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Notes on the topology of folds

Dedicated to Professor Haruo Suzuki on his 60th birthday

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1. Introduction.

In this paper we study smooth maps $f: M^n \to N^p$ of n-manifolds into p-manifolds ($n \geqq p$) having only fold singular points and find some obstructions to the existence of such maps. In [**15**], Thom showed that, for generic maps $f: M^n \to \boldsymbol{R}^2 (n \geqq 2)$, the number of cusp singular points has the same parity as the euler number of M^n (see also [**7**]); in particular, there are no smooth maps $f: M^n \to \boldsymbol{R}^2$ having only fold singular points if the euler number of M^n is odd. Thom also showed that if $n-p+1$ is odd and the $(n-p+2)$-th Stiefel-Whitney class of M^n is non-zero, then there are no smooth maps $f: M^n \to \boldsymbol{R}^p$ having only fold singular points. Our main results of this paper are some generalizations of Thom's results.

In §3, we shall show the following.

THEOREM 1. *Let M^n be a closed manifold with odd euler number and N^p an even-dimensional manifold with $w_{p-1}(N^p)=0$ and $w_p(N^p)=0$ $(n \geqq p \geqq 2)$, where $w_i(N^p) \in H^i(N^p; \boldsymbol{Z}/2\boldsymbol{Z})$ denotes the i-th Stiefel-Whitney class of N^p. Then there exist no smooth maps $f: M^n \to N^p$ having only fold singular points.*

THEOREM 2. *Let N^p be a stably parallelizable manifold. Suppose that $n-p+1 (\geqq 1)$ is odd and that $w_i(M^n) \neq 0$ for some $i \geqq n-p+2$. Then there exist no smooth maps $f: M^n \to N^p$ having only fold singular points.*

Theorems 1 and 2 are consequences of a more general result (Proposition 3.2), which we shall prove by directly constructing a certain bundle map φ: $TM^n \oplus \varepsilon^1 \to TN^p$, where ε^1 is the trivial line bundle over M^n. Unfortunately, Theorems 1 and 2 do not hold if $n-p+1$ is even. In fact, we shall give an explicit example of a smooth map $f: M^4 \to \boldsymbol{R}^3$ with only fold singular points such that M^4 has odd euler number (Example 3.7). However, if we restrict ourselves to *simple* maps, we have the following.

THEOREM 3. *Let M^n be a closed orientable manifold with odd euler number and N^p an orientable manifold with $w_{p-1}(N^p)=0$ and $w_p(N^p)=0$ $(n \geqq p \geqq 2)$. Then* *there exist no simple smooth maps $f: M^n \to N^p$ having only fold singular points.*

Recall that a smooth map $f: M^n \to N^p$ having only fold singular points is *simple* if every component of the fiber $f^{-1}(q)$ contains at most one singular point ($^\forall q \in N^p$) (see also § 3). Simple maps have the property that their Stein factorizations ([8]) are easy to handle.

In § 2, we shall show the following, which shows us that there do exist some obstructions even if $n-p+1$ is even and the maps are non-simple.

THEOREM 4. *Let M^4 be a smooth closed 4-manifold such that $H_*(M^4; \boldsymbol{Z}) \cong H_*(\boldsymbol{CP}^2; \boldsymbol{Z})$. Then there exist no smooth maps $f: M^4 \to \boldsymbol{R}^3$ having only fold singular points.*

We shall prove Theorem 4, using a result of Sakuma [**12**], which is peculiar to 4-dimensions.

Here we note that in this paper we are concerned with necessary conditions for the existence of a smooth map with only fold singular points and that we do not touch on sufficient conditions. For this problem, see [**6**, **2**, **3**], in which it is shown that if the euler number of M^n is even, then it admits a smooth map into $\boldsymbol{R}^2$ with only fold singular points. See also the literatures cited in the remark in [**4**, § 1].

Throughout the paper, all manifolds, fiber bundles and maps are differentiable of class C^∞ unless otherwise indicated. All manifolds are paracompact and Hausdorff.

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2. Normal bundles of fold components.

Let $f: M^n \to N^p$ be a smooth map, where M^n and N^p are n- and p-dimensional manifolds without boundary respectively ($n \geqq p$). We denote by $S(f)$ the set of the singular points of f and call it the *singular set* of f. A point $q \in S(f)$ is called a *fold singular point* (or a *fold point*) if there exist local coordinates $(x_1, \cdots, x_n)$ centered at q and $(y_1, \cdots, y_p)$ centered at $f(q)$ such that f has the form:

$$y_i \circ f = x_i \qquad (i \leqq p-1)$$
$$y_p \circ f = -x_p^2 - \cdots - x_{p+\lambda-1}^2 + x_{p+\lambda}^2 + \cdots + x_n^2$$

for some λ ($0 \leqq \lambda \leqq n-p+1$), which is called the *index* of q. (Note that the index is well-defined if we consider that λ and $(n-p+1)-\lambda$ represent the same index.) If, in addition, $\lambda=0$ or $n-p+1$, we call q a *definite fold point*; otherwise, we call q an *indefinite fold point*. We set $S_d(f) = \{$definite fold points of $f\}$ and

$S_i(f)=\{$indefinite fold points of $f\}$. We say that a component S of $S(f)$ is a *fold component* if all the points in S are fold singular points. Note that a fold component also has its own index if we consider that λ and $(n-p+1)-\lambda$ represent the same index. Note also that a fold component S is a $(p-1)$-dimensional submanifold of M^n and that $f|S: S\to N^p$ is always an immersion. Furthermore, note that if f is stable, $f|S_d(f)\cup S_i(f)$ is an immersion with normal crossings.

Let S be a fold component of a smooth map $f: M^n\to N^p$. We define the normal bundle $\nu(f_S)$ of the immersion $f_S=f|S: S\to N^p$ so that the following sequence of vector bundles over S is exact:

$$0 \longrightarrow TS \xrightarrow{i} f_S^*TN^p \longrightarrow \nu(f_S) \longrightarrow 0\,,$$

where $i: TS\to f_S^*TN^p$ is the canonical inclusion. Note that $\nu(f_S)$ is a line bundle over S. When $\nu(f_S)$ is trivial, a specific trivialization $\nu(f_S)\xrightarrow{\cong} S\times \boldsymbol{R}$ is called an *orientation* of $\nu(f_S)$ and if an orientation is given we say that $\nu(f_S)$ is *oriented*. Note that if $\nu(f_S)$ is trivial it has exactly two orientations. Furthermore, $\nu(f_S)$ is trivial if and only if $w_1(\nu(f_S))(\in H^1(S;\boldsymbol{Z}/2\boldsymbol{Z}))$ vanishes. When $\nu(f_S)$ is oriented, the index λ of S is a well-defined integer.

Our purpose of this section is to study the structure of the normal bundle $\nu(S)$ of a fold component S in M^n and to prove Theorem 4 as an application.

In this paper, we adopt the convention that the index of a quadratic function is the number of its negative eigenvalues.

LEMMA 2.1. *Let S be a fold component of a smooth map $f: M^n\to N^p$. Then we have a smooth map $d^2f: \nu(S)\to\nu(f_S)$ such that $\pi_2\circ d^2f=\pi_1$ and that $d^2f|\nu_q(S): \nu_q(S)\to\nu_q(f_S)$ is a quadratic function of index λ for $^\forall q\in S$, where π_1: $\nu(S)\to S$ and $\pi_2: \nu(f_S)\to S$ are the bundle projections, $\nu_q(S)=\pi_1^{-1}(q)$, $\nu_q(f_S)=\pi_2^{-1}(q)(\cong\boldsymbol{R})$ and λ is the index of S with respect to f.*

PROOF. For $^\forall q\in S$, we have a well-defined quadratic map d^2f_q: $\mathrm{Ker}(df_q)\to \mathrm{Coker}(df_q)$([**1**]). We see easily that $\mathrm{Ker}(df_q)$ and $\mathrm{Coker}(df_q)$ are canonically identified with $\nu_q(S)$ and $\nu_q(f_S)$ respectively. Hence, we have a smooth well-defined fiber-wise quadratic map $d^2f: \nu(S)\to\nu(f_S)$. Furthermore we see easily that the index of the quadratic function d^2f_q agrees with the index of q with respect to f, using the definition of a fold singular point. □

LEMMA 2.2. *Let S be a fold component of a smooth map $f: M^n\to N^p$. If $\nu(f_S)$ is non-trivial, then we have $2\lambda=n-p+1$, where λ is the index of S.*

PROOF. Take a point $q\in S$. Since $\nu(f_S)$ is non-trivial, we see that $d^2f_q\circ\varphi=a\cdot d^2f_q$ for some linear isomorphism $\varphi: \nu_q(S)\to\nu_q(S)$ and for some $a<0$. Hence,

$d^2 f_q$ and $-d^2 f_q$ have the same index with respect to a fixed isomorphism $\nu_q(f_S) \xrightarrow{\cong} \boldsymbol{R}$. Thus we have $\lambda = n-p+1-\lambda$. □

One of the main results of this section is the following observation.

PROPOSITION 2.3. *Let $f: M^n \to N^p$ $(n>p)$ be a smooth map and S a fold component of f of index λ $(0<\lambda<n-p+1)$. Then the structure group of the normal bundle $\nu(S)$ of S in M^n can be reduced to*

$$G = \begin{cases} O(\lambda)\times O(n-p+1-\lambda) & (2\lambda \neq n-p+1) \\ \{O(\lambda)\times O(\lambda)\} \rtimes \boldsymbol{Z}/2\boldsymbol{Z} & (2\lambda = n-p+1) \end{cases}$$

$(G \subset O(n-p+1))$, where we identify $O(\lambda)$ with

$$\left\{\begin{pmatrix} P & 0 \\ 0 & I_{n-p+1-\lambda} \end{pmatrix} \in O(n-p+1);\ P \in O(\lambda)\right\}$$

and $O(n-p+1-\lambda)$ with

$$\left\{\begin{pmatrix} I_\lambda & 0 \\ 0 & Q \end{pmatrix} \in O(n-p+1);\ Q \in O(n-p+1-\lambda)\right\},$$

and $\{O(\lambda)\times O(\lambda)\} \rtimes \boldsymbol{Z}/2\boldsymbol{Z}$ is the subgroup of $O(2\lambda)$ generated by $O(\lambda)\times O(\lambda)$ and

$$\begin{pmatrix} 0 & I_\lambda \\ I_\lambda & 0 \end{pmatrix} \in O(2\lambda)$$

$(I_k \in O(k)$ is the identity matrix$)$.

PROOF. First, we consider the case $2\lambda \neq n-p+1$. Then by Lemma 2.2, $\nu(f_S)$ is trivial. We fix an orientation of $\nu(f_S)$. Then by Lemma 2.1 we see that there exists a smooth map $\nu(S) \to \boldsymbol{R}$ such that on each fiber $\nu_q(S)$ $(q \in S)$ it is a quadratic function of index λ. Thus, we see that the structure group of $\nu(S)$ is reduced to

$$O(\lambda, n-p+1-\lambda) = \{\varphi \in GL(n-p+1);\ Q_\lambda \circ \varphi = Q_\lambda\},$$

where $Q_\lambda: \boldsymbol{R}^{n-p+1} \to \boldsymbol{R}$ is the quadratic function of index λ defined by $Q_\lambda(x_1, \cdots, x_{n-p+1}) = -x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_{n-p+1}^2$. It is known that the maximal compact subgroup of $O(\lambda, n-p+1-\lambda)$ is $O(\lambda)\times O(n-p+1-\lambda) = G$. Thus the structure group of $\nu(S)$ reduces to G.

When $2\lambda = n-p+1$, by a similar argument, we see that the structure group of $\nu(S)$ is reduced to

$$G' = \{\varphi \in GL(n-p+1);\ Q_\lambda \circ \varphi = aQ_\lambda \text{ for some } a \in \boldsymbol{R}-\{0\}\}.$$

It is easily seen that $G' \cong (O(\lambda, \lambda) \rtimes \boldsymbol{Z}/2\boldsymbol{Z}) \times \boldsymbol{R}_+$, where $\boldsymbol{R}_+ = \{a \in \boldsymbol{R};\ a>0\}$. Furthermore, we can show that the maximal compact subgroup of G' is $G=$

$\{O(\lambda)\times O(\lambda)\}\rtimes \boldsymbol{Z}/2\boldsymbol{Z}$. Thus the structure group of $\nu(S)$ reduces to G. □

REMARK 2.4. Existence of a reduction of the structure group of a k-plane bundle ν to $O(\lambda)\times O(k-\lambda)$ is equivalent to the existence of a λ-plane field on ν ([**14**]).

Next we confine ourselves to the case where $n-p+1=2$. For $r\geqq 2$, set $C_r=\{\exp((2\pi\sqrt{-1}/r)k)\in \boldsymbol{C};\ k\in \boldsymbol{Z}\}\subset U(1)=SO(2)$, which is a cyclic group of order r.

LEMMA 2.5. *Let ξ: $E\to X$ be a 2-plane bundle whose structure group is reduced to $C_r\subset SO(2)$. If $H^2(X;\ \boldsymbol{Z})$ has no r-torsion, then ξ is trivial as a vector bundle.*

PROOF. Let BC_r and $BSO(2)$ be the classifying spaces of C_r-bundles and $SO(2)$-bundles respectively. We have a canonical map ι: $BC_r\to BSO(2)$. Let α_ξ: $X\to BSO(2)$ be the classifying map of ξ. Then by the hypothesis, α_ξ factors through ι; i.e., $\alpha_\xi\simeq\iota\circ\beta_\xi$ for some β_ξ: $X\to BC_r$ ($\simeq$denotes a homotopy). Let $e\in H^2(BSO(2);\ \boldsymbol{Z})\cong \boldsymbol{Z}$ be the generator. Then the euler class $e(\xi)$ of ξ is equal to $\beta_\xi^*\circ\iota^*(e)$. On the other hand, it is known that $H^2(BC_r;\ \boldsymbol{Z})\cong C_r$. Hence $re(\xi)=\beta_\xi^*(r\iota^*(e))=0$. Since $H^2(X;\ \boldsymbol{Z})$ has no r-torsion, we have $e(\xi)=0$, which implies that ξ is a trivial 2-plane bundle. □

PROPOSITION 2.6. *Let f: $M^n\to N^{n-1}$ be a smooth map with M^n orientable and S a fold component of f of index* 1.

(1) *If S is orientable and $H^2(S;\ \boldsymbol{Z})$ has no* 4*-torsion, then the normal bundle $\nu(S)$ of S in M^n is trivial.*

(2) *If S is non-orientable, let ω: $\tilde{S}\to S$ be the orientation double cover and suppose $H^2(\tilde{S};\ \boldsymbol{Z})$ has no* 4*-torsion. Then the induced bundle $\omega^*\nu(S)$ over $\tilde{S}$ is trivial.*

PROOF. First we consider (1). Since S and M^n are orientable, we see that the normal bundle $\nu(S)$ of S in M^n is also orientable; i.e., its structure group is reduced to $SO(2)$. Hence, by Proposition 2.3, the structure group of $\nu(S)$ is reduced to $SO(2)\cap\{(O(1)\times O(1))\rtimes \boldsymbol{Z}/2\boldsymbol{Z}\}=C_4$. Then, by Lemma 2.5, $\nu(S)$ is trivial. We can prove (2) in a similar way. □

In particular, for f: $M^4\to N^3$, we have the following.

COROLLARY 2.7. *Let f: $M^4\to N^3$ be a smooth map with M^4 oriented and S a compact fold component of f of index* 1. *Then the self intersection number $S\cdot S$ of S in M^4 vanishes.*

PROOF. If S is orientable, the result is obvious by Proposition 2.6, since $H^2(S;\ \boldsymbol{Z})\cong \boldsymbol{Z}$. When S is non-orientable, let ω: $\tilde{S}\to S$ be the orientation double cover. Then we have $S\cdot S=(1/2)\tilde{S}\cdot\tilde{S}$, where $\tilde{S}\cdot\tilde{S}$ is the self intersection number of the zero section of $\omega^*\nu(S)\to\tilde{S}$ in $\omega^*\nu(S)$ which is oriented by the pull back of that of $\nu(S)$. By Proposition 2.6, we have $\tilde{S}\cdot\tilde{S}=0$, since $H^2(\tilde{S};\ \boldsymbol{Z})\cong\boldsymbol{Z}$. $\square$

Now we prove Theorem 4 in the introduction. Let M^4 be a closed 4-manifold with $H_*(M^4;\ \boldsymbol{Z})\cong H_*(\boldsymbol{C}P^2;\ \boldsymbol{Z})$ and suppose $f: M^4\to\boldsymbol{R}^3$ is a smooth map with only fold singular points. We have $S(f)=S_d(f)\cup S_i(f)$, where $S_d(f)$ is the set of the definite fold points (index 0 or 2) and $S_i(f)$ is the set of the indefinite fold points (index 1). Note that $S_d(f)$ and $S_i(f)$ are (possibly disconnected) closed 2-manifolds smoothly embedded in M^4. Note also that M^4 is orientable, since $H_4(M^4;\ \boldsymbol{Z})\cong\boldsymbol{Z}$. We orient M^4 arbitrarily. Then by Corollary 2.7, we have $S_i(f)\cdot S_i(f)=0$. Furthermore, since $H_1(M^4;\ \boldsymbol{Z})\cong H_1(\boldsymbol{C}P^2;\ \boldsymbol{Z})=0$, we have $\sigma(M^4)\equiv-S(f)\cdot S(f)(\mathrm{mod}\,4)$ by Sakuma [**12**], where $\sigma(M^4)$ denotes the signature of M^4. Since $S(f)\cdot S(f)=S_d(f)\cdot S_d(f)+S_i(f)\cdot S_i(f)$, we have $S_d(f)\cdot S_d(f)\equiv-\sigma(M^4)(\mathrm{mod}\,4)$. Furthermore, since $H_2(M^4;\ \boldsymbol{Z})\cong H_2(\boldsymbol{C}P^2;\ \boldsymbol{Z})\cong\boldsymbol{Z}$, the intersection form of M^4 is represented by the matrix (ε), where $\varepsilon=\pm1(=\sigma(M^4))$.

LEMMA 2.8. *$S_d(f)$ is an orientable 2-manifold.*

PROOF. Let S be a component of $S_d(f)$. Then by Lemma 2.2, the normal bundle $\nu(f_S)$ of the immersion $f_S=f|S: S\to\boldsymbol{R}^3$ is trivial. Since $\boldsymbol{R}^3$ is orientable, so is S. $\square$

Thus the homology class $[S_d(f)]$ represented by $S_d(f)$ is an element of $H_2(M^4;\ \boldsymbol{Z})$. Let $[S_d(f)]=l\xi$, where $\xi\in H_2(M^4;\ \boldsymbol{Z})(\cong\boldsymbol{Z})$ is a generator and $l\in\boldsymbol{Z}$. Then we have $S_d(f)\cdot S_d(f)=l^2\xi\cdot\xi=\varepsilon l^2$. Hence we have $\varepsilon l^2\equiv-\varepsilon(\mathrm{mod}\,4)$, which implies $l^2\equiv-1(\mathrm{mod}\,4)$. However this equation has no integer solutions, which is a contradiction. This completes the proof of Theorem 4.

As a consequence of Theorem 4, we deduce an interesting result concerning stable maps $f: M^4\to\boldsymbol{R}^3$. Let $f: M^4\to\boldsymbol{R}^3$ be a stable map. Then it is known that for ${}^\forall q\in S(f)$ there exist local coordinates (x_1, x_2, x_3, x_4) centered at q and (y_1, y_2, y_3) centered at $f(q)$ such that f has the form:

$$y_i\circ f = x_i \qquad (i=1, 2)$$
$$y_3\circ f = x_3^{k+1}+\sum_{i=1}^{k-1} x_i x_3^{k-i}\pm x_4^2$$

for some $k\,(k=1, 2, 3)$([**15, 10**]). Such a point q is called a singular point of A_k-type. Moreover, the singular point q is called a fold singular point if $k=1$, a *cusp singular point* if $k=2$, and a *swallow tail* if $k=3$. Let $A_k(f)$ be the set of the singular points of f of A_k-type. Then it is easily checked that $A_k(f)$ are submanifolds of M^4 of dimension $3-k$. We also note that since the dimension pair (4, 3) is nice in the sense of Mather [**9**], the set of the stable maps are dense in the space $C^\infty(M^4, N^3)$ of the smooth proper maps $M^4\to N^3$.

COROLLARY 2.9. *Let M^4 be a smooth closed* 4*-manifold such that $H_*(M^4; \boldsymbol{Z}) \cong H_*(CP^2; \boldsymbol{Z})$. Then every stable map $f: M^4 \to \boldsymbol{R}^3$ has cusp singular points.*

PROOF OF COROLLARY 2.9. It is well-known that $\overline{A_k(f)} = \cup_{i \geqq k} A_i(f)$. Now suppose that f has no cusp points; i.e., $A_2(f) = \emptyset$. Then we have $\emptyset = \overline{A_2(f)} = A_2(f) \cup A_3(f)$. Thus f has no swallow tails, either. Hence, f has only fold points as its singularities. This contradicts Theorem 4. □

REMARK 2.10. Using Corollary 2.9, we see easily that a stable map $f: M^4 \to N^3$ always has definite fold points, indefinite fold points and cusp singular points if M^4 is as in Corollary 2.9 and N^3 is an open orientable 3-manifold. Note that every open orientable 3-manifold can be immersed into $\boldsymbol{R}^3$([**16**]).

COROLLARY 2.11. *Let M^4 be a* 4*-manifold as in Theorem* 4. *Then there exist no smooth maps $f: M^4 \to \boldsymbol{R}^p$ $(2 \leqq p \leqq 4)$ with only fold singular points.*

PROOF. For $p=3$, this is a consequence of Theorem 4. For $p=4$, this is proved using a result of Èliašberg [**2**] (see also [**11**]) and the fact that M^4 is not stably parallelizable. For $p=2$, the result follows from [**15, 7**], since the euler number of M^4 is odd. □

REMARK 2.12. A smooth map $f: M^n \to \boldsymbol{R}^1$ with only fold singular points is a Morse function; thus, such maps as in Corollary 2.11 always exist for $p=1$.

REMARK 2.13. Every closed orientable 3-manifold admits smooth maps into $\boldsymbol{R}^p (1 \leqq p \leqq 3)$ with only fold singular points ([**2, 6**]).

3. Maps having only fold singular points.

In this section, we consider smooth maps $f: M^n \to N^p$ $(n \geqq p)$ all of whose singularities are fold singular points. This type of maps has been investigated in some detail by Fukuda [**4**], who studied the relationships between the euler numbers of M^n and $S(f)$ and between their Stiefel-Whitney classes, in the case where $N^p = \boldsymbol{R}^p$. Here, we study from a point of view different from Fukuda's.

LEMMA 3.1. *Let $f: M^n \to N^p$ $(n \geqq p)$ be a smooth map having only fold singular points. If the normal bundle of the immersion $f|S(f): S(f) \to N^p$ is trivial, the $(n+1)$-plane bundle $TM^n \oplus \varepsilon^1$ is isomorphic to $f^*TN^p \oplus \eta$ for some $(n-p+1)$-plane bundle η over M^n, where ε^1 is the trivial line bundle over M^n.*

PROOF. We construct a fiber-wise surjective linear map $\varphi: TM^n \oplus \varepsilon^1 \to TN^p$ such that the diagram

$$\begin{array}{ccc} TM^n \oplus \varepsilon^1 & \xrightarrow{\varphi} & TN^p \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M^n & \xrightarrow{f} & N^p \end{array}$$

commutes, where π and π' are the bundle projections. By the hypothesis, the normal bundle of the immersion $f|S(f)$ is trivial. We orient it arbitrarily. For $q \in S(f)$, there exist an open neighborhood U_q in M^n of q and a smooth map $\phi_q: U_q \to TN^p$ such that $\pi' \circ \phi_q = f|U_q$ and that $\phi_q(x) \in T_{f(x)}N^p$ is a non-zero vector normal to $df_x(T_x(S(f)))$ consistent with the orientation chosen above $({}^\forall x \in S(f) \cap U_q)$. Using $\phi_q (q \in S(f))$ and the partition of unity, we can construct a smooth map $\tilde{\phi}: M^n \to TN^p$ such that $\pi' \circ \tilde{\phi} = f$ and $\tilde{\phi}(x) \notin df_x(T_x(S(f)))$ for ${}^\forall x \in S(f)$. Then define $\varphi: TM^n \oplus \varepsilon^1 \to TN^p$ by $\varphi(v, (x, a)) = df_x(v) + a\tilde{\phi}(x)$, where $v \in T_xM^n$, $x \in M^n$ and $a \in \boldsymbol{R}$. Then φ is the desired fiber-wise surjective linear map. Then φ induces a fiber-wise surjective linear map $\bar{\varphi}: TM^n \oplus \varepsilon^1 \to f^*TN^p$ such that $\pi'' \circ \bar{\varphi} = \pi$, where $\pi'': f^*TN^p \to M^n$ is the bundle projection. Define the $(n-p+1)$-plane bundle η over M^n by $\eta = \operatorname{Ker} \bar{\varphi}$. Then it is easy to see that $TM^n \oplus \varepsilon^1 \cong f^*TN^p \oplus \eta$. $\square$

PROPOSITION 3.2. *Let $f: M^n \to N^p (n \geqq p)$ be as in Lemma* 3.1. *If $w_i(N^p) = 0 ({}^\forall i \geqq k)$ for some $k (k < p)$, then we have $w_j'(M^n) = 0 ({}^\forall j \geqq (n-p+1)+k)$.*

PROOF. By Lemma 3.1, we have

$$w_j(M^n) = w_j(TM^n \oplus \varepsilon^1) = \sum_{l=0}^{n+1} f^*w_l(N^p) \cup w_{j-l}(\eta),$$

where η is an $(n-p+1)$-plane bundle as in Lemma 3.1. When $l \geqq k$, we have $f^*w_l(N^p) = 0$ by our hypothesis. When $l < k$ and $j \geqq (n-p+1)+k$, we have $j-l > n-p+1$; hence, $w_{j-l}(\eta) = 0$. We have, therefore, $w_j(M^n) = 0$ for ${}^\forall j \geqq (n-p+1)+k$. $\square$

REMARK 3.3. By [15], we have $[S(f)]^* = w_{n-p+1}(\eta)$ in Proposition 3.2, where $[S(f)]^*$ is the Poincaré dual of the homology class $[S(f)] \in H_{p-1}(M^n; \boldsymbol{Z}/2\boldsymbol{Z})$ represented by $S(f)$. In particular, if N^p is stably parallelizable, we have $w_{n-p+1}(M^n) = [S(f)]^*$.

Now we prove Theorem 2 in the introduction. Suppose $f: M^n \to N^p$ is a smooth map with only fold singular points. Since $n-p+1$ is odd, the normal bundle of the immersion $f|S(f)$ is trivial by Lemma 2.2. Hence, by Proposition 3.2, we have $w_j(M^n) = 0 ({}^\forall j \geqq n-p+2)$. This is a contradiction. This completes the proof of Theorem 2.

Here we note that, for $N^p = \boldsymbol{R}^p$ Theorem 2 can also be deduced using Theorem 2(b) in [4] and Lemma 2.2.

COROLLARY 3.4. *Let $f: M^n \to N^p (n \geqq p \geqq 2)$ be a smooth map having only fold singular points. If $w_{p-1}(N^p)=0$, $w_p(N^p)=0$ and M^n is a closed manifold with odd euler number, then the normal bundle of the immersion $f|S(f)$ is non-trivial. In particular, if N^p is orientable, $S(f)$ is non-orientable.*

PROOF. If the normal bundle of the immersion $f|S(f)$ is trivial, then by Proposition 3.2 we have $w_n(M^n)=0$. Hence we have

$$\chi(M^n) \equiv \langle w_n(M^n), [M^n] \rangle \equiv 0 \pmod 2,$$

where $\chi(M^n)$ denotes the euler number of M^n. This contradicts our hypothesis. □

REMARK 3.5. If $N^p=\boldsymbol{R}^p$, Corollary 3.4 is a result of Sakuma [**12**], who showed it using results of [**4**].

REMARK 3.6. In the situation of Corollary 3.4, n must be even, since M^n is a closed manifold of odd euler number, and p must be odd, since $n-p+1$ is odd (cf. Lemma 2.2).

Now we prove Theorem 1 in the introduction. Since M^n is a closed manifold with odd euler number, $n=\dim M^n$ must be even. Thus, $n-p+1$ is odd. Suppose $f: M^n \to N^p$ is a smooth map with only fold singular points. Then by Lemma 2.2, the normal bundle of the immersion $f|S(f)$ is trivial. This contradicts Corollary 3.4. This completes the proof of Theorem 1.

Here we note that, for $N^p=\boldsymbol{R}^p$, Theorem 1 can also be deduced using Theorem 2(a) of [**4**] and Lemma 2.2.

A smooth map as in Corollary 3.4 does exist. We give an example as follows.

EXAMPLE 3.7. Let $h: \boldsymbol{RP}^2 \to \boldsymbol{R}$ be the Morse function defined by

$$h([x:y:z]) = \frac{x^2-y^2}{x^2+y^2+z^2},$$

where $[x:y:z]$ denotes the homogeneous coordinate of $\boldsymbol{RP}^2$. The function h has exactly 3 critical points $[0:1:0]$, $[0:0:1]$ and $[1:0:0]$ with indices 0, 1 and 2 respectively. Let $\gamma: \boldsymbol{RP}^2 \to \boldsymbol{RP}^2$ be the involution defined by $\gamma([x:y:z]) =[y:x:z]$. Note that $h\circ\gamma=-h$ and that $\gamma([0:0:1])=[0:0:1]$ and $\gamma([0:1:0])=[1:0:0]$. Then define another involution $\alpha: S^2\times\boldsymbol{RP}^2 \to S^2\times\boldsymbol{RP}^2$ by $\alpha(u, v)=(-u, \gamma(v))$, where $-u\in S^2$ is the antipodal point of $u\in S^2$. Since α is fixed point free, the quotient space $M^4=S^2\times\boldsymbol{RP}^2/\alpha$ is a closed (non-orientable) 4-manifold (in fact, M^4 is an $\boldsymbol{RP}^2$-bundle over $\boldsymbol{RP}^2$). Furthermore, let $\beta: S^2\times\boldsymbol{R} \to S^2\times\boldsymbol{R}$ be the involution defined by $\beta(u, a)=(-u, -a)$. Then the quotient space $\nu(\boldsymbol{R}P^2)=S^2\times\boldsymbol{R}/\beta$ is a line bundle over $\boldsymbol{R}P^2$, which is orientable as an open 3-manifold. Since the diagram

$$\begin{array}{ccc} S^2\times\boldsymbol{R}P^2 & \xrightarrow{\mathrm{id}\times h} & S^2\times\boldsymbol{R} \\ \alpha\downarrow & & \downarrow\beta \\ S^2\times\boldsymbol{R}P^2 & \xrightarrow{\mathrm{id}\times h} & S^2\times\boldsymbol{R} \end{array}$$

commutes, we have a smooth map $g: M^4\to\nu(\boldsymbol{R}P^2)$ induced by $\mathrm{id}\times h$. Furthermore, since $\nu(\boldsymbol{R}P^2)$ is an open orientable 3-manifold, there exists an immersion $\eta: \nu(\boldsymbol{R}P^2)\to\boldsymbol{R}^3$ ([**16**]). Then define the smooth map $f: M^4\to\boldsymbol{R}^3$ by $f=\eta\circ g$. Then it is easily seen that all the singularities of f are fold points and that $\chi(M^4)=(1/2)\chi(S^2\times\boldsymbol{R}P^2)=1$. In fact, $S(f)$ consists of two components $S_d(f)$ and $S_i(f)$, where $S_d(f)\cong S^2$ is the set of the definite fold points and $S_i(f)\cong\boldsymbol{R}P^2$ is the set of the indefinite fold points.

Example 3.7 shows that Theorems 1 and 2 do not hold if $n-p+1$ is even in general.

DEFINITION 3.8 [8]. Let $f: M^n\to N^p\,(n\geqq p)$ be a smooth map with only fold singular points. A point $q\in S(f)$ is said to be *simple* if the component of $f^{-1}(f(q))$ containing q intersects $S(f)$ only at q. Furthermore, f is said to be *simple* if all its singularities are simple.

For example, the map of Example 3.7 is simple. Note that if f is *stable*, $f|S(f)$ is an immersion with normal crossings. Hence, the non-simple points are contained in a locally finite union of codimension 1 submanifolds of $S(f)$. In particular, if $f|S(f)$ is an embedding, f is simple. We also note that a smooth map $f: M^n\to N^p\,(n\geqq p)$ all of whose singularities are definite fold points is simple. Such a map is called a *special generic map* (see, for example, [**11**]).

An important property of simple maps is that their Stein factorizations ([8]) are quite simple (see the proof of the following proposition), which enables us to study their global topology.

The following proposition shows that it is difficult to construct an example as in Example 3.7 with M^4 orientable.

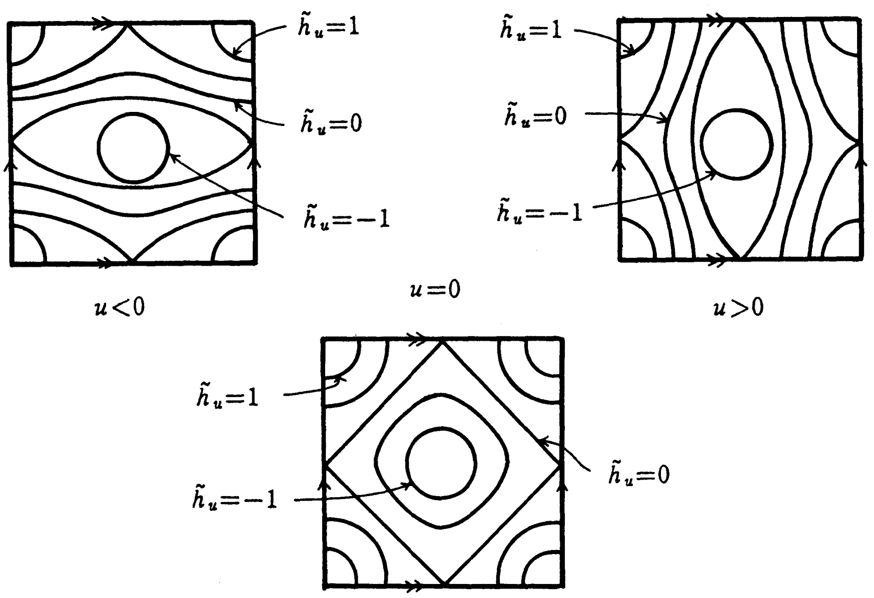
PROPOSITION 3.9. *Let $f: M^n\to N^p\,(n\geqq p)$ be a proper smooth map with only fold singular points. If f is simple, N^p is orientable, and $S(f)$ is non-orientable, then M^n is non-orientable.*

PROOF. Suppose M^n is orientable and let $S\subset S(f)$ be a non-orientable component of $S(f)$. Let $q_f: M^n\to W_f$ be the Stein factorization of f ([8]): i.e., for $q, q'\in M^n$ we define $q\sim q'$ if $f(q)=f(q')$ and q and q' belong to the same connected component of $f^{-1}(f(q))=f^{-1}(f(q'))$, and let $q_f: M^n \to W_f = M^n/\sim$ be the quotient map. Note that $W_f - q_f(S(f))$ admits a natural structure of a p-manifold. Furthemore, since f is simple, $q_f|S(f)$ is an embedding and the regular neighborhood $N(\Sigma)$ of $\Sigma=q_f(S)$ in W_f is homeomorphic to an I-bundle over Σ or a Y-bundle over Σ, where $I=[-1, 1]$ and $Y=\{r\exp((2\pi\sqrt{-1}/3)k)\in \boldsymbol{C};\ 0\leqq r\leqq 1,\ k=0, 1, 2\}$ (see [8]). (Note that [8] treats only the case $p=2$; however, the same argument works for general p.) Let $\pi': N(\Sigma)\to\Sigma$ be the bundle projection. Set $X=q_f^{-1}(N(\Sigma))$, which is an orientable n-manifold with boundary. Then $\pi: X\to\Sigma$ defined by $\pi=\pi'\circ q_f|X$ is a fiber bundle over Σ with fiber T, where T is the transverse manifold at $q\in S$ ($\dim T=n-p+1$); i.e., T is the component of $f^{-1}(J)$ containing q, where J is a sufficiently small arc embedded in N^p which passes through $f(q)$ and is transverse to $df_q(T_q(S(f)))$ (see [8, p. 9]). Note that ∂T consists of two components ∂_+T and ∂_-T and that there exists a Morse function $g: T\to\boldsymbol{R}$ with $g(T)=[-1, 1]$, $g^{-1}(-1)=\partial_-T$, $g^{-1}(1)=\partial_+T$ such that it has exactly one critical point. Take a smoothly embedded simple closed curve C in Σ which is orientation reversing and set $V=\pi^{-1}(C)$. Then $\pi|V: V\to C$ is a T-bundle over $C(\cong S^1)$. Since X is orientable, T is orientable. Furthermore, the normal bundle of C in Σ is non-orientable, which implies that the normal bundle of V in X is non-orientable. Hence, V is non-orientable. Let $\gamma: T\to T$ be the geometric monodromy of the bundle $\pi|V$; i.e., γ is a diffeomorphism such that $V\cong T\times[0, 1]/(x, 1)\sim(\gamma(x), 0)$. Since V is non-orientable, γ is orientation reversing. In particular we have $\sigma(T)=0$. Furthermore since the normal bundle of the immersion $f|S$ restricted to $q_f^{-1}(C)$ is non-trivial, we see that $\gamma(\partial_+T)=\partial_-T$ and $\gamma(\partial_-T)=\partial_+T$. Set $Z=T\cup_\varphi\partial_-T\times[-1, 1]$, where the diffeomorphism $\varphi: \partial_-T\times\{\pm 1\}\to\partial T$ is defined by $\varphi|\partial_-T\times\{-1\}=\mathrm{id}: \partial_-T\times\{-1\}\to\partial_-T$ and $\varphi|\partial_-T\times\{1\}=\gamma|\partial_-T: \partial_-T\times\{1\}\to\partial_+T$. Then Z is a closed *orientable* $(n-p+1)$-manifold. We have $\chi(Z)=\chi(T)+\chi(\partial_-T)-\{\chi(\partial_-T)+\chi(\partial_+T)\}$. By the existence of the Morse function $g: T\to\boldsymbol{R}$ with exactly one critical point, we have $\chi(T)=\chi(\partial_-T)\pm 1$. Hence we have $\chi(Z)=\pm 1$. Furthermore, by the Novikov additivity, we have $\sigma(Z)=\sigma(T)+\sigma(\partial_-T\times[-1, 1])=0$. (Note that $\dim T=n-p+1$ is even by Lemma 2.2.) However, there exist no closed orientable manifolds Z with $\chi(Z)$ odd and $\sigma(Z)$ even by the Poincaré duality, which is a contradiction. This completes the proof. $\square$

Now we prove Theorem 3 in the introduction. Suppose $f: M^n\to N^p$ is a simple map. Then by Proposition 3.9, $S(f)$ is orientable. Thus the normal bundle of the immersion $f|S(f)$ is trivial. This contradicts Corollary 3.4. This completes the proof of Theorem 3.

Example 3.7 shows that Theorem 3 does not hold for non-orientable manifolds M^n in general.

In Proposition 3.9, the hypothesis that f be simple is essential as the following example shows.



Levels of $\tilde{h}_u$

Figure 1.

EXAMPLE 3.10. We construct a smooth map $f: M^4 \to \boldsymbol{R}^3$ with M^4 closed and orientable and $S(f)$ non-orientable such that f has only fold points as its singularities. Let T_2 be the torus with 2 open disks removed. Let $g: I \times T_2 \to I \times J$ be the map as in Proposition 1 of [8, p. 33] ($I=J=[-1, 1]$); i.e., $g(u, x) =(u, h_u(x))$ and $h_u: T_2 \to J$ is a certain Morse function with exactly 2 critical points for $u \neq 0$. Set $T=T_2 \cup D_+^2 \cup D_-^2$, where $D_\pm^2$ are the unit 2-disks in $\boldsymbol{R}^2$ and D_+^2 (resp. D_-^2) is attached to $h_u^{-1}(1)$ (resp. $h_u^{-1}(-1)$). T is diffeomorphic to the torus. Define $\tilde{g}: I \times T \to I \times \tilde{J}$ ($\tilde{J}=[-2, 2]$) by

$$\tilde{g}|I \times T_2 = g$$

$$\tilde{g}(u, x) = (u, 2-\|x\|^2) \qquad (x \in D_+^2)$$

$$\tilde{g}(u, x) = (u, \|x\|^2-2) \qquad (x \in D_-^2).$$

Then $\tilde{g}(u, x)=(u, \tilde{h}_u(x))$, where $\tilde{h}_u: T \to \tilde{J}$ is a Morse function for $u \neq 0$ (see Figure 1). Define the orientation reversing diffeomorphism $\eta: T \to T$ by $\eta(l, m) =(-m+1/2, -l+1/2)$, where we identify T with $\boldsymbol{R}^2/\boldsymbol{Z}^2$. Then we have $\gamma \circ \tilde{g} =\tilde{g} \circ \delta$, where $\gamma: I \times \tilde{J} \to I \times \tilde{J}$ is defined by $\gamma(u, t)=(-u, -t)$ and $\delta: I \times T \to I \times T$ is defined by $\delta(u, x)=(-u, \eta(x))$, since we have $\tilde{h}_{-u}(\eta(x))=-\tilde{h}_u(x)$. Let $M'= D_1^2 \times T \cup_{\varphi_1} S^1 \times I \times T \cup_{\varphi_2} D_2^2 \times T$, and $X'=D_1^2 \times \tilde{J} \cup_{\psi_1} S^1 \times I \times \tilde{J} \cup_{\psi_2} D_2^2 \times \tilde{J}$, where D_1^2 and D_2^2 are 2-disks and $\varphi_1: \partial D_1^2 \times T \to S^1 \times \{-1\} \times T$, $\varphi_2: \partial D_2^2 \times T \to S^1 \times \{1\} \times T$, $\psi_1: \partial D_1^2 \times \tilde{J} \to S^1 \times \{-1\} \times \tilde{J}$ and $\psi_2: \partial D_2^2 \times \tilde{J} \to S^1 \times \{1\} \times \tilde{J}$ are the identities ($M' \cong$

$S^2\times T$, $X'\cong S^2\times\tilde{J}$). Furthermore define $H\colon M'\to X'$ by

$$H(z, x)=(z, \tilde{h}_{-1}(x))\in D_1^2\times\tilde{J} \qquad ((z, x)\in D_1^2\times T)$$
$$H(\theta, u, x)=(\theta, u, \tilde{h}_u(x))\in S^1\times I\times\tilde{J} \qquad ((\theta, u, x)\in S^1\times I\times T)$$
$$H(z, x)=(z, \tilde{h}_1(x))\in D_2^2\times\tilde{J} \qquad ((z, x)\in D_2^2\times T).$$

Note that $H'\colon M'\xrightarrow{H}X'\hookrightarrow \boldsymbol{R}^3$ is a smooth map with only fold singular points and that $S_d(H')\cong S^2\amalg S^2$ and $S_i(H')\cong S^2\amalg S^2$. Define the smooth involutions $\alpha\colon M'\to M'$ and $\beta\colon X'\to X'$ by

$$\alpha(z, x)=(-z, \eta(x))\in D_2^2\times T \qquad ((z, x)\in D_1^2\times T)$$
$$\alpha(\theta, u, x)=(-\theta, \delta(u, x))\in S^1\times I\times T \qquad ((\theta, u, x)\in S^1\times I\times T)$$
$$\alpha(z, x)=(-z, \eta(x))\in D_1^2\times T \qquad ((z, x)\in D_2^2\times T)$$

and by

$$\beta(z, t)=(-z, -t)\in D_2^2\times\tilde{J} \qquad ((z, t)\in D_1^2\times\tilde{J})$$
$$\beta(\theta, u, t)=(-\theta, \gamma(u, t))\in S^1\times I\times\tilde{J} \qquad ((\theta, u, t)\in S^1\times I\times\tilde{J})$$
$$\beta(z, t)=(-z, -t)\in D_1^2\times\tilde{J} \qquad ((z, t)\in D_2^2\times\tilde{J}).$$

Note that α exchanges the two components of $S_d(H')$ and that α maps each component of $S_i(H')$ to itself orientation reversingly. Then we see that α and β are orientation preserving involutions without fixed points and that $H\circ\alpha=\beta\circ H$. Thus we have a well-defined smooth map $\bar{H}\colon M\to X$ which is induced by H, where $M=M'/\alpha$ and $X=X'/\beta$. Note that M is a closed orientable 4-manifold (in fact, it is diffeomorphic to a T^2-bundle over $\boldsymbol{RP}^2$). Since X is an orientable 3-manifold with boundary, there exists an immersion $\zeta\colon X\to\boldsymbol{R}^3$ ([**16**]). Set $f=\zeta\circ\bar{H}\colon M\to\boldsymbol{R}^3$. Then we see that f has only fold singular points. Furthermore, we see that $S_d(f)\cong S^2$ and $S_i(f)\cong\boldsymbol{RP}^2\amalg\boldsymbol{RP}^2$. Of course, f is not simple. We also note that $\chi(M)=0$. We do not know if there exists an example of Corollary 3.4 with M^n orientable.

EXAMPLE 3.11. We give an example which shows that Proposition 3.9 does not hold if N^p is non-orientable. Let $h\colon S^2\to\boldsymbol{R}$ be the standard height function defined by $h(x, y, z)=z$ and $\beta\colon S^2\to S^2$ the involution defined by $\beta(x, y, z)=(-x, y, z)((x, y, z)\in S^2\subset\boldsymbol{R}^3)$. Note that $h\circ\beta=h$. Define another involution $\alpha\colon S^2\times S^2\to S^2\times S^2$ by $\alpha(u, v)=(-u, \beta(v))\,((u, v)\in S^2\times S^2)$. Then α is orientation preserving and has no fixed points. Furthermore define the involution $\gamma\colon S^2\times\boldsymbol{R}\to S^2\times\boldsymbol{R}$ by $\gamma(u, t)=(-u, t)$ $((u, t)\in S^2\times\boldsymbol{R})$. Then γ is orientation reversing and has no fixed points. Since the diagram

$$\begin{array}{ccc} S^2\times S^2 & \xrightarrow{\mathrm{id}\times h} & S^2\times \boldsymbol{R} \\ \alpha\downarrow & & \downarrow\gamma \\ S^2\times S^2 & \xrightarrow{\mathrm{id}\times h} & S^2\times \boldsymbol{R} \end{array}$$

commutes, we have a well-defined smooth map $f: M^4\to N^3$ induced by $\mathrm{id}\times h$, where $M^4=S^2\times S^2/\alpha$ and $N^3=S^2\times \boldsymbol{R}/\gamma$. It is easily seen that f is a simple map; in fact, it is a special generic map. Furthermore, we see that $S(f)\cong \boldsymbol{RP}^2\amalg\boldsymbol{RP}^2$, that M^4 is orientable and that $N^3(\cong \boldsymbol{RP}^2\times\boldsymbol{R})$ is non-orientable. Thus Proposition 3.9 does not hold if N^p is non-orientable.

In [**11**], we showed that a closed orientable manifold which admits a special generic map (i.e. a smooth map with only definite fold points as its singularities) into an open manifold is zero in the smooth oriented cobordism ring. Note that a special generic map is always simple. For maps with indefinite folds, we have the following, which shows us that there are additional obstructions to the existence of a simple map $f: M^n\to N^{n-1}$ for any N^{n-1}.

PROPOSITION 3.12. *Let $f: M^n\to N^{n-1}$ be a smooth map with only fold singular points, where M^n is a closed orientable manifold. If f is simple, then M^n is zero in the smooth oriented cobordism ring; in particular, its signature vanishes and its euler number is even.*

PROOF. Let $q_f: M^n\to W_f$ be the Stein factorization of f (see the proof of Proposition 3.9). Set $\Sigma=q_f(S(f))$ and let $N(\Sigma)$ be the regular neighborhood of Σ in W_f. Furthermore put $R=W_f-\mathrm{Int}\,N(\Sigma)$. Then $q_f|q_f^{-1}(R): q_f^{-1}(R)\to R$ is an S^1-bundle over the smooth manifold R. Let N_d (resp. N_i) be the regular neighborhood of $q_f(S_d(f))$ (resp. $q_f(S_i(f))$) in W_f. Then we have $N(\Sigma)=N_d\cup N_i$. It is easily seen that $q_f^{-1}(N_d)$ is diffeomorphic to a D^2-bundle over $S_d(f)$. Furthermore, since M^n is orientable and f is simple, $q_f^{-1}(N_i)$ is diffeomorphic to an S^2_3-bundle over $S_i(f)$ (cf. [8]), where S^2_3 is the 2-sphere with 3 open 2-disks removed. Now let Z^{n+1} be the D^2-bundle over R associated with the S^1-bundle $q_f^{-1}(R)\to R$. Since M^n is orientable, so is Z^{n+1}. Note that ∂Z^{n+1} is diffeomorphic to $q_f^{-1}(R)\cup q_f^{-1}(N_d)\cup A$, where A is a $(D^2\amalg D^2\amalg D^2)$-bundle over $S_i(f)$ associated with the $(S^1\amalg S^1\amalg S^1)$-bundle $\partial(q_f^{-1}(N_i))$. Set $X'=A\cup_{\partial A}q_f^{-1}(N_i)$, which is an S^2-bundle over $S_i(f)$. Note that X' is an orientable n-manifold. Since the natural inclusion

$$O(3)\hookrightarrow \mathrm{Diff}\,S^2$$

is a weak homotopy equivalence ([**13**]), we see that there exists a D^3-bundle X^{n+1} over $S_i(f)$ such that $\partial X^{n+1}\cong X'$. Set $\tilde{M}^{n+1}=Z^{n+1}\cup_A X^{n+1}$. Then $\tilde{M}^{n+1}$ is a smooth compact orientable $(n+1)$-manifold with $\partial\tilde{M}^{n+1}\cong M^n$. This completes the proof. $\square$

REMARK 3.13. We do not know if in Proposition 3.12 the hypothesis that f be simple is essential or not. We also note that Proposition 3.12 does not hold for $f: M^n\to N^p$ with $p<n-1$ in general. For example, there exists a simple map $M^4\to \boldsymbol{R}^2$ with $\sigma(M^4)\neq 0$ (see [5]).

Finally we note that there does exist a gap between simple maps and non-simple maps. For example, we will show, in a forthcoming paper, that if f: $M^3\to N^2$ is a simple map then the closed 3-manifold M^3 is a so-called graph manifold. By Levine [6], every closed 3-manifold admits a smooth map into $\boldsymbol{R}^2$ with only fold singular points. Thus, non-graph manifolds, which are known to exist, are examples of 3-manifolds which admit smooth maps into $\boldsymbol{R}^2$ with only fold singular points but not simple ones.

References

[1] T. Bröcker, Differentiable germs and catastrophes, Translated by L. Lander, London Math. Soc. Lect. Note Ser., **17**, Cambridge Univ. Press, 1975.
[2] J. M. Èliašberg, On singularities of folding type, Math. USSR Izv., **4** (1970), 1119-1134.
[3] J. M. Èliašberg, Surgery of singularities of smooth mappings, Math. USSR Izv., **6** (1972), 1302-1326.
[4] T. Fukuda, Topology of folds, cusps and Morin singularities, in "A Fete of Topology", (eds. Y. Matsumoto, T. Mizutani and S. Morita), Academic Press, 1987, pp. 331-353.
[5] M. Kobayashi, Simplifying certain mappings from simply connected 4-manifolds into the plane, preprint.
[6] H. Levine, Elimination of cusps, Topology, **3** (suppl. 2) (1965), 263-296.
[7] H. Levine, Mappings of manifolds into the plane, Amer. J. Math., **88** (1966), 357-365.
[8] H. Levine, Classifying immersions into $\boldsymbol{R}^4$ over stable maps of 3-manifolds into $\boldsymbol{R}^2$, Lecture Notes in Math., **1157**, Springer, 1985.
[9] J. N. Mather, Stability of C^∞ mappings: VI. The nice dimensions, in "Proc. Liverpool Singularities-Symposium I", Lect. Notes in Math., **192**, Springer, 1971, pp. 207-253.
[10] B. Morin, Formes canoniques des singularités d'une application différentiables, C. R. Acad. Sc. Paris, **260** (1965), 5662-5665, 6503-6506.
[11] O. Saeki, Topology of special generic maps of manifolds into Euclidean spaces, to appear in Topology Appl..
[12] K. Sakuma, On special generic maps of simply connected $2n$-manifolds into $\boldsymbol{R}^3$, to appear in Topology Appl..
[13] S. Smale, Diffeomorphisms of the 2-sphere, Proc. Amer. Math. Soc., **10** (1959), 621-626.
[14] N. Steenrod, Topology of fibre bundles, Princeton Univ. Press, 1951.
[15] R. Thom, Les singularités des applications différentiables, Ann. Inst. Fourier

(Grenoble), **6** (1955-56), 43-87.
[16] J. H. C. Whitehead, The immersions of an open 3-manifold in Euclidean 3-space, Proc. London Math. Soc., **11** (1961), 81-90.

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