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Symmetric plane curves with nodes and cusps

Dedicated to Professor Heisuke Hironaka on his 60th birthday

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§1. Introduction.

In [**Z1**], Zariski considered the family of projective curves of degree 6 with 6 cusps on a conic. This family is defined by: $f(X, Y, Z)=f_2(X, Y, Z)^3+f_3(X, Y, Z)^2=0$ where f_i is a homogeneous polynomial of degree i, $i=2, 3$. He showed that the fundamental group $\pi_1(\boldsymbol{P}^2-C)$ is isomorphic to the free product $\boldsymbol{Z}_2*\boldsymbol{Z}_3$ for a generic member of this family. He also proved that the fundamental group of the complement of a curve of degree 6 with 6 cusps which are not on a conic is not isomorphic to $\boldsymbol{Z}_2*\boldsymbol{Z}_3$. In fact, we will show in §5 that this fundamental group is abelian. Zariski also studied a curve of degree 4 with 3 cusps as a degeneration of the first family in [**Z1**] and he claims that the complement of such a curve has a non-commutative finite fundamental group of order 12. We will reprove this assertion (§3 Theorem (3.12)).

The purpose of this note is to construct systematically plane curves with nodes and cusps which are defined by symmetric polynomials $f(x, y)$. A symmetric polynomial $f(x, y)$ can be written as a polynomial $h(u, v)$ where $u=x+y$ and $v=xy$. In this expression, the degree of h in v is half of the original degree and the calculation of the fundamental group becomes comparatively easy. Let $p: \boldsymbol{C}^2 \to \boldsymbol{C}^2$ be the two-fold branched covering defined by $p(x, y)=(u, v)$. The branching locus is the discriminant variety $D=\{u^2-4v=0\}$. Let $C=\{h(u, v)=0\}$ and $\tilde{C}=p^{-1}(C)$. Under a certain condition, the homomorphism $p_*: \pi_1(\boldsymbol{C}^2-\tilde{C}) \to \pi_1(\boldsymbol{C}^2-C)$ is an isomorphism (Theorem (2.3), §2). Symmetric polynomials give enough models for the cuspidal curves with small degree. In fact, we will give examples of symmetric plane curves of the following type and we will compute their fundamental groups.

(1) Symmetric curve of degree 4 with 3 cusps (Theorem (3.12), §3).
(2) Symmetric curve of degree 5, with 4 cusps (Theorem (3.14), §3).
(3) Symmetric curve of degree 6, with conical 6 cusps (Theorem (4.5), §4).
(4) Symmetric curve of degree 6, with non-conical 6 cusps (Theorem (5.8), §5).

The fundamental groups of the complement of the above examples (2), (4) are abelian. We also discuss the degenerations of the above curves. For example, we will show that the symmetric curves of degree 5 with 4 cusps can be degenerated to a curve with 5 cusps (§3). The conical 6 cuspidal curve of degree 6 can be degenerated to a curve with d nodes and $6+s$ cusps for any d, s with $d+s\leqq 3$ (§4). We will also prove that the moduli space of curves of degree 4 with 3 cusps is an irreducible surface. All the curves which we treat in this paper except in §6 are defined over the real numbers and the essential information can be obtained from their real graphs. In §6, we will give explicit examples of a maximal nodal curve and a cuspidal curve of degree n which has asymptotically $n^2/4$ cusps. The three cuspidal curves of degree 4 is also treated in [**D-L**]. I would like to thank Professors M. Namba and H. Tokunaga for pointing out an error in the first version and Professor A. Dimca for the information about [**D-L**].

§2. Symmetric covering.

Let $p: \boldsymbol{C}^2 \to \boldsymbol{C}^2$ be the two-fold covering mapping defined by $p(x, y)=(u, v)$ where $u=x+y$, $v=xy$. This is branched along the discriminant variety: $D=\{(u, v);\ g(u, v)=0\}$ where $g(u, v)=u^2-4v$. As u and v are elementary symmetric polynomials, we refer $p: \boldsymbol{C}^2 \to \boldsymbol{C}^2$ as the symmetric covering. Hereafter we consider the symmetric weight: $\deg u=1$, $\deg v=2$ unless otherwise stated. Thus $g(u, v)$ is a weighted homogeneous polynomial of degree 2 under the symmetric weight. Let $h(u, v)$ be a reduced polynomial of degree n (under the symmetric weight) and let $C=\{(u, v)\in \boldsymbol{C}^2;\ h(u, v)=0\}$. We denote the inverse image $p^{-1}(C)$ of C by $\tilde{C}$. The defining equation of $\tilde{C}$ is $p^*h(x, y)=h(x+y, xy)=0$. Note that $p^*h(x, y)$ is a polynomial of degree n in x and y. We say that C is *symmetrically regular at infinity* if

$$(R_\infty) \qquad \{(u, v) \in \boldsymbol{C}^2;\ h_n(u, v) = g(u, v) = 0\} = \emptyset$$

where h_n is the weighted homogeneous part of degree n of h. The geometric meaning of (R_∞) is the following. First, under the condition (R_∞), the compactification of $\tilde{C}$ and the line $\tilde{D}=\{X-Y=0\}$ in $\boldsymbol{P}^2$ do not intersect at infinity i.e., on the infinite line $Z=0$. Secondly,

LEMMA (2.1). *Assume that C is symmetrically regular at infinity. Let g_C: $C\to \boldsymbol{C}$ be the restriction of the function $g(u, v)=u^2-4v$ to C. Then the number of the fiber $g_C^{-1}(c)$, counting the multiplicity, is constant for $c\in \boldsymbol{C}$.*

PROOF. Assume the contrary. Then there is a sequence P_ν, $\nu=1, 2, \cdots$ of C such that $g(P_\nu)$ is bounded and $\|P_\nu\|\to\infty$. We apply the Curve Selection

Lemma ([**M**], [**H**]) to find a real analytic curve $(u(t), v(t))$, $0<t<1$ so that $u(t)$, $v(t)$ can be expanded in a Laurent series at $t=0$ and (1) $h(u(t), v(t))\equiv 0$, (2) $\lim_{t\to 0} g(u(t), v(t))=c$ for some $c\in \boldsymbol{C}$ and (3) $\lim_{t\to 0}\|(u(t), v(t))\|=\infty$. Let $u(t)=at^p+$(higher terms) and $v(t)=bt^q+$(higher terms) be the respective Laurent series. Here a (respectively b) is non-zero unless $u(t)\equiv 0$ (resp. $v(t)\equiv 0$). We consider the leading terms of $h(u(t), v(t))$ and $g(u(t), v(t))$. Let $P={}^t(p, q)$ and $X=(a, b)$. For a given polynomial f, $f_P(u, v)$ denotes the leading part of f with respect to the weight P and $f_P(u, v)$ is a weighted homogeneous polynomial of degree $d(P; f)$. This is a usual notation. See for instance [**O4**]. Note that

$$g(u(t), v(t))=\begin{cases} a^2t^{2p}+(\text{higher terms}) & \text{if } 2p<q \\ (a^2-4b)t^{2p}+(\text{higher terms}) & \text{if } 2p=q \\ -4bt^q+(\text{higher terms}) & \text{if } 2p>q. \end{cases}$$

Therefore the assumption (2) and (3) can not be satisfied simultaneously unless $g_P=g$ and $g(a, b)=0$. Namely $X\in \boldsymbol{C}^{*2}$, $P={}^t(c, 2c)$ for some negative number c and $a^2-4b=0$. On the other hand, the assumption (1) implies that $h_P(a, b)=0$. As $h_P=h_n$, we get a contradiction to the assumption (R_∞). Q. E. D.

(A) Correspondence of fundamental groups.

We consider the fundamental groups $\pi_1(\boldsymbol{C}^2-C)$ and $\pi_1(\boldsymbol{C}^2-\tilde{C})$ and their relation. Hereafter we always fix a suitable base point and we omit it.

LEMMA (2.2). *Assume that C is symmetrically regular at infinity.*

(i) *If C meets transversely with D, the canonical homomorphism*

$$\phi=(\phi_1, \phi_2)\colon \pi_1(\boldsymbol{C}^2-C\cup D)\longrightarrow \pi_1(\boldsymbol{C}^2-C)\times\pi_1(\boldsymbol{C}^2-D)$$

is an isomorphism where ϕ_1 and ϕ_2 are induced by the respective inclusion mappings.

(ii) *The homomorphism $g_\#\colon \pi_1(\boldsymbol{C}^2-D)\to\pi_1(\boldsymbol{C}^*)\cong \boldsymbol{Z}$ is an isomorphism and the composition homomorphism $\psi\colon \pi_1(\boldsymbol{C}^2-C\cup D)\xrightarrow{\phi_2}\pi_1(\boldsymbol{C}^2-D)\xrightarrow{g_\#}\boldsymbol{Z}$ is the rotation number:*

$$\psi(\omega)=\frac{1}{2\pi i}\int_\omega \frac{dg}{g}, \qquad \omega\in\pi_1(\boldsymbol{C}^2-C\cup D).$$

(iii) *The image of $p_\#\colon \pi_1(\boldsymbol{C}^2-\tilde{C}\cup\tilde{D})\to\pi_1(\boldsymbol{C}^2-C\cup D)$ consists of the loops ξ with even rotation number $\psi(\xi)$.*

PROOF. Note that (u, g) is a global system of coordinates. Let $\Sigma=\{c_1, \cdots, c_k\}$ be the set of the critical value of $g_C\colon C\to \boldsymbol{C}$. Then $g\colon \boldsymbol{C}^2-g_C^{-1}(\Sigma)\to \boldsymbol{C}-\Sigma$ is a locally trivial fibration by virtue of Lemma (2.1) and $0\notin\Sigma$ by the transversality assumption. By van Kampen Theorem ([**K**]), the homomorphism $\iota\colon \pi_1(g^{-1}(c)-g^{-1}(c)\cap C)\to\pi_1(\boldsymbol{C}^2-C)$ is surjective for any $c\notin\Sigma$. Note that

$\pi_1(g^{-1}(c)-g^{-1}(c)\cap C)$ is a free group of rank n. We fix a system of generators $\rho_1, \cdots, \rho_n$. As $g: (\boldsymbol{C}^2, C)\to \boldsymbol{C}$ has no critical point at infinity by Lemma (2.1), the generating relations of $\rho_1, \cdots, \rho_n$ as the generators of $\pi_1(\boldsymbol{C}^2-C)$ are given by the monodromy relations around $c=c_1, \cdots, c_k$. The generators of $\pi_1(\boldsymbol{C}^2-C\cup D)$ are given by $\rho_1, \cdots, \rho_n$ and ρ where ρ is represented by a small loop which goes around D outside of the intersection $D\cap C$. In particular, we have $\phi(\rho)=(e, a)$ where a is the canonical generator of $\pi_1(\boldsymbol{C}^2-D)$. The generating relations are given by the same monodromy relations at $c=c_1, \cdots, c_k$ and the commutation relation of ρ with other generators: $[\rho, \rho_i]=e$, $i=1, \cdots, n$. The last commutation relations follows from the topological triviality of the projection $g: (\boldsymbol{C}^2, C)\to \boldsymbol{C}$ near $c=0$. Now the first assertion (i) follows immediately. The assertion (ii) follows also from the observation that $g: \boldsymbol{C}^2-D\to \boldsymbol{C}^*$ is a homotopy equivalence. The assertion (iii) is also clear as the image of $p_\#$: $\pi_1(\boldsymbol{C}^2-\tilde{C}\cup\tilde{D})\to\pi_1(\boldsymbol{C}^2-C\cup D)$ is a normal subgroup of index 2 and $p^*g(x, y)=(x-y)^2$. Q.E.D.

We remark here that the transversality of C and D does not imply the generic intersection as projective curves. In fact, the number of the intersection poins $C\cap D$ in $\boldsymbol{C}^2$ is not $2\deg C$ but $\deg C$. Thus the assertion (i) does not follow from [**O-S**]. We fix an element $\rho\in\pi_1(\boldsymbol{C}^2-C\cup D)$ where ρ is represented by a small loop which goes around D outside of the intersection $C\cap D$. By the above isomorphism, $\phi(\rho)=(e, 1)$ where e is the unit element of $\pi_1(\boldsymbol{C}^2-C)$. Let $\tilde{D}$ be the inverse image of the discriminant variety D. Note that $\tilde{D}=\{x-y=0\}$ and the defining polynomial $p^*g(x, y)=(x-y)^2$ is not reduced. The following theorem says that we can compute the fundamental group $\pi_1(\boldsymbol{C}^2-\tilde{C})$ from $\pi_1(\boldsymbol{C}^2-C)$ in a certain case.

THEOREM (2.3). *Let C be a curve which is symmetrically regular at infinity.*

(i) *The canonical homomorphism $p_\#: \pi_1(\boldsymbol{C}^2-\tilde{C})\to\pi_1(\boldsymbol{C}^2-C)$ is surjective.*

(ii) *If the homomorphism $\phi=(\phi_1, \phi_2): \pi_1(\boldsymbol{C}^2-C\cup D)\to\pi_1(\boldsymbol{C}^2-C)\times\pi_1(\boldsymbol{C}^2-D)$ is isomorphic, in particular if C meets transversely with D in the base space $\boldsymbol{C}^2$, the above homomorphism $p_\#: \pi_1(\boldsymbol{C}^2-\tilde{C})\to\pi_1(\boldsymbol{C}^2-C)$ is bijective.*

PROOF. We consider the commutative diagram:

$$\begin{array}{ccc}
\pi_1(\boldsymbol{C}^2-\tilde{C}\cup\tilde{D}) & \xrightarrow{p'_\#} & \pi_1(\boldsymbol{C}^2-C\cup D) \\
\downarrow \tilde{\iota} & & \downarrow \iota \\
\pi_1(\boldsymbol{C}^2-\tilde{C}) & \xrightarrow{p_\#} & \pi_1(\boldsymbol{C}^2-C)
\end{array}$$

The horizontal maps are induced by the projection p and the vertical maps are induced by the respective inclusion maps. It is obvious that the vertical maps are surjective. Take any loop $\omega\in\pi_1(\boldsymbol{C}^2-C)$. Choose $\omega'\in\pi_1(\boldsymbol{C}^2-C\cup D)$ so that $\iota(\omega')=\omega$. The loop ω' can be lifted to a loop by p if and only if the rotation number $\psi(\omega')$ is even. (Of course, ω' is always liftable as a path.) Thus either ω' or $\omega'\rho$ can be lifted to a loop ω''. Therefore $p_2(\tilde{\iota}(\omega''))=\omega$. Thus $p_\#$ is surjective. Now we prove the injectivity of $p_\#$ assuming that ϕ is an isomorphism. Let $\sigma\in\pi_1(\boldsymbol{C}^2-\tilde{C})$ be an arbitrary element and take an element $\sigma'\in\pi_1(\boldsymbol{C}^2-\tilde{C}\cup\tilde{D})$ which is mapped to σ by $\tilde{\iota}$. Assume that $p_\#(\sigma)=e$. Then by Lemma (2.2), $p'_\#(\sigma')=\rho^{2k}$ for some even integer $2k$. Thus σ' is represented by the lift of ρ^{2k} as $p'_\#$ is injective. This corresponds obviously to the unit element e by $\tilde{\iota}$. Thus σ is trivial in $\pi_1(\boldsymbol{C}^2-C)$. Q.E.D.

If $C\cap D$ has at least one transversal intersection, the canonical homomorphism $\phi=(\phi_1,\phi_2)$: $\pi_1(\boldsymbol{C}^2-C\cup D)\to\pi_1(\boldsymbol{C}^2-C)\times\pi_1(\boldsymbol{C}^2-D)$ is often isomorphic.

(B) Correspondence of singularities.

Now we consider the correspondence of the singularities of C and $\tilde{C}$. For the calculation's sake we use the coordinates (u, g) in the base space of p: $\boldsymbol{C}^2\to\boldsymbol{C}^2$ and the coordinate (u, l) in the source where $g=u^2-4v$, $u=x+y$ and $l=x-y$. In § 3, we simply write $\sqrt{g}$ instead of l. In these coordinates, the projection p is simply defined by $p(u, l)=(u, l^2)$ and the discriminant variety D is the horizontal line $\{g=0\}$. Let $h(u, g)$ be the defining polynomial of C. Then $\tilde{C}$ is defined by $\tilde{h}(u, l)=0$ where $\tilde{h}(u, l)=h(u, l^2)$. Let $\boldsymbol{w}\in C$. Assume first that $\boldsymbol{w}\notin C\cap D$. Then $p^{-1}(\boldsymbol{w})$ consists of two points, say $\tilde{\boldsymbol{w}}_1$ and $\tilde{\boldsymbol{w}}_2$. As p is locally isomorphic, the germs $(\tilde{C}, \tilde{\boldsymbol{w}}_i)$, $i=1, 2$ are isomorphic to the germ $(C, \boldsymbol{w})$.

Now we assume that $\boldsymbol{w}\in C\cap D$ and let $p^{-1}(\boldsymbol{w})=\tilde{\boldsymbol{w}}$. In the above coordinates, we can write $\boldsymbol{w}=(\alpha, 0)=\tilde{\boldsymbol{w}}$ for some $\alpha\in\boldsymbol{C}$. We calculate the differentials:

$$\frac{\partial h}{\partial u}(u, l)=\frac{\partial h}{\partial u}(u, l^2),\qquad \frac{\partial\tilde{h}}{\partial l}(u, l)=2l\frac{\partial h}{\partial g}(u, l^2). \tag{2.4}$$

Thus $\tilde{\boldsymbol{w}}$ is a singular point of $\tilde{C}$ if and only if

$$\frac{\partial h}{\partial u}(\alpha, 0)=0. \tag{2.5}$$

This implies the following.

PROPOSITION (2.6). *$\tilde{\boldsymbol{w}}$ is a singular point of $\tilde{C}$ if and only if*

(i) *$\boldsymbol{w}$ is a singular point of C, or*

(ii) *$\boldsymbol{w}$ is a regular point of C and C is tangent to D at $\boldsymbol{w}$.*

Recall that $\boldsymbol{w}$ is called a *cusp* singularity if C is locally isomorphic to the curve $\xi^2+\zeta^3=0$ for a system of coordinates (ξ, ζ) centered at $\boldsymbol{w}$. This is a generic property in the class of the singularity with the condition $H(h)(\boldsymbol{w})=0$

where $H(h)(\boldsymbol{w})$ is the Hessian of h at $(u, g)=\boldsymbol{w}$. We give a criterion for a given singularity to be a cusp singularity. Let (ξ, ζ) be a local coordinate system centered at $\boldsymbol{w}$ and let $\hat{h}(\xi, \zeta)=h(u(\xi, \zeta), g(\xi, \zeta))$. Let $\mathcal{M}$ be the maximal ideal of $\mathcal{O}_{C^2, \boldsymbol{w}}$.

PROPOSITION (2.7). *Assume that* $\boldsymbol{w}$ *is a singular point of* C *and* $\hat{h}(\xi, \zeta)\equiv a\xi^2$, $a\neq 0$ *modulo* $\mathcal{M}^3$. *Then* $\boldsymbol{w}\in C$ *is a cusp singularity if and only if* $\hat{h}(\xi, \zeta)$ *contains the monomial* ζ^3 *with a non-zero coefficient.*

PROOF. The necessity follows from the fact that the local Milnor number is 2. The proof for the sufficiency is easily obtained by the standard argument of the generalized Morse lemma. Q. E. D.

Now we consider the Hessian of $\tilde{h}$ at $\tilde{\boldsymbol{w}}=(\alpha, 0)$ assuming $\tilde{\boldsymbol{w}}$ is a singular point of $\tilde{C}$. From (2.4), we have

$$H(\tilde{h})(\tilde{\boldsymbol{w}}) = 2\frac{\partial h}{\partial g}(\alpha, 0)\frac{\partial^2 h}{\partial u^2}(\alpha, 0). \tag{2.8}$$

Let $\mu(C, D; \boldsymbol{w})$ be the intersection multiplicity of C and D at $\boldsymbol{w}$. We claim that

LEMMA (2.9). *Assume that* $\boldsymbol{w}\in C\cap D$ *and let* $\tilde{\boldsymbol{w}}$ *as above. Then*

(i) $\tilde{\boldsymbol{w}}\in\tilde{C}$ *is an ordinary double point if and only if* $\boldsymbol{w}$ *is a regular point of* C *with* $\mu(C, D; \boldsymbol{w})=2$.

(ii) $\tilde{\boldsymbol{w}}\in\tilde{C}$ *is a cusp singularity if and only if* $\boldsymbol{w}$ *is a regular point of* C *with* $\mu(C, D; \boldsymbol{w})=3$.

PROOF. As a coordinate system centered at $\boldsymbol{w}$, we can take (u_α, g) where $u_\alpha=u-\alpha$. Recall that $\mu(C, D; w)=\mathrm{val}_{u_\alpha}k(u_\alpha)$ where $k(u_\alpha)=h(u_\alpha+\alpha, 0)$. Thus

$$\mu(C, D; \boldsymbol{w}) = s \Longleftrightarrow \frac{d^i k}{du_\alpha^i}(0)=\frac{\partial^i h}{\partial u^i}(\alpha, 0)\begin{cases} =0 & \text{for } i<s \text{ and} \\ \neq 0 & \text{for } i=s. \end{cases}$$

In particular we have $\mu(C, D; \boldsymbol{w})\geqq 2$ if $\tilde{\boldsymbol{w}}$ is a singular point. On the other hand, by (2.8) we have the equivalence

$$\begin{aligned} \tilde{\boldsymbol{w}}: \text{ordinary double point} &\Longleftrightarrow \frac{\partial h}{\partial u}(\alpha, 0)=0, \quad H(\tilde{h})(\tilde{\boldsymbol{w}})\neq 0 \\ &\Longleftrightarrow \frac{\partial h}{\partial u}(\alpha, 0)=0, \ \frac{\partial h}{\partial g}(\alpha, 0)\neq 0, \quad \frac{\partial^2 h}{\partial u^2}(\alpha, 0)\neq 0. \end{aligned}$$

The last condition implies that $\boldsymbol{w}\in C$ is a regular point and $\mu(C, D; \boldsymbol{w})=2$. This proves the assertion (i).

Now we prove the assertion (ii). Let $s=\mu(C, D; \boldsymbol{w})$ and assume that $\partial h/\partial u(\alpha, 0)=0$. Let $h_\alpha(u_\alpha, g)=h(u_\alpha+\alpha, g)$. Then $h_\alpha=0$ is a defining equation of C. By the assumption, we can write

$$h_\alpha(u_\alpha, g) = u_\alpha^s U + g^j V$$

where $U, V \in \mathcal{O}_{C^2, w}$, $j \geqq 1$ and U is a suitable unit in $\mathcal{O}_{C^2, w}$. Then the defining equation of $\tilde{C}$ is:

$$p^* h_\alpha(u_\alpha, l) = u_\alpha^s p^* U + l^{2j} p^* V = 0 .$$

Thus using Proposition (2.7), we can see easily that $\tilde{\boldsymbol{w}} \in \tilde{C}$ is a cusp singularity if and only if $j=1$, $s=3$ and V is a unit. This implies that $\boldsymbol{w} \in C$ is a regular point and $\mu(C, D; \boldsymbol{w})=3$. Q. E. D.

DEFINITION (2.10). Recall that a regular point P of a curve C is called a *flex of order* k if the intersection multiplicity of C and the tangent line at P is $(k+2)$ ([**Z1**]). We call a regular point P of C a *D-flex of order* k if $P \in C \cap D$ and the intersection multiplicity of C and D at P is $k+2$. Hereafter we call an ordinary double point simply a *node*.

The following corollary follows immediately from Lemma (2.9).

COROLLARY (2.11). *Let $C=\{h(u, g)=0\}$ be a curve in the base space and let $\tilde{C}=p^{-1}(C)$. We assume that the singular points of C are either nodes or cusps and there is no singular point of C on the intersection $C \cap D$. Let $d(C)$ and $s(C)$ be the number of the nodes and cusps of C respectively and let $d(\tilde{C})$ and $s(\tilde{C})$ be the number of nodes and cusps of $\tilde{C}$ respectively. We also assume that $\mu(C, D; P) \leqq 3$ for any $P \in C \cap D$. Let $t_2(C)$ and $t_3(C)$ be the number of the D-flex of order 0 and of order 1 respectively. Then the lifted curve $\tilde{C}$ has only nodes and cusps and we have*

$$d(\tilde{C}) = 2d(C) + t_2(C), \qquad s(\tilde{C}) = 2s(C) + t_3(C).$$

§ 3. Construction of cuspidal curves.

In this section, we consider irreducible projective curves with many cusps. Let $F(X, Y, Z)$ be an irreducible homogeneous polynomial of degree n and let $C=\{(X; Y; Z) \in \boldsymbol{P}^2;\ F(X, Y, Z)=0\}$ be the corresponding projective curve. For convenience, we assume that the intersection of C with the infinite line $Z=0$ is generic. Namely $F(X, Y, 0)=0$ consists of n distinct points and we consider hereafter the affine equation $f(x, y)=0$ of C where $f(x, y)=F(x, y, 1)$. We assume that C has only nodes and cusps as its singular points. Let $d(C)$ and $s(C)$ be the number of nodes and cusps respectively. We first recall the known bounds for $d(C)$ and $s(C)$. Suppose that C is non-singular. Then by the Plücker's formula, the genus of C is $(n-1)(n-2)/2$. For the general case, we deform the curve by $C_t=\{f(x, y)=t\}$. For any sufficiently small t, C_t is non-singular. Let C' be the non-singular model of $C=C_0$. Then the Euler-Poincaré characteristic $\chi(C'')$ satisfies $\chi(C')=\chi(C_t)+2(d(C)+s(C))$. Thus by considering the genus of C', we have

$$d(C),\quad s(C)\leqq d(C)+s(C)\leqq\frac{(n-1)(n-2)}{2}. \tag{3.1}$$

The second equality holds if and only if C is rational. If C is rational, by Plücker's formula for the dual curve, $s(C)$ satisfies:

$$s(C)\leqq\frac{3(n-2)}{2}\qquad(C:\text{ rational}). \tag{3.2}$$

We refer to [**B**] for the detail about these things. See also [**W**]. For a non-rational curve, the number $s(C)$ may be much bigger but we do not know the maximum of $s(C)$ for a generic n. For $n=4, 5, 6$, $s=3, 5, 9$ is the maximum respectively. See §§3, 4, 6. Let $P_1, \cdots, P_s$ be the cusps of C. We say that $\{P_1, \cdots, P_s\}$ are *independent* if for any $P_1', \cdots, P_s'$ which are sufficiently near to $P_1, \cdots, P_s$ respectively, there exists an irreducible curve C' of degree n which has cusps at $P=P_1', \cdots, P_s'$. Note that the necessary condition for a curve $\{f(x, y)=0\}$ to have a cusp singularity at a given point $P=(\alpha, \beta)$ is given by three linear equations and one quadratic equation in the coefficients of $f(x, y)$:

$$f(\alpha, \beta)=\frac{\partial f}{\partial x}(\alpha, \beta)=\frac{\partial f}{\partial x}(\alpha, \beta)=H(f)(\alpha, \beta)=0. \tag{3.3}$$

Therefore counting the number of coefficients of $f(x, y)$, we get the following estimation for the independent cusps:

$$s(C)\leqq\frac{n(n+3)}{8}\qquad\text{for independent cusps.} \tag{3.4}$$

The following example shows that the number of cusps which are not independent may be much bigger.

EXAMPLE (3.5). Let $n_2=n-2[n/2]$ and $n_3=n-3[n/3]$ and let C be the curve defined by the following Join type polynomial

$$f(x, y)=n_2(x)\prod_{i=1}^{[n/2]}(x-\alpha_i)^2-\delta\prod_{k=1}^{n_3}(y-\gamma_k)\prod_{j=1}^{[n/3]}(y-\beta_j)^3$$

where $n_2(x)=1$ or $x-\alpha_0$ according to n is even or odd respectively. For a generic choice of $\{\delta, \alpha_0, \cdots, \alpha_{[n/2]}, \gamma_1, \cdots, \gamma_{n_3}, \beta_1, \cdots, \beta_{[n/3]}\}$, C has $[n/2][n/3]$ cusps $\{(\alpha_i, \beta_j);\ i=1, \cdots, [n/2], j=1, \cdots, [n/3]\}$. Thus asymptotically, we can put $n^2/6$ cusps. In the case of $n_3=2$, we can replace $\prod_{k=1}^{n_3}(y-\gamma_k)$ by $(y-\gamma)^2$. Then our curve also obtains $[n/2]$ nodes: $\{(\alpha_i, \gamma);\ 1\leqq i\leqq[n/2]\}$. If we take special α_i, $1\leqq i\leqq[n/2]$, γ, β_j, $1\leqq j\leqq[n/3]$, we can put more nodes or cusps. See §4 and §6. These cusps are not independent. The following table shows the above estimations.

n	3	4	5	6	7	8	9	10	11	12
$[3(n-2)/2]$	1	3	4	6	7	9	10	12	13	15
$[n(n+3)/8]$	1	3	5	6	8	11	13	16	19	22
$[n/2][n/3]$	1	2	2	6	6	8	12	15	15	24
$(n-1)(n-2)/2$	1	3	6	10	15	21	28	36	45	55

Table (3.A)

Hereafter we consider the case that $f(x, y)$ is a symmetric polynomial. We use the systems of coordinates (u, g) in the base space and (u, l) in the source space as in §2. For brevity's sake, we simply denote $\sqrt{g}$ instead of l. Thus $u=x+y$ and $\sqrt{g}=x-y$. Note that g is a weighted homogeneous coordinate of weight 2. Let $h(u, g)$ be a polynomial of degree n under the symmetric weight as in §2 and let $C=\{(u, v);\ h(u, g)=0\}$. We assume that C is symmetrically regular at infinity as before. We study the curve $\tilde{C}$ of degree n which is the inverse image of C by $p: \boldsymbol{C}^2\to\boldsymbol{C}^2$. Its defining polynomial is $f(u, \sqrt{g})=p^*h(u, \sqrt{g})=h(u, g)$ where $g=\sqrt{g}^2$. We also assume that $h_n(u, g)=0$ has no multiple roots. This says that the infinite line $Z=0$ is generic with respect to $\tilde{C}$. The number of free coefficients of $h(u, v)$ is $[n/2]([n/2]+2)$ for n even and $[n/2]^2+3[n/2]+1$ for n odd. Thus by the same argument as above, we have an estimation

$$s(C)\leqq\begin{cases}\dfrac{[n/2]([n/2]+2)}{4} & n:\ \text{even}\\[2ex] \dfrac{[n/2]^2+3[n/2]+1}{4} & n:\ \text{odd}\end{cases}$$

for the number of the independent cusps of C. Of course, this estimation is asymptotically equivalent to (3.4) for $s(\tilde{C})$. One advantage of the study of symmetric curves $\tilde{C}$ is that we can read almost all information about $\tilde{C}$ from the information about C and the intersection $C\cap D$. On the other hand if C is defined by a polynomial $h(u, g)$ of symmetric degree n, the degree of h in the variable g in the usual sense is $[n/2]$. Thus the number of the generators of the fundamental group $\pi_1(\boldsymbol{C}^2-C)$ can be half of the generators of the fundamental group $\pi_1(\boldsymbol{C}^2-\tilde{C})$.

(A) Admissible change of coordinates.

Now we consider the change of coordinates in the base space which does not change the symmetric degree. As deg $g=2$, we can not carry out a general linear change of coordinates without changing the symmetric degree but a change of coordinates of the following type does not change the symmetric degree of C.

$$\Phi(u, g) = (U, G); \quad U = \alpha u+\beta, \quad G = \gamma g+\delta u^2+\varepsilon u+\zeta, \quad \alpha, \gamma \in C^*$$

In the case of $\delta=0$ (respectively $\delta\neq 0$), we call Φ *an admissible linear change of coordinates* (resp. *an admissible quadratic change of coordinates*). An admissible linear or quadratic change of coordinates changes nothing about the curve C or its complement $\boldsymbol{C}^2-C$ up to an isomorphism but *the lifted curves* $\tilde{C}$ *and* $\Phi(\tilde{C})$ *are not necessarily isomorphic* if the intersection of C and D changes. In fact, the following proposition says that we can always put one node or cusp in $\tilde{C}$ if C and D are transverse.

PROPOSITION (3.6). (I) *Assume that* C *and* D *are transverse. Then*

(i) *there is an admissible linear change of coordinates* Φ *so that the curve* $\Phi(C)$ *gets a D-flex of order* 0 *in the new coordinates and*

(ii) *there exists also an admissible quadratic change of coordinates* Φ *so that* $\Phi(C)$ *gets a D-flex of order* 1 *in the new coordinates.*

(II) *Assume that* C *has a single D-flex of order* 0. *Then we can change this flex into a D-flex of order* 1 *by an admissible quadratic change of coordinates.*

(III) *The above changes of coordinates can be done in a family of admissible change of coordinates* Φ_t *with* Φ_0 *being identity.*

PROOF. Let $P\in C$ be a regular point where $\partial h/\partial g(P)\neq 0$. Then the tangent line L_P at P can be written as $g-\alpha u+\beta=0$. For almost all P, the intersection multiplicity of C and L_P is 2. So assume that $\mu(C, L_P; P)=2$ and let $\Phi(u, g)=(U, G)$ where $U=u$, $G=g-\alpha u-\beta$ be new coordinates. As $\mu(\Phi(C), D; \Phi(P))=\mu(C, L_P; P)$, it is obvious that $\Phi(C)$ gets a D-flex of order 0 in this coordinates. This proves (i). For the assertion (ii), we consider a quadratic change of coordinates $\Phi(u, g)=(U, G)$ where $U=u$, $G=g-\gamma u^2-\alpha u-\beta$ where $g=\alpha u+\beta$ is the tangent line of C at P. Let $E=\{g-\gamma u^2-\alpha u-\beta=0\}$. It is easy to see that there is a unique $\gamma\in\boldsymbol{C}$ such that $\mu(C, E; P)\geqq 3$ and the equality holds for almost all P. We assume $\mu(C, E; P)=3$ and we consider the above quadratic change of coordinates. Then $\Phi(C)$ gets a D-flex of order 1 in this system of coordinates. This proves the assertion (ii). If C has some nodes or cusps before the above change of coordinates, we can choose $P\in C$ so that the tangent line L_P or parabola E does not pass through the singularities. Assume that D is simply tangent to C at $(\alpha, 0)$. Then we can take a quadratic change of coordinates $U=u$, $G=g+\beta(u-\alpha)^2$ for a suitable β to change this D-flex of order 0 into a D-flex of order $\geqq 1$. If P is not generic in the sense of (I-ii), we take the similar quadratic change of coordinates centered at a sufficiently near regular point $P'\in C$. The assertion (III) is almost trivial. Q.E.D.

Now we study several examples of cuspidal curves of degree n for small n in detail. A symmetric curve of degree 3 with one cusp is simply given by the lifting of a curve C: $h(u, g)=0$ with one D-flex of order 1. For example, we can take $C=\{h(u, g)=(u+1)g-u^3\}$.

(B) Maximal cuspidal curve of degree 4.

We first construct a curve $A=\{h(u, g)=0\}$ of degree 4 which has 1 cusp singularity at $\boldsymbol{w}\in A-D$ and a D-flex $\boldsymbol{w}'\in A\cap D$ of order 1. In the notation of Corollary (2.11), A has the invariants $s=1$ and $t_3=1$. For such a curve, we have $s(\tilde{A})=3$ and the above Table (3.A) says that $\tilde{A}$ is a rational curve. The determination of the defining polynomial $h(u, g)$ is much simpler if we choose the singular point and D-flex point in special position. Thus we take $\boldsymbol{w}=(1, 1)$ and $\boldsymbol{w}'=(0, 0)$. We first consider the condition for $\boldsymbol{w}$ to be a cusp singularity. Write first

(3.7) $$h(u, g) = h_{(4)}(u)+h_{(2)}(u)(g-1)+\gamma(g-1)^2$$

where $\{h_{(i)}(u);\ i=2, 4\}$ are polynomials of u with $\deg h_{(i)}\leqq i$. As $\boldsymbol{w}=(1, 1)$ is a singular point of A, we have

(3.8) $$h_{(4)}(1) = \frac{dh_{(4)}}{du}(1) = h_{(2)}(1) = 0.$$

The condition for $\boldsymbol{w}$ being a cusp is:

(3.9) $$H(h)(w) = 2\frac{d^2h_{(4)}}{du^2}(1)\ \gamma-\left(\frac{dh_{(2)}}{du}(1)\right)^2 = 0.$$

The condition (3.9) is a quadratic equation. By (3.8), we can write

(3.10) $$h_{(4)}(u) = (u-1)^2(au^2+bu+c), \qquad h_{(2)}(u) = (u-1)(du+e).$$

Then (3.9) is equivalent to:

$$4(a+b+c)\gamma-(d+e)^2 = 0.$$

Now the condition that $\mu(A, D;\ \boldsymbol{w}')=3$ is equivalent to $\operatorname{val} h(u, 0)=3$. Thus

$$c+e+\gamma = 0, \qquad -2c+b+d-e = 0, \qquad a-2b+c-d = 0.$$

The solution space is 1-dimensional. For instance, we can take

(3.11) $$A:\ h(u, g) = (u-1)^3(3u+5)-6(u-1)^2(g-1)-(g-1)^2 = 0.$$

Figure (3.B) shows the real plane sections of A and $\tilde{A}$ respectively.

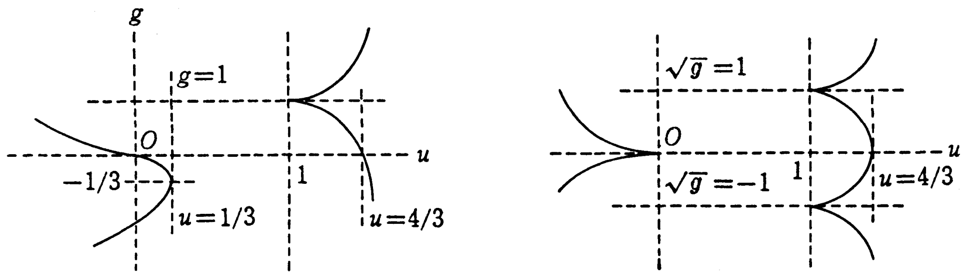


Figure (3.B) A: left, $\tilde{A}$: right

Now we consider the fundamental groups $\pi_1(\boldsymbol{C}^2-A)$ and $\pi_1(\boldsymbol{C}^2-\tilde{A})$. Zariski claims in [**Z1**] that three cuspidal curves of degree 4 are the exceptional rational curves whose complements have a non-commutative fundamental group of order 12. We will reprove this assertion. In fact, as the moduli space of curves of degree 4 with three cusps is irreducible (see Appendix (3.A)), the fundamental group of the complement of any curve of degree 4 with three cusps is isomorphic to the group described in the following.

THEOREM (3.12). *The fundamental groups* $\pi_1(\boldsymbol{C}^2-\tilde{A})$ *is isomorphic to the group*

$$\langle \rho, \xi;\ \rho\xi\rho = \xi\rho\xi,\ \rho^2 = \xi^2 \rangle$$

and $\pi_1(\boldsymbol{P}^2-\tilde{A})$ *is isomorphic to the finite non-abelian group of order* 12:

$$\langle \rho, \xi;\ \rho\xi\rho = \xi\rho\xi,\ \rho^2\xi^2 = e \rangle .$$

PROOF. We consider the fundamental group $\pi_1(\boldsymbol{C}^2-A)$ and $\pi_1(\boldsymbol{C}^2-\tilde{A})$ simultaneously. Let $q: (\boldsymbol{C}^2, A)\to \boldsymbol{C}$ be the projection into the u-coordinate and let $\tilde{q}: (\boldsymbol{C}^2, \tilde{A})\to \boldsymbol{C}$ be the composition $\tilde{q}=q\circ p$. We consider the pencil $\{q^{-1}(\alpha);\ \alpha\in \boldsymbol{C}\}$ and $\{\tilde{q}^{-1}(\alpha);\ \alpha\in \boldsymbol{C}\}$. There are only two critical values $u=1/3$ and $u=1$ for $q: \boldsymbol{C}^2-A\to \boldsymbol{C}$. As $h(u, 0)=u^3(3u-4)$, we get two more critical values $u=0, 4/3$ for the pencil $\{\tilde{q}^{-1}(\alpha)\}$. See Figure (3.B). We take a system of generators ξ_1, ξ_2 for $\pi_1(\boldsymbol{C}^2-A)$, in $q^{-1}(1/3-\varepsilon)$ where ε is small enough. As a system of generators for $\pi_1(\boldsymbol{C}^2-\tilde{A})$, we take $\rho_1, \rho_2, \rho_1', \rho_2'$ as in Figure (3.C). For the simplicity of Figures which follow, we assume hereafter *every small loop is oriented counterclockwise* unless otherwise stated. The monodromy relation around $u=1/3$ gives the relation:

$$(R_1)\qquad \begin{cases} \xi_1 = \xi_2 & \text{for } A \\ \rho_1 = \rho_2,\quad \rho_1' = \rho_2' & \text{for } \tilde{A}. \end{cases}$$

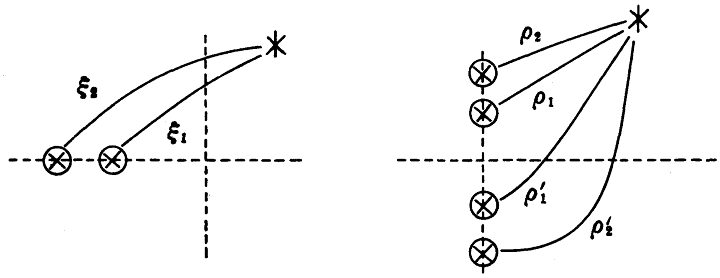


Figure (3.C) $(u=1/3-\varepsilon)$

Thus we have that $\pi_1(C^2-A;\ w_0)\cong \boldsymbol{Z}$. The monodromy relation around $u=0$ for $\tilde{A}$ gives the following cusp relation for $\tilde{A}$:

(R_2) $$\rho_1\rho_1'\rho_1=\rho_1'\rho_1\rho_1'.$$

For the sake of the calculation of the monodromy relations around $u=1$ and $u=4/3$, we show in Figure (3.D) how the two intersection points $A\cap q^{-1}(u)$ (resp. the four intersection points $\tilde{A}\cap\tilde{q}^{-1}(u)$) move homotopically when u moves from $u=1/3+\varepsilon$ to $u=1-\varepsilon$.

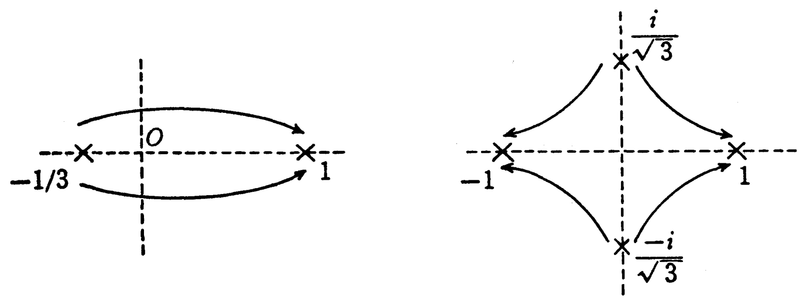
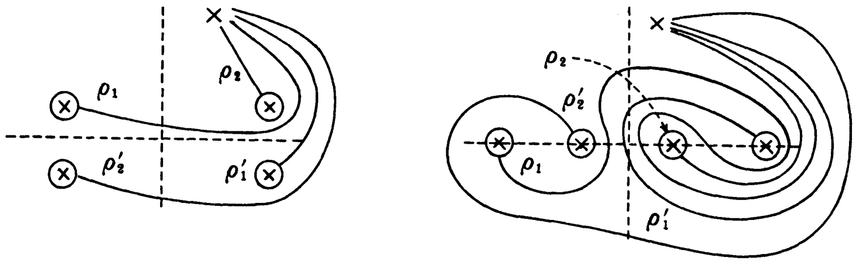


Figure (3.D)

From $u=1/3-\varepsilon$ to $u=1/3+\varepsilon$ or from $u=1-\varepsilon$ to $u=1+\varepsilon$, u moves on the circle $|u-1/3|=\varepsilon$ or $|u-1|=\varepsilon$ clockwise. The essential point here is that two points of $q^{-1}(u)\cap A$ (resp. four points $\tilde{q}^{-1}(u)\cap\tilde{A}$) do not cross the real axis (resp. the real axis and the imaginary axis) during the motion of u from $u=1/3+\varepsilon$ to $u=1-\varepsilon$ and they are symmetric with respect to the real axis (resp. the real axis and the imaginary axis). Figure (3.E) shows how our generators are deformed in the fibers $\tilde{q}^{-1}(1-\varepsilon)$ and $\tilde{q}^{-1}(4/3-\varepsilon)$.

Strictly speaking, each loop in a different fiber has a temporary base point in that fiber. This base point is joined to the original base point through the triviality of the fibering structure over the fixed path. Thus the monodromy relation around $u=1$ can be easily computed as:



$\tilde{q}^{-1}(1-\varepsilon)$ $\tilde{q}(4/3-\varepsilon)$

Figure (3.E)

$$(R_3) \qquad \begin{cases} \rho_2\rho_1'\rho_2 = \rho_1'\rho_2\rho_1' \\ (\rho_1'^{-1}\rho_1\rho_1')\rho_2'(\rho_1'^{-1}\rho_1\rho_1') = \rho_2'(\rho_1'^{-1}\rho_1\rho_1')\rho_2' \end{cases}$$

It is easy to see that these relations are derived from (R_1) and (R_2). Finally the monodromy relation at $u=4/3$ gives

$$(R_4) \qquad \rho_2 = (\rho_1'^{-1}\rho_1\rho_1')\rho_2'(\rho_1'^{-1}\rho_1\rho_1')^{-1}$$

which reduces to $\rho_1^2=(\rho_1')^2$ by (R_1) and (R_2). Thus writing $\rho=\rho_1=\rho_2$ and $\xi=\rho_1'=\rho_2'$, $\pi_1(\boldsymbol{C}^2-\tilde{A})$ is isomorphic to the group

$$\langle \rho, \xi;\ \rho\xi\rho = \xi\rho\xi,\ \rho^2 = \xi^2 \rangle$$

as desired. For the fundamental group $\pi_1(\boldsymbol{P}^2-\tilde{A})$ we add the vanishing relation of the big circle: $\rho_2\rho_1\rho_1'\rho_2'=e$. Thus $\pi_1(\boldsymbol{P}^2-\tilde{A})$ is represented as

$$\langle \rho, \xi;\ \rho\xi\rho = \xi\rho\xi,\ \rho^2 = \xi^2,\ \rho^2\xi^2 = e \rangle .$$

Now the relation $\rho^2=\xi^2$ is derived from the other relations as

$$\rho^2 = (\rho\xi\rho)^2 = (\xi\rho\xi)^2 = \xi^2 .$$

This is a finite non-abelian group of order 12 which is studied by [**Z1**].

Q.E.D.

(C) Cuspidal curves of degree 5.

We consider an irreducible curve B of degree 5 defined by $h(u, g)=0$. We can have at most 2 independent cusps as we have only 11 free coefficients. So we consider the case that B has only two cusps at $(1, 1)$ and $(-1, 1)$. We first write $h(u, g)$ as

$$h(u, g) = h_{(5)}(u) + h_{(3)}(u)(g-1) + h_{(1)}(u)(g-1)^2$$

where $\{h_{(i)}(u);\ i=1, 3, 5\}$ are polynomials of u with $\deg h_{(i)}(u) \leq i$. The singularity condition gives the divisibility:

$$(u^2-1)^2 \mid h_{(5)}(u), \qquad (u^2-1) \mid h_{(3)}(u) .$$

Considering the cusp condition, we can choose as $h(u, g)$ for instance:

(3.13) $$h(u, g) = 2u(g-1)^2+2(u^2-1)(u+1)(g-1)+(u^2-1)^2(u+1).$$

(The moduli of curves with two cusps at (1, 1) and (−1, 1) is three dimensional.) Figure (3.F) is the graph of the section of B with the real plane.

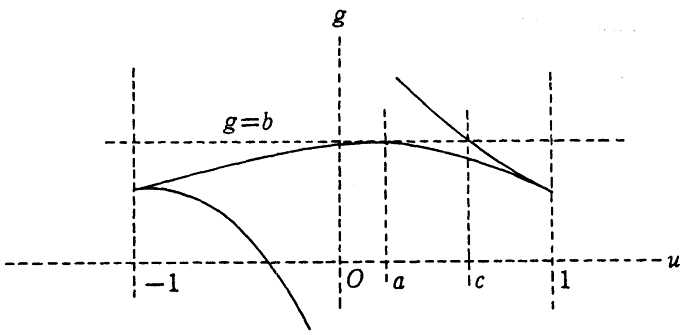


Figure (3.F)

By the Rolle theorem, there exists a positive number b such that the line L: $g=b$ is tangent to B, say at $\boldsymbol{w}=(a, b)$. As the curve B intersects with the g-axis at (0, 3/2) and $\partial h/\partial g(0, 3/2)=-2<0$, we can see that $a>0$. As B is defined over $\boldsymbol{R}$, we can easily see from Figure (3.F) that $\mu(B, L; \boldsymbol{w})=2$. Now let us consider the change of coordinates $\Psi(u, g)=(U, G)$ where $U=u$ and $G=g-b$ and let $B'=\Psi(B)$. This is a typical admissible linear change of coordinates and it does not change anything about the curve B. However by the choice of b, the new curve B' has the invariant $s=2$ and $t_2=1$. Thus $\tilde{B}'$ has four cusps and a node. Now we consider the fundamental group. We assert that

THEOREM (3.14). *$\pi_1(\boldsymbol{C}^2-B)$, $\pi_1(\boldsymbol{C}^2-B')$, $\pi_1(\boldsymbol{C}^2-\tilde{B})$ and $\pi_1(\boldsymbol{C}^2-\tilde{B}')$ are isomorphic to $\boldsymbol{Z}$. Therefore $\pi_1(\boldsymbol{P}^2-\tilde{B}')$ and $\pi_1(\boldsymbol{P}^2-\tilde{B})$ are isomorphic to $\boldsymbol{Z}_5$.*

PROOF. The second assertion is immediate from the first assertion ([**O1**]). So we will show the first assertion. As $\tilde{B}'$ is obtained as the degeneration of $\tilde{B}$ under the family $\tilde{B}_{5,t}$ where $B_{5,t}=\{h(u, g+tb)=0\}_{0\le t\le 1}$, $\pi_1(\boldsymbol{C}^2-\tilde{B})$ is a quotient group of $\pi_1(\boldsymbol{C}^2-\tilde{B}')$. Thus we may consider only the case of B'. Instead of considering $\tilde{B}'$, we will show that $\pi_1(\boldsymbol{C}^2-B'\cup D)\cong\boldsymbol{Z}\times\boldsymbol{Z}$ in the base space. Then by Theorem (2.3) the commutativity of $\pi_1(\boldsymbol{C}^2-\tilde{B}')$ also follows. As $\pi_1(\boldsymbol{C}^2-B'\cup D)$ is isomorphic to $\pi_1(\boldsymbol{C}^2-B\cup L)$ by the definition where L is the line $g=b$, we will show that $\pi_1(\boldsymbol{C}^2-B\cup L)\cong\boldsymbol{Z}$. We consider again the pencil $\{q^{-1}(\alpha);\ \alpha\in\boldsymbol{C}\}$. We have four critical fibers at $u=\pm 1$, a, c where c is the u-coordinate of the another intersection of B and L. Note that $0<c<1$.

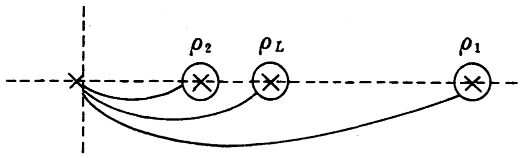


Figure (3.G) $(u=\varepsilon)$

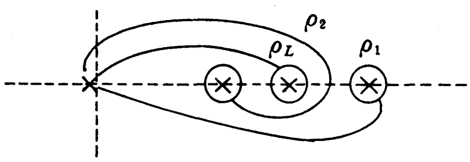


Figure (3.H) $(u=\gamma-\varepsilon)$

We take the generators ρ_1, ρ_2 and ρ_L of $\pi_1(\boldsymbol{C}^2-B\cup L)$ in the fiber $q^{-1}(\varepsilon)$ as in Figure (3.G). The loop ρ_L goes around the intersection with the line L. We consider the monodromy relations. We move the fiber $q^{-1}(u)$ along the real axis. At the critical points, we take the upper circle. First we consider the monodromy around $u=0$. We can solve the equation $h(u, g)=0$ as

$$g-1=\frac{1-u^2}{2u}\{u+1\pm\sqrt{1-u^2}\}$$

where the value of $\sqrt{1-u^2}$ is chosen so that it takes 1 at $u=0$. When u goes around the small circle $|u|=\varepsilon$ counterclockwise, one of the intersection $q^{-1}(u)\cap B$ which is given by

$$g-1=\frac{1-u^2}{2u}\{u+1-\sqrt{1-u^2}\}$$

is analytic and stays almost fixed near $g=3/2$ and the intersection $q^{-1}(u)\cap L$ does not move. The other intersection described by

$$g-1=\frac{1-u^2}{2u}\{u+1+\sqrt{1-u^2}\}$$

goes around the origin clockwise near infinity. Thus we get the relation:

(R_1) $$\rho_1(\rho_L\rho_2)=(\rho_L\rho_2)\rho_1.$$

This is a commutation relation which we call *a strange monodromy relation at infinity*. Usually a commutation relation is obtained around an ordinary double point. However the fiber $q^{-1}(0)$ does not pass through any ordinary double point! We will give another example of a strange relation later (see Appendix (3.B)). We do not need the monodromy relation for $u=a$ for the present purpose. We deform the fiber $q^{-1}(u)$ from $u=\varepsilon$ to $u=\gamma-\varepsilon$ along the real axis. At $u=a$, we take the upper semi-circle of $|u-a|=\varepsilon$. See Figure (3.H). The monodromy relation at $u=\gamma$ is the commutation relation of two elements $(\rho_L\rho_2)^{-1}\rho_1(\rho_L\rho_2)$ and ρ_L. The first element is equal to ρ_1 by (R_1). Thus we get:

(R_2) $$\rho_1\rho_L=\rho_L\rho_1$$

Combining this with (R_1), we get the commutation relation:

(R_3) $$\rho_1\rho_2=\rho_2\rho_1.$$

Now the monodromy relation at $u=1$ modulo (R_1) and (R_2) is the cusp relation: $\rho_1\rho_2\rho_1=\rho_2\rho_1\rho_2$ which reduces by (R_3) to:

(R_4) $$\rho_1=\rho_2$$

Now we can easily see that $\pi_1(\boldsymbol{C}^2-B\cup L)$ is isomorphic to $\boldsymbol{Z}\times\boldsymbol{Z}$. Q.E.D.

Now we consider a quadratic change of coordinates at (0, 3/2) by

$$\Psi(u, g)=(U, G) \text{ where } U=u,\ G=g-\frac{1}{2}u^2+\frac{u}{4}-\frac{3}{2}$$

Let $B''=\Psi(B)$. The lifted curve $\tilde{B}''$ gets 5 cusps. As $\tilde{B}''$ has already 5 cusps and a rational curve can get at most 4 cusps by (3.2), we see that 5 is the maximum number of cusps for the irreducible curves of degree 5. The fundamental group $\pi_1(\boldsymbol{C}^2-\tilde{B}'')$ is also abelian. The proof is parallel. Figure (3.I) shows the intersection of the parabola $E=g-u^2/2+u/4-3/2$ and B.

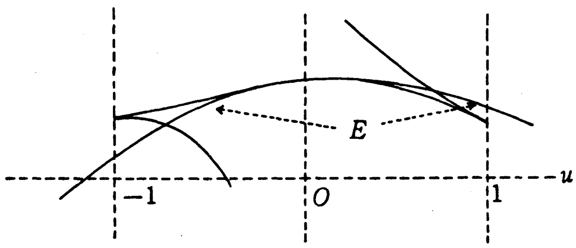


Figure (3.I)

APPENDIX (3.A). In this appendix we prove the irreducibility of the moduli space of curves of degree 4 with three cusps. First we consider a curve $f(X, Y, Z)=0$ of degree 4 with three cusps P_1, P_2, P_3. Note that P_1, P_2 and P_3 are not colinear. In fact assume that they are colinear. Taking a linear change of coordinates if necessary, we assume that $P_1=(\alpha_1, 0)$, $P_2=(\alpha_2, 0)$, $P_3=(\alpha_3, 0)$ in the affine coordinate space $\boldsymbol{C}^2=\boldsymbol{P}^2-\{Z=0\}$. Then in this affine coordinate, it is necessary that f is divisible by $\prod_{i=1}^{3}(x-\alpha_i)^2$ which is ridiculous as $\deg f=4$. We consider the subvariety $\mathcal{M}'$ of $\boldsymbol{P}^{14}\times(\boldsymbol{P}^2)^3$ defined by

$$\mathcal{M}'=\{(f, P_1, P_2, P_3);\ \deg f=4,\ C=\{f=0\} \text{ irreducible},\ P_1, P_2, P_3\in C \text{ cusps}\}.$$

$\mathcal{M}'$ has the canonical projection $\pi: \mathcal{M}'\to(\boldsymbol{P}^2)^3-\Delta$ where Δ is the set of ordered colinear points. The symmetric group $\mathcal{S}_3$ canonically acts on $\mathcal{M}'$ and the quotient space $\mathcal{M}=\mathcal{M}'/S_3$ is the moduli space which we are interested in. To prove the irreducibility of $\mathcal{M}$, it is enough to prove the irreducibility of $\mathcal{M}'$. Let P_1, P_2, P_3 be given three points which are not colinear. Let $\mathcal{M}(P_1, P_2, P_3)=\pi^{-1}(P_1, P_2, P_3)$. This is the moduli space of curves with three cusps at prescribed positions P_1, P_2, P_3. First we assert

PROPOSITION (3.15). *$\pi: \mathcal{M}'\to(\boldsymbol{P}^2)^3-\Delta$ is a locally trivial fibration.*

PROOF. Let $Z=0$ be a fixed infinite line and let $U=\boldsymbol{P}^2-\{Z=0\}$. We show that $\pi: \mathcal{M}'\to(\boldsymbol{P}^2)^3-\Delta$ is trivial over $U^3-\Delta$. We use the affine coordinate. For given $P_1=(\alpha_1, \beta_1)$, $P_2=(\alpha_2, \beta_2)$, $P_3=(\alpha_3, \beta_3)$, there is a unique linear change of coordinates $\Phi(x, y)=(l_1(x, y), l_2(x, y))$ so that $\Phi(P_1)=O$, $\Phi(P_2)=E_1$, $\Phi(P_3)=E_2$ where $O=(0, 0)$, $E_1=(1, 0)$ and $E_2=(0, 1)$. This map Φ depends algebraically on P_1, P_2, P_3 and Φ gives an isomorphism $\Psi: \mathcal{M}(P_1, P_2, P_3)\to\mathcal{M}(O, E_1, E_2)$. Thus it gives the triviality and the assertion follows immediately. Q.E.D.

We study the typical fiber $\mathcal{M}(O, E_1, E_2)$.

PROPOSITION (3.16). *$\mathcal{M}(O, E_1, E_2)$ is an irreducible rational surface.*

PROOF. We prove the assertion by a direct calculation. Let

$$f(x, y)=\sum_{i+j\leq 4} a_{i,j}x^i y^j$$

We have 15 coefficients and our moduli space $\mathcal{M}(O, E_1, E_2)$ is a subvariety of $\boldsymbol{P}^{14}$ whose homogeneous coordinates are $\{a_{i,j};\ i+j\leqq 4\}$. As the curve $f=0$ has three cusps at (0, 0), (1, 0) and (0, 1), the following conditions are necessary:

$$\begin{aligned} a_{0,0}=a_{1,0}=a_{0,1}=a_{3,0}+2a_{2,0}=a_{4,0}-a_{2,0}=a_{0,3}+2a_{0,2}\\ =a_{0,4}-a_{0,2}=a_{1,1}+a_{2,1}+a_{3,1}=a_{1,1}+a_{1,2}+a_{1,3}=0 \end{aligned} \tag{3.16.1}$$

$$4a_{2,0}a_{0,2}=a_{1,1}^2 \tag{3.16.2}$$

$$4a_{2,0}(a_{0,2}+a_{1,2}+a_{2,2})=(a_{1,1}+2a_{2,1}+3a_{3,1})^2 \tag{3.16.3}$$

$$4a_{0,2}(a_{2,0}+a_{2,1}+a_{2,2})=(a_{1,1}+2a_{1,2}+3a_{1,3})^2 \tag{3.16.4}$$

We denote the projective variety defined by (3.16.1)~(3.16.4) by $\mathcal{N}$. First we isomorphically project the variety $\mathcal{N}$ into $\boldsymbol{P}^7$ whose homogeneous coordinates are $a_{2,0}, a_{1,1}, a_{1,2}, a_{1,3}, a_{0,2}, a_{2,1}, a_{2,2}, a_{3,1}$. Now we take the following new linear coordinate: $a_{2,0}, a_{1,1}, a_{0,2}, a_{2,2}, b_1, b_2, c_1, c_2$ where

$$b_1=a_{1,1}+a_{2,1}+a_{3,1},\qquad b_2=a_{1,1}+2a_{2,1}+3a_{3,1}$$
$$c_1=a_{1,1}+a_{1,2}+a_{1,3},\qquad c_2=a_{1,1}+2a_{1,2}+3a_{1,3}$$

Again we project our variety $\mathcal{N}$ into $\boldsymbol{P}^5$ whose coordinates are $a_{2,0}, a_{1,1}, a_{0,2}, a_{2,2}, b_2, c_2$ and our variety is described by three quadratic equations:

(3.16.5) $$b_2^2 = 4a_{2,0}(a_{0,2}-c_2-2a_{1,1}+a_{2,2})$$

(3.16.6) $$c_2^2 = 4a_{0,2}(a_{2,0}-b_2-2a_{1,1}+a_{2,2})$$

(3.16.7) $$a_{1,1}^2 = 4a_{2,0}a_{0,2}$$

From (3.16.5) and (3.16.6), we obtain

$$4(b_2^2-a_{1,1}^2)a_{0,2}^2+4a_{0,2}a_{1,1}^2(c_2-b_2)-(c_2^2-a_{1,1}^2)a_{1,1}^2 = 0\,.$$

This equality splits into two irreducible factors:

(3.16.8) $$2(b_2+a_{1,1})a_{0,2}-a_{1,1}(a_{1,1}+c_2) = 0$$

(3.16.9) $$2(b_2-a_{1,1})a_{0,2}-a_{1,1}(a_{1,1}-c_2) = 0$$

Thus the variety $\mathcal{N}$ is the union of two rational surfaces:

(3.16.10) $$\mathcal{N}_1 = \{(a_{2,0},\, a_{1,1},\, a_{0,2},\, a_{2,2},\, b_2,\, c_2) \in \boldsymbol{P}^5;\ (3.16.5)\sim(3.16.8)\}$$

(3.16.11) $$\mathcal{N}_2 = \{(a_{2,0},\, a_{1,1},\, a_{0,2},\, a_{2,2},\, b_2,\, c_2) \in \boldsymbol{P}^5;\ (3.16.5)\sim(3.16.7),\ (3.16.9)\}$$

To see their rationalities, we use the affine coordinates: $U=\{a_{1,1}\neq 0\}$ and $x=b_2/a_{1,1}$, $y=c_2/a_{1,1}$, $z=a_{0,2}/a_{1,1}$, $v=a_{2,0}/a_{1,1}$, $w=a_{2,2}/a_{1,1}$. Then we have

$$\begin{aligned}\mathcal{N}_1\cap U &= \{(s,\, t,\, z,\, v,\, w);\ 2(s+1)z-(1+t)=0,\ 4zv=1,\ 4v(z-2-t-w)=s^2\}\\ &\cong \{(s,\, t,\, z)\in \boldsymbol{C}^3;\ 2(s+1)z-(1+t)=0,\ z\neq 0\}\\ \mathcal{N}_2\cap U &= \{(s,\, t,\, z,\, v,\, w);\ 2(s-1)z-(1-t)=0,\ 4zv=1,\ 4v(z-2-t-w)=s^2\}\\ &\cong \{(s,\, t,\, z)\in \boldsymbol{C}^3;\ 2(s+1)z-(1+t)=0,\ z\neq 0\}\end{aligned}$$

This expression implies the irreducibility and rationality of $\mathcal{N}_1$ and $\mathcal{N}_2$ respectively. To complete the proof, we have to prove the following.

SUBLEMMA (3.17). (1) *$\mathcal{N}_1$ is the variety of the square of the polynomials of degree 2 which pass through O, E_1, E_2.*

(2) *The closure of $\mathcal{M}(O, E_1, E_2)$ is $\mathcal{N}_2$.*

PROOF. The curves of degree 2 which pass through O, E_1, E_2 are given by $k(x, y)=ax(x-1)+by(y-1)+cxy$. Let $f(x, y)=k(x, y)^2$. It is a direct calculation to see that $f\in\mathcal{N}_1$. The dimension of this non-reduced polynomial is also 2. Thus $\mathcal{N}_1$ consists of these polynomials. To prove the second assertion, we notice that the symmetric curve defined by $\tilde{A}=\{p^*h(u, \sqrt{g})=0\}$ where $h(u, g)$ is defined by (3.7), (3.9), (3.10) has three cusps at O, E_1, E_2. Therefore we have that $p^*h\in\mathcal{N}_2$. As the example (3.11) gives a smooth point of $\mathcal{N}_2$, the assertion (2) follows.

APPENDIX (3.B). We give an example of the strange monodromy relation. Let us consider a non-singular affine plane curve $C=\{(x, y)\in\boldsymbol{C}^2;\ y(xy-1)=0\}$.

C has two non-singular components which are tangent at infinity. Now we claim that $\pi_1(C^2-C)\cong Z\times Z$. In fact, the pencil $\{x=\alpha\}$ has only one critical value $\alpha=0$ which comes from infinity. We have the same strange monodromy relation. Let $X=C^2-C$. As a subvariety of $C\times C^*$, X can be described by $X=\{(x, t)\in C^2;\ x-t\neq 0, t\neq 0\}=C^2-\{t(x-t)=0\}$ where $t=y^{-1}$. This last expression explains well why $\pi_1(X)$ is abelian.

APPENDIX (3.C). In this appendix, we give a rough sketch of the proof of the irreducibility of the moduli space of the symmetric curve of order 5 with 4 cusps. As we only consider the symmetric curve, we consider the moduli space

$$\mathcal{M}' = \{(C, P, Q);\ \deg C = 4,\ P, Q:\ \text{cusps}\}$$

of the curves $C=\{h(u, g)=0\}$ in the base space with two cusps P, Q. It is easy to see that P, Q are not on a vertical line $\{u=c\}$. Let $\mathcal{M}(P, Q)$ be the moduli space of curves of degree 4 with two cusps at P, Q being fixed. There is a unique linear change of coordinates Φ: $U=au+b$, $G=cg+du+e$ with $a, c\neq 0$ so that $\Phi(P)=(-1, 1)$ and $\Phi(Q)=(1, 1)$. Thus $\mathcal{M}'$ is a product of $\mathcal{M}((-1, 1), (1, 1))$ and $C^2\times C^2-\Delta$ where Δ is the subset of two points of $C^2\times C^2$ which are on a vertical line. By an easy calculation, we can show that $\mathcal{M}((-1, 1), (1, 1))$ is isomorphic to the irreducible rational 3-variety:

$$\{(s_1, s_2, s_3, s_4, s_5, s_6)\in C^6;\ 4s_1s_2=s_3^2,\ 4s_4s_5=s_6^2\}$$

As the calculation is completely parallel to that of Appendix (3.A), we omit the detail.

§ 4. Cuspidal curves of degree 6.

In this section, we consider the curve of degree six. By (3.4), we can have at most 6 independent cusps for a symmetric curve $\tilde{C}$ of degree 6. Assume that $\tilde{C}$ has six cusps. As a curve C in the base space, there are obviously two possible cases by Corollary (2.11).

Type I. C has three cusps ($s=3$).

Type II. C has two cusps and two D-flex of order 1 ($s=2$, $t_3=2$).

First note that any symmetric curve $k(x, y)=0$ of degree 2 is a lifting of a curve of symmetric degree 2 in the base space. A curve of symmetric degree 2 in the base space is a parabola with vertical axis: $E=\{k(u, g)=au^2+bu+cg+d=0\}$. This containts two parallel vertical lines $(u-\alpha)(u-\beta)=0$ or a horizontal line $g-\gamma=0$ as special cases. Hereafter we call a curve of symmetric degree 2 a symmetric conic. For any given three points $P, Q, R\in C^2$, we can find a symmetric conic E which contains P, Q, R. Thus the six cusps of $\tilde{C}$ of Type I lies on the conic $p^{-1}(E)$. This is the class which was studied by Zariski in

[**Z1**]. See also [**O2**]. Assume that C is of Type II and let Q_1, Q_2 be two D-flex points of order 1 and let P_1, P_2 be the cusps of C. There are two different families depending on whether these four points are on a symmetric conic or not. These two families have in fact completely different natures.

Type (II.1). The four points P_1, P_2, Q_1, Q_2 are on a symmetric conic.

Type (II.2). The four points P_1, P_2, Q_1, Q_2 are not on a symmetric conic. Note that the case (II.2) occurs for a generic choice of P_1, P_2 and Q_1, Q_2. Thus Type (II.2) corresponds to the case of non-conical six cusps in [**Z1**]. The Join type construction is very useful to give nice examples in the cases of Type I or Type (II.1).

CASE I. Conical 6 cuspidal curves of degree 6.

We consider the symmetric curves of degree 6 with three cusps. First we observe that for given three points, the moduli of curves with three cusps at prescribed points has dimension 3. (There exists no curve of degree 6 with two cusps in a vertical line.) Thus the dimension of the moduli space is $3+2\times 3=9$ and the moduli space is described by

$$h_2(u, g)^3+h_3(u, g)^2=0 \tag{4.1}$$

where

$$h_2(u, g)=t_0g^2+t_1u^2+t_2u+t_3$$
$$h_3(u, g)=(t_4u+t_5)g+t_6u^3+t_7u^2+t_8u+t_9$$

First we propose to study a subfamily C_I of Type I which is defined by a Join type polynomial:

$$C_I:\ h(u, g)=(g-\gamma)^3-\delta(u-\alpha_1)^2(u-\alpha_2)^2(u-\alpha_3)^2 \tag{4.2}$$

where $\gamma, \delta\in \boldsymbol{C}^*$ and $\alpha_1, \alpha_2, \alpha_3$ are mutually distinct. C_I has three cusp singularities: (α_1, γ), (α_2, γ), (α_3, γ). Now we consider the possible D-flex. Let $\phi(u)=\delta(u-\alpha_1)^2(u-\alpha_2)^2(u-\alpha_3)^2$. If C_I is tangent to D at $(\beta, 0)$, then β must satisfy:

$$\begin{cases} \dfrac{d\phi}{du}(\beta)=\delta\Pi_{i=1}^3(\beta-\alpha_i)(3\beta^2-2(\alpha_1+\alpha_2+\alpha_3)\beta+\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1) \\ \phi(\beta)=-\gamma^3 \end{cases} \tag{4.3}$$

Let β_1 and β_2 be the roots of the first equation:

$$3u^2-2(\alpha_1+\alpha_2+\alpha_3)u+\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1=0 \tag{4.4}$$

There are four possible subcases which give non-isotopic $\tilde{C}_I$ of type I.

(I.1) $\phi(\beta_j)\neq-\gamma^3$ for $j=1, 2$. In this case, D is transverse to C_I. $\tilde{C}_I$ has only six cusps.

(I.2) $\beta_1 \neq \beta_2$ and $\phi(\beta_1)=-\gamma^3$ and $\phi(\beta_2)\neq-\gamma^3$. In this case, we have a tangent $(0, \beta_1)$ to the discriminant curve. Thus $s=3$, $t_2=1$. The lifted curve $\tilde{C}_I$ has six cusps and one node.

(I.3) $\beta_1 \neq \beta_2$ and $\phi(\beta_1)=\phi(\beta_2)=-\gamma^3$. In this case, $\tilde{C}_I$ gets two nodes besides 6 cusps.

(I.4) $\beta_1=\beta_2$ and $\phi(\beta_1)=-\gamma^3$. In this case, $\tilde{C}_I$ has 7 cusps.

There exist curves of Type (I.i) for $i=1, \cdots, 4$ and a curve of type (I.(i+1)) can be a degeneration of a family of curves of the preceding type (I.i) for $i=1, \cdots, 3$.

THEOREM (4.5). *Let C_I be a curve defined by* (4.2). *Then* $\pi_1(\boldsymbol{C}^2-C_I)\cong\pi_1(\boldsymbol{C}^2-\tilde{C}_I)$ *and they are isomorphic to the braid group B_3. The fundamental group $\pi_1(\boldsymbol{P}^2-\tilde{C}_I)$ is isomorphic to $\boldsymbol{Z}_2 * \boldsymbol{Z}_3$.*

PROOF. The last statement is proved by Zariski [**Z1**]. To give a clear geometric image, we first study a curve of type (I.3), though it is enough to consider the case of (I.4) for the proof of the assertion. Let us consider the following simple curve:

$$C_I:\ h(u, g)=(g+1)^3-\frac{27}{4}u^2(u^2-1)^2. \tag{4.6}$$

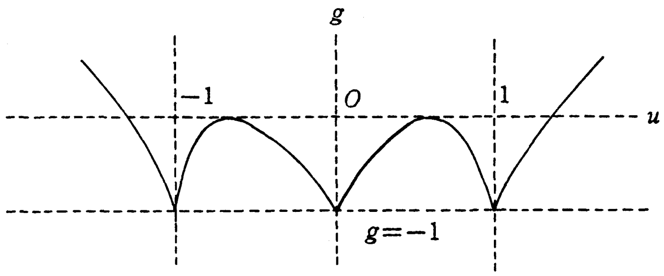


Figure (4.J)

C_I has also two D-flexes of order 0: $(\pm 1/\sqrt{3}, 0)$. Figure (4.J) is the real graph of C_I. We consider the pencil $\{q^{-1}(c)=\{u=c\}\}$ for $c\in\boldsymbol{C}$. We first consider $\pi_1(\boldsymbol{C}^2-C_I)$. It has three critical fibers $u=\pm 1$ and $u=0$. We take a system of generators ρ_1, ρ_2, ρ_3 in the fiber $q^{-1}(1+\varepsilon)$ (ε: small enough) as in Figure (4.K). The monodromy relation at $u=1$ is given by

$$\rho_1=\rho_3, \qquad \rho_1\rho_2\rho_1=\rho_2\rho_1\rho_2 \tag{R_1}$$

We move the fiber along the real axis. It is easy to see that the other monodromy relations at $u=0$ and $u=1$ are the same with (R_1). Thus we have that

$$\pi_1(\boldsymbol{C}^2-C_I)=\langle\rho_1, \rho_2;\ \rho_1\rho_2\rho_1=\rho_2\rho_1\rho_2\rangle \tag{4.7}$$

This is the standard representation of the Braid group of three strings B_3([**A**]).

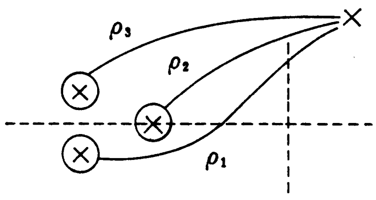


Figure (4.K) ($u=1+\varepsilon$)

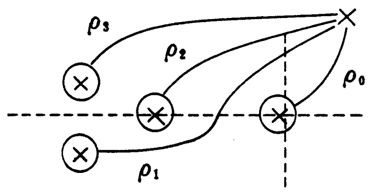


Figure (4.L) ($u=1+\varepsilon$)

Now we consider the fundamental group $\pi_1(\boldsymbol{C}^2-C_I\cup D)$. As generators, we have to add one more loop ρ_0 in the same fiber $q^{-1}(1+\varepsilon)$ as in Figure (4.L). We have to consider also monodromy relations around $u=\pm 2/\sqrt{3}$ where $\boldsymbol{C}_I$ intersects transversally with $D=\{g=0\}$ and the monodromy relation around $u=\pm 1/\sqrt{3}$ where our curve is simply tangent with D. The monodromy relation at $u=2/\sqrt{3}$ is:

$$(R_2) \qquad \rho_0\rho_2=\rho_2\rho_0$$

The monodromy relation at $u=1$ is the same with the above (R_1). Now to consider the monodromy relation around $u=1/\sqrt{3}$ and $u=0$, we deform our generators along the real axis from $u=1-\varepsilon$ to $u=\varepsilon$. At the critical fiber, we take upper half semi-circle. First we get the following relation at $u=1/\sqrt{3}$: $\rho_1=\tau^2\rho_1\tau^{-2}$ where $\tau=\rho_1\rho_0$. Namely

$$(R_3) \qquad \rho_1\rho_0\rho_1\rho_0=\rho_0\rho_1\rho_0\rho_1.$$

Our generators are deformed in the fiber $q^{-1}(\varepsilon)$ as in Figure (4.M). To see the monodromy relation at $u=0$, we consider the loop $\rho_1'=\rho_0^{-1}\rho_1\rho_0$ instead of ρ_1. See Figure (4.M). Then monodromy relation at $u=0$ gives

$$(R_4) \qquad \rho_1'=\rho_3, \qquad \rho_2\rho_3\rho_2=\rho_3\rho_2\rho_3$$

By virtue of (R_1), this relation implies the commutation relation:

$$(R_5) \qquad \rho_0\rho_1=\rho_1\rho_0.$$

Thus combining (R_5) with (R_2), we conclude that ρ_0 is in the center of $\pi_1(\boldsymbol{C}^2-C_I\cup D)$. It is obvious that ρ_0 is the generator of the infinite cyclic group $\pi_1(\boldsymbol{C}^2-D)$. Therefore we have the isomorphism:

$$\phi:\ \pi_1(\boldsymbol{C}^2-C_I\cup D)\cong\pi_1(\boldsymbol{C}^2-C_I)\times\pi_1(\boldsymbol{C}^2-D)$$

Now by Theorem (2.3), we get the isomorphism:

$$p_{\#}:\ \pi_1(\boldsymbol{C}^2-\tilde{C}_I)\longrightarrow\pi_1(\boldsymbol{C}^2-C_I)$$

which proves the first assertion.

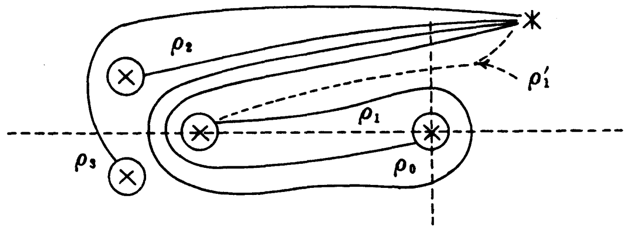


Figure (4.M)

For the fundamental group $\pi_1(\boldsymbol{P}^2-\tilde{C}_I)$ in the projective space, we need one more relation:

$$(R_5) \qquad \rho_3\rho_2\rho_1\rho_3\rho_2\rho_1 = e$$

which corresponds to the vanishing of a big circle in $\tilde{q}(1+\varepsilon)$. Replacing ρ_3 by ρ_1, we get

$$(4.8) \qquad \pi_1(\boldsymbol{P}^2-\tilde{C}_I) = \langle \rho_1, \rho_2;\ \rho_1\rho_2\rho_1 = \rho_2\rho_1\rho_2,\ \ (\rho_1\rho_2)^3 = e\rangle$$

We consider the element $\omega=\rho_1\rho_2\rho_1$, $\zeta=\rho_1^{-1}\omega$ for the determination of the group structure. Then we have the isomorphisms:

$$\begin{aligned}\pi_1(\boldsymbol{P}^2-\tilde{C}_I) &= \langle \rho_1, \rho_2;\ \rho_1\rho_2\rho_1 = \rho_2\rho_1\rho_2,\ (\rho_2\rho_1)^3 = e\rangle \\ &= \langle \rho_1, \rho_2, \omega;\ \omega = \rho_1\rho_2\rho_1,\ \omega^2 = (\rho_1^{-1}\omega)^3 = e\rangle \\ &= \langle \rho_1, \rho_2, \omega, \zeta;\ \omega^2 = \zeta^3 = e,\ \omega = \rho_1\rho_2\rho_1,\ \zeta = \rho_1^{-1}\omega\rangle \\ &= \langle \rho_1, \rho_2, \omega, \zeta;\ \omega^2 = \zeta^3 = e,\ \rho_1 = \omega\zeta^{-1},\ \rho_2 = \zeta^2\omega^{-1}\rangle \\ &= \langle \omega, \zeta;\ \omega^2 = \zeta^3 = e\rangle \\ &= \boldsymbol{Z}_2 * \boldsymbol{Z}_3 .\end{aligned}$$

This completes the proof of Theorem (4.5) except for Type (I.4).

Now we consider the curve of type (I.4) by the following example:

$$(4.9) \qquad C_I' = \{(u, g);\ h(u, g) = (g+4)^3-(u^2+3)^2(u-3)^2\}\,.$$

Figure (4.N) is the real graph of C_I'. As two cusps $(\pm\sqrt{3}i, -4)$ are not visible in the real section, we have to pay more attention to the calculation of the fundamental group $\pi_1(\boldsymbol{C}^2-C_I'\cup D)$. Singular fibers are $u=\pm\sqrt{3}i$, 3, $u=1$ and $u=\alpha_1, \alpha_2, \bar{\alpha}_2$ where $\alpha_1=1+\sqrt[3]{16}$ and $\alpha_2=1+\sqrt[3]{16}\exp(2\pi i/3)$. The last three fibers pass each of the three transverse intersections with D. We take a system of generators $\rho_0, \rho_1, \rho_2, \rho_3$ of $\pi_1(\boldsymbol{C}^2-C_I'\cup D)$ in the fiber $q^{-1}(3+\varepsilon)$ as in Figure (4.O).

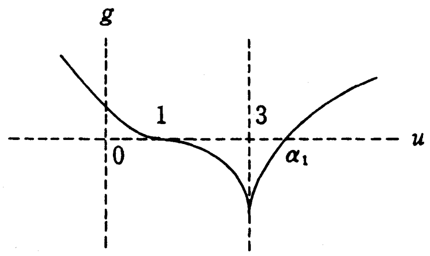


Figure (4.N)

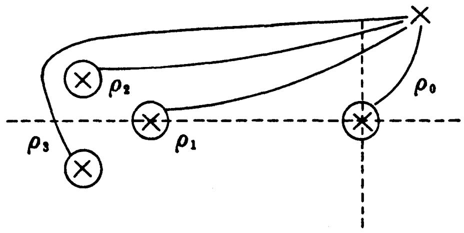


Figure (4.O) ($u=3+\varepsilon$)

We study the monodromy relations around $u=3$, $1+\alpha$, 1 and $\sqrt{3}i$. The other monodromy relations are not necessary for our purpose. First the monodromy relation at $u=\alpha_1$ is:

(R_1) $$\rho_0\rho_1=\rho_1\rho_0.$$

The monodromy relation at $u=3$ is the cusp relation:

(R_2) $$\rho_1=\rho_3, \qquad \rho_1\rho_2\rho_1=\rho_2\rho_1\rho_2$$

To see the monodromy at $u=1$, we first deform our generators along the real axis to the fiber $q^{-1}(1+\varepsilon)$. See figure (4.P). At $u=3$ we take the upper semicircle. We introduce a loop ρ_3' in the fiber $q^{-1}(1+\varepsilon)$ by

(R_3) $$\rho_3'=(\rho_3\rho_2\rho_1)^{-1}\rho_1(\rho_3\rho_2\rho_1)$$

By (R_2), we have that

(R_3') $$\rho_3'=\rho_2$$

The monodromy relation around $u=1$ is:

(R_4) $$\rho_3'=\tau^3\rho_3'\tau^{-3}, \qquad \tau=\rho_3'\rho_0$$

Using (R_3'), we can reduce the relation (R_4) to

(R_4') $$(\rho_2\rho_0)^3=(\rho_0\rho_2)^3$$

Compare this relation with (R_3) in $\pi_1(\boldsymbol{C}^2-C_I)$.

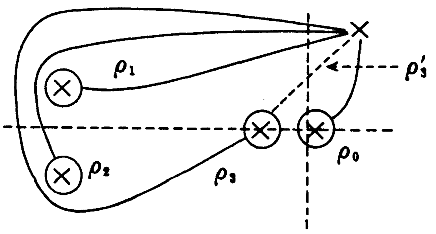


Figure (4.P) ($u=1+\varepsilon$)

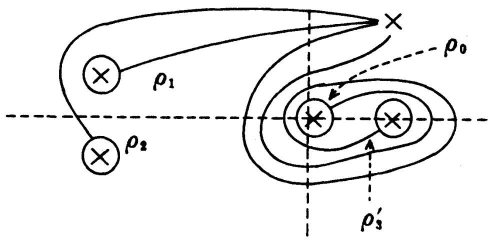


Figure (4.Q) ($u=0$)

Now we study the monodromy relation at $u=\sqrt{3}i$. First in the fiber $q^{-1}(0)$, our generators are deformed as in Figure (4.Q). Now we move u along the imaginary axis to $u=\sqrt{3}i$. The motion of three intersections $C_I'\cap q^{-1}(u)$ is a bit complicated. Let $\phi(u)=(u^2+3)^2(u-3)^2$. Consider the path $u(t)=ti$, $0\leqq t\leqq(\sqrt{3}-\varepsilon)$. Then we can see easily that (1) $|\phi(u(t))|$ is monotone decreasing and (2) $\arg(\phi(u(t)))$ decreases monotonically from 0 to $\arg(4-2\sqrt{3}i)$. From this information, we can see that the three intersection points move as the arrows in Figure (4.R). We use the loops ρ_3'' and ρ_0'' instead of ρ_3' and ρ_0 where

$$(R_5)\qquad \begin{cases} \rho_3' = \tau^2\rho_3''\tau^{-2} \\ \rho_0 = \tau\rho_0''\tau^{-1} \end{cases}$$

and $\tau=\rho_3'\rho_0=\rho_0''\rho_3''=\rho_2\rho_0$. The cusp relation at $u=\sqrt{3}i$ gives: $\rho_0''\rho_3''\rho_0''^{-1}=\rho_2$. Rewriting this in the original generators, we get

$$(R_6)\qquad \rho_0\rho_2=\rho_2\rho_0$$

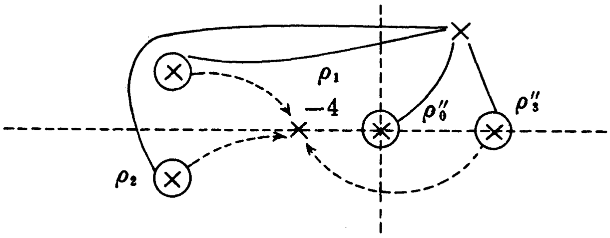


Figure (4.R)

By (R_1), (R_2) and (R_6), ρ_0 is in the center. Therefore by Theorem (2.3), we have the isomorphism: $\pi_1(\boldsymbol{C}^2-\tilde{C}_I')\cong\pi_1(\boldsymbol{C}^2-C_I')$. Obviously the isotopy type of the complement $\boldsymbol{C}^2-C_I'$ does not depend on Types (I.1)$\sim$(I.4). Thus $\pi_1(\boldsymbol{C}^2-\tilde{C}_I')\cong\pi_1(\boldsymbol{C}^2-C_I')\cong\boldsymbol{Z}$. Q. E. D.

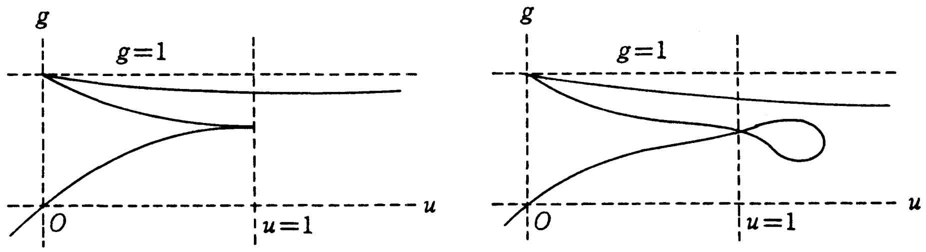
The importance of the curve $\tilde{C}_I'$ is that $\tilde{C}_I'$ is also a degeneration of non-conical 6 cuspidal curves which will be studied in the next section. In fact, any five cusps among the 6 cusps which go down to one of the three cusps and the cusp which project to the D-flex $(1, 0)$ is not conical. We can deform the curve $\tilde{C}_I'$ keeping these singularities and push out the last cusp to get non-conical 6 cuspidal curves.

Further degeneration.

We consider the further degeneration of C_I. Taking the result of Zariski ([**Z2**]) for the fundamental group of the complement of rational curves into account, we can see that C_I can not be degenerated into a rational curve. Therefore C_I can get at most 3 nodes or cusps. In fact, these degenerations are all possible. For instance, let

$$(4.10)\qquad C_I(t):\ w^3+\left\{\frac{8}{27(3t-1)}u\left(u^2-\frac{3}{2}(1+t)u+3t\right)+w\right\}^2$$

where $w=g-1$. The following is the graph of $C_I(1)$: $w^3+\{(4/27)u(u^2-3u-3)+w\}^2=0$ and $C_I(3/2)$: $w^3+\{(16/189)u(u^2-(15/4)u+(9/2)+w\}^2=0$. It is easy to see that $C_I(t)$ has 3 cusps at $(0, 1)$, $(\alpha(t), 1)$, $(\beta(t), 1)$ and a node for $t\neq 1, 1/3, 3$ (respectively a cusp for $t=1$) at $(1, 5/9)$ where $\alpha(t)$ and $\beta(t)$ are roots of $u^2-3/2(1+t)u+3t=0$. Taking an admissible quadratic change of coordinates, we can also put a D-flex of order 0 or 1 as we like. For example, the horizontal line $g-8/9=0$ is tangent to $C_I(1)$ with intersection multiplicity 3. So we can simply take the linear change of coordinates: $U=u$, $G=g-8/9$ to put one more cusp in $\widetilde{C_I}(1)$. It is easy to see that $\pi_1(\boldsymbol{C}^2-\widetilde{C_I(3/2)})\cong\pi_1(\boldsymbol{C}^2-C_I(3/2))\cong B_3$ using the pencils $\{u=\theta\}$. On the other hand, the fundamental group $\pi_1(\boldsymbol{P}^2-\widetilde{C_I}(1))$ seems much bigger than $\boldsymbol{Z}_2*\boldsymbol{Z}_3$. See Appendix (4.A).



$C_I(1)$ $C_I(3/2)$

Figure (4.S)

CASE (II.1). We consider the curve C_{II} with two cusps and two D-flexes of Type (II.1). The generic curve in this class is described by the polynomial

$$(4.11)\qquad C_{II}:\ gh_2(u, g)^2+j_2(u, g)^3=0$$

where $h_2(u, g)=a_0g+a_1u^2+a_2u+a_3$ and $j_2(u, g)=b_0g+b_1u^2+b_2u+b_3$. C_{II} has two cusps at $h_2=j_2=0$ and two D-flexes at $j_2(u, 0)=g=0$. It has seven free parameters. We give an easy example below and leave the detail to the reader.

$$C_{II}=\{(u, g)\in\boldsymbol{C}^2;\ h(u, g)=g(g-1)^2+b(u^2-1)^3=0\}$$

Figure (4.T) is the real graph of C_{II}. A curve $\tilde{C}_I$ of type I and a curve $\widetilde{C_{II}}$ of type (II.1) can be joined by a family of curves with conical six cusps in the source space breaking the symmetricity. Thus the fundamental group is same as in Case I. This can be also checked directly. Note that $\pi_1(\boldsymbol{C}^2-C_{II})\cong\boldsymbol{Z}$. This can be checked easily by the pencils $\{g=c\}$. Thus C_{II} gives an example of symmetric a curve C for which

$$p_{\#}: \pi_1(\boldsymbol{C}^2-\tilde{C}) \longrightarrow \pi_1(\boldsymbol{C}^2-C) \cong \boldsymbol{Z}$$

is not bijective.

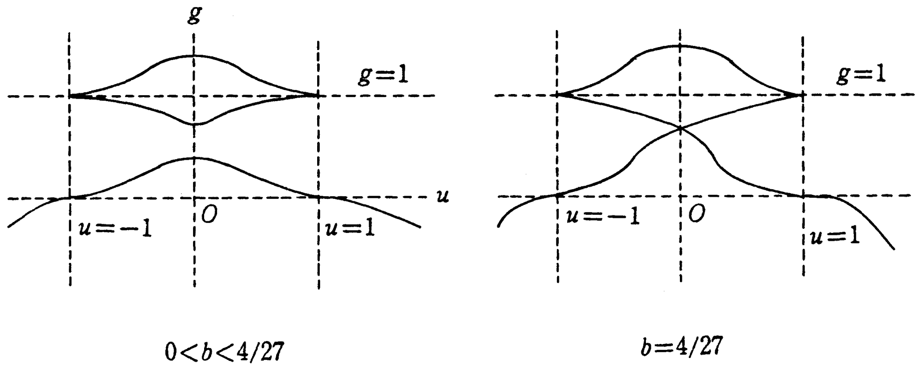


Figure (4.T)

We close this section by a conjecture.

CONJECTURE. *The fundamental group of the complement of a curve does not change by a degeneration which puts only nodes.*

APPENDIX (4.A). We consider the fundamental group of the complement of

$$\widetilde{C_I(1)}:\ w^3+\left\{\frac{4}{27}u(u^2-3u+3)+w\right\}^2=0, \qquad w=g-1$$

through the pencils $\{\sqrt{g}=c\}$. We can rewrite the equation as $w^3+\{4/27(u-1)^3+w+(4/27)\}^2=0$. Thus it is easy to see that we have the following singular fibers: $\sqrt{g}=\pm 1$, $\pm 2\sqrt{2}/3$, $\pm\sqrt{5}/3$. The pencil $\sqrt{g}=\pm 1$ passes three cusps and $\{\sqrt{g}=\pm\sqrt{5}/3\}$ passes the seventh and the eighth cusps $(1, \pm\sqrt{5}/3)$ respectively. The line $\{\sqrt{g}=\pm 2\sqrt{2}/3\}$ is tangent to $\widetilde{C_I(1)}$ at $(1, \pm 2\sqrt{2}/3)$. We take generators $\rho_1, \cdots, \rho_6$ in the pencil $\sqrt{g}=2\sqrt{2}/3+\varepsilon$ as in Figure (4.U). We get following relations.

(R_0) $\quad \rho_4\rho_1\rho_2\rho_3\rho_5\rho_6=e$

(R_1) $\quad \rho_1=\rho_2=\rho_3$ at $\sqrt{g}=\pm 2\sqrt{2}/3$

(R_2) $\quad \{\rho_1, \rho_4\}=\{\rho_2, \rho_5\}=\{\rho_5^{-1}\rho_3\rho_5, \rho_6\}=e$ at $\sqrt{g}=\pm 1$

(R_3) $\quad \rho_2^{-1}\rho_4\rho_2=(\rho_5^{-1}\rho_3\rho_5)\rho_6(\rho_5^{-1}\rho_3\rho_5)^{-1}$, $\{\rho_2^{-1}\rho_4\rho_2, \rho_5\}=e$ at $\sqrt{g}=\pm\sqrt{5}/3$

where we use the notation $\{a, b\}=abab^{-1}a^{-1}b^{-1}$ for the cusp relation.

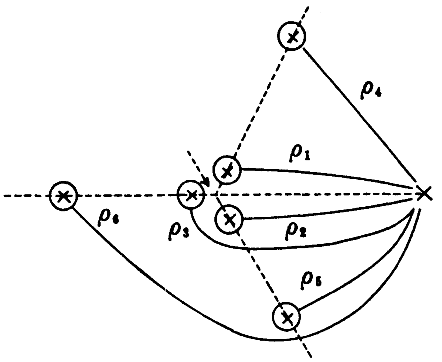


Figure (4.U) ($\sqrt{g}=2\sqrt{2}/3+\varepsilon$)

The equality $g\{a, b\}g^{-1}=\{gag^{-1}, gbg^{-1}\}$ is often useful. We introduce the new elements $\xi_1=\rho_5^{-1}\rho_1\rho_5$, $\xi_2=\rho_5$ and $\xi_3=\xi_1\rho_6\xi_1^{-1}$. Then we can write the old generators as

$$\rho_1=\rho_2=\rho_3=\xi_2\xi_1\xi_2^{-1}, \quad \rho_4=(\xi_2\xi_1\xi_2^{-1})\xi_3(\xi_2\xi_1\xi_2^{-1})^{-1}, \quad \rho_6=\xi_1^{-1}\xi_3\xi_1$$

and the generating relations in the new generators are:

(R_0') $\quad (\xi_2\xi_1\xi_3)^2=e \quad$ from (R_0)

(R_4) $\quad \{\xi_1, \xi_2^{-1}\xi_3\xi_2\}=e \quad$ from $\{\rho_1, \rho_4\}=e$

(R_5) $\quad \{\xi_1, \xi_2\}=\{\xi_1, \xi_3\}=\{\xi_3, \xi_2\}=e \quad$ from $(R_2), (R_3)$

Now (R_4) follows from other relations. In fact, using $(R_0')+(R_5)$

$$\xi_1(\xi_2^{-1}\xi_3\xi_2)\xi_1=\xi_1\xi_2^{-1}\xi_1^{-1}\xi_2^{-1}\xi_3^{-1}=\xi_2^{-1}\xi_1^{-1}\xi_3^{-1}$$

$$(\xi_2^{-1}\xi_3\xi_2)\xi_1(\xi_2^{-1}\xi_3\xi_2)=\xi_2^{-1}\xi_1^{-1}\xi_2^{-1}\xi_3^{-1}\xi_2^{-1}\xi_3\xi_2=\xi_2^{-1}\xi_1^{-1}\xi_3^{-1}$$

As (R_0') is equivalent to $(\xi_3\xi_2\xi_1)^2=e$, we have proved that

(4.12) $\pi_1(\boldsymbol{P}^2-\widetilde{C_I}(1))=\langle\xi_1, \xi_2, \xi_3;\ \{\xi_1, \xi_2\}=\{\xi_2, \xi_3\}=\{\xi_1, \xi_3\}=(\xi_3\xi_2\xi_1)^2=e\rangle$

Using the representation (4.8), we get a canonical surjective homomorphism

$$\phi:\ \pi_1(\boldsymbol{P}^2-\widetilde{C_I}(1))\longrightarrow \boldsymbol{Z}_2*\boldsymbol{Z}_3=\langle\eta, \zeta;\ \{\eta, \zeta\}=e, (\eta\zeta)^3=e\rangle$$

by $\phi(\xi_1)=\phi(\xi_3)=\eta$ and $\phi(\xi_2)=\zeta$. We assert

PROPOSITION (4.11). *$\phi:\ \pi_1(\boldsymbol{P}^2-\widetilde{C_I}(1))\rightarrow\boldsymbol{Z}_2*\boldsymbol{Z}_3$ is surjective but not injective.*

PROOF. The surjectivity is obvious. We will show that $\xi_1\xi_3^{-1}\in Ker(\phi)$ is a non-trivial element of the kernel of ϕ. To show this, we consider the quotient group G' of $\pi_1(\boldsymbol{P}^2-\widetilde{C_I}(1))$ by putting another relation $\xi_2=\xi_3$. Then by easy calculation, we can see that

$$\begin{aligned} G' &= \langle \bar{\xi}_1, \bar{\xi}_2;\ \{\bar{\xi}_1, \bar{\xi}_2\} = (\bar{\xi}_1\bar{\xi}_2{}^2)^2 = e\rangle \\ &= \langle a, b, \bar{\xi}_1, \bar{\xi}_2;\ a^2 = b^3 = e, \bar{\xi}_1 = a(ba^{-1})^2, \bar{\xi}_2 = ab^{-1}\rangle \\ &= \boldsymbol{Z}_2 * \boldsymbol{Z}_3 \end{aligned}$$

Thus the image of $\xi_1\xi_3^{-1}$ in G' is equal to $a(ba^{-1})^3 \neq e$. This completes the proof. Q. E. D.

§5. Non-conical six cuspidal curves.

In this section, we consider the curve of Type (II.2) in §4. The description of the moduli space seems to be complicated. The main reason is the fact that such a curve does not have a good symmetry. For a technical reason, we consider one-parameter family of curves F_t of degree 6 with two cusps P_1, P_2 and two D-flexes $Q_{1,t}$, $Q_{2,t}$ of order 1 where $P_1=(0, 1)$, $P_2=(1, 1)$ and $Q_{1,t}=(-t, 0)$, $Q_{2,t}=(t, 0)$ for $t \in \boldsymbol{C}$. We have chosen four points P_1, P_2, $Q_{1,t}$, $Q_{2,t}$ so that they are not on a conic. As F_t has singularities at P_1 and P_2, we can write

$$\begin{aligned} h_t(u, g) = {} & u^2(u-1)^2(u^2+bu+c)+u(u-1)(du^2+eu+f)(g-1) \\ & +(\alpha u^2+\beta u+\gamma)(g-1)^2+\delta(g-1)^3 \end{aligned} \tag{5.1}$$

We normalize the coefficient of u^6 by the homogenuity. The cusp conditions at P_1 and P_2 are:

$$4c\gamma-f^2 = 0, \qquad 4(1+b+c)(\alpha+\beta+\gamma)-(d+e+f)^2 = 0 \tag{5.2}$$

The condition for $Q_{1,t}$ and $Q_{2,t}$ to be D-flexes of order 1 is:

$$h_t(u, 0) = (u^2-t^2)^3. \tag{5.3}$$

Therefore the coefficients must satisfy the equalities:

$$\begin{aligned} & b = 2,\ c-d = 3(1-t^2),\ e = 2-c-3(1-t^2), \\ & f = -\beta,\ \alpha = 1-\beta-3t^2+3t^4,\ \gamma = \delta-t^6 \end{aligned} \tag{5.4}$$

As the moduli is one dimensional, we may assume that $\alpha+\beta+\gamma=0$. This assumption implies by (5.2) that

$$\alpha+\beta+\gamma = 0, \qquad d+e+f = 0. \tag{5.5}$$

This assumption is equivalent to the constancy of the tangential cone of the cusp at P_2. Thus solving these equalities, we get

$$\begin{aligned} & \gamma = -1+3t^2-3t^4, \quad \delta = (t^2-1)^3, \quad \beta = -f = -4+6t^2 \\ & \alpha = 5-9t^2+3t^4, \quad c = f^2/4\gamma, \quad d = c-3+3t^2, \quad e = -1-c+3t^2 \end{aligned} \tag{5.6}$$

The diffeomorphism type of $\boldsymbol{C}^2-\tilde{F}_t$ does not depend on the choice of a generic

t but for the sake of the calculation of the fundamental group $\pi_1(\boldsymbol{C}^2-\tilde{F}_t)$, we choose $t=t_0=\sqrt{2/3}$ for which the calculation seems to be simpler. Thus we study the curve F_{t_0}. The curve F_{t_0} is defined by

$$(5.7)\quad u^2(u-1)^2(u^2+2u)-u^2(u-1)^2(g-1)+\frac{1}{3}(u^2-1)(g-1)^2-\frac{1}{27}(g-1)^3=0.$$

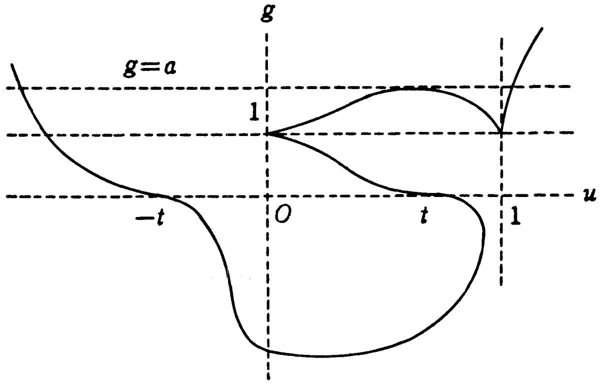


Figure (5.V)

Figure (5.V) shows the graph of the real section of F_{t_0}. We believe that F_{t_0} is non-singular outside of two cusps P_1, P_2, but to prove the smoothness, we have to take care of the discriminant polynomial of h_{t_0} as a polynomial of g which is a polynomial of degree 12 in the variable u. Instead of proving it, we will be satisfied by showing that *F_t is smooth outside its two cusps P_1, P_2 except for a finite number of t's.* We will give a proof of this assertion in Appendix (5.A). We assert

THEOREM (5.8). *$\pi_1(\boldsymbol{P}^2-\tilde{F}_{t_0})\cong\boldsymbol{Z}_6$ and $\pi_1(\boldsymbol{C}^2-\tilde{F}_{t_0})\cong\boldsymbol{Z}$. Thus the same assertion is true for generic F_t.*

PROOF. The commutativity of $\pi_1(\boldsymbol{C}^2-\tilde{F}_{t_0})$ follows immediately from the commutativity of $\pi_1(\boldsymbol{P}^2-\tilde{F}_{t_0})$ as the affine fundamental group $\pi_1(\boldsymbol{C}^2-\tilde{F}_{t_0})$ is abelian if and only if $\pi_1(\boldsymbol{P}^2-\tilde{F}_{t_0})$ is abelian by [O1]. As $F_{t_0}\cap D$ has only two D-flexes of order 1, the calculation of $\pi_1(\boldsymbol{C}^2-F_{t_0}\cup D)$ seems more difficult. Thus we consider the fundamental group $\pi_1(\boldsymbol{P}^2-\tilde{F}_{t_0})$ directly through the horizontal line sections $\tilde{F}_{t_0}\cap\{\sqrt{g}=s\}$, $s\in\boldsymbol{C}$. Note that the pair of spaces $(\boldsymbol{C}^2, \tilde{F}_{t_0})\cap\{\sqrt{g}=s\}$ is identical with $(\boldsymbol{C}^2, F_{t_0})\cap\{g=s^2\}$. In this sense, we only consider the real line sections of F_{t_0} with $g=s^2$, $s^2\geqq 0$. First we take our generators $\rho_1, \cdots, \rho_6$ in the line $\sqrt{g}=1+\varepsilon$ as in Figure (5.WI). (ε is a positive number which is small enough.)

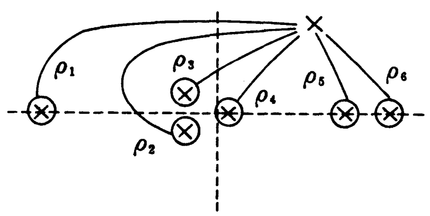


Figure (5.WI) ($\sqrt{g}=1+\varepsilon$)

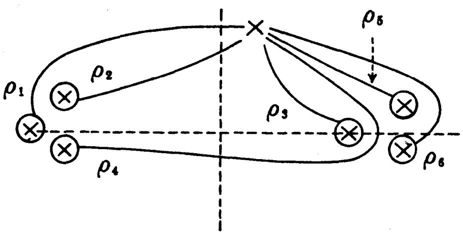


Figure (5.WII) ($\sqrt{g}=\varepsilon$)

As the global vanishing relation we have:

(R_0) $$\rho_1\rho_2\rho_3\rho_4\rho_5\rho_6=e$$

The monodromy relation around $\sqrt{g}=1$ is a pair of cusp relations:

(R_1) $$\rho_2=\rho_4,\qquad \rho_2\rho_3\rho_2=\rho_3\rho_2\rho_3,\qquad \rho_5\rho_6\rho_5=\rho_6\rho_5\rho_6$$

Let $a>1$ be the real number for which the horizontal line $g=a$ is simply tangent to F_{t_0} as in Figure (5.V). The monodromy relation at $\sqrt{g}=\sqrt{a}$ is:

(R_2) $$\rho_4=\rho_5.$$

Now we move the line $\sqrt{g}=s$ from $s=1+\varepsilon$ to $s=0$. From $s=1+\varepsilon$ to $s=1-\varepsilon$, we go clockwise around the small circle $|s-1|=\varepsilon$. We assert

ASSERTION. *At the level of* $\sqrt{g}=\varepsilon$, *our generators are deformed as in Figure* (5.WII).

This assertion is not obvious and we give a proof in the Appendix (5.B). Recall that Q_{1,t_0}, Q_{2,t_0} are cusps for $\tilde{F}_{t_0}$. The monodromy relation around $\sqrt{g}=0$ is:

(R_3) $$\begin{cases}\rho_1=\rho_3\rho_4\rho_3^{-1}, & \rho_2\rho_1\rho_2=\rho_1\rho_2\rho_1\\ \rho_3=\rho_6, & \rho_5\rho_3\rho_5=\rho_3\rho_5\rho_3\end{cases}$$

Now we have enough relations to prove the commutativity. We can take ρ_2 and ρ_3 as generators. Then we eliminate the generators $\rho_1, \rho_4, \rho_5, \rho_6$ using $(R_1)\sim(R_3)$ and we get:

$$\begin{aligned}e&=\rho_1\rho_2\rho_3\rho_4\rho_5\rho_6 &&\text{by } (R_0)\\ &=(\rho_3\rho_2\rho_3^{-1})\rho_2\rho_3\rho_2\rho_2\rho_3 &&\text{by } (R_1)\sim(R_3)\\ &=\rho_3\rho_2\rho_3^{-1}(\rho_3\rho_2\rho_3)\rho_2\rho_3 &&\text{by } (R_1)\\ &=\rho_3\rho_2(\rho_3\rho_2\rho_3)\rho_3 &&\text{by } (R_1)\\ &=\rho_3^2\rho_2\rho_3^3 &&\text{by } (R_1).\end{aligned}$$

Thus we get the relation: $\rho_2=\rho_3^{-5}$. Putting this in the cusp relation (R_1):

$\rho_2\rho_3\rho_2=\rho_3\rho_2\rho_3$, we get $\rho_3^6=e$. This implies that $\pi_1(\boldsymbol{P}^2-\tilde{F}_{t_0})\cong\boldsymbol{Z}_6$ as $H_1(\boldsymbol{P}^2-\tilde{F}_{t_0})=\boldsymbol{Z}_6$. This completes the proof. Q. E. D.

Among the curves of the family $\{F_t\}$, there are several interesting degenerations.

(I) For $t=1$, F_1 is degenerated into double of line $u=1$ and a curve A of degree 4:

$$A:\ h(u, g)=(g-1)^2+u(u-2)(g-1)-u^2(u^2+2u-1)$$

which has one cusp $P_1=(0, 1)$ and one D-flex of order 1 at $Q_{1,1}$. A is a curve considered in (B), § 3.

(II) Let $t=0$. Then F_0 is defined by

$$u^2(u-1)^2(u^2+2u-4)+u(u-1)(-7u^2+3u+4)(g-1)$$
$$+(5u^2-4u-1)(g-1)^2-(g-1)^3=0$$

The cusp P_1 is degenerated in a union of two parabolas (but we can only see an isolated point in the real graph as the local equation is equivalent $-(g-1+2u)^2-7u^4=0$. Two D-flexes are degenerated in one D-flex of order 4 at $O=(0, 0)$. Its real graph is as Figure (5.X).

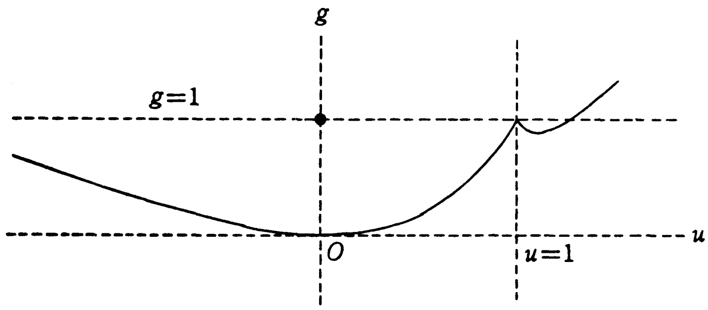


Figure (5.X)

(III) Comparing the graph of F_0 and F_{t_0}, we can see that there is $0<\theta<t_0$ so that D-flex $Q_{2,\theta}$ is degenerated into a cusp which is locally equivalent to $(u-t)^3-g^2=0$. Thus F_θ has three cusps P_1, P_2, $Q_{2,\theta}$ and one D-flex $Q_{1,\theta}$. (The cusp $Q_{2,\theta}$ is on the discriminant line D.) The lifting has 5 cusps and a cusp of general type which is locally isomorphic to $\{(u-t)^3-\sqrt{g}^4=0\}$. Considering the one parameter change of coordinates $\Psi_s=(U_s, G_s)$ $U_s=u$, $G_s=g-s$, we can also consider F_θ as a degeneration of 3 cuspidal curves $\Psi_s(F_{t_0})$, $s\in\boldsymbol{C}$ of Type I.

APPENDIX (5.A). We prove that F_t has only two singularities P_1, P_2 except for a finite number of t's. Let $\mathfrak{F}=\{(u, g, t)\in\boldsymbol{C}^2\times\boldsymbol{C};\ h(u, g, t)=0\}$. First we show that *the singurarity of* $\mathfrak{F}$ *on* $\mathfrak{F}\cap\{t=\sqrt{2/3}\}$ *is only* P_1 *and* P_2: Assume that $(u, g, \sqrt{2/3})$ be a singular point of $\mathcal{F}$. Then we have

$$(5.9)\quad h(u, g, \sqrt{2/3}) = \frac{\partial h}{\partial u}(u, g, \sqrt{2/3}) = \frac{\partial h}{\partial g}(u, g, \sqrt{2/3}) = \frac{\partial h}{\partial t}(u, g, \sqrt{2/3}) = 0.$$

By (5.6), the last equality gives:

$$(5.10)\qquad \frac{1}{3}(g-1)^3+(-5u^2+6u-1)(g-1)^2+u(u-1)(3u^2+3u-6) = 0.$$

Assume that $u(u-1)(g-1)=0$. We can see easily that there is no singularity other than P_1, P_2. Thus we assume that $u(u-1)(g-1)\neq 0$. By (5.10) and $\partial h/\partial g(u, g, \sqrt{2/3})=0$, we get $g-1=2u$. Substituting this in (5.9) and (5.10), we conclude that there exists no common solution.

Assume that F_t has a singularity which is different from P_1 and P_2 for any $t\in \boldsymbol{C}$. Then using Curve Selection Lemma ([**M**], [**H**]), we can take an analytic family of singularities R_t of F_t for $t_0-\varepsilon\leqq t\leqq t_0+\varepsilon$. Taking the differential of $h(R_t, t)\equiv 0$, we see that (R_t, t) is a singular point of $\mathcal{F}$. Assume that $\lim_{t\to t_0}\|R_t\|$ is finite. By the above consideration, we must have $\lim_{t\to t_0}R_t=P_1$ or P_2. However by the constancy of the local sum of the Milnor numbers, this is impossible. Thus R_t must disappear at infinity when t goes to t_0. However this is also impossible because the family $\{\tilde{F}_t;\ t_0-\varepsilon\leqq t\leqq t_0+\varepsilon\}$ is regular at infinity for a sufficiently small ε. As the closure of the set $\{t\in \boldsymbol{C}\}$ such that F_t has a singularity which is different from P_1, P_2 is algebraic, this completes the proof of the assertion. Q.E.D.

APPENDIX (5.B). We study how the six intersection points $F_{t_0}\cap\{\sqrt{g}=s\}$ move when s moves on the real interval $[0, 1]$ from $s=1$ to 0. Let $u_1(s), \cdots, u_6(s)$ be the intersection $\tilde{F}_{t_0}\cap\{\sqrt{g}=s\}$. We parametrize $u_1(s), \cdots, u_6(s)$ so that they are continuous for $0\leqq s\leqq 1$ and $u_1(1)<0$, $u_2(1)=u_3(1)=u_4(1)=0$ and $u_5(1)=u_6(1)=1$, $u_1(s)$, $u_4(s)$ are real and $u_2(s)=\overline{u_3(s)}$ and $u_5(s)=\overline{u_6(s)}$. Our assertion on Figure (5.W) is immediate from the following proposition.

PROPOSITION (5.11). *$u_1(s), \cdots, u_6(s)$ do not cut the imaginary axis for $0<s<1$. In particular, $\Re(u_2(s))$, $\Re(u_3(s))<0$ and $\Re(u_5(s))$, $\Re(u_6(s))>0$ for $0<s<1$.*

PROOF. First rewrite (5.7) as

$$(5.7)'\quad \begin{aligned} &u^6-u^4(3+g-1)+u^3\Big(\frac{2}{27}+2(g-1)\Big)+\\ &u^2\Big(\frac{1}{3}(g-1)^2-(g-1)\Big)+\Big(-\frac{1}{27}(g-1)^3+\frac{1}{3}(g-1)^2\Big)=0\end{aligned}$$

Assume that (u, s^2) satisfies (5.7)′ for some $0<s<1$ and some u which is a pure imaginary complex number. Taking the imaginary part of (5.7), we get $g-1$

$=-1/27$. Substituting this in (5.7)′, we get

$$u^6-\frac{80}{27}u^4+\frac{82}{3^7}u^2-\frac{1}{27^3}\left(\frac{1}{27}+9\right)=0$$

However if u is a pure imaginary complex number, every term in the left side is negative which contradicts the equality. Q. E. D.

§ 6. Further remarks.

In this section, we consider the bounds for $d(C)$ or $s(C)$ for a curve C defined by a join type polynomial equation $C: f(x)-g(y)=0$ where $\deg f=\deg g=n$. Let Σ_f and Σ_g be the critical points of the respective polynomial mappings $f: \boldsymbol{C}\to\boldsymbol{C}$ and $g: \boldsymbol{C}\to\boldsymbol{C}$ and let $\Delta_f=f(\Sigma_f)$ and $\Delta_g=g(\Sigma_g)$. The singularity of C consists of those points (α, β) where $\alpha\in\Sigma_f$ and $\beta\in\Sigma_g$ with $f(\alpha)=g(\beta)$. In [**O3**], we have studied the case that $\Delta_f\cap\Delta_g$ consists of a single point. For the generic case, the curve C need not be irreducible. Let $\Delta=\Delta_1\cup\Delta_g$ and c be a fixed point such that $c\in\boldsymbol{C}-\Delta$. Then $\pi_1(\boldsymbol{C}-\Delta, c)$ acts on $f^{-1}(c)$ and $g^{-1}(c)$ by the respective monodromies of the covering $f: \boldsymbol{C}-f^{-1}(\Delta)\to\boldsymbol{C}-\Delta$ and $g: \boldsymbol{C}-g^{-1}(\Delta)\to\boldsymbol{C}-\Delta$. Thus it also acts on the product $f^{-1}(c)\times g^{-1}(c)$. As C is irreducible if and only if the complement of any finite points is path-connected, the following is obvious.

PROPOSITION (6.1). *C is irreducible if and only if the action of $\pi_1(\boldsymbol{C}-\Delta)$ on $f^{-1}(c)\times g^{-1}(c)$ is transitive.*

Let X be a finite set and let $\mathcal{S}(X)$ be the symmetric group of X. The above actions of $\pi_1(\boldsymbol{C}-\Delta)$ on $f^{-1}(c)$ and on $g^{-1}(c)$ are induced by the canonical homomorphisms

$$\phi_f: \pi_1(\boldsymbol{C}-\Delta)\longrightarrow\mathcal{S}(f^{-1}(c)) \tag{6.2}$$

$$\phi_g: \pi_1(\boldsymbol{C}-\Delta)\longrightarrow\mathcal{S}(g^{-1}(c)) \tag{6.3}$$

For a permutation $\sigma\in\mathcal{S}(X)$, let $|\sigma|=\{x\in X;\ \sigma(x)\neq x\}$. For given k elements $x_1, \cdots, x_k\in X$, we denote the cyclic permutation σ of order k which is defined by $\sigma(x_i)=x_{i+1}$ $(i=1, \cdots, k,\ x_{k+1}=x_1)$ and $\sigma(x)=x$ for $x\neq x_1, \cdots, x_k$ by $(x_1, x_2, \cdots, x_k)$. We use the following simple lemma which is essentially the same as Lemma (1) in [**T**].

LEMMA (6.4). *Let σ and τ be cyclic permutations of order s and t and assume that $\sigma\cap\tau$ consists of a single element. Then $\sigma\tau$ is a cyclic permutation of order $s+t-1$ with $|\sigma\tau|=|\sigma|\cap|\tau|$.*

Let $\Delta_f=\{\alpha_1, \cdots, \alpha_k\}$ and let $f^{-1}(\alpha_i)\cap\Sigma_f=\{x_{i,1}, \cdots, x_{i,p_i}\}$. Let $m_{i,j}$ be the multiplicity of the solution $x=x_{i,j}$ of the equation $df/dx(x)=0$. Of course, we must have the equality:

(6.5) $$\sum_{i=1}^{k}\sum_{j=1}^{p_i} m_{i,j}=n-1$$

Let $\rho_1, \cdots, \rho_k$ be a system of generators of $\pi_1(\boldsymbol{C}-\Delta_f)$ where ρ_i is represented by a small loop which goes around α_i, $i=1, \cdots, k$. Then the image $\phi_f(\rho_i)$ can be expressed as $\phi_f(\rho_i)=\sigma_{i,1}\cdots\sigma_{i,p_i}$ for mutually commuting cyclic permutations $\sigma_{i,j}$ of order $m_{i,j}+1$. The commutativity is the result of the disjointness $|\sigma_{i,j}|\cap|\sigma_{i,j'}|=\emptyset$ for $j\neq j'$. Let X_f^s be the orbit space of $f^{-1}(c)$ by the action of the subgroup generated by $\{\sigma_{i,j};\ 1\leqq j\leqq p_i,\ 1\leqq i\leqq s\}$. Let $|X_f^s|$ be the cardinality of X_f^s. As f is a polynomial of degree n, the product $\rho_1\cdots\rho_k$ induces a cyclic permutation of order n. Thus $|X_f^k|=1$. On the other hand, the cyclic permutation $\sigma_{i,j}$ has a unique non-trivial orbit which contains $m_{i,j}+1$ points. Thus we have $|X_f^s|\geqq n-\sum_{i=1}^{s}\sum_{j=1}^{p_i}m_{i,j}$. Taking the fact that f has $n-1$ critical points counting the multiplicity, we can easily see that

(6.6) $$|X_f^s|=n-\sum_{i=1}^{s}\sum_{j=1}^{p_i}m_{i,j}, \qquad s=1, \cdots, k.$$

In particular, $|\sigma_{i,j}|\cap|\sigma_{k,l}|$ contains at most a point for $(i, j)\neq(k, l)$. Now we consider the explicit construction using the result of Thom [**T**].

(A) Maximal nodal curve.

We first consider a curve C which has the maximum number of nodes i.e., $d(C)=(n-1)(n-2)/2$. Recall that the Chebycheff polynomial $T_n(x)$ of degree n is defined by $T_n(x)=\cos n\arccos x$. See for instance [**Kr**]. It has two critical values $-1, 1$ and $T_n^{-1}(-1)\cap\Sigma_{T_n}$ (respectively $T_n^{-1}(1)\cap\Sigma_{T_n}$) consists of $[n/2]$ (resp. $n-[n/2]-1$) simple critical points. We start from the curve C': $T_n(x)-T_n(-y)=0$. It has $2[n/2](n-[n/2]-1)$ nodes. Note that $2[n/2](n-[n/2]-1)>(n-1)(n-2)/2$. This has too many nodes! The reason is, of course, that C' is not irreducible. We take $g(y)=T_n(-y)$. As $f(x)$, we perturb $T_n(x)$ a little so that it has three critical values $\{-1, 1, a\}$ where each fiber $f^{-1}(-1)$, $f^{-1}(a)$ and $f^{-1}(1)$ has $[n/2]-1$, 1 and $n-[n/2]-1$ simple critical points respectively. The existence of such a polynomial is due to Thom [**T**]. Let C: $f(x)-g(y)=0$.

PROPOSITION (6.7). *C is an irreducible curve with $d(C)=(n-1)(n-2)/2$.*

PROOF. We take the generators ρ_1, ρ_2, ρ_3 of the fundamental group $\pi_1(\boldsymbol{C}-\Delta)$ as in Figure (6.Y). ρ_1, ρ_2 and ρ_3 correspond to the critical values -1, 1 and a respectively. Under a suitable numbering of the respective fibers $f^{-1}(c)=\{x_1, \cdots, x_n\}$ and $g^{-1}(c)=\{y_1, \cdots, y_n\}$, we can assume that

$$\phi_f(\rho_1)=\sigma_1\cdots\sigma_{[n/2]-1},\quad \phi_f(\rho_2)=\tau_1\cdots\tau_{n-[n/2]-1},\quad \phi_f(\rho_3)=\sigma_{[n/2]}$$
$$\phi_g(\rho_1)=\tau_1'\cdots\tau_{n-[n/2]-1}',\quad \phi_g(\rho_2)=\sigma_1'\cdots\sigma_{[n/2]}',\quad \phi_g(\rho_3)=e$$

where $\sigma_i=(x_{2i-1}, x_{2i})$, $\tau_j'=(y_{2j}, y_{2j+1})$, $\tau_j=(x_{2j}, x_{2j+1})$ and $\sigma_i'=(y_{2i-1}, y_{2i})$ for $1\leqq i\leqq[n/2]$ and $1\leqq j\leqq n-[n/2]-1$.

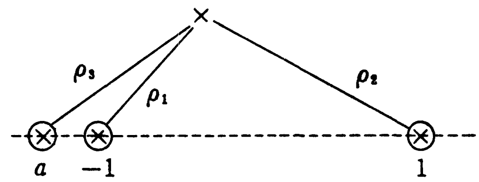


Figure (6.Y)

First assume that $n=2m$. C has $(m-1)^2+(m-1)m=(n-1)(n-2)/2$ nodes. By Lemma (6.4), the product $\rho_1\rho_2$ acts on $f^{-1}(c)$ as a cyclic permutation of $\{x_1, \cdots, x_{2m-1}\}$ (not in this order) and it acts as a cyclic permutation of order n on $g^{-1}(c)$. Thus $(\rho_1\rho_2)^n$ acts as a cyclic permutation of order $n-1$ on $f^{-1}(c)$ and it acts trivially on $g^{-1}(c)$. Take an arbitrary point $(x_i, y_j)\in f^{-1}(c)\times g^{-1}(c)$. We will prove that the orbit of this point contains (x_1, y_1). First by the action of $(\rho_1\rho_2)^t$ for some t, we can send this point to (x_k, y_1) for some k. Secondly using the action of ρ_3 is $k=n$, we may assume that $k\leqq 2m-1$. Now again by the action of $(\rho_1\rho_2)^{nl}$ for some l, we can send this point (x_k, y_1) to (x_1, y_1). Next assume that $n=2m+1$. C has $(m-1)m+m^2=(n-1)(n-2)/2$ nodes. Then $\rho_1\rho_2$ acts on $f^{-1}(c)$ as $\xi\tau_m$ where ξ is a cyclic permutation of $2m-1$ elements $\{x_1, \cdots, x_{2m-1}\}$ and $\tau_m=(x_{2m}, x_{2m+1})$. Its action on $g^{-1}(c)$ is the same as above. Figure (6.Z) shows the corresponding braids of the covering map $f: \boldsymbol{C}-f^{-1}(\Delta_f)\to\boldsymbol{C}-\Delta_f$.

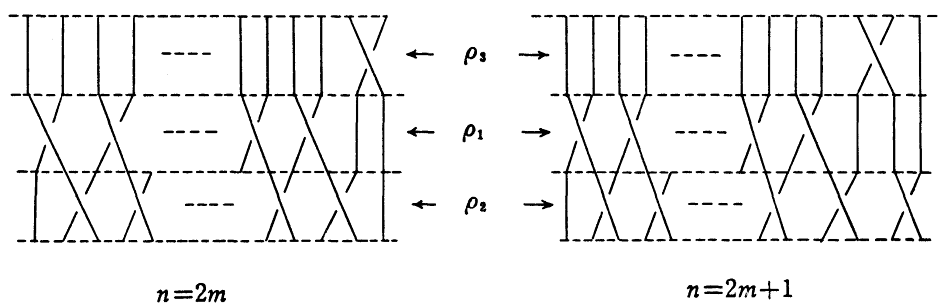


Figure (6.Z)

For a given point (x_i, y_j), we can first send it to (x_k, y_1) as in the case $n=2m$ using a power of $\rho_1\rho_2$. If $k=2m$, we take the action of ρ_3 on this point. If $k=2m+1$, we take the action of $\rho_2\rho_3\rho_2$ on this point which sends this point to (x_{2m-1}, y_1). In any case, we can assume that $k\leqq 2m-1$. As $2m+1$ and $2m-1$

are coprime, the action of the powers of $(\rho_1\rho_2)^{2m+1}$ is transitive on $\{(x_1, y_1), \cdots, (x_{2m-1}, y_1)\}$. Therefore the orbit of (x_i, y_j) contains (x_1, y_1) which completes the proof. Q. E. D.

The fundamental group of the complement of a nodal curve is abelian by [**F**] and [**D**].

(B) Cuspidal curves.

Now we consider the cuspidal curves. Let $g(y)$ be a polynomial of degree n with $\Delta_g=\{-1, 1\}$ where $g^{-1}(-1)\cap\Sigma_g$ consists of $[n/3]$ critical points $\{\eta_1, \cdots, \eta_{[n/3]}\}$ of multiplicity 2 and $g^{-1}(1)\cap\Sigma_g$ consists of $n-2[n/3]-1$ simple critical points $\{\eta_{[n/3]+1}, \cdots, \eta_{n-[n/3]-1}\}$. Let $s=[(n-1-[n/2])/2]$ and $t=n-1-[n/2]-2s$. Let $f(x)$ be a polynomial of degree n with $\Delta_f=\{-1, 1\}$ or $\{-1, 1, a\}$ according to $t=0$ or $t=1$ respectively. We assume that $f^{-1}(-1)\cap\Sigma_f$ consists of $[n/2]$ simple critical points $\{\xi_1, \cdots, \xi_{[n/2]}\}$ and $f^{-1}(1)\cap\Sigma_f$ consists of s critical points $\{\xi_{[n/2]+1}, \cdots, \xi_{[n/2]+s}\}$ of multiplicity 2. In the case of $t=1$, $f^{-1}(a)\cap\Sigma_f$ consists of a single simple critical point. Now we consider the curve $C: f(x)-g(y)=0$. By the construction, C has $[n/3][n/2]+(n-[n/3]-1)s$ cusps (ξ_i, η_j) with $1\leqq i\leqq[n/2]$, $1\leqq j\leqq[n/3]$ or $[n/2]+1\leqq i\leqq[n/2]+s$, $[n/3]+1\leqq j\leqq n-[n/3]-1$. We assert

PROPOSITION (6.8). *C is irreducible and C has asymptotically $n^2/4$ cusps.*

PROOF. The second assertion follows from the observation that $[n/3][n/2]+(n-2[n/3]-1)s\approx n^2/6+(n/3)(n/4)=n^2/4$. Now we show the irreducibility. We take the generators ρ_1, ρ_2 (and ρ_3 if $t\neq 0$) of $\pi_1(\boldsymbol{C}-\Delta)$. Here ρ_1, ρ_2 correspond to the critical value $-1, 1$ respectively and ρ_3 corresponds to the critical value a. See Figure (6.Y). By (6.6), we can write

$$\phi_f(\rho_1)=\sigma_1\cdots\sigma_{[n/2]},\quad \phi_f(\rho_2)=\tau_1\cdots\tau_s$$
$$\phi_g(\rho_1)=\eta_1\cdots\eta_{[n/3]},\quad \phi_g(\rho_2)=\lambda_1\cdots\lambda_{n-2[n/3]-1}$$
$$\phi_f(\rho_3)=\xi,\ \phi_g(\rho_3)=e\qquad \text{if}\quad t=1$$

Here $\sigma_1, \cdots, \sigma_{[n/2]}, \lambda_1, \cdots, \lambda_{n-2[n/3]-1}$ and ξ are cyclic permutations of order 2 and $\eta_1, \cdots, \eta_{[n/3]}$ and $\tau_1, \cdots, \tau_s$ are cyclic permutations of order 3 which satisfy

$$|\sigma_i|\cap|\tau_j|\neq\emptyset \qquad \text{if and only if}\quad 2j-1\leqq i\leqq 2j+1$$
$$|\eta_i|\cap|\lambda_j|\neq\emptyset \qquad \text{if and only if}\quad j\leqq i\leqq j+1$$
$$|\tau_j|\cap|\xi|=\emptyset,\quad |\sigma_i|\cap|\xi|\neq\emptyset \qquad \text{if and only if}\quad i=[n/2]$$

Under a suitable numbering of the fibers $f^{-1}(c)=\{x_1, \cdots, x_n\}$ and $g^{-1}(c)=\{y_1, \cdots, y_n\}$, we can assume for example that

$$\sigma_i=(x_{2i-1}, x_{2i})\left(i\leqq\left[\frac{n}{2}\right]\right), \qquad \tau_j=(x_{4j-2}, x_{4j-1}, x_{4j+1})(j\leqq s)$$

$$\eta_k=(y_{3k-2}, y_{3k-1}, y_{3k})\left(k\leqq\left[\frac{n}{3}\right]\right), \qquad \lambda_l=(y_{3l}, y_{3l+1})\left(l\leqq n-2\left[\frac{n}{3}\right]-2\right)$$

$$\lambda_{n-2[n/3]-1}=\begin{cases}(y_{3(n-2[n/3]-1)}, y_{3(n-2[n/3]-1)+1}) & \text{for } n\equiv 0,\ 1 \text{ modulo } 3\\ (y_{n-1}, y_n) & \text{for } n\equiv 2 \text{ modulo } 3\end{cases}$$

$$\xi=\begin{cases}(x_{n-2}, x_{n-1}) & \text{for } n\equiv 0 \text{ modulo } 4\\ (x_{n-1}, x_n) & \text{for } n\equiv 3 \text{ modulo } 4\end{cases}$$

Assume first that $t=0$. Then $n=4m+2$ or $n=4m+1$ for some integer m. Using Lemma (6.4) and the following expression

$$\begin{aligned}\phi_f(\rho_1\rho_2)&=\sigma_1\cdots\sigma_{[n/2]}\tau_1\cdots\tau_s\\ &=\begin{cases}(\cdots((\sigma_1\sigma_2\sigma_3\tau_1)\sigma_4\sigma_5\tau_2)\cdots)\sigma_{2m}\sigma_{2m+1}\tau_m & n=4m+2\\ (\cdots((\sigma_1\sigma_2\sigma_3\tau_1)\sigma_4\sigma_5\tau_2)\cdots)\sigma_{2m}\tau_m & n=4m+1\end{cases}\end{aligned}$$

we can easily see that $\phi_f(\rho_1\rho_2)$ is a cyclic permutation of order n. Assume that $t=1$. Then we have $n=4m+3$ or $4m$. By the same argument, we can see that $\phi_f(\rho_1\rho_2\rho_3)$ is a cyclic permutation of order n. On the other hand, $\phi_g(\rho_1\rho_2)=\phi_g(\rho_1\rho_2\rho_3)$ is always a cyclic permutation of order n in the both cases. Note that $\phi_f(\rho_1^3\rho_2^2)$ or $\phi_f(\rho_1^3\rho_2^2\rho_3)$ also acts as a permutation of order n on $f^{-1}(c)$ but their actions on $g^{-1}(c)$ are trivial. Now we are ready to prove the transitivity of the action. Take an arbitrary point $(x_i, y_j)\in f^{-1}(c)\times g^{-1}(c)$. By the action of a suitable power of $\rho_1\rho_2$ or $\rho_1\rho_2\rho_3$, we can first send this point to (x_k, y_1) for some k. Then using the action of a power of $\phi_f(\rho_1^3\rho_2^2)$ or $\phi_f(\rho_1^3\rho_2^2\rho_3)$, we can send this point to (x_1, y_1). This completes the proof.

Q. E. D.

References

[A] E. Artin, Theory of braids, Ann. of Math., **48** (1947), 101-126.

[B] E. Brieskorn and H. Knörrer, Ebene Algebraische Kurven, Birkhäuser, Basel-Boston-Stuttgart, 1981.

[D] P. Deligne, Le groupe fondamental du complément d'une courbe plane n'ayant que des points doubles ordinaires est abélien, Séminaire Bourbaki, **543** (1979/80).

[D-L] I. Dolgachev and A. Libgober, On the fundamental group of the complement to a discriminant variety, in Algebraic Geometry, Lecture Notes in Math., **862**, Springer, Berlin-Heidelberg-New York, 1980, pp. 1-25.

[F] W. Fulton, On the fundamental group of the complement of a node curve, Ann. of Math., **111** (1980), 407-409.

[H] H. Hamm, Lokale topologische Eigenschaften komplexer Räume, Math. Ann., **191** (1971), 235-252.

[K] E. R. van Kampen, On the fundamental group of an algebraic curve, Amer. J. Math., **55** (1933), 255-260.

[Kr] V.I. Krylov, Approximate calculation of integrals (translation), Macmillan, New York, 1962.

[M] J. Milnor, Singular Points of Complex Hypersurface, Ann. Math. Stud., Princeton Univ. Press, Princeton, 1968.

[O1] M. Oka, On the fundamental group of a reducible curve in $\boldsymbol{P}^2$, J. London Math. Soc. (2), **12** (1976), 239-252.

[O2] M. Oka, Some plane curves whose complements have non-abelian fundamental groups, Math. Ann., **218** (1975), 55-65.

[O3] M. Oka, On the fundamental group of the complement of certain plane curves, J. Math. Soc. Japan, **30** (1978), 579-597.

[O4] M. Oka, Principal zeta-function of non-degenerate complete intersection variety, J. Fac. Sci. Univ. Tokyo, **37** (1990), 11-32.

[O-S] M. Oka and K. Sakamoto, Product theorem of the fundamental group of a reducible curve, J. Math. Soc. Japan, **30** (1978), 599-602.

[T] R. Thom, L'équivalence d'une fonction différentiable et d'une polynome, Topology, **3** (1965), 297-307.

[W] R.J. Walker, Algebraic Curves, Springer, New York-Heidelberg-Berlin, 1949.

[Z1] O. Zariski, On the problem of existence of algebraic functions of two variables possessing a given branch curve, Amer. J. Math., **51** (1929), 305-328.

[Z2] O. Zariski, On the Poincaré group of rational plane curves, Amer. J. Math., **58** (1929), 607-619.

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