\mathrm{ns}}$ to derive a contradiction. By (2.4.1) and (2.4.2) we assume that

(2.4.3) $$\{d\varphi;\ \varphi \in I_{C_\lambda}\} \subset I_{C_\lambda} \Omega^1_X + O_X\{d\phi;\ \phi \in I_C\}.$$

Let $x_1, \cdots, x_n$ be a system of local coordinates of $X$ at $p$ such that $I_{C_\lambda} = (x_1, \cdots, x_{n-1})$. Since $I_C \subset I_{C_\lambda}$ and $I_C \neq I_{C_\lambda}$, we have

$$I_C = (x_1, \cdots, x_m, \phi_1, \cdots, \phi_l)$$

for some $\phi_i \in I_{C_\lambda} \cap m_p^2 = I_{C_\lambda} m_p$, and $m < n-1$. Since $\Omega^1_X$ is freely generated by $dx_i$ $(1 \leqq i \leqq n)$, we have by (2.4.3)

$$dx_j \in I_{C_\lambda} dx_j + m_p dx_j \ (m+1 \leqq j \leqq n-1),$$

which is a contradiction. Hence $l(\mathrm{Ker}(\Omega^1_C \otimes O_{C_\lambda} \to \Omega^1_{C_\lambda})) \geqq 1$ for $\lambda \in \Lambda_{\mathrm{ns}}$. This proves the first inequality of (2.4). The second inequality follows readily from the first inequality and (2.3). q.e.d.

The following theorem and corollary are clear from (2.2)-(2.4).

(2.5) THEOREM. *$l(Q''_C) \geqq l(Q'_C)$ for any irreducible curve $C$. Equality holds if and only if $C$ is nonsingular. If $C$ is singular, then $l(Q''_C) \geqq l(Q'_C)+2$.*

(2.6) COROLLARY. (2.6.1) *If $(C, p)$ is irreducible, then $e(Q''_C, q) \geqq e(Q'_C, q)$ for the unique point $q$ above $p$. Equality holds if and only if $(C, p)$ is nonsingular. If $(C, p)$ is singular, then $e(Q''_C, q) \geqq e(Q'_C, q)+2$.*

(2.6.2) *Under the same notation and assumption in (2.4), let $q$ be a point of the normalization $\tilde{C}_\lambda$ of $C_\lambda$ above $p$. Then*

$$e(Q''_C, q) \geqq e(Q'_C, q)+1,\ e(Q'_C, q)=0 \qquad \textit{for}\ \ \lambda \in \Lambda_{\mathrm{ns}},$$

$$e(Q''_C, q) \geqq l(Q''_\lambda) \geqq e(Q'_C, q)+2 \qquad \textit{for}\ \ \lambda \in \Lambda_{\mathrm{s}}.$$

## Appendix. Subadjunction formula.

(2.A) THEOREM (SUBADJUNCTION FORMULA). *Let $X$ be a smooth algebraic variety of dimension $n$, $D_i$ a reduced irreducible divisor of $X$ $(1 \leqq i \leqq m)$. Assume* *that the scheme-theoretic complete intersection $\tau=D_1\cap\cdots\cap D_m$ has an irreducible component $Z=Z_{\mathrm{red}}$ of dimension $n-m$ along which $\tau$ is reduced generically. Let $\nu\colon Y\to Z$ be the normalization of $Z$. Then there exists an effective Weil divisor $\Delta$ of $Y$ such that*

(2.A.1) $$K_Y=\nu^*(K_X+D_1+\cdots+D_m)-\Delta$$

(2.A.2) $\mathrm{supp}(\nu_*\Delta)$ *is the union of all the Weil divisors of $Z$ whose supports are contained in either* $\mathrm{Sing}\,Z$ *or one of the irreducible components of $\tau$ other than $Z$.*

We note that the canonical sheaf $K_Y$ is the unique torsion free sheaf on the normal variety $Y$ given by $K_Y=i_*(\Omega^n_{Y\setminus\mathrm{Sing}\,Y})$, where $i\colon Y\setminus\mathrm{Sing}(Y)\to Y$ is the inclusion.

The condition (2.A.2) implies that $\mathrm{supp}\,\Delta=\phi$ if and only if $Z$ is smooth in codimension one and moreover $Z$ intersect the irreducible components of $\tau$ other than $Z$ along some subvarieties of at most $(n-m-2)$ dimension.

PROOF OF (2.A). The proof is almost the same as those of (1.8) and (1.9). Let $U=Y\setminus\mathrm{Sing}\,Y$, $V=\nu(U)$ and $V'=V\setminus\mathrm{Sing}\,V$, $U'=\nu^{-1}(V')$. Let $I_{D_i}$ (resp. $I$) be the ideal sheaf of $O_X$ defining $D_i$ (resp. $Z$) and let $I_\tau=I_{D_1}+\cdots+I_{D_m}$. So we note $\sqrt{I_{D_i}}=I_{D_i}$ and $\sqrt{I}=I$. Now we consider the exact sequences

(2.A.3) $$I/I^2\longrightarrow\Omega^1_X\otimes O_Z\longrightarrow\Omega^1_Z\longrightarrow 0$$

(2.A.4) $$\nu^*(I/I^2)\otimes O_U\longrightarrow\nu^*(\Omega^1_X)\otimes O_U\longrightarrow\nu^*(\Omega^1_Z)\otimes O_U\longrightarrow 0.$$

Since $U'\cong V'$ and $V'$ is nonsingular, the first homomorphism in (2.A.4) is injective over $U'$. Hence denoting by $[F]$ the quotient of $F$ by $O_U$-torsions in $F$, we infer an exact sequence,

(2.A.5) $$0\longrightarrow[\nu^*(I/I^2)\otimes O_U]\longrightarrow\nu^*(\Omega^1_X)\otimes O_U\longrightarrow\nu^*(\Omega^1_Z)\otimes O_U\longrightarrow 0.$$

Since $\tau$ is reduced generically along $Z$, we have a natural injective homomorphism $\eta$

$$\nu^*(I_\tau/I_\tau^2)\otimes O_U\overset{\rho}{\cong}\bigoplus_{i=1}^{m}O_U(-\nu^*D_i)\xrightarrow{\eta}[\nu^*(I/I^2)\otimes O_U]$$

where we can prove that $\rho$ is an isomorphism in the same manner as in (1.4). Let $Q_U$ be the cokernel of $\eta$. Then we have an exact sequence

(2.A.6) $$0\longrightarrow\bigoplus_{i=1}^{m}O_U(-\nu^*D_i)\xrightarrow{\eta}[\nu^*(I/I^2)\otimes O_U]\longrightarrow Q_U\longrightarrow 0.$$

On the other hand we have an exact sequence

(2.A.7) $$0\longrightarrow Q''_U\longrightarrow\nu^*\Omega^1_Z\otimes O_U\xrightarrow{\lambda}\Omega^1_U\longrightarrow Q'_U\longrightarrow 0$$

where $Q''_U$ (resp. $Q'_U$) is $\operatorname{Ker}\lambda$ (resp. $\operatorname{Coker}\lambda$). Now take an arbitrary prime Weil divisor $B$ of $Y$ contained in one of the supports of $Q_U$, $Q'_U$ and $Q''_U$. We define $e(F, B)$ to be the length of a torsion sheaf $F$ at a generic point of $B$ as a $k(B)$-module. Then $e(Q_U, B)$, $e(Q'_U, B)$ and $e(Q''_U, B)$ are essentially the same as the invariants $e(Q_C, q)$, $e(Q'_C, q)$ and $e(Q''_C, q)$ defined in (1.8) and (2.1). By (2.6) we have

$$e(Q''_U, B) \geqq e(Q'_U, B). \tag{2.A.8}$$

Moreover by (2.A.7), (2.A.5) and (2.A.6), we have

$$\begin{aligned} K_U = \det \Omega^1_U &\cong \det(\nu^*\Omega^1_Z \otimes O_U) - \sum_B (e(Q''_U, B) - e(Q'_U, B))B \\ &\cong \det(\nu^*\Omega^1_X \otimes O_U) - \det[\nu^*(I/I^2) \otimes O_U] - \sum_B (e(Q''_U, B) - e(Q'_U, B))B \\ &\cong \nu^*K_X + \sum_{i=1}^m \nu^*D_i - \sum_B e(Q_U, B)B - \sum_B (e(Q''_U, B) - e(Q'_U, B))B. \end{aligned}$$

Let $\Delta := \sum_B (e(Q_U, B) + e(Q''_U, B) - e(Q'_U, B))B$. Then we have (2.A.1). Moreover if $Z$ is singular along a prime Weil divisor $C$, then in view of (2.6) $e(Q''_U, B) \geqq e(Q'_U, B)+1$ for any prime Weil divisor $B$ of $Y$ lying over $C$. (Note that $B$ may not be birational to $C$.) If $Z$ intersects one of the irreducible components of $\tau$ other than $Z$ along a prime Weil divisor $C$, then by the definition $e(Q_U, B) \geqq 1$ for any prime Weil divisor $B$ lying over $C$. Thus we have (2.A.2).

q.e.d.

It is easy to see that (2.A) has a counterpart in the complex analytic category.

## § 3. Proofs of (0.2) and (0.3).

(3.1) THEOREM. *Let $X$ be a complete nonsingular algebraic variety (or a compact complex manifold) of dimension $n$. Assume that $c_1(X) = dc_1(L)$ $(d \geqq n+1)$ and $h^0(X, L) \geqq n$. If general $(n-1)$-members of $|L|$ intersect rationally outside* $\mathrm{Bs}|L|$, *then $X \cong \boldsymbol{P}^n$.*

PROOF. Our proof of (3.1) consists of two steps. First we prove (3.1) in (3.1.1)-(3.1.7) under the assumption $h^0(X, L) \geqq n+1$. Next we disprove the possibility of $h^0(X, L) = n$ in (3.1.8)-(3.1.10).

First we prove

(3.1.1) CLAIM. *Let $N = h^0(X, L) - 1 \geqq n$ and $f: X \to \boldsymbol{P}^N$ be the rational map associated with $|L|$. Let $\bar{X} := \overline{f(X \setminus B)}$. Then $d = n+1$, $N = n$ and $\bar{X} \cong \boldsymbol{P}^n$.*

PROOF. We use the same notation $l_V = C_V + B$ as in (1.1). Let $\mathcal{H} = H^0(X, L)$, $V$ a general $(n-1)$-dimensional subspace of $\mathcal{H}$.

First we prove $\dim \bar{X}=n$. By the assumption, $\dim \bar{X} \geqq n-1$. Assume $\dim \bar{X}=n-1$. By (1.10) and (1.11), $d=n+1$ and if $V$ is general enough, $C_V$ is a disjoint union of nonsingular rational curves $C_i$ ($1 \leqq i \leqq r \deg \bar{X}$) with $LC_i=1$ and $f(C_i \backslash B)$ a point, where $r$ is the number of irreducible components of a general fiber of $f$. Let $C=C_1$. If $\mathrm{Bs}|L|_C$ is empty, then $LC=1$ implies $\dim \bar{X}=n$, a contradiction. Hence by (1.10), $\mathrm{Bs}|L|_C=\{p\}$ for some point $p$ of $C$. Since $p$ is isolated in $B$ by (1.10), $p$ is contained in any $C_i$. However $C$ is a connected component of $l_V$ by (1.10), whence $r=\deg \bar{X}=1$. Therefore $N=n-1$ and $\bar{X} \cong \boldsymbol{P}^{n-1}$, which contradicts $N \geqq n$. It follows that $\dim \bar{X}=n$. Therefore for $V$ general enough, $C_V$ is a disjoint union of smooth rational curves $C_i$ with $LC_i=1$. Since $LC_i=\deg(f_{|C_i}) \deg \bar{X}+\deg \mathrm{Bs}|L|_{C_i}$, we have $\deg(f_{|C_i})=1$, $\deg \bar{X}=1$ and $\mathrm{Bs}|L|_{C_i}=\emptyset$. Therefore we have $N=n$ and $\bar{X} \cong \boldsymbol{P}^n$. q.e.d.

(3.1.2) Let $\mathcal{H}:=H^0(X, L)$ and $G=\mathrm{Grass}(n-1, \mathcal{H})$. Then we define

$$P=\{([V], x) \in G \times X;\ s(x)=0 \quad \text{for any} \quad s \in V\}.$$

Then by the assumption there exists an irreducible component $P_0$ of $P$ such that $pr_G(P_0)=G$, $pr_X(P_0)$ is not contained in $B$. Let $\pi_0$ (resp. $\rho_0$) be the natural projection from $P_0$ onto $G$ (resp. into $X$). For general $W \in G$, $C_W$ has an irreducible component $C$ ($\cong \boldsymbol{P}^1$). We may assume by (1.10) that $\rho_0(\pi_0^{-1}[W])$ contains $C$ as a connected component.

Let $C'$ be an irreducible component of $\pi_0^{-1}([W])$ mapped onto $C$, $z$ a general point of $C'$, $x=\rho_0(z)$. Since $C'$ is smooth at $z$, so is $P_0$ at $z$. Now we recall canonical isomorphisms;

$$T_z(P_0) \cong T_{[W]}G \oplus T_x(C) \cong (\mathcal{H}/W)^{\oplus(n-1)} \oplus T_x(C),$$

$$T_x(X) \cong (N_{C/X})_x \oplus T_x(C) \cong (L_C)_x^{\oplus(n-1)} \oplus T_x(C).$$

Let $p$ be a point of $C$, $\mathcal{H}(-p):=\{s \in \mathcal{H};\ s(p)=0\}$, $G(-p):=\mathrm{Grass}(n-1, \mathcal{H}(-p))$. Since $\mathrm{Bs}|L|_C=\emptyset$ by (1.10), $G(-p)$ is a smooth proper subvariety of $G$ by the natural morphism induced from the inclusion $\mathcal{H}(-p) \subset \mathcal{H}$. We also see,

$$T_z(G(-p) \times X) \cong T_{[W]}G(-p) \oplus T_x(X) \cong (\mathcal{H}(-p)/W)^{\oplus(n-1)} \oplus T_x(X).$$

It follows that $G(-p) \times X$ and $P_0$ intersect transversally at $z$. Therefore the intersection $P_0 \cap (G(-p) \times X)$ is smooth at $z$. Let $S_0$ be the unique irreducible component of $P_0 \cap (G(-p) \times X)$ passing through $z$. Then we see

$$T_z(S_0) \cong T_{[W]}G(-p) \oplus T_x(C) \cong (\mathcal{H}(-p)/W)^{\oplus(n-1)} \oplus T_x(C).$$

Since $\mathcal{H}(-p)/W$ is mapped onto $L_x$ for $p \in C$ general, $T_z(S_0)$ is mapped onto $T_x(X)$ in the natural manner. Hence $\rho_0(S_0)=X$.

(3.1.3) We choose a general $W_0 \in G$ and take an irreducible component

$C_0\,(\cong \boldsymbol{P}^1)$ of $C_{W_0}$ which is a connected component of $\rho_0(\pi_0^{-1}[W_0])$ as in (3.1.2). We choose and fix a general point $p$ of $C_0$ and we define

$$Y = \{([V],\ x) \in G(-p)\times X;\ s(x) = 0 \quad \text{for any} \quad s \in V\}.$$

Let $Y=\cup_{i=0}^{b} Y_i$ be the decomposition of $Y$ into irreducible components, $Y_i\,(0\leqq i\leqq e)$ all the components such that $pr_{G(-p)}(Y_i)=G(-p)$, $pr_X(Y_i)$ is not contained in $B$. By (3.1.2), we have $e\geqq 0$. Let $p_i$ (resp. $q_i$) be the natural projection from $Y_i$ onto $G(-p)$ (resp. into $X$). We may assume $S_0\subset Y_0$ under the notation of (3.1.2). For general $W\in G(-p)$, let $C_W=\sum_{i=0}^{a} C_W^i$ be the decomposition of $C_W$ into irreducible components where $C_W^i$ is a rational curve $(0\leqq i\leqq a)$ and $C_W^0$ is by (1.10) the unique component containing the point $p$. We may assume that $q_0(p_0^{-1}[W])$ contains $C_W^0$ as a connected component.

(3.1.4) CLAIM. *Any general fibre* $p_0^{-1}([V])$ *is irreducible.*

PROOF. Consider the Stein factorization of $p_0$

$$\begin{tikzcd} Y_0 \arrow[rr, "p_0"] \arrow[dr, "\xi"'] & & G(-p) \\ & \tilde{G}(-p) \arrow[ur, "\eta"'] & \end{tikzcd}$$

We note that $p_0$: $Y_0\to G(-p)$ has a section $\sigma_0$ defined by $\sigma_0([V])=([V],\ p)$. Hence we have a morphism $\xi\cdot\sigma_0$: $G(-p)\to\tilde{G}(-p)$ such that $\eta\cdot\xi\cdot\sigma_0=\mathrm{id}_{G(-p)}$. As $\eta$ is finite, we have $\dim\tilde{G}(-p)=\dim G(-p)$. Since $G(-p)$ is complete, we have $\tilde{G}(-p)=\xi\cdot\sigma_0(G(-p))$, and $\eta$ is an isomorphism. Therefore any general fibre of $p_0$ is irreducible. q.e.d.

Next we prove

(3.1.5) CLAIM. $q_i(Y_i)=X$ *for* $0\leqq i\leqq e$.

PROOF. Let $C'$ be an irreducible component of $p_i^{-1}([W])$, $C''=q_i(C')$. Since $pr_X(Y_i)$ is not contained in $B$ by assumption, $C''$ is an irreducible component of $C_W$ for $W$ general so that $C''$ is $\boldsymbol{P}^1$ by (1.10) and $\mathrm{Bs}|L|_{C''}=\emptyset$ by the proof of (3.1.1). Hence by (3.1.1) the natural homomorphism of $\mathcal{H}$ into $H^0(C'',\ L_{C''})$ induces an isomorphism $\mathcal{H}/W\cong H^0(C'',\ L_{C''})$. Any point $q\in C''$ determines a unique $n$-dimensional subspace $\mathcal{H}(-q)$ of $\mathcal{H}$ containing $W$. Conversely any $n$-dimensional linear subspace $V$ of $\mathcal{H}$ containing $W$ determines a unique point $q'$ of $C''$ with $\mathcal{H}(-q')=V$. This correspondence is bijective.

The curve $C'$ is mapped isomorphically onto $C''$ by $q_i$ because $W$ is general. Let $z$ be a general point of $C'$, $x=q_i(z)$. Now we have canonical isomorphisms;

$$T_z(Y_i) \cong T_{[W]}G(-p)\oplus T_x(C'') \cong (\mathcal{H}(-p)/W)^{\oplus(n-1)}\oplus T_x(C''),$$

$$T_x(X) \cong (N_{C''/X})_x \oplus T_x(C'') \cong (L_{C''})_x^{\oplus(n-1)} \oplus T_x(C'').$$

First we consider the case $i=0$, $C''=C_W^0$. Since $S_0 \subset Y_0$ and $\rho_0(S_0)=X$ under the notation in (3.1.2), we have $\rho_0(Y_0)=X$.

Next we consider the case $C''=C_W^i$, $i>0$. As we observed above, the natural homomorphism $\mathcal{H}(-p) \to H^0(C'', L_{C''})$ has a one-dimensional image. Hence $\mathcal{H}(-p)$ has a unique base point $p'$ on $C''$, so that the image of $\mathcal{H}(-p)/W$ generates the line bundle $L_{C''}$ everywhere except at $p'$. So by choosing $z \in C'$ with $x=q_i(z) \neq p'$, we see that

$$(dq_i)_*\colon T_z(Y_i) \longrightarrow T_x(X)$$

is surjective. This shows that $q_i(Y_i)=X$. q.e.d.

(3.1.6) CLAIM.

(3.1.6.1) *$f$ is birational.*

(3.1.6.2) *$C_V$ is irreducible for general $V \in G(-p)$.*

PROOF. (3.1.6.1) follows from (3.1.1), (3.1.6.2) and (1.10) easily. So we prove (3.1.6.2). By (3.1.4) it suffices to prove $e=0$ under the notation in (3.1.3). Let $C_V=\sum_{i=0}^{a} C_V^i$ be the decomposition of $C_V$ into irreducibe components for $V \in G(-p)$ general, where $C_V^0$ is the unique irreducible component of $C_V$ passing through $p$. Assume $e>0$. Then $a>0$. Take and fix $j$ $(1 \leqq j \leqq e)$. By (3.1.5) $q_j(Y_j)=X$. This implies that for any general $V \in G(-p)$, there exists $V' \in G(-p)$ such that $C_V^0 \cap C_{V'}^j \neq \emptyset$. Let $C'=C_V^0$, $C''=C_{V'}^j$. We may assume that $C' \cap C''=\{p', \cdots\}$, $p' \neq p$ for a sufficiently general $V'$ with $C_V^0 \cap C_{V'}^j \neq \emptyset$. Let $|m_p L|$ be the linear subsystem of $|L|$ consisting of members of $|L|$ passing through the point $p$. If $D \in |m_p L|$ contains $l_{V'}$, then it contains $p$ and $p'$, whence $C' \subset D$ because $LC'=1$. This shows that $C_{V'}$ contains $C'=C_V^0$. Since $C_{V'}^0$ is the unique irreducible component of $C_{V'}$ containing $p$, we have $C'=C_V^0=C_{V'}^0$. But $C'$ intersects $C''=C_{V'}^j$, which contradicts (1.10). Hence $e=0$ and $C_V$ is irreducible for general $V \in G(-p)$ by (3.1.4). q.e.d.

By (3.1.6) we have a birational morphism $f\colon X \backslash B \to \boldsymbol{P}^n$. Let $\hat{X}$ be the normalization of the closure in $X \times \boldsymbol{P}^n$ of the graph of $f$, $\hat{f}\colon \hat{X} \to \boldsymbol{P}^n$ and $h\colon \hat{X} \to X$ the natural morphisms. Let $\hat{B}=h^{-1}(B)$ and $\hat{B}^*$ be the minimal subvariety of $\hat{X}$ containing $\hat{B}$ such that $\hat{f}$ is unramified on $\hat{X} \backslash \hat{B}^*$. Let $B^*=h(\hat{B}^*)$, $R=\hat{f}(\hat{B})$, and $R^*=\hat{f}(\hat{B}^*)$. We note that $\hat{B}^*=h^{-1}(B^*)=\hat{f}^{-1}(R^*)$, $\hat{X} \backslash \hat{B} \cong X \backslash B$, $X \backslash B^* \cong \hat{X} \backslash \hat{B}^* \cong \boldsymbol{P}^n \backslash R^*$.

(3.1.7) CLAIM. *$B^*=B=\emptyset$ and $X \cong \boldsymbol{P}^n$.*

PROOF. Assume the contrary. Hence $R^* \neq \emptyset$. Then we can choose a line $l$ which is not contained in $R^*$ and meets $R^*$. Hence we can choose (not necessarily general) $W\in\text{Grass}(n-1, \mathcal{H})$ such that $l_W$ is pure one dimensional and irreducible nonsingular outside $B^*$ and the closure of $f(l_W\backslash B^*)$ is $l$. Let $q$ be a point of $l\cap R^*$, $C$ the unique irreducible component of $l_W$ with $\overline{f(C\backslash B^*)}=l$. Let $\hat{C}$ be the proper transform of $C$ by $h^{-1}$. Then $\hat{C}\cup\hat{f}^{-1}(q)$ is a connected subset of $\hat{X}$ intersecting $\hat{B}^*$, whence $C\cup h(\hat{f}^{-1}(q))$ is a connected subset of $l_W$ intersecting $B^*$. By (1.10) $C\cong \boldsymbol{P}^1$ and it is a connected component of $l_W$. Hence $h(\hat{f}^{-1}(q))\subset C$. Since $\hat{f}^{-1}(q)$ is connected, this implies that $h(\hat{f}^{-1}(q))$ is a unique point of $C\cap B^*$. Let $p:=h(\hat{f}^{-1}(q))$. If $p\in B^*\backslash B$, then $q=f(p)$ and $\hat{f}^{-1}(q)$ is a single point because $\hat{X}\backslash\hat{B}\cong X\backslash B$. However by the definition of $\hat{B}^*$, $\dim\hat{f}^{-1}(q)>0$, a contradiction. Therefore $p\in B$. Then $p=h(\hat{f}^{-1}(q))=C\cap B$ by (1.10).

Since $LC=1$, this implies that $f(C\backslash B)$ is a point, which contradicts $\overline{f(C\backslash B^*)}=l$. Therefore $R^*=\emptyset$. Hence $B=\hat{B}=\emptyset$, $B^*=\hat{B}^*=\emptyset$. It follows that $f$ is defined and unramified everywhere on $X$. Consequently the birational morphism $f$ is an isomorphism. This completes the proof of (3.1) under the assumption $h^0(X, L)\geqq n+1$. q.e.d.

In what follows, we assume that $h^0(X, L)=n$. We derive a contradiction in (3.1.10). Let $f\colon X\to\boldsymbol{P}^{n-1}$ be the rational map associated with $|L|$, $Y$ the closure of $f(X\backslash B)$. By the assumption $\dim Y\geqq n-1$, whence $Y\cong\boldsymbol{P}^{n-1}$. Let $\hat{X}$ be the normalization of the closure in $X\times Y$ of the graph of $f$, $\hat{f}\colon\hat{X}\to Y$ and $h\colon\hat{X}\to X$ the natural morphisms. Let $\hat{B}=h^{-1}(B)$.

(3.1.8) CLAIM. *$d=n+1$ and $\hat{f}^{-1}(y)\cong\boldsymbol{P}^1$ for any general $y\in Y$.*

PROOF. Let $V\in\text{Grass}(n-1, \mathcal{H})$ be general. Then by (1.10) and (1.11), $d=n+1$ and $C_V$ is a disjoint union of smooth rational curves $C_i$ $(0\leqq i\leqq r)$ with $LC_i=1$. Since $f(C_i\backslash B)$ is a point $y\in Y$, we have $\deg \text{Bs}|L|_{C_i}=1$, whence there is a point $p_i\in C_i$ such that $\text{Bs}|L|_{C_i}=\{p_i\}$. By (1.10), $p_i$ is an isolated point of $B$. Therefore $p_0\in C_i$ for any $i$ if $V$ is general. Since $C_i$ is a connected component of $l_V$, this implies that $C_V$ is irreducible.

Let $y\in Y$ be general. Then $V_y\in\text{Grass}(n-1, \mathcal{H})$ is uniquely determined by the condition that $f(l_{V_y}\backslash B)=y$. Therefore $C_{V_y}$ is irreducible for $y$ general. Since $\hat{X}\backslash\hat{B}\cong X\backslash B$, $\hat{f}^{-1}(y)$ is irreducible outside $\hat{B}$. Since $\dim\hat{B}\leqq\dim Y=n-1$, no irreducible components of $\hat{f}^{-1}(y)$ are contained in $\hat{B}$ for $y$ general. Hence $\hat{f}^{-1}(y)$ is irreducible for $y$ general. This proves (3.1.8). q.e.d.

(3.1.9) CLAIM. *Let $R:=\{y\in Y;\ \hat{f}^{-1}(y)$ is not smooth$\}$. Let $l^*$ be a general line of $Y$ not contained in $R$. Then $\hat{f}^{-1}(l^*)\cong\boldsymbol{F}_1$ and $h(\hat{f}^{-1}(l^*))\cong\boldsymbol{P}^2$.*

PROOF. Let $\hat{Z}$ be a unique irreducible component of $\hat{f}^{-1}(l^*)$ with $\hat{Z}_y:=\hat{Z}\cap\hat{f}^{-1}(y)\cong\boldsymbol{P}^1$ for general $y\in l^*$. Let $Z=h(\hat{Z})_{\text{red}}$. The line $l^*$ corresponds to an $(n-2)$-dimensional subspace $U$ of $\mathcal{H}$ with $f(l_U\backslash B)\subset l^*$, where $l_U=\cap_{s\in U}D_s$. See §1. The surface $Z$ is an irreducible component of $l_{U,\text{red}}$.

Let $\nu$: $T \to Z$ be the normalization, $\sigma$: $S \to T$ the minimal resolution of $T$. Let $g=\nu \cdot \sigma$. Then there exist by (2.A) or [**5**, Corollary (18)] an effective Weil divisor $\Delta$ on $T$, effective Cartier divisors $E$ and $G$ on $S$ with no common components such that the canonical sheaves $K_T$ and $K_S$ are given by

$$K_T = \nu^*(K_X+(n-2)L)-\Delta, \qquad K_S = g^*(K_X+(n-2)L)-E-G$$

with $\sigma_*(E)=\Delta$, $\sigma_*(G)=0$. Moreover by (2.A) there exists a finite subset $\Sigma_0$ of $S$ such that $g$ is an isomorphism over $S \backslash \Sigma$ where $\Sigma:=\sigma^{-1}(\Delta) \cup \sigma^{-1}(\mathrm{Sing}\, T) \cup \Sigma_0$. Clearly $\Sigma$ contains $\mathrm{supp}(E+G)$. Note that if $E=0$, then $Z$ has no singularities along curves and no curve intersection with the irreducible components of $l_U$ other than $Z$. This follows from (2.A) and (2.6).

Since $Z \not\subset \mathrm{Bs}|L|$, $g^*L$ is effective. Since $S$ is projective, we have $P_m(S)=0$, whence $S \cong \boldsymbol{P}^2$ or $S$ has a pencil of rational curves $F \cong \boldsymbol{P}^1$ with $(F^2)_S=0$. (Note that if $X$ is non-Kählerian, then $S$ can be in class VII. See (3.4) below.) Let $H=g^*D \in g^*|L|$ for a general member $D \in |L|$. By Bertini's theorem, $\mathrm{Sing}\, Z$ is contained in $\mathrm{Bs}|L|$, whence $g(\mathrm{supp}(E+G)) \subset \mathrm{Bs}|L|$. This implies that $E_{\mathrm{red}}+G_{\mathrm{red}} \subset H_{\mathrm{red}}$. Assume that $S$ has a pencil of rational curves $F \cong \boldsymbol{P}^1$ with $(F^2)_S=0$. Then we have,

$$-2=K_SF+F^2=K_SF=-(3H+E+G)F$$

because $d=n+1$. It follows that $HF=0$, $(E+G)F=2$. However this contradicts $E_{\mathrm{red}}+G_{\mathrm{red}} \subset H_{\mathrm{red}}$. Therefore $S \cong T \cong \boldsymbol{P}^2$ and $G=0$. Since $E_{\mathrm{red}} \subset H_{\mathrm{red}}$ and $K_S=-3H-E$, we see that $O_S(H) \cong O_{\boldsymbol{P}^2}(1)$, $E=0$ and that $\Sigma$ is finite. Since $E=0$, $Z$ has by (2.A) at worst isolated singularities.

Next we prove that $Z$ is a connected component of $l_U$. Let $H:=g^*(D) \in g^*|L|$ and let $V \in \mathrm{Grass}(n-1, \mathscr{H})$ be a subspace of $\mathscr{H}$ corresponding to $D \cap l_U$. Then since $S \backslash \Sigma \cong Z \backslash g(\Sigma)$, $C:=g(H)=D \cap Z$ is a reduced curve component of $l_V$. We have

$$1=(H^2)_S=(g^*(L)H)_S=(Lg_*(H))_X=(LC)_X\,.$$

It follows from (1.10) that $C \cong \boldsymbol{P}^1$ and $C \cap B=\{p_0\}$ and that $C$ is a connected component of $l_V$. Hence $Z \cap B=Z \cap D \cap B=C \cap B=\{p_0\}$. Since $g(\Sigma) \subset B$, we see $g(\Sigma)=\{p_0\}$. Assume that $Z$ intersects another irreducible component $Z'$ of $l_U$. Then $\dim Z' \geqq 2$, $\dim l_V \cap Z' \geqq 1$ and $Z \cap Z' \subset g(\Sigma)=\{p_0\}$. Therefore $p_0 \in l_V \cap Z' \subset l_V$. This contradicts that $C$ is a connected component of $l_V$. Thus $Z$ is a connected component of $l_U$.

Therefore $l_U$ is a proper complete intersection along $Z$ such that $(l_U)_{\mathrm{red}} \cong Z$ along $Z$. Hence $l_U$ is Gorenstein and reduced generically along $Z$ so that it is reduced along $Z$. Hence $l_U \cong Z$ along $Z$. Since the Gorenstein surface $Z$ has at worst isolated singularities, it is normal, whence $S \cong Z$. In particular, $Z$ is smooth everywhere.

Meanwhile since $p_0$ is isolated in $B$, there exists a closed subset $A$ of $B$ such that $D_1\cap\cdots\cap D_n=p_0+A$, and $p_0\notin A$, where $D_i\in|L|$ is chosen general. In fact, this is true scheme-theoretically at $p_0$ by (1.10). This implies that $n$ equations defining $D_i$ form a local coordinate system at $p_0$. Let $Q_{p_0}(X)$ be the blowing-up of $X$ with $p_0$ center, $\mathcal{E}:=Q_{p_0}(p_0)$ the exceptional divisor. Then we have a rational map $\hat{h}$ from $Q_{p_0}(X)$ to $Y$ induced from $f$, which is a morphism near $\mathcal{E}$. It follows that $\hat{X}\cong Q_{p_0}(X)$ near $\mathcal{E}$. Therefore $\hat{Z}$ is smooth everywhere. In what follows we view $\mathcal{E}$ as a divisor of $\hat{X}$ by the above isomorphism. Then $\mathcal{E}=h^{-1}(p_0)$. Clearly $\hat{f}_{|\mathcal{E}}=\hat{h}_{|\mathcal{E}}$: $\mathcal{E}\to Y$ is an isomorphism. Since $p_0$ is isolated in $B$, $\mathcal{E}$ is disjoint from the irreducible components of $\hat{B}$ other than $\mathcal{E}$.

Next we prove that $\hat{Z}\cong\boldsymbol{F}_1$. We note $Z\backslash\{p_0\}\cong\hat{Z}\backslash\hat{Z}\cap\mathcal{E}$ and $\hat{f}(\hat{Z})=l^*$. Since $\mathcal{E}\cong Y$, we have $\hat{Z}\cap\mathcal{E}\cong\hat{f}(\hat{Z}\cap\mathcal{E})\cong l^*\cong\boldsymbol{P}^1$. Hence $\hat{Z}\cong\boldsymbol{F}_1$.

Finally we prove $\hat{Z}=\hat{f}^{-1}(l^*)$. In view of (3.1.8), $f^{-1}(l^*)$ is connected. Hence it suffices to prove that $\hat{Z}$ is a connected component of $\hat{f}^{-1}(l^*)$. Assume the contrary. Note that $\hat{Z}$ is a unique irreducible component of $\hat{f}^{-1}(l^*)$ outside $\hat{B}$. Let $\hat{B}'$ be an irreducible component of $\hat{B}$ other than $\mathcal{E}$ such that $\hat{Z}\cap\hat{B}'\neq\emptyset$. Then $h(\hat{Z}\cap\hat{B}')\subset Z\cap B=\{p_0\}$, whence $\hat{Z}\cap\hat{B}'(\neq\emptyset)\subset\mathcal{E}$. It follows that $\hat{B}'\cap\mathcal{E}\neq\emptyset$. However $\mathcal{E}$ is disjoint from $\hat{B}'$, a contradiction. q.e.d.

(3.1.10) CLAIM. *$X\cong\boldsymbol{P}^n$ and $\hat{X}\cong\boldsymbol{P}(O_Y(1)\oplus O_Y)$.*

PROOF. First we prove $R=\emptyset$. Assume the contrary. Then we can choose a line $l^*$ of $Y$ not contained in $R$ but intersecting $R$. We can apply the same argument as in (3.1.9) to a general line $l^*$ with $l^*\cap R\neq\emptyset$. Hence $\hat{f}^{-1}(l^*)\cong\boldsymbol{F}_1$ by (3.1.9), whence $\hat{f}^{-1}(y)\cong\boldsymbol{P}^1$ for any $y\in l^*$. This contradicts $l^*\cap R\neq\emptyset$. Hence $R=\emptyset$.

Therefore $\hat{f}^{-1}(y)\cong\boldsymbol{P}^1$ for any $y\in Y$. Hence $\hat{X}\cong\boldsymbol{P}(O_Y(a)\oplus O_Y)$ for some $a\geqq0$. By (3.1.9), $\hat{X}\times_Y l^*\cong\hat{f}^{-1}(l^*)\cong\boldsymbol{F}_1$ so that $a=1$. Hence $X\cong\boldsymbol{P}^n$. q.e.d.

In (3.1.8)-(3.1.10) we assume $h^0(X, L)=n$, which contradicts (3.1.10). This completes the proof of (3.1). q.e.d.

(3.2) THEOREM. *Let $X$ be a complete nonsingular algebraic variety (or a Moishezon manifold) of dimension $n$ with $b_2=1$, and $L$ a line bundle on $X$. Assume that $c_1(X)=dc_1(L)$ $(d\geqq n+1)$ and $h^0(X, L)\geqq n$. If general $(n-1)$-members of $|L|$ intersect outside* $\mathrm{Bs}|L|$, *then $X\cong\boldsymbol{P}^n$.*

PROOF. Let $B=\mathrm{Bs}|L|$. Let $l_W=\cap_{s\in W}D_s$ for general $W\in\mathrm{Grass}(n-1, H^0(X, L))$, and $C_W=l_W-B$. See §1. Let $f$: $X\backslash B\to\boldsymbol{P}^N$ be the rational map associated with $|L|$ where $N+1=h^0(X, L)$, and $Y$ the closure of $f(X\backslash B)$. Then by the assumption, $\dim Y\geqq n-1$. Assume $\dim Y=n-1$. Then the union of

$C_W = l_W - B$ contains an open dense subset of $X$ when $[W]$ ranges over a Zariski open dense subset of Grass$(n-1, H^0(X, L))$. If $LC_W=0$, then $C_W \cap B = \emptyset$ by (1.11). Hence $mLC_W = 0$, $\mathrm{Bs}|mL| \cap C_W = \emptyset$ for any $m>0$. Consequently the rational map $f_m$ associated with $|mL|$ contracts $C_W$ to a point, and $\dim f_m(X \backslash \mathrm{Bs}|mL|) < n$. However since $b_2=1$, the Moishezon assumption on $X$ implies that $\dim f_m(X \backslash \mathrm{Bs}|mL|) = n$ for suitable $m$. This is a contradiction. Hence there is an irreducible component $C_W^i$ of $C_W$ such that $LC_W^i>0$, whence $C_W^i \cong \boldsymbol{P}^1$ by (1.10). Thus general $(n-1)$-members of $|L|$ intersect rationally. Consequently $X \cong \boldsymbol{P}^n$ by (3.1). q.e.d.

REMARK. The above proof of (3.2) shows that the assumption $b_2=1$ can be replaced by the condition $\kappa(X, L)=n$.

(3.3) THEOREM. *Let $X$ be a complete nonsingular algebraic* 3*-fold* (*or a Moishezon* 3*-fold*), *$L$ a line bundle on $X$. Assume that $c_1(X)=dc_1(L)$ $(d \geqq 4)$ and $h^0(X, L) \geqq 2$. Then $X \cong \boldsymbol{P}^3$.*

PROOF. Let $M$(resp. $F$) be a moving part (resp. a fixed part) of $|L|$. By Bertini's theorem, we choose a general member $D=Z_1+\cdots+Z_r$ of $|M|$ where $Z_i$ is reduced irreducible and smooth outside $\mathrm{Bs}|M|$. Let $Z=Z_1$ and let $\nu$: $Y \to Z$ be the normalization, $f: S \to Y$ the minimal resolution of $Y$. Let $g=\nu \cdot f$. Then there exist by (2.A) or [**5**, Corollary (18)] an effective Weil divisor $\Delta$ on $Y$, effective Cartier divisors $E$ and $G$ on $S$ with no common components such that the canonical sheaves $K_Y$ and $K_S$ are given by

$$K_Y = \nu^*(K_X+L) - \Delta, \qquad K_S = g^*(K_X+L) - E - G$$

with $f_*(E)=\Delta$, $f_*(G)=0$. By (2.A) there exists a finite subset $\Sigma_0$ of $S$ such that $g$ is an isomorphism over $S \backslash \Sigma$ where $\Sigma := f^{-1}(\Delta) \cup f^{-1}(\mathrm{Sing}\, Y) \cup \Sigma_0$. Note that $\Sigma$ contains $\mathrm{supp}(E+G)$.

Then by the same argument as in (3.1.9), we see that $d=4$, $S \cong Y \cong \boldsymbol{P}^2$, $O_S(g^*L) \cong O_{P^2}(1)$, $E=G=0$ and that $\Sigma$ is finite. Since $E=0$, $Z$ has by (2.A) at worst isolated singularities. Since $Z$ is Gorenstein, $Z$ is normal, whence $S \cong Y \cong Z \cong \boldsymbol{P}^2$. Moreover $Z$ is a connected component of $D+F$. In fact, since $\dim X=3$, $F \cap Z$ and $Z_i \cap Z$ $(i \geqq 2)$ are either a curve or empty. $E=0$ shows that $F \cap Z = Z_i \cap Z = \emptyset$ $(i \geqq 2)$. Assume $r \geqq 2$. Since $Z_i$ and $Z$ are algebraically equivalent and $H^1(Z, O_Z)=0$, we have $O_{P^2}(1) \cong O_Z(Z) \cong O_Z(Z_i) \cong O_Z$ by $Z_i \cap Z = \emptyset$, which is a contradiction. Hence $r=1$ and $D$ is irreducible.

Since $O_Z(M) \cong O_Z(Z) \cong O_{P^2}(1)$, we have $h^0(X, L)=h^0(X, M)=h^0(Z, O_Z(Z))+1 =4$ by $h^1(X, O_X)=0$. We also have $(M^3)_X=(M_Z^2)_Z=1$ and $\mathrm{Bs}|M|=\mathrm{Bs}|M|_Z= \mathrm{Bs}|O_Z(M)|=\emptyset$ so that we have a surjective birational morphism $f: X \to \boldsymbol{P}^3$. We also have $-4M-4F=K_X=f^*(K_{P^3})+\mathrm{Jac}_f=-4M+\mathrm{Jac}_f$ for the exceptional divisor $\mathrm{Jac}_f$ of $f$. It follows that $F=\mathrm{Jac}_f=0$ and $X \cong \boldsymbol{P}^3$. q.e.d.

(3.4) EXAMPLE. For any pair $(d, p)$ with $d \geqq 3$ and $p \geqq 1$, there exist infinitely many *non-Kählerian* 3-folds $X$ (Hopf 3-folds) with $c_1(X)=dc_1(L)$, $h^0(X, L)=p+1$. We define

$$X = \boldsymbol{C}^3 \backslash (0, 0, 0)/\{g^n;\ n \in \boldsymbol{Z}\}$$

where $g$ is a transformation of $\boldsymbol{C}^3$ defined by $g: (x, y, z) \rightarrow (\alpha^{dp-2}x+y^{dp-2}, \alpha y, \alpha z)$ for $\alpha \in \boldsymbol{C}^*$, $|\alpha|<1$. Let $S$ be a divisor $\{y=0\}$ of $X$. Then we see that $S$ is a primary Hopf surface with all plurigenera $P_m(S)=0$. We also see that $K_X=-dpS$, $h^0(X, pS)=p+1$.

(3.5) THEOREM. *Let $X$ be a Moishezon 4-fold, and $L$ a line bundle on $X$. Assume that* $\mathrm{Pic}\, X=\boldsymbol{Z}L$, $c_1(X)=dc_1(L)$ $(d \geqq 5)$ *and* $h^0(X, L) \geqq 4$. *Then* $X \cong \boldsymbol{P}^4$.

PROOF. Let $h: X \rightarrow \boldsymbol{P}^N$ be a rational map associated with $|L|$, and $W$ the closure of $h(X \backslash \mathrm{Bs}|L|)$, where $N=h^0(X, L)-1$. Let $e=\deg W$. Then $e \geqq N+1-\dim W$. If $\dim W=1$, then $e=1$, $N=1$ by $\mathrm{Pic}\, X=\boldsymbol{Z}L$, which contradicts $N \geqq 3$. Therefore $\dim W \geqq 2$. Hence by choosing general $D$ and $D' \in |L|$, we have *a reduced component* $Z$ of $\tau:=D \cap D'$ outside $\mathrm{Bs}|L|$. Then by the proof of (3.1.7) or (3.3), $Z \cong \boldsymbol{P}^2$, $L_Z \cong O_{\boldsymbol{P}^2}(1)$ and $Z \cap \mathrm{Bs}|L|$ is at most a line in $\boldsymbol{P}^2$.

If $Z \cap \mathrm{Bs}|L|$ is finite, then $\tau \cap D''$ has *a reduced curve-component* $Z \cap D'' \cong \boldsymbol{P}^1$ outside $\mathrm{Bs}|L|$ for $D'' \in |L|$ general. In this case, $X \cong \boldsymbol{P}^4$ by (3.2). Hence we may assume that $C:=Z \cap \mathrm{Bs}|L| \cong \boldsymbol{P}^1$. We assume $\dim W=2$. Then $e \geqq N-1 \geqq 2$. By choosing general $D$ and $D' \in |L|$, we have *er* irreducible components $Z_1, \cdots, Z_{er}$ outside $\mathrm{Bs}|L|$, where $r$ is the number of irreducible components of a general fiber $h^{-1}(w)$ $(w \in W)$. By the proof of (3.1.7) or (3.3), we see that $Z_i \cong \boldsymbol{P}^2$ and that $Z_i \cap Z_j$ is finite for $i \neq j$. (In fact, we see moreover that $Z_i$ is a connected component of $\tau:=D \cap D'$ because $\tau$ is Gorenstein.) However $Z_i$ contains $C$ for any $i$, whence $e=1$, $r=1$ and $N=2$, which contradicts $N \geqq 3$. Hence $\dim W \geqq 3$. Therefore $D \cap D' \cap D''$ has *a reduced curve component* $Z \cap D'' \cong \boldsymbol{P}^1$ outside $\mathrm{Bs}|L|$. Hence by (3.2), $X \cong \boldsymbol{P}^4$. Therefore it is impossible that $Z \cap \mathrm{Bs}|L| \cong \boldsymbol{P}^1$. This completes the proof of (3.5). q.e.d.

## § 4. Complex manifolds homeomorphic to $\boldsymbol{P}^n_{\boldsymbol{C}}$.

(4.1) PROPOSITION. *Let $X$ be a compact complex manifold homeomorphic to $\boldsymbol{P}^n$. If $\chi(X, O_X) \geqq 1$, then there is a holomorphic line bundle $L$ on $X$ whose Chern class $c_1(L)$ generates $H^2(X, \boldsymbol{Z}) \cong \boldsymbol{Z}$. If $h^1(X, O_X)=0$, $\chi(X, O_X) \geqq 1$ and $h^0(X, L) \geqq n$ and if general $(n-1)$-members $|L|$ intersect rationally outside* $\mathrm{Bs}|L|$, *then $X \cong \boldsymbol{P}^n$.*

PROOF. Let $\delta$ be a generator of $H^2(X, \boldsymbol{Z})$ $(\cong \boldsymbol{Z})$ with $\delta^n=1$. Since the second Stiefel-Whitney class $w_2(=c_1(X) \bmod 2)$ is a topological invariant, we have $c_1(X)=(n+1+2s)\delta$ for an integer $s$. Then by [**3**, p. 208], we have

$$\chi(X, O_X)=\binom{n+s}{s}=(n+s)(n+s-1)\cdots(n+1)/n!.$$

By $\chi(X, O_X)\geqq 1$, we see $s\geqq 0$ or that $n$ is even and $s\leqq -n-1$. Hence in particular $c_1(X)\neq 0$ and $H^1(X, O_X^*)\neq\{1\}$.

Now we consider an exact sequence

$$0\longrightarrow H^1(X, O_X)\longrightarrow H^1(X, O_X^*)\xrightarrow{c_1} H^2(X, \boldsymbol{Z})\longrightarrow H^2(X, O_X).$$

Since $c_1(X)\neq 0$ and $H^2(X, O_X)$ is torsion free, $c_1$ is surjective. Hence there exists a line bundle $L$ on $X$ with $c_1(L)=\delta$. Assume $s\leqq -n-1$, and $h^0(X, L)\geqq n$. By $h^1(X, O_X)=0$, we have $K_X=-(n+1+2s)L$, $-(n+1+2s)\geqq n+1$. Consequently $h^0(X, \Omega_X^n)\geqq h^0(X, L)\geqq n$, which contradicts $h^0(X, \Omega_X^n)\leqq b_n\leqq 1$. Hence $s\geqq 0$, and (4.1) follows from (3.1). q.e.d.

(4.2) THEOREM. *Let $X$ be a Moishezon manifold homeomorphic to $\boldsymbol{P}^n$, and $L$ a line bundle on $X$ with $L^n=1$. Assume that $h^0(X, L)\geqq n$. If general $(n-1)$-members of $|L|$ intersect outside* $\mathrm{Bs}|L|$, *then $X\cong\boldsymbol{P}^n$.*

PROOF. Since $X$ is Moishezon, the Hodge spectral sequence $E_1^{pq}=H^p(X, \Omega_X^q)$ with abutment $H^{p+q}(X, \boldsymbol{C})$ degenerates at $E_1$ terms [**19**, p. 99]. Hence we have $H^q(X, O_X)=0\,(q>0)$, $\chi(X, O_X)=1$, $\mathrm{Pic}\,X:=H^1(X, O_X^*)\cong H^2(X, \boldsymbol{Z})\cong H^2(\boldsymbol{P}^n, \boldsymbol{Z})\cong\boldsymbol{Z}$. Therefore $K_X=-(n+1)L$ by the proof of (4.1). Hence $X\cong\boldsymbol{P}^n$ by (3.2).

q.e.d.

(4.3) THEOREM [**10**]. *Let $X$ be a compact complex 3-fold homeomorphic to $\boldsymbol{P}^3$, and $L$ a line bundle on $X$ with $L^3=1$. Assume that $h^1(X, O_X)=0$ and $h^0(X, L)\geqq 2$. Then $X\cong\boldsymbol{P}^3$.*

PROOF. This is a corollary to (3.1) or (3.3). The proof is almost the same as [**11**, (9.1)]. It is easy to see that $h^3(X, O_X)=0$, $\chi(X, O_X)\geqq 1$. By the proof of (4.1), $c_1(X)=dc_1(L)$ for some $d\geqq 4$. By using $h^1(X, O_X)=0$ and $h^0(X, L)\geqq 2$, we see that $h^2(X, pL)=h^1(X, -(p+4)L)=0$ for $p>0$. Then we see that $h^0(X, L)\geqq 4$, and that $X$ is Moishezon by Riemann-Roch theorem. By (3.1) or (3.3), $X\cong\boldsymbol{P}^3$. q.e.d.

REMARK. A somewhat stronger theorem has been obtained in [**11**, (9.1)], which however follows from (4.3) easily.

## § 5. Moishezon fourfolds homeomorphic to $\boldsymbol{P}_{C}^{4}$.

The purpose of this section is to prove:

(5.1) THEOREM. *Let $X$ be a Moishezon 4-fold homeomorphic to $\boldsymbol{P}^4$, and $L$*

*a line bundle on $X$ with $L^4=1$. Assume that $h^0(X, L)\geqq 3$. Then $X\cong \boldsymbol{P}^4$.*

Our proof of (5.1) is completed in (5.4).

(5.2) LEMMA. *Under the assumptions in* (5.1), *let $D$ and $D'$ be distinct members of $|L|$, $\tau$ the scheme-theoretic complete intersection $D\cap D'$. Then we have*

(5.2.1) $$\operatorname{Pic} X = \boldsymbol{Z}L, \qquad K_X \cong -5L,$$

(5.2.2) $$H^p(X, -qL) = 0 \qquad (p = 0,\ q > 0, \text{ or } p > 0,\ 0 \leqq q \leqq 4)$$

(5.2.3) $$H^p(D, -qL_D) = 0 \qquad (p = 0,\ q > 0 \text{ or } p > 0,\ 0 \leqq q \leqq 3)$$

(5.2.4) $$H^0(X, O_X) \cong H^0(D, O_D) \cong H^0(\tau, O_\tau) \cong \boldsymbol{C},$$

(5.2.5) $$|L|_D = |L_D| \quad \text{and} \quad |L|_\tau = |L_\tau|.$$

PROOF. The proof of (5.2.1) is similar to [**10**]. The vanishing (5.2.2) of $H^p(X, -qL)$ for $p\neq 2$ is proved in the same way as in [**10**]. Since $X$ is homeomorphic to $\boldsymbol{P}^4$, we have

$$\chi(X, -qL) = \chi(\boldsymbol{P}^4, O_{\boldsymbol{P}^4}(-q)) = \frac{1}{24}\prod_{i=1}^{4}(q-i)$$

for any $q$ in view of (5.2.1). This proves the vanishing of $H^2(X, -qL)$ for $0\leqq q\leqq 5$. The remaining assertions are easy to prove. q.e.d.

(5.3) LEMMA. *Let $D$ and $D'$ be general members of $|L|$, and let $\tau=D\cap D'$. Let $Z=Z_{\mathrm{red}}$ be a reduced component of $\tau$, that is, an irreducible component of $\tau$ along which $\tau$ is reduced generically. If $Z\not\subset \operatorname{Bs}|L|$, then $\tau\cong Z\cong \boldsymbol{P}^2$ and $L_\tau\cong O_{\boldsymbol{P}^2}(1)$.*

PROOF. Let $g\colon S\to Z$ be the minimal resolution of the normalization of $Z$. Then there exist by (2.A) or [**5**, Corollary (18)] effective Cartier divisors $E$ and $G$ on $S$ with no common components such that the canonical sheaf $K_S$ is given by

$$K_S = g^*(K_X+2L)-E-G$$

with $f_*(G)=0$, etc. as in the proof of (3.3). There exists a finite subset $\Sigma_0$ of $S$ such that $g_{|S\setminus\Sigma}$ is an isomorphism where $\Sigma:=f^{-1}(\Delta)\cup f^{-1}(\operatorname{Sing} Y)\cup\Sigma_0$. Then $\Sigma$ contains $\operatorname{supp}(E+G)$.

We have $c_1(S)=3c_1(g^*L)+c_1(E+G)$. Since $h^0(X, L)\geqq 3$ and $Z\not\subset\operatorname{Bs}|L|$, $g^*L$ is effective. Since $S$ is projective, we have $P_m(S)=0$, whence $S\cong\boldsymbol{P}^2$ or $S$ is ruled. Let $H\in g^*|L|$. Then by the same argument as in (3.3), we see that $S\cong Y\cong\boldsymbol{P}^2$, $E=G=0$, $O_S(H)\cong O_{\boldsymbol{P}^2}(1)$ and that $\Sigma$ is finite. By $E=0$ and (2.A), $Z$ has at worst isolated singularities. There exists $D''\in|L|$ such that

$g^*(Z\cap D'')=H$ by the choice of $H$. Let $l=D\cap D'\cap D''$ be a scheme-theoretic complete intersection. Since $g^*D''=H\cong\boldsymbol{P}^1$ and $g$ is an isomorphism on $S\backslash\Sigma$, we have $H\backslash\Sigma\cong C\backslash g(\Sigma)$, so that $C:=g(H)_{\mathrm{red}}$ is *a reduced curve component* of $l$, that is, $l$ is reduced generically along $C$. $C$ is isomorphic to $Z\cap D''$ on $(Z\backslash g(\Sigma))\cap D''$. Namely $I_C=\sqrt{I_C}=I_l$ along $C\cap(Z\backslash g(\Sigma))$. We have

$$1=(H^2)_S=(g^*(L)H)_S=(Lg_*(H))_X=(LC)_X\,.$$

Therefore we can apply (1.10) to $X$, $C$ and $l$ to infer that $C\cong\boldsymbol{P}^1$ is a connected component of $l$ and that $C\cong l$ along $C$. If $\mathrm{Sing}\,\tau_{\mathrm{red}}$ is nonempty, then $\mathrm{Sing}\,\tau_{\mathrm{red}}\subset\mathrm{Bs}|L|$. Hence $Z\cap\mathrm{Sing}\,\tau_{\mathrm{red}}\subset Z\cap D''(=g(H)_{\mathrm{red}})$. Consequently $Z\cap\mathrm{Sing}\,\tau_{\mathrm{red}}\subset C$. As $C$ is a connected component of $l$, this shows that $Z$ *is a connected component of* $\tau$. In fact, if not, there is an irreducible component $Z'(\neq Z)$ of $\tau$ meeting $Z$. Then we choose a point $p\in Z\cap Z'$. We note that $Z\cap Z'$ is finite by $E=0$. Hence since $p\in Z\cap\mathrm{Sing}\,\tau_{\mathrm{red}}\subset C$, $Z'\cap D''$ contains an irreducible component (a curve or a surface) of $l$ meeting $C$. This contradicts that $C$ is a connected component of $l$.

However $h^0(\tau, O_\tau)=1$ by (5.2). Hence $Z\cong\tau_{\mathrm{red}}$. As $\tau$ is Gorenstein and reduced generically along $Z$, $\tau$ is reduced everywhere and $\tau\cong Z$. Since a prime Cartier divisor $C$ of $Z$ is smooth, so is $Z$ along $C$. As $\mathrm{Sing}\,Z\subset Z\cap\mathrm{Sing}\,\tau_{\mathrm{red}}\subset C$, it follows that $Z$ is smooth everywhere. Thus we see $\boldsymbol{P}^2\cong S\cong Y\cong Z\cong\tau$.

q.e.d.

(5.4) Completion of the Proof of (5.1). Now it is easy to prove (5.1). By (5.2.5), $\mathrm{Bs}|L|_\tau=\mathrm{Bs}|L_\tau|=\mathrm{Bs}|O_{P^2}(1)|=\emptyset$. We have also $h^0(X, L)=h^0(\tau, L_\tau)+2=5$ and $L^4=(H^2)_S=1$. Consequently $X\cong\boldsymbol{P}^4$ by an easy argument. q.e.d.

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