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Curvature homogeneous spaces with a solvable Lie group as homogeneous model

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Introduction.

In this paper (M, g) will denote a C^{∞} Riemannian manifold with Riemann curvature tensor R defined by

$$R_{XY}=D_{[X,Y]}-[D_X, D_Y]$$

where D denotes the Levi Civita connection and X, Y are tangent vector fields.

(M, g) is said to be *curvature homogeneous* if for each pair of points p and q there exists a (linear) isometry F between the tangent spaces T_pM and T_qM such that

$$F^*(R_q)=R_p\,,$$

or equivalently, there exists an orthonormal basis of T_pM and one of T_qM such that the corresponding components of R_p and R_q are the same. This turns out to be equivalent to the following: the Riemann curvature tensor R of (M, g) viewed as an equivariant map from the total space of the orthonormal frame bundle OM of (M, g) into the space $\mathcal{R}(V)$ of algebraic curvature tensors over $V=\boldsymbol{R}^n$, $n=\dim M$, maps OM into a single $O(n)$-orbit of $\mathcal{R}(V)$. (See [**7**] for more details.) Further, a homogeneous Riemannian space (M', g') with Riemann curvature tensor R' is called a *homogeneous model* for (M, g) if $R(OM)\subset R'(OM')$.

The notion of a curvature homogeneous Riemannian space has been introduced by Singer in 1960 [**13**]. Sekigawa [**12**] and Takagi [**17**] produced the first examples of irreducible complete Riemannian manifolds which are curvature homogeneous but not locally homogeneous. All these examples have a symmetric model, i.e., their Riemann curvature tensor satisfies

$$R_{XY}\cdot R=0$$

where the operators R_{XY} act as derivations on R. Riemannian manifolds satisfying the last algebraic condition are said to be *semi-symmetric*. Many examples

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which are not curvature homogeneous are known.

This broader class of all semi-symmetric spaces has been studied extensively by Szabó [**14**], [**15**], [**16**]. A pure existence theorem in [**15**] shows that the semi-symmetric spaces in each dimension $\geqq 3$ depend on arbitrary functions of two variables and functions of one variable. Thus one can expect more examples of curvature homogeneous spaces inside this class. Motivated by Szabó's results, and by a conjecture of M. Gromov (see [**2**], [**6**], [**7**], [**18**], [**19**] for more details), the authors constructed in [**6**], [**7**] new explicit examples of manifolds of the previous type which are curvature homogeneous. Also a complete study of the isometry classes is given there. These examples are obtained in a geometric way by deforming a flat right invariant metric on a semidirect product of $\boldsymbol{R}$ and $\boldsymbol{R}^{n-1}$.

In this paper we construct new examples of curvature homogeneous spaces by deforming a flat right invariant metric on a Lie group which is a semidirect product of $\boldsymbol{R}^p$ and $\boldsymbol{R}^q$. In this way we obtain now spaces which are *curvature homogeneous but not semi-symmetric*. All of our examples are non-compact. Actually, the only known examples of compact curvature homogeneous Riemannian spaces which are not locally homogeneous are the non-homogeneous examples of isoparametric hypersurfaces in spheres described in [**4**]. It would be interesting to find other examples in case they exist.

In this context it is worthwhile to mention some recent results of K. Yamato on three-dimensional curvature homogeneous spaces [**21**]. Some of his results suggest that it could be very difficult to produce *compact* three-dimensional examples, but he was able to construct new *complete* irreducible curvature homogeneous metrics on $\boldsymbol{R}^3$ with three distinct principal Ricci curvatures. We shall show here that all these examples have homogeneous models.

It is also remarkable that the isoparametric hypersurfaces in real space forms are not the only curvature homogeneous hypersurfaces. K. Tsukada constructed in [**20**] an example of a four-dimensional hypersurface in the hyperbolic space $\boldsymbol{H}^5(-1)$ which is neither locally homogeneous nor isoparametric. This manifold is isometric to $(SO(3)/K)\times\boldsymbol{R}^+$, $K=\boldsymbol{Z}_2\times\boldsymbol{Z}_2$, endowed with a suitable cohomogeneity one Riemannian metric. The embedding of this manifold in $\boldsymbol{H}^5(-1)$ has type number two and it is uniquely determined up to local congruence. We shall show that this example does not admit a homogeneous model. To our knowledge, this is the first example of a curvature homogeneous space without any homogeneous model.

The paper is organized as follows. In Section 1 and Section 2 we construct our new examples. We focus on some of their curvature properties in Section 3. Section 4 is devoted to the study of the homogeneous model spaces. After giving a summary of the main results in Section 5, we concentrate on the irreducibility and the completeness of a special class of examples of Section 6. In Section 7 we treat the isometry classes of these examples. We finish this paper by giving some new results about Yamato's and Tsukada's examples in Section 8.

1. The group $\boldsymbol{R}^p \ltimes \boldsymbol{R}^q$ with the flat invariant metric g_0.

A (connected) Lie group G admits a *flat* left or right invariant Riemannian metric g_0 if and only if its Lie algebra $\mathfrak{g}$ is the *semidirect sum* of two mutually orthogonal Abelian subalgebras $\mathfrak{h}$ and $\mathfrak{k}$ where $\mathfrak{h}$ acts on $\mathfrak{k}$ by skew-symmetric endomorphisms (see [**9**, p. 298]). In other words $\mathfrak{g}$ splits as

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k} \tag{1.1}$$

where $\mathfrak{h}$ is an Abelian subalgebra of $\mathfrak{g}$ and $\mathfrak{k}$ an Abelian ideal. Moreover, $\mathfrak{h}$ and $\mathfrak{k}$ are orthogonal and, for each X belonging to $\mathfrak{h}$, the operator

$$\mathrm{ad}_X : \mathfrak{k} \longrightarrow \mathfrak{k} : Y \longmapsto \mathrm{ad}_X Y = [X, Y] \tag{1.2}$$

is skew-symmetric with respect to g_0.

Now, let $p = \dim \mathfrak{h}$ and $q = \dim \mathfrak{k}$. Then we can realize the universal covering group of G as a semidirect product $\boldsymbol{R}^p \ltimes \boldsymbol{R}^q$ of the two additive groups $\boldsymbol{R}^p$ and $\boldsymbol{R}^q$ in the following way. We identify $\mathfrak{h}$ with $\boldsymbol{R}^p$ and $\mathfrak{k}$ with $\boldsymbol{R}^q$ by choosing an orthonormal basis of $\mathfrak{h}$ and of $\mathfrak{k}$. Further, let A be the $\mathfrak{so}(q)$-valued linear form defined on $\mathfrak{h} = \boldsymbol{R}^p$ by

$$A(W) = -\mathrm{ad}_W, \qquad W \in \mathfrak{h}. \tag{1.3}$$

Then the product in $\boldsymbol{R}^p \ltimes \boldsymbol{R}^q$ is given by the following rule:

$$(W, X)(W', X') = (W + W', e^{-A(W)}X' + X). \tag{1.4}$$

We denote by w^i, x^α and $A^\alpha_\beta(W)$ the components of the vectors W, X and the entries of the matrix $A(W)$, respectively. (Here and in the sequel, the Latin indices $i, j, \cdots$ run from 1 to p, and the Greek indices $\alpha, \beta, \cdots$ from $p+1$ to $p+q$.) The entries A^α_β of A are linear forms. Hence one may write

$$A^\alpha_\beta(W) = w^i A^\alpha_{i\beta}, \quad \text{or} \quad A(W) = w^i A_i \tag{1.5}$$

where $A_i \in \mathfrak{so}(q)$ for $i = 1, \cdots, p$.

Now, an easy computation shows that the differential one-forms

$$\begin{cases} \theta^i = dw^i, \\ \theta^\alpha = dx^\alpha + A^\alpha_{i\beta} x^\beta dw^i \end{cases} \tag{1.6}$$

form a basis for the space of the *right* invariant differential one-forms on the Lie group $\boldsymbol{R}^p \ltimes \boldsymbol{R}^q$. Moreover, (1.7) $$g_0=\sum_{i=1}^{p}\theta^i\otimes\theta^i+\sum_{\alpha=p+1}^{p+q}\theta^\alpha\otimes\theta^\alpha$$

is a *flat right invariant* Riemannian metric on this group.

The exterior derivatives of the forms θ^i and θ^α are easily computed from (1.6). We have

(1.8) $$\begin{cases} d\theta^i=0, \\ d\theta^\alpha=-A^\alpha_{i\beta}\theta^i\wedge\theta^\beta. \end{cases}$$

Further, the dual basis is given by the right invariant vector fields

(1.9) $$\begin{cases} e_i=\dfrac{\partial}{\partial w_i}-A^\alpha_{i\beta}x^\beta\dfrac{\partial}{\partial x^\alpha}, \\ e_\alpha=\dfrac{\partial}{\partial x^\alpha}. \end{cases}$$

From this we get at once

(1.10) $$\begin{cases} [e_i, e_j]=0, \\ [e_\alpha, e_\beta]=0, \\ [e_i, e_\alpha]=A^\beta_{i\alpha}e_\beta. \end{cases}$$

Finally, we note that the endomorphisms A_i commute, i.e.,

$$A_iA_j=A_jA_i, \qquad i, j=1, \cdots, p,$$

as follows immediately from the Jacobi identity and (1.3).

2. The deformation of the metric g_0 and the curvature homogeneous examples.

Now, we consider on the group manifold $\boldsymbol{R}^p\ltimes\boldsymbol{R}^q$ the Riemannian metric

(2.1) $$g=\sum_{i=1}^{p}\omega^i\otimes\omega^i+\sum_{\alpha=p+1}^{p+q}\omega^\alpha\otimes\omega^\alpha$$

where

(2.2) $$\begin{cases} \omega^i=f^i\theta^i, \qquad f^i>0 \\ \omega^\alpha=\theta^\alpha \end{cases}$$

for some positive differentiable functions f^i defined on $\boldsymbol{R}^{p+q}$. The dual orthonormal frame is given by the vector fields

(2.3) $$\begin{cases} E_i=(f^i)^{-1}e_i, \\ E_\alpha=e_\alpha. \end{cases}$$

Then, from (2.2) and (1.8), we get

$$\text{(2.4)}\qquad \begin{cases} d\omega^i = (f^i)^{-1} df^i \wedge \omega^i = (f^i)^{-1}\{E_j(f^i)\omega^j + E_\beta(f^i)\omega^\beta\}\wedge\omega^i, \\ d\omega^\alpha = d\theta^\alpha = -(f^i)^{-1} A^\alpha_{i\beta}\omega^i\wedge\omega^\beta. \end{cases}$$

As usual, we compute the *connection forms* ω^A_B and the *curvature forms* Ω^A_B by using the Cartan structural equations. A straightforward computation yields

$$\text{(2.5)}\qquad \begin{cases} \omega^\alpha_\beta = \sum_i (f^i)^{-1} A^\alpha_{i\beta}\omega^i, \\ \omega^i_\beta = (f^i)^{-1} E_\beta(f^i)\omega^i, \\ \omega^i_j = (f^i)^{-1} E_j(f^i)\omega^i - (f^j)^{-1} E_i(f^j)\omega^j, \end{cases}$$

and

$$\text{(2.6)}\qquad \Omega^\alpha_\beta = 0,$$

$$\text{(2.7)}\qquad \Omega^i_\beta = -(f^i)^{-1}\{\sum_j E_\beta E_j(f^i)\omega^i\wedge\omega^j - \sum_\alpha E_\alpha E_\beta(f^i)\omega^\alpha\wedge\omega^i\},$$

$$\begin{aligned} \text{(2.8)}\qquad \Omega^i_j = & -(f^i f^j)^{-1} g(df^i, df^j)\omega^i\wedge\omega^j \\ & +(f^i)^{-1}\sum_k \{E_k E_j(f^i) - (f^k)^{-1} E_k(f^i) E_j(f^k)\}\omega^k\wedge\omega^i \\ & -(f^j)^{-1}\sum_k \{E_k E_i(f^j) - (f^k)^{-1} E_k(f^j) E_i(f^k)\}\omega^k\wedge\omega^j \\ & +(f^i)^{-1}\sum_\alpha E_\alpha E_j(f^i)\omega^\alpha\wedge\omega^i - (f^j)^{-1}\sum_\alpha E_\alpha E_i(f^j)\omega^\alpha\wedge\omega^j. \end{aligned}$$

Here $g(df^i, df^j)$ denotes the inner product of the one-forms df^i and df^j induced by the metric g.

Of course, the metric g is in general no longer right invariant nor homogeneous. Nevertheless, it follows from (2.6), (2.7) and (2.8) that this metric is *curvature homogeneous* if the following *sufficient* conditions are satisfied:

$$\text{(2.9)}\qquad E_\alpha E_\beta(f^i) = \lambda^i_{\alpha\beta} f^i,$$

$$\text{(2.10)}\qquad E_\beta E_j(f^i) = \mu^i_{\beta j} f^i, \quad \text{for } i\neq j,$$

$$\text{(2.11)}\qquad g(df^i, df^j) = \nu^{ij} f^i f^j, \quad \text{for } i\neq j,$$

$$\text{(2.12)}\qquad (f^i)^{-1}\{E_k E_j(f^i) - (f^k)^{-1} E_k(f^i) E_j(f^k)\} = \sigma_{ijk}, \quad i\neq j,\ k\neq i,$$

where $\lambda^i_{\alpha\beta}$, $\mu^i_{\beta j}$, ν^{ij} and σ_{ijk} are constants.

We will not solve the system (2.9)-(2.12) completely. For our purposes it will be sufficient to provide a particular solution which produces an interesting example. In order to do this, we note that (2.9) may be rewritten in the form

$$\text{(2.13)}\qquad \frac{\partial^2 f^i}{\partial x^\alpha \partial x^\beta} = \lambda^i_{\alpha\beta} f^i.$$

The integrability conditions show that the matrices $\lambda^i = (\lambda^i_{\alpha\beta})$ must have rank one. Therefore, their entries may be written as

$$\lambda^i_{\alpha\beta}=\mu^i c^i_\alpha c^i_\beta \tag{2.14}$$

where μ^i and c^i_α are constant. Then (2.13) may be solved explicitly. Recall that we look for positive solutions on the whole of $\boldsymbol{R}^{p+q}$. Therefore, we may suppose that the constants μ^i are positive and equal to 1. In this case, the general solution of (2.13), and hence of (2.9), is given by

$$f^i=a^i(w)e^{\Sigma_\alpha c^i_\alpha x^\alpha}+b^i(w)e^{-\Sigma_\alpha c^i_\alpha x^\alpha}, \tag{2.15}$$

where $a^i(w)$ and $b^i(w)$ are arbitrary functions of $w=(w^1, \cdots, w^p)$ such that $f^i>0$.

In what follows we consider only the *special case*

$$e_j(f^i)=0 \tag{2.16}$$

for all $i\neq j$, and do not pursue the search for the general solution. These conditions (2.16) are equivalent to

$$\begin{cases} \dfrac{\partial a^i}{\partial w^j}-A^\alpha_{j\beta}x^\beta c^i_\alpha a^i(w)=0, \\ \dfrac{\partial b^i}{\partial w^j}+A^\alpha_{j\beta}x^\beta c^i_\alpha b^i(w)=0, \end{cases} \tag{2.17}$$

for all $i\neq j$. Hence, we must have

$$\begin{cases} \dfrac{\partial a^i}{\partial w^j}=\dfrac{\partial b^i}{\partial w^j}=0, \qquad i\neq j, \\ \sum A^\alpha_{j\beta}c^i_\alpha=0, \qquad i\neq j. \end{cases} \tag{2.18}$$

Then (2.10) and (2.12) are also satisfied and (2.11) reduces to

$$\sum_\alpha E_\alpha(f^i)E_\alpha(f^j)=\nu^{ij}f^if^j, \qquad i\neq j. \tag{2.19}$$

On the other hand, we have

$$E_\alpha(f^i)=c^i_\alpha g^i, \qquad i=1, \cdots, p \tag{2.20}$$

where

$$g^i=a^i(w)e^{\Sigma_\alpha c^i_\alpha x^\alpha}-b^i(w)e^{-\Sigma_\alpha c^i_\alpha x^\alpha}. \tag{2.21}$$

Therefore, we must have

$$(\sum_\alpha c^i_\alpha c^j_\alpha)g^ig^j=\nu^{ij}f^if^j \tag{2.22}$$

for $i\neq j$. In what follows we always suppose

$$\sum_\alpha c^i_\alpha c^j_\alpha\neq 0.$$

(Note that $c^i_\alpha=0$, $\alpha=p+1, \cdots, p+q$, for some i gives a trivial product case.) Then the ratios g^ig^j/f^if^j must be constant. This is possible if and only if

$a^i(w)=0$ or $b^i(w)=0$ identically.

In the sequel we suppose $b^i(w)=0$ for $i=1, \cdots, p$. Then we get

PROPOSITION 2.1. *Let* g *be a metric given by* (2.1), (2.2) *and* (1.6) *such that*

$$f^i = a^i(w^i)e^{\Sigma_\alpha c_\alpha^i x^\alpha}, \qquad i=1, \cdots, p, \tag{2.23}$$

where each $a^i(w^i)$ *is an arbitrary positive function of the variable* w^i *and the* c_α^i *are constants. If these constants satisfy*

$$\sum_\alpha A_{j\beta}^\alpha c_\alpha^i = 0 \qquad \text{for } i \neq j,\ i, j=1, \cdots, p, \tag{2.24}$$

then the metric g *is curvature homogeneous.*

It is useful to notice that, if we consider the vectors

$$c^i = (c_{p+1}^i, \cdots, c_{p+q}^i)$$

as elements of $\boldsymbol{R}^q=\mathfrak{k}$, then (2.24) is equivalent to

$$A_i(c^j) = 0, \tag{2.25}$$

where the A_i are the skew-symmetric operators introduced in Section 1. In other words, the vectors c^j must belong to the kernel of A_i for $i \neq j$.

Moreover, we stress the fact that we always suppose that the scalar products

$$\langle c^i, c^j \rangle = \sum_\alpha c_\alpha^i c_\alpha^j \tag{2.26}$$

are different from zero.

3. Further properties of the curvature homogeneous examples.

In this section we continue the study of the deformed metric under the hypotheses of Proposition 2.1. First, from (2.6), (2.7) and (2.8) we get

$$\begin{cases} \Omega_\beta^\alpha = 0, \\ \Omega_\beta^i = c_\beta^i(\sum_\alpha c_\alpha^i \omega^\alpha) \wedge \omega^i, \\ \Omega_j^i = -\langle c^i, c^j \rangle \omega^i \wedge \omega^j \end{cases} \tag{3.1}$$

where

$$\langle c^i, c^j \rangle \neq 0.$$

We use these formulas to determine the *Riemann curvature tensor* of g which is given by

$$R = -2 \sum_{A,B} \Omega_A^B \otimes \omega^A \wedge \omega^B, \qquad A, B=1, \cdots, p+q. \tag{3.2}$$

We obtain

$$R=-2\sum_{i,j}\langle c^i, c^j\rangle\omega^i\wedge\omega^j\otimes\omega^i\wedge\omega^j-4\sum_{i,\alpha,\beta}c_\alpha^i c_\beta^i\omega^i\wedge\omega^\alpha\otimes\omega^i\wedge\omega^\beta. \tag{3.3}$$

So, the possible non-zero components of R are

$$\begin{cases} R_{ijij}=-\langle c^i, c^j\rangle, & i\neq j, \\ R_{i\alpha i\beta}=-c_\alpha^i c_\beta^i. \end{cases} \tag{3.4}$$

Further, it follows easily from (3.4) that the *Ricci tensor* ρ is given by

$$\rho=-\sum_{i,k}\langle c^i, c^k\rangle\omega^i\otimes\omega^i-\sum_{\alpha,\beta,k}c_\alpha^k c_\beta^k\omega^\alpha\otimes\omega^\beta, \tag{3.5}$$

and the *scalar curvature* τ by

$$\tau=-\|\sum_i c^i\|^2-\sum_i\|c^i\|^2. \tag{3.6}$$

Further, we compute the covariant derivatives with respect to the Levi Civita connection D of the metric g. First, from (2.5) we have

$$\begin{cases} \omega_\beta^\alpha=\sum_i(f^i)^{-1}A_{i\beta}^\alpha\omega^i, \\ \omega_\beta^i=c_\beta^i\omega^i, \\ \omega_j^i=0. \end{cases} \tag{3.7}$$

Hence,

$$\begin{cases} D_X\omega^i=-\omega^i(X)\sum_\beta c_\beta^i\omega^\beta, \\ D_X\omega^\alpha=\sum_j c_\alpha^j\omega^j(X)\omega^j-\sum_{i,\beta}(f^i)^{-1}A_{i\beta}^\alpha\omega^i(X)\omega^\beta \end{cases} \tag{3.8}$$

and

$$\begin{cases} D_XE_i=-\omega^i(X)\sum_\beta c_\beta^i E_\beta, \\ D_XE_\alpha=\sum_j c_\alpha^j\omega^j(X)E_j+\sum_{j,\beta}(f^j)^{-1}A_{j\alpha}^\beta\omega^j(X)E_\beta. \end{cases} \tag{3.9}$$

Taking into account (3.5), (3.8) and (2.24), we can now compute easily the covariant derivative $D\rho$ of the Ricci tensor. We get

$$\begin{aligned} D\rho=&\sum_{i,k,\beta}\langle c^i, c^k\rangle(c_\beta^i-c_\beta^k)(\omega^i\otimes\omega^\beta\otimes\omega^i+\omega^i\otimes\omega^i\otimes\omega^\beta) \\ &+\sum_{i,\alpha,\beta,\gamma}c_\alpha^i c_\beta^i A_{i\gamma}^\alpha(f^i)^{-1}(\omega^i\otimes\omega^\gamma\otimes\omega^\beta+\omega^i\otimes\omega^\beta\otimes\omega^\gamma). \end{aligned} \tag{3.10}$$

Therefore, the possible non-zero components of $D\rho$ are

$$\begin{cases} D_i\rho_{\beta i}=\sum_k\langle c^i, c^k\rangle(c_\beta^i-c_\beta^k), \\ D_i\rho_{\gamma\beta}=\sum_\alpha c_\alpha^i c_\beta^i A_{i\gamma}^\alpha(f^i)^{-1}. \end{cases} \tag{3.11}$$

From this we get

PROPOSITION 3.1. *Let $A_i(c^i)\neq 0$ for some $i\in\{1, \cdots, p\}$. Then $\|D\rho\|$ is not constant and the Riemannian manifold $(\boldsymbol{R}^{p+q}, g)$ is not locally homogeneous.*

PROOF. We have

$$\|D\rho\|^2 = 2\sum_{i,\beta}(D_i\rho_{\beta i})^2+\sum_i \|c^i\|^2\|A_i(c^i)\|^2(f^i)^{-2}. \tag{3.12}$$

Moreover, $\sum_{i,\beta}(D_i\rho_{\beta i})^2$ is constant.

It is important to note that (3.3) shows that the curvature tensor R does not depend on the operators A_i but only on the constant c^i_α. So, by changing the A_i and by keeping fixed the c^i_α, we get examples of metrics on $\boldsymbol{R}^{p+q}$ with the same curvature. In particular, we may put $A_i=0$ for all $i=1, \cdots, p$. We denote by g' the Riemannian metric obtained in this way. It is defined by (2.1) and (2.2) where

$$\theta^i = dw^i, \qquad \theta^\alpha = dx^\alpha.$$

In this case, the semidirect product $\boldsymbol{R}^p\ltimes\boldsymbol{R}^q$ is just the direct product of $\boldsymbol{R}^p$ and $\boldsymbol{R}^q$ and g_0 is the standard Euclidean metric of $\boldsymbol{R}^{p+q}$. Further, from (3.8) we get that the covariant derivatives of the one-forms ω^A are linear combinations with *constant* coefficients of $\omega^A\otimes\omega^B$. From this and from (3.3) it follows that the metric g' on $\boldsymbol{R}^{p+q}$ is *infinitesimally homogeneous* in the sense of Singer [**13**]. Therefore, it is locally homogeneous. Actually, it is locally isometric to a homogeneous Riemannian space—*one of the model spaces*—which we will determine and study in the next section.

4. The model spaces.

In contrast to what we did in Section 1, we consider now the Lie group $\bar{G}$ which is a semidirect product of $\boldsymbol{R}^q\ltimes\boldsymbol{R}^p$ where $\boldsymbol{R}^q$ is acting on $\boldsymbol{R}^p$ as follows:

$$\alpha(X)W = e^{-C(X)}W \tag{4.1}$$

where

$$C(X) = \begin{pmatrix} \sum_\alpha c^1_\alpha x^\alpha & 0 & \cdots & 0 \\ 0 & \sum_\alpha c^2_\alpha x^\alpha & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \sum_\alpha c^p_\alpha x^\alpha \end{pmatrix}. \tag{4.2}$$

The product law of $\bar{G}$ is given by

$$(W_0, X_0)(W, X) = (W_0+e^{-C(X_0)}W, X_0+X). \tag{4.3}$$

Hence, the left translations are given by

$$(4.4)\qquad \begin{cases} w^i \circ L_{(w_0, x_0)} = e^{-\Sigma_\alpha c_\alpha^i x_0^\alpha} w^i + w_0^i, \\ x^\alpha \circ L_{(w_0, x_0)} = x^\alpha + x_0^\alpha. \end{cases}$$

It follows that a basis of the space of left invariant one-forms on $\bar{G}$ is given by

$$(4.5)\qquad \begin{cases} \bar{\omega}^i = \bar{f}^i dw^i, \\ \bar{\omega}^\alpha = dx^\alpha, \end{cases}$$

where

$$(4.6)\qquad \bar{f}^i = a^i e^{\Sigma_\alpha c_\alpha^i x^\alpha}$$

for some non-zero constants a^i.

The first structural equations of $\bar{G}$ are

$$(4.7)\qquad \begin{cases} d\bar{\omega}^i = -c_\alpha^i \bar{\omega}^i \wedge \bar{\omega}^\alpha, \\ d\bar{\omega}^\alpha = 0. \end{cases}$$

Further, let $\bar{g}$ be the left invariant Riemannian metric on $\bar{G}$ given by

$$(4.8)\qquad \bar{g} = \sum_i \bar{\omega}^i \otimes \bar{\omega}^i + \sum_\alpha \bar{\omega}^\alpha \otimes \bar{\omega}^\alpha.$$

Since (4.7) are, mutatis mutandis, exactly the same as the first structural equations of the locally homogeneous metric g', defined at the end of Section 3, both metrics have the "same" connection forms. Namely, we have

$$(4.9)\qquad \bar{\omega}_\beta^\alpha = 0, \qquad \bar{\omega}_\beta^i = c_\beta^i \bar{\omega}^i, \qquad \bar{\omega}_j^i = 0$$

(compare with (3.7) where $A_i = 0$). Hence, the Riemannian curvature tensor $\bar{R}$ of the metric $\bar{g}$ is given by

$$(4.10)\qquad \begin{aligned} \bar{R} = & -2\sum_{i,j} \langle c^i, c^j \rangle \bar{\omega}^i \wedge \bar{\omega}^j \otimes \bar{\omega}^i \wedge \bar{\omega}^j \\ & -4\sum_{i,\alpha,\beta} c_\alpha^i c_\beta^i \bar{\omega}^i \wedge \bar{\omega}^\alpha \otimes \bar{\omega}^i \wedge \bar{\omega}^\beta. \end{aligned}$$

Therefore, $(\bar{G}, \bar{g})$ is the desired *model space* for all the metrics introduced in Proposition 2.1 with the corresponding constants c_α^i.

Note that the spaces $(\boldsymbol{R}^{p+q}, g')$ (see the end of Section 3) and $(\bar{G}, \bar{g})$ are *locally isometric* since they have the "same" connection forms. Moreover, if the functions $a^i(w^i)$ are bounded from below by positive constants

$$(4.11)\qquad (a^i(w^i))^2 \geqq (a^i)^2 > 0, \qquad 1 \leqq i \leqq p,$$

then also the metric g' is *complete*. In fact, the intrinsic distance corresponding to g' is bounded from below by that of $\bar{g}$ (for more details, see for example [7]). In that case, $(\boldsymbol{R}^{p+q}, g')$ and $(\bar{G}, \bar{g})$ are *globally isometric*.

Further, the covariant derivatives of the forms $\bar{\omega}^i$ and $\bar{\omega}^\alpha$ with respect to the Levi Civita connection $\bar{D}$ of $\bar{g}$ are given by

$$(4.12)\qquad \begin{cases} \bar{D}_X\bar{\omega}^i = -\bar{\omega}^i(X)\sum c_\alpha^i\bar{\omega}^\alpha, \\ \bar{D}_X\bar{\omega}^\alpha = \sum_j c_\alpha^j\bar{\omega}^j(X)\bar{\omega}^j. \end{cases}$$

Using (4.12) and (4.10), a straightforward computation yields the following expression for the covariant derivative of the curvature tensor:

$$(4.13)\qquad \bar{D}\bar{R} = 4\sum_{i,j,\beta}\langle c^i, c^j\rangle(c_\beta^i - c_\beta^j)\bar{\omega}^i\otimes\{\bar{\omega}^\beta\wedge\bar{\omega}^j\otimes\bar{\omega}^i\wedge\bar{\omega}^j + \bar{\omega}^i\wedge\bar{\omega}^j\otimes\bar{\omega}^\beta\wedge\bar{\omega}^j\}.$$

Because $\langle c^i, c^j\rangle \neq 0$ for all i, j according to our assumption (see Section 2), we obtain

PROPOSITION 4.1. *If the vectors c^i are not all equal, then the model space $(\bar{G}, \bar{g})$ is not symmetric.*

REMARK. (4.3) also implies the following general statement: *A model space $(\bar{G}, \bar{g})$ is symmetric if and only if the following condition holds: for each pair (i, j) with $i<j$ we have either $\langle c^i, c^j\rangle=0$ or $c^i=c^j$.*

A more algebraic method to prove this result is to show that $(\bar{G}, \bar{g})$ is locally symmetric if and only if

$$(4.14)\qquad R_{XY}\cdot R = 0$$

for all X, Y. In fact, it has been proved in [**15**] that *a homogeneous semi-symmetric space is always locally symmetric.*

Next, we consider the possible irreducibility of a model space $(\bar{G}, \bar{g})$ (under the same hypothesis about $\langle c^i, c^j\rangle$).

PROPOSITION 4.2. *Let $p\geqq q$. If $\mathrm{rank}(c_\alpha^i)=q$, then $(\bar{G}, \bar{g})$ is irreducible.*

PROOF. We consider the Riemann curvature tensor $\bar{R}$ as a two-form with values in the Lie algebra $\mathfrak{so}(p+q)$. Then we have

$$(4.15)\qquad \bar{R} = -2\sum_{i,j}\langle c^i, c^j\rangle\bar{\omega}^i\wedge\bar{\omega}^j\otimes E_{ij} - 4\sum_{i,\alpha,\beta}c_\alpha^i c_\beta^i\bar{\omega}^i\wedge\bar{\omega}^\alpha\otimes E_{i\beta},$$

where E_{AB}, $1\leqq A<B<p+q$, are the skew-symmetric matrices all of whose entries are zero except those of place (A, B), and (B, A) respectively, which are $+1$, and -1 respectively. Further, recall that

$$(4.16)\qquad [E_{AB}, E_{BC}] = E_{AC}$$

for $A\neq B\neq C\neq A$.

First we see from (4.15) that E_{ij} belongs to the holonomy algebra $\mathfrak{hol}(\bar{G}, \bar{g})$

for all pairs (i, j), $i \neq j$, because $\langle c^i, c^j \rangle \neq 0$. Further, we also derive from (4.15) that

$$c^i_\alpha \sum_\beta c^i_\beta E_{i\beta} \in \mathfrak{hol}(\bar{G}, \bar{g}) \tag{4.17}$$

for every i and α. Since for each i there exists at least one index α for which $c^i_\alpha \neq 0$ (because $c^i \neq 0$ for $1 \leqq i \leqq p$), (4.17) yields

$$\sum_\beta c^i_\beta E_{i\beta} \in \mathfrak{hol}(\bar{G}, \bar{g}), \qquad 1 \leqq i \leqq p. \tag{4.18}$$

Then we also have

$$[E_{ji}, \sum_\beta c^i_\beta E_{i\beta}] = \sum_\beta c^i_\beta E_{j\beta} \in \mathfrak{hol}(\bar{G}, \bar{g}) \tag{4.19}$$

for all i, j. By a renumeration of our basis of $\boldsymbol{R}^p$, we can achieve that $\det(c^i_\beta) \neq 0$ for $i=1, 2, \cdots, q$; $\beta=p+1, \cdots, p+q$. Then (4.19) yield

$$E_{j\beta} \in \mathfrak{hol}(\bar{G}, \bar{g}) \tag{4.20}$$

for all j and β. Finally, using (4.16) and (4.20), we see that the operators $E_{\alpha\beta}$ also belong to $\mathfrak{hol}(\bar{G}, \bar{g})$ and hence

$$\mathfrak{hol}(\bar{G}, \bar{g}) = \mathfrak{so}(p+q).$$

This implies that $(\bar{G}, \bar{g})$ is irreducible.

5. Summary.

Before we return to the study of the curvature homogeneous examples we summarize the main result we obtained up to now.

THEOREM 5.1. *Let g be the metric on $\boldsymbol{R}^{p+q}(w, x)$ given by*

$$g = \sum_i \omega^i \otimes \omega^i + \sum_\alpha \omega^\alpha \otimes \omega^\alpha \tag{5.1}$$

where

$$\begin{cases} \omega^i = f^i \theta^i = f^i dw^i, \\ \omega^\alpha = \theta^\alpha = dx^\alpha + A^\alpha_{i\beta} x^\beta dw^i, \end{cases} \tag{5.2}$$

$i=1, \cdots, p$, $\alpha, \beta=p+1, \cdots, p+q$, $p \geqq 2$ *and*

$$f^i = a^i(w^i) e^{\Sigma_\alpha c^i_\alpha x^\alpha}, \qquad c^i_\alpha = \text{const}, \ a^i(w^i) > 0. \tag{5.3}$$

Further, let the operators A_i: $\boldsymbol{R}^q \to \boldsymbol{R}^q$ given by

$$A_i(\xi_{p+1}, \cdots, \xi_{p+q}) = (\sum_\alpha \xi_\alpha A^\alpha_{i p+1}, \cdots, \sum_\alpha \xi_\alpha A^\alpha_{i p+q}) \tag{5.4}$$

be skew-symmetric with respect to the standard metric of $\boldsymbol{R}^q$ *and suppose they all commute. If the vectors*

$$c^i = (c^i_{p+1}, \cdots, c^i_{p+q}) \tag{5.5}$$

satisfy the following conditions:

(i) c^j *belongs to the kernel of* A_i *for* $i \neq j$;

(ii) *there exists at least one index* i *such that* $A_i(c^i) \neq 0$;

(iii) *the inner products* $\langle c^i, c^j \rangle = \sum_\alpha c^i_\alpha c^j_\alpha$ *are different from zero,*

then $(\boldsymbol{R}^{p+q}, g)$ *is a non-homogeneous curvature homogeneous Riemannian manifold with a non-symmetric homogeneous model. The corresponding model space is one of the homogeneous spaces* $(\bar{G}, \bar{g})$ *treated in Section* 4.

REMARK. We see easily that we cannot combine the conditions in Theorem 5.1 with those of Proposition 4.2 which would assure the irreducibility of a model space $(\bar{G}, \bar{g})$ (and of the metric g too). In fact, if $p \geqq q$, $\operatorname{rank}(c^i_\alpha) = q$ and if (i) of Theorem 5.1 holds, then we always have $\dim \ker A_i \geqq q-1$ and hence all the skew-symmetric operators must vanish because $\operatorname{rank} A_i = q - \dim \ker A_i \leqq 1$, and $\operatorname{rank} A_i$ is even. Therefore, $(\boldsymbol{R}^{p+q}, g)$ would be locally homogeneous as showed before. Thus we have to consider *reducible* model spaces as we will do in the next sections.

6. Irreducibility and completeness for a special class of examples.

In this section we shall construct a family of non-homogeneous examples with a model space $(\bar{G}_1, \bar{g}_1)$ which is a Riemannian product of the form $(\bar{G}, \bar{g}) \times \boldsymbol{R}$.

THEOREM 6.1. *Let* $(\bar{G} = \boldsymbol{R}^{q-1} \ltimes \boldsymbol{R}^p, \bar{g})$ *be a model space as in Section* 4 *where* $p \geqq q-1 \geqq 2$, $\operatorname{rank}(c^i_\beta) = q-1$ *and* $\langle c^i, c^j \rangle \neq 0$ *for all* $i, j = 1, \cdots, p$. *Further, suppose that for some* $j \in \{1, \cdots, p\}$

$$\operatorname{span}(c^1, \cdots, \hat{c}^j, \cdots, c^p) \subsetneqq \operatorname{span}(c^1, \cdots, c^p) \quad \text{in } \boldsymbol{R}^{q-1}. \tag{6.1}$$

Then there is a non-empty family of non-homogeneous spaces $(\boldsymbol{R}^{p+q}, g)$ *with the same curvature tensor as* $(\bar{G}, \bar{g}) \times (\boldsymbol{R}, \langle\, ,\rangle_{\mathrm{can}})$ *(and thus curvature homogeneous). All spaces of this family are irreducible and complete.*

PROOF. First, note that our conditions imply that $(\bar{G}, \bar{g})$ is irreducible and non-symmetric. On the other hand we can see that the new model space $(\bar{G}, \bar{g}) \times (\boldsymbol{R}, \langle\,\rangle_{\mathrm{can}})$ is isometric to a model space $(\bar{G}_1 = \boldsymbol{R}^q \ltimes \boldsymbol{R}^p, \bar{g}_1)$ from Section 4 such that $c^i \in \boldsymbol{R}^q$ with $c^i_{p+q} = 0$ for $i = 1, \cdots, p$. Thus, we shall construct non-homogeneous examples $(\boldsymbol{R}^{p+q}, g)$ in the class given in Theorem 5.1 with the additional conditions $c^i_{p+q} = 0$.

A. Construction of the non-homogeneous family.

After a renumeration of the vectors $c^1, \cdots, c^p \in \boldsymbol{R}^q$ we can suppose that

$$\det(c^i_\beta) \neq 0 \qquad \text{for} \begin{cases} i = 1, \cdots, q-1, \\ \beta = p+1, \cdots, p+q-1 \end{cases}$$

and (6.1) is satisfied for $j=1$. We define an orthonormal basis $(e_1, \cdots, e_q)$ of $\boldsymbol{R}^q$ as follows: $(e_2, \cdots, e_{q-1})$ is an orthonormal basis of $\mathrm{span}(c^2, \cdots, c^{q-1}) = \mathrm{span}(c^2, \cdots, c^p)$, e_1 is a unit vector in $\mathrm{span}(c^1, \cdots, c^{q-1})$ orthogonal to $\mathrm{span}(c^2, \cdots, c^{q-1})$ and $e_q = {}^t(0, \cdots, 0, 1)$. We define a skew-symmetric operator A_1 on $\boldsymbol{R}^q$ by the formulas

$$A_1(e_1) = e_q, \qquad A_1(e_q) = -e_1, \qquad A_1(e_j) = 0 \quad \text{otherwise.}$$

We see that $A_1(c^1) \neq 0$ and $A_1(c^i) = 0$ for $i = 2, \cdots, p$.

Further, we define

$$A_2 = \cdots = A_p = 0.$$

Theorem 5.1 implies that the space $(\boldsymbol{R}^{p+q}, g)$ given by (5.1)-(5.4) is a curvature homogeneous space with the model space $(\bar{G}, \bar{g}) \times (\boldsymbol{R}, \langle\,,\rangle_{can})$. Proposition 3.1 implies that $(\boldsymbol{R}^{p+q}, g)$ is not locally homogeneous.

B. Irreducibility.

From the proof of Proposition 4.2 it follows that the holonomy algebra of $(\boldsymbol{R}^{p+q}, g)$ contains all E_{ij}, $E_{i\beta}$, $E_{\alpha\beta}$ for $i, j = 1, \cdots, p$ and $\alpha, \beta = p+1, \cdots, p+q-1$.

Further, from (3.3) and (3.9) we obtain easily

$$(D_{E_1}R)(E_1, E_\beta) = 2(f^1)^{-1}\{c^1_\beta \sum_{\gamma,\delta} c^1_\gamma A^\gamma_{1\delta} E_{1\delta} + \sum_{\gamma,\delta} c^1_\gamma c^1_\delta A^\gamma_{1\beta} E_{1\delta}\} \tag{6.2}$$

where the coefficient of E_{1p+q} is non-zero for some β because $c^1 \neq 0$ and

$$\sum_\alpha A^\alpha_{1p+q} c^1_\alpha = \langle A_1(c^1), e_q\rangle = -\langle A_1(e_q), c^1\rangle = \langle e_1, c^1\rangle \neq 0. \tag{6.3}$$

Hence, $E_{1p+q} \in \mathfrak{hol}(\boldsymbol{R}^{p+q}, g)$ and then we derive that

$$E_{AB} \in \mathfrak{hol}(\boldsymbol{R}^{p+q}, g)$$

for all $A < B$, $A, B = 1, \cdots, p+q$. This yields the irreducibility.

C. Completeness.

In the matrix notation of [7] we can write

(6.4)
$$g=\sum_{i=1}^{p}(f^{i})^{2}dw^{i}\otimes dw^{i}+{}^{t}\theta\otimes\theta,$$

where

$$\theta=dX+A_{1}Xdw^{1},\qquad X=\begin{pmatrix}x^{p+1}\\ \vdots\\ x^{p+q}\end{pmatrix}$$

and $A_1=(A^{\alpha}_{1\beta})$, α, $\beta=p+1,\cdots,p+q$.

Now, put

$$U=e^{w^{1}A_{1}}X,\qquad U=\begin{pmatrix}u^{p+1}\\ \vdots\\ u^{p+q}\end{pmatrix}.$$

Then

(6.5)
$$dU=e^{w^{1}A_{1}}\theta,\qquad {}^{t}dU\otimes dU={}^{t}\theta\otimes\theta$$

and hence,

(6.6)
$$g=\sum_{i=1}^{p}(f^{i})^{2}dw^{i}\otimes dw^{i}+\sum_{\alpha=p+1}^{p+q}du^{\alpha}\otimes du^{\alpha}.$$

Here (5.3) and $X=e^{-w^{1}A_{1}}U$ imply

(6.7)
$$f^{i}=a^{i}(w^{i})e^{\Sigma c^{i}_{\alpha}P^{\alpha}_{\beta}(w^{1})u^{\beta}},$$

where $(P^{\alpha}_{\beta}(w^{1}))=\exp(-w^{1}A_{1})$ is an orthogonal matrix function and hence $|P^{\alpha}_{\beta}(w^{1})|\leq 1$ for all α, β.

Now, in the same way as in [**7**, Section 11] (see also [**12**, Prop. 2.1]) we see first that the metric

(6.8)
$$g_{1}=(f^{1})^{2}dw^{1}\otimes dw^{1}+\sum_{\alpha=p+1}^{p+q}du^{\alpha}\otimes du^{\alpha}$$

(which is that of a *generalized warped product*) is complete on $\boldsymbol{R}^{p+1}$ if $0<a<a_{1}(w^{1})<b$ holds. But $(\boldsymbol{R}^{p+q}, g)$ is then the usual *iterated* warped product of $(\boldsymbol{R}^{p+1}, g_{1})$ and $(\boldsymbol{R}, \langle\ ,\ \rangle_{can})$ (made $q-1$ times). Hence, $(\boldsymbol{R}^{p+q}, g)$ is complete [**3**]. Thus, the inequality of the form

$$0<a<a^{1}(w^{1})<b$$

is the only limitation for our family of examples to ensure the completeness.

7. The isometry classes of the special examples.

We consider now the examples constructed in Section 6 and study their isometry classes. Before doing this, we recall that following (3.5), the Ricci tensor ρ may be written as

$$\rho=\sum_{i=1}^{p} b_i\omega^i\otimes\omega^i+\sum_{\alpha,\beta=p+1}^{p+q} b_{\alpha\beta}\omega^\alpha\otimes\omega^\beta$$

where

$$\begin{cases} b_i=-\langle c^i, \sum_{k=1}^{p} c^k\rangle, & i=1, \cdots, p, \\ b_{\alpha\beta}=-\langle c_\alpha, c_\beta\rangle, & \alpha, \beta=p+1, \cdots, p+q. \end{cases} \tag{7.2}$$

The quadratic form $Q=\sum b_{\alpha\beta}\omega^\alpha\otimes\omega^\beta$ can be transformed into its diagonal form

$$\sum_{\alpha=p+1}^{p+q}\lambda_\alpha\eta^\alpha\otimes\eta^\alpha$$

where $\lambda_{p+q}=0$. To do this, we only have to take in (2.3) a new orthonormal (sub)basis

$$\tilde{E}_\alpha=\sum p_\alpha^\beta E_\beta, \qquad \alpha=p+1, \cdots, p+q,$$

consisting of eigenvectors of Q. This change is metric preserving. Hence, from now on, we may assume

$$\rho=\sum_{i=1}^{p} b_i\omega^i\otimes\omega^i+\sum_{\alpha=p+1}^{p+q-1}\lambda_\alpha\omega^\alpha\otimes\omega^\alpha. \tag{7.3}$$

Now, we call our model space to be *generic* if all scalar products $\langle c^i, \sum_k c^k\rangle$, $i=1, \cdots, p$, and all eigenvalues of the matrix $(\langle c_\alpha, c_\beta\rangle)$, $\alpha, \beta=p+1, \cdots, p+q-1$, are mutually different and non-zero.

Then we have

PROPOSITION 7.1. *For a generic model space, the isometry classes of the family constructed in Section 6 depend on one arbitrary function of one variable.*

PROOF. We first fix a generic model space and compare two non-homogeneous examples $(\boldsymbol{R}^{p+q}, g)$, $(\boldsymbol{R}^{p+q}, \tilde{g})$ with the same operator $A_1=(A_{1\alpha}^\beta)$. Next, let φ be an isometry between both spaces. Then $\varphi^*\tilde{\rho}=\rho$ and this yields at once

$$\varphi^*\tilde{\omega}^i=\pm\omega^i, \qquad \varphi^*\tilde{\omega}^\alpha=\pm\omega^\alpha. \tag{7.4}$$

Comparing now $\|\tilde{D}\tilde{\rho}\|\circ\varphi=\|D\rho\|$, we get from (3.12)

$$\tilde{f}^1\circ\varphi=f^1$$

(note that $D\rho$ involves only f^1). In particular we get

$$\varphi^*\tilde{\omega}^1=\varepsilon\omega^1$$

and hence

$$d\tilde{w}^1=\varepsilon dw^1, \qquad \varepsilon=\pm 1.$$

From (7.4) we then also get

$$d\tilde{x}^\alpha+\varepsilon\sum A^\alpha_{1\beta}\tilde{x}^\beta dw^1=\varepsilon_\alpha(dx^\alpha+\sum A^\alpha_{1\beta}x^\beta dw^1), \tag{7.5}$$

with $\varepsilon_\alpha=\pm 1$ and for $\alpha=p+1, \cdots, p+q$. This yields

$$d(\tilde{x}^\alpha-\varepsilon_\alpha x^\alpha)=-\varepsilon\sum_\beta A^\alpha_{1\beta}(\tilde{x}^\beta-\varepsilon\varepsilon_\alpha x^\beta)dw^1 \tag{7.6}$$

and hence

$$\tilde{x}^\alpha-\varepsilon_\alpha x^\alpha=\varphi_\alpha(w^1), \tag{7.7}$$

$$\sum_\beta A^\alpha_{1\beta}(\tilde{x}^\beta-\varepsilon\varepsilon_\alpha x^\beta)=-\varepsilon\varphi'_\alpha(w^1), \tag{7.8}$$

$\alpha=p+1, \cdots, p+q$.

We now prove that $A^\alpha_{1\beta}\neq 0$ always implies $\varepsilon_\beta=\varepsilon\varepsilon_\alpha$. Indeed, fix an index α and let K and L, respectively, be the subsets of indices in $\{p+1, \cdots, p+q\}$ for which $\varepsilon_\beta=\varepsilon\varepsilon_\alpha$, or $\varepsilon_\beta=-\varepsilon\varepsilon_\alpha$ respectively. Then (7.8) together with (7.7) yields

$$\sum_{\beta\in L} A^\alpha_{1\beta}(\tilde{x}^\beta+\varepsilon_\beta x^\beta)=-\varepsilon\varphi'_\alpha(w^1)-\sum_{\gamma\in K} A^\alpha_{1\gamma}\varphi_\gamma(w^1), \tag{7.9}$$

and substituting (7.7) once more, we get

$$2\sum_{\beta\in L} A^\alpha_{1\beta}\varepsilon_\beta x^\beta=-\varepsilon\varphi'_\alpha(w^1)-\sum_{\gamma=p+1}^{p+q} A^\alpha_{1\gamma}\varphi_\gamma(w^1). \tag{7.10}$$

This should be satisfied on the whole of $\boldsymbol{R}^{p+q}$. But, if some $A^\alpha_{1\beta}$ is non-zero for $\beta\in L$, we get a relation between the independent variables $x^{p+1}, \cdots, x^{p+q}, w^1$, which is a contradiction.

Then (7.8) may be rewritten in the form

$$\varphi'_\alpha=-\varepsilon\sum_\beta A^\alpha_{1\beta}\varphi_\beta, \qquad \alpha=p+1, \cdots, p+q. \tag{7.11}$$

Putting

$$Z=\begin{pmatrix}\varphi_{p+1}(w^1)\\ \vdots\\ \varphi_{p+q}(w^1)\end{pmatrix}$$

we get a matrix differential equation

$$Z'=-\varepsilon A_1 Z \tag{7.12}$$

with the general solution

$$Z=e^{-\varepsilon A_1 w^1}Z_0, \tag{7.13}$$

that is,

$$\varphi_\alpha(w^1)=\sum_\beta P^\alpha_\beta(w^1)z^\beta_0 \tag{7.14}$$

where $(P^\alpha_\beta(w^1))$ is an orthogonal matrix and z^β_0 are real numbers. This yields

$$\tilde{x}^{\alpha} = \varepsilon_{\alpha} x^{\alpha} + \sum_{\beta} P^{\alpha}_{\beta}(w^1) z^{\beta}_{0}, \qquad \alpha = p+1, \cdots, p+q. \tag{7.15}$$

Then, $\tilde{f}^1 \circ \varphi = f^1$ and $d\tilde{w}^1 = \varepsilon dw^1$ imply

$$\tilde{a}_1(\varepsilon w^1 + r) e^{\Sigma c^1_{\alpha} \tilde{x}^{\alpha}} = a_1(w^1) e^{\Sigma c^1_{\alpha} x^{\alpha}}$$

which means

$$\tilde{a}_1(\varepsilon w^1 + r) e^{\Sigma c^1_{\alpha} \varepsilon_{\alpha} x^{\alpha}} e^{\Sigma c^1_{\alpha} P^{\alpha}_{\beta}(w^1) z^{\beta}_{0}} = a_1(w^1) e^{\Sigma c^1_{\alpha} x^{\alpha}}, \tag{7.16}$$

where r is a constant. Hence $\varepsilon_{\alpha} = 1$ for $\alpha = p+1, \cdots, p+q$ whenever $c^1_{\alpha} \neq 0$, and

$$\tilde{a}_1(\varepsilon w^1 + r) = a_1(w^1) e^{-\Sigma c^1_{\alpha} P^{\alpha}_{\beta}(w^1) z^{\beta}_{0}}. \tag{7.17}$$

Because the matrix $(A^{\alpha}_{1\beta})$ was fixed in advance, the matrix $(P^{\alpha}_{\beta}(w^1))$ is also well-defined (up to taking the transpose). We see that the function $\tilde{a}_1(w^1)$ belongs to a family containing the function $a_1(w^1)$ and depending on $q+1$ arbitrary parameters and the sign ε.

We conclude that, roughly speaking, the isometry classes of our special non-homogeneous examples depend on one arbitrary function $a_1(w^1)$ of one variable.

REMARK. The isometry classes do not depend on other arbitrary functions because, for $i=2, \cdots, p$, we can always make the reduction of $f^2, \cdots, f^p$ to fixed functions when introducing the new variables $\hat{w}^i = \hat{w}^i(w^i)$ by

$$d\hat{w}^i = a^i(w^i) dw^i, \qquad i = 2, \cdots, p.$$

Then (5.3) is reduced to $f^i = e^{\Sigma c^i_{\alpha} x^{\alpha}}$ for $i > 1$.

8. Appendix: The examples of Yamato and Tsukada.

In this final section we give some additional results about the curvature homogeneous examples of Yamato and Tsukada. In particular we focus on the existence problem of a homogeneous model space. In this context we note that up to now and to our knowledge there was no example known of a curvature homogeneous Riemannian manifold whose curvature tensor is not that one of a homogeneous Riemannian space. We will show that for Yamato's examples there exist model spaces but, as we will see, the example of Tsukada has no homogeneous model at all.

A. The examples of Yamato.

K. Yamato constructed the first example of a three-dimensional non-homogeneous curvature homogeneous Riemannian manifold with three distinct (constant) principal Ricci curvatures ρ_1, ρ_2, ρ_3 [21]. Not all possible triples (ρ_1, ρ_2, ρ_3)

are admitted. Namely, if we put

$$A=\frac{\rho_1+\rho_2-\rho_3}{2}, \qquad B=\frac{\rho_1-\rho_3}{\rho_3-\rho_2}, \qquad C=-\frac{(\rho_1+\rho_2)(\rho_3-\rho_2)^2}{(\rho_2-\rho_1)^2}, \tag{8.1}$$

we have the restrictions

$$A>0, \qquad C>0, \qquad A+BC>0. \tag{8.2}$$

Yamato's explicit example is the complete space $(\boldsymbol{R}^3(x, y, z), \tilde{g})$ where $\tilde{g}$ is given by

$$\tilde{g}=\sum_{i=1}^{3}\omega^i\otimes\omega^i$$

with

$$\omega^1=dx-\{x\phi(z)+yf(z)\}dz,$$
$$\omega^2=dy-\{xg(z)+By\phi(z)\}dz,$$
$$\omega^3=dz.$$

Here $\phi(z)$ is a solution of the differential equation

$$\frac{d\phi}{dz}+(1+B)(\phi^2-C)=0$$

satisfying $\phi^2<C$ and f, g are the functions chosen so that

$$f^2-g^2=2\{\rho_1+(1+B)C\},$$
$$(f+g)^2=4(A+B\phi^2),$$
$$f+g>0.$$

For these manifolds we have

PROPOSITION 8.1. *Let $\tilde{g}$ be a Riemannian metric on $\boldsymbol{R}^3$ of Yamato's type with principal Ricci curvatures ρ_1, ρ_2, ρ_3. Then $\rho_3<0$, $\rho_1\rho_2<0$ and $\tilde{g}$ has the same curvature as a three-dimensional unimodular Lie group endowed with a suitable left invariant Riemannian metric g'.*

PROOF. From (8.1) we obtain, assuming $B\neq 1$,

$$\rho_1=(B+1)(B-1)^{-1}\{2A+C(B+1)\}, \tag{8.3}$$

$$\rho_2=-(B+1)(B-1)^{-1}\{2A+BC(B+1)\}, \tag{8.4}$$

$$\rho_3=-(B+1)^2C-2A. \tag{8.5}$$

Note that $B\neq -1$ since the ρ_i are distinct. Moreover, $B\neq 1$ always holds since otherwise $A=-C=1/2\rho_3$ and $A+BC=0$ in contrast to the conditions (8.2).

From (8.5) and (8.2) we get $\rho_3<0$. On the other hand (8.2), (8.3) and (8.4) yield $\rho_1\rho_2<0$. Therefore the signature of the Ricci tensor is $(+, -, -)$.

Now, from [**9**] we know that a unimodular three-dimensional Lie group G endowed with a left invariant metric g' admits an orthonormal frame (e_1, e_2, e_3) whose elements are left invariant vector fields satisfying

$$[e_1, e_2] = \lambda_3 e_3, \qquad [e_2, e_3] = \lambda_1 e_1, \qquad [e_3, e_1] = \lambda_2 e_2, \tag{8.6}$$

where the λ_i, $i=1, 2, 3$ are constants. The corresponding principal Ricci curvatures are given by

$$\rho_{11} = 2\mu_2\mu_3, \qquad \rho_{22} = 2\mu_3\mu_1, \qquad \rho_{33} = 2\mu_1\mu_2 \tag{8.7}$$

where

$$2\mu_i = (\lambda_1+\lambda_2+\lambda_3)-\lambda_i, \tag{8.8}$$

$i=1, 2, 3$. Solving (8.7) for μ_i, $i=1, 2, 3$, we get

$$2\mu_1^2 = \frac{\rho_{22}\rho_{33}}{\rho_{11}}, \qquad 2\mu_2^2 = \frac{\rho_{33}\rho_{11}}{\rho_{22}}, \qquad 2\mu_3^2 = \frac{\rho_{11}\rho_{22}}{\rho_{33}}. \tag{8.9}$$

It is clear that these equations always admit a solution if $\rho_{ii}=\rho_i$, $i=1, 2, 3$ and $\rho_3<0$, $\rho_1\rho_2<0$. This gives the desired metric g'.

B. The example of Tsukada.

Finally, we turn to Tsukada's four-dimensional example mentioned in the introduction. In case that it admits a homogeneous model (M', g'), the universal covering of it must be a symmetric space or a Riemannian group space since every four-dimensional simply connected homogeneous Riemannian manifold is a symmetric space or a Lie group with a left invariant metric on it (see, for example, [**1**]). The case of a symmetric model can be excluded since it follows easily from Tsukada's explicit formulas that the Riemann curvature tensor R does not satisfy $R_{XY}\cdot R=0$. So we are left with the possible existence of a Lie group as model space. We shall now prove that such a model space cannot exist.

First of all we recall the basic property of the curvature of Tsukada's example (M, g): in each tangent space $T_m M$ there exists an orthonormal basis for which the components R_{ijkl} of the Riemann curvature tensor are given by

$$\begin{cases} R_{1212} = 3, \\ R_{ijij} = -1 \quad \text{for each couple } (i, j) \text{ such that } 1\leqq i<j\leqq 4,\ (i, j)\neq(1, 2), \\ R_{ijkl} = 0 \quad \text{whenever at least three indices are different.} \end{cases} \tag{8.10}$$

Now, let $\mathfrak{g}$ be the Lie algebra of the supposed model space $(M', g')=(G, g')$

and let (e_1, e_2, e_3, e_4) be an orthonormal basis of $(\mathfrak{g}, \langle , \rangle)$ for which the curvature components of (G, g') satisfy (8.10). Let $\omega^1, \cdots, \omega^4$ be the invariant one-forms which are dual to $e_1, \cdots, e_4$, i.e., $g'=\sum \omega^i \otimes \omega^i$. The structural equations of Cartan are given by

$$d\omega^i + \sum_m \omega^i_m \wedge \omega^m = 0, \qquad \omega^i_j + \omega^j_i = 0, \tag{8.11}$$

$$d\omega^i_j + \sum_m \omega^i_m \wedge \omega^m_j = \frac{1}{2} \sum_{k,l} R_{ijkl} \omega^k \wedge \omega^l \tag{8.12}$$

where $i, j, k, l \in \{1, 2, 3, 4\}$. Further, we have

$$\omega^i_j = \sum_k \Gamma^i_{jk} \omega^k, \qquad i, j = 1, 2, 3, 4 \tag{8.18}$$

where Γ^i_{jk} are constant coefficients (the components of the Riemannian connection).

By differentiation of (8.12) we obtain

$$\sum_{k,l,m} (R_{mjkl}\omega^m_i + R_{imkl}\omega^m_j + R_{ijml}\omega^m_k + R_{ijkm}\omega^m_l) \wedge \omega^k \wedge \omega^l = 0,$$

and substituting from (8.13) we get the (reduced) second Bianchi identity

$$\mathfrak{S}_{h,k,l}[\sum_m (R_{mjkl}\Gamma^m_{ih} + R_{imkl}\Gamma^m_{jh} + R_{ijml}\Gamma^m_{kh} + R_{ijkm}\Gamma^m_{lh})] = 0, \tag{8.14}$$

$$1 \leqq h < k < l \leqq 4, \qquad 1 \leqq i, \ j \leqq 4.$$

By a routine calculation, using the conditions (8.10) and the skew-symmetry $\Gamma^j_{ik} + \Gamma^i_{jk} = 0$, we obtain the following relations for the connection coefficients:

$$\begin{aligned} &\Gamma^i_{jk} = 0 \quad \text{whenever } 1 \leqq i \leqq 2 \text{ and } 3 \leqq j, \ k \leqq 4, \\ &\Gamma^3_{11} + \Gamma^3_{22} = 0, \\ &\Gamma^4_{11} + \Gamma^4_{22} = 0. \end{aligned} \tag{8.15}$$

Hence, (8.13) takes the form

$$\begin{aligned} \omega^1_2 &= a\omega^1 + b\omega^2 + c\omega^3 + f\omega^4, \\ \omega^3_4 &= \bar{a}\omega^1 + \bar{b}\omega_2 + \bar{c}\omega^3 + \bar{f}\omega^4, \\ \omega^1_3 &= \alpha\omega^1 + \beta\omega^2, \\ \omega^1_4 &= \bar{\alpha}\omega^1 + \bar{\beta}\omega^2, \\ \omega^2_3 &= \gamma\omega^1 - \alpha\omega^2, \\ \omega^2_4 &= \delta\omega^1 - \bar{\alpha}\omega^2, \end{aligned} \tag{8.16}$$

with constant coefficients.

We shall now rewrite (8.12) in the more explicit form using (8.13):

(8.17) $$\sum_m \{\Gamma^i_{jm}(\Gamma^m_{kl}-\Gamma^m_{lk})+(\Gamma^i_{mk}\Gamma^m_{jl}-\Gamma^i_{ml}\Gamma^m_{jk})\} = R_{ijkl}.$$

If we substitute here the reduced forms (8.16) we obtain a system of 36 quadratic equations for the 14 unknowns $a, b, c, f, \bar{a}, \bar{b}, \bar{c}, \bar{f}, \alpha, \beta, \gamma, \delta, \bar{\alpha}, \bar{\beta}$. For the sake of brevity, we shall denote by $E(i_0, j_0, k_0, l_0)$ the equation of (8.17) corresponding to the fixed choice of indices $(i, j, k, l)=(i_0, j_0, k_0, l_0)$, $1\leqq i_0, j_0, k_0, l_0\leqq 4$. We will show that this system of algebraic equations is inconsistent.

First, the equations $E(1, 3, 2, 3)$ and $E(2, 3, 1, 3)$ give

(8.18) $$2\alpha c-\bar{\beta}\bar{c} = 0, \qquad 2\alpha c-\delta\bar{c} = 0$$

and hence $(\bar{\beta}-\delta)\bar{c}=0$. Next, the equations $E(1, 4, 2, 4)$ and $E(2, 4, 1, 4)$ give

(8.19) $$2\bar{\alpha}f+\beta\bar{f} = 0, \qquad 2\bar{\alpha}f+\gamma\bar{f} = 0$$

and so $(\beta-\gamma)\bar{f}=0$. Since the equation $E(3, 4, 3, 4)$ means that $\bar{c}^2+\bar{f}^2=1$, there are just three possible cases:

i) $\bar{c} = 0$, $\bar{f}^2 = 1$, $\beta = \gamma$;

ii) $\bar{c}^2 = 1$, $\bar{f} = 0$, $\bar{\beta} = \delta$;

iii) $\bar{f}\bar{c} \neq 0$, $\beta = \gamma$, $\bar{\beta} = \delta$.

We treat these cases separately:

—the case i)

The equation $E(1, 2, 3, 4)$ means that $c\bar{c}+f\bar{f}=0$, i.e., $f=0$. Then (8.19) implies $\beta=\gamma=0$. The equation $E(2, 3, 2, 3)$ then yields $\alpha^2=1$ and (8.18) implies $c=0$. From $E(1, 4, 2, 3)$ we get $\bar{\beta}=0$. Comparing the equations $E(1, 4, 1, 4)$ and $E(2, 4, 2, 4)$ we get $\alpha\bar{f}=0$, a contradiction.

—the case ii)

From the equation $c\bar{c}+f\bar{f}=0$ we get $c=0$ and from (8.18) we obtain $\bar{\beta}=\delta=0$. The equation $E(1, 4, 1, 4)$ then yields $\bar{\alpha}^2=1$ and (8.19) implies $f=0$. Then the equation $E(1, 3, 2, 4)$ implies $\beta=0$. Comparing the equations $E(1, 3, 1, 3)$ and $E(2, 3, 2, 3)$ we get $\bar{\alpha}\bar{c}=0$, a contradiction.

—the case iii)

The equation $E(1, 2, 1, 2)$ implies $\alpha^2+\gamma^2+\bar{\alpha}^2+\delta^2=3+a^2+b^2$. Comparing the equation $E(1, 3, 1, 3)$, $E(2, 3, 2, 3)$ we get $\alpha^2+\gamma^2=1$, and comparing the equations $E(1, 4, 1, 4)$, $E(2, 4, 2, 4)$ we get $\bar{\alpha}^2+\delta^2=1$. Hence $2=3+a^2+b^2$, a contradiction.

This finishes the proof of our claim.

NOTE. The main purpose of [**20**] is the determination of the curvature homogeneous hypersurfaces in a real space form $M^{n+1}(c)$ (of constant sectional curvature c). There, Tsukada proved that such a hypersurface has constant sectional curvature c, or is isoparametric or has type number two at each point. In the last case $n=4$ or $n=3$ if $c<0$, and $n=3$ if $c>0$. Moreover, for $n=4$ (and $c<0$) it is locally isometric to the example discussed above. Therefore, Tsukada obtained a complete description of the curvature homogeneous hypersurfaces of dimension $n \geqq 4$ of real space forms. For $n=3$ the problem is still open. Actually, it is not known if there exist curvature homogeneous hypersurfaces in $S^4(1/r^2)$ or $\boldsymbol{H}^4(-1/r^2)$ which are neither locally homogeneous nor isoparametric. In any case, if such a non-trivial hypersurface exists in $\boldsymbol{H}^4(-1/r^2)$, it cannot be compact. In fact, every curvature homogeneous hypersurface must have constant scalar curvature. Therefore, if $c \leqq 0$, a beautiful result of Ros-Montiel and Korevaar (see [**10**], [**11**], [**5**]) then implies that M^3 is congruent to a geodesic sphere and therefore is locally homogeneous.

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