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Homogenization of cadlag processes

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1. Introduction.

Let L be a d-dimensional Lévy type operator:

$$(1.1)\qquad Lf(x)=\frac{1}{2}\sum_{i,j=1}^{d}a_{ij}(x)\partial_{x_i}\partial_{x_j}f(x)+\sum_{i=1}^{d}b_i(x)\partial_{x_i}f(x)$$
$$+\int_{R^d}\left\{f(x+y)-f(x)-\sum_{i=1}^{d}y_i\partial_{x_i}f(x)\right\}\nu(x, dy),$$

where $\partial_{x_i}=\partial/\partial x_i$, $a(x)=(a_{ij}(x))$ is a nonnegative definite symmetric $d\times d$ matrix, $b(x)=(b_i(x))$ is a d-vector, and $\nu(x, dy)$ is a Lévy measure on $\boldsymbol{R}^d$ for each $x\in\boldsymbol{R}^d$: $\nu(x, \{0\})=0$ and $\int_{\boldsymbol{R}^d}|y|^2/(1+|y|^2)\nu(x, dy)<\infty$, $x\in\boldsymbol{R}^d$. Denote by $\{X^L(t)\}$ a cadlag process on $\boldsymbol{R}^d$ governed by L. Here a cadlag process means a Markov process whose sample paths are right continuous and have left hand limits. In this paper we will consider a homogenization problem associated with $\{X^L(t)\}$. Namely, under the condition of periodicity of $a(x)$, $b(x)$ and $\nu(x, dy)$ in x and some additional condition, we will study to what process the scaled process $\{\varepsilon X^L(t/\varphi(\varepsilon))\}$ converges as $\varepsilon\downarrow 0$ with some suitable scaling function φ.

Horie, Inuzuka and Tanaka [3] has already investigated the same problem in the case where $d=1$, $a(x)\equiv 0$ and Lévy measure is absolutely continuous with respect to the Lebesgue measure. More precisely, let

$$(1.2)\qquad Af(x)=b(x)f'(x)+\int_{-\infty}^{\infty}\{f(x+y)-f(x)-yf'(x)\}c(x, y)n(y)dy,$$

where $b(x)$ and $c(x, y)$ are periodic in x with period 1 and c is strictly positive, and $n(y)=\gamma_-|y|^{-1-\alpha_0}$ $(y<0)$, $=\gamma_+y^{-1-\alpha_0}$ $(y>0)$, for some $\alpha_0\in(1, 2)$ and nonnegative numbers γ_-, γ_+ with $\gamma_-+\gamma_+>0$. If there exist the limits $c_\pm=\lim_{r\to\pm\infty}(1/r)\int_0^r dy\int_T c(x, y)\mu(dx)$, μ being the invariant probability measure of the cadlag process $\{\mathfrak{X}^A(t)\}$ on $\boldsymbol{T}\equiv\boldsymbol{R}/\boldsymbol{Z}$ induced by $\{X^A(t)\}$, then the scaled cadlag process $\{\varepsilon X^A(t/\varepsilon^{\alpha_0})\}$ converges to a stable process $\{X^{A^*}(t)\}$ in law as $\varepsilon\downarrow 0$. The generator A^* of the process $\{X^{A^*}(t)\}$ is given by

$$(1.3)\qquad A^*f(x)=\int_{-\infty}^{\infty}\{f(x+y)-f(x)-yf'(x)\}c^*(y)n(y)dy,$$

where $c^*(y)=c_-1_{(-\infty,0)}(y)+c_+1_{(0,\infty)}(y)$.

Their result is still applicable to the case where there exist the limits $\tilde{c}_\pm(x)=\lim_{y\to\pm\infty}c(x,y)|y|^{\delta_0}$ for some $\delta_0>0$. However, in this case, c^* in (1.3) vanishes. This fact means that the scaling $x\mapsto\varepsilon x$ is too fast as compared with the scaling $t\mapsto t/\varepsilon^{\alpha_0}$. In fact, as will be seen in Section 4 later, in this case the scalings must be $x\mapsto\varepsilon x$ and $t\mapsto t/\varepsilon^{\alpha_0+\delta_0}$ and A^* is given as (1.3) with $c_\pm$ and the exponent α_0 in $n(y)$ replaced by $\tilde{c}_\pm\equiv\int_T\tilde{c}_\pm(x)\mu(dx)$ and $\alpha_0+\delta_0$ respectively.

An observation as above shows that homogenization of cadlag processes is much different from that of diffusion processes (see [**2**], [**12**] for the latter). In homogenization of cadlag processes large jumps have an effect on the limit process. Hence we have to do suitable scalings according to a given Lévy measure. Moreover these scalings suggest that the generator of the limit process is determined by a part of the given Lévy measure which is corresponding to the largest jump. These will be verified in Section 3.

In Section 2 we will summarize some properties of a cadlag process governed by L. The construction of such process was already investigated by many authors. It was mainly discussed as the martingale problem under the assumption that the diffusion matrix is positive definite ([**4**], [**14**]), vanishes ([**5**], [**6**]), or is nonnegative definite ([**9**], [**10**], [**11**]). In each case various conditions are imposed for the Lévy measure ν. In this paper we will construct cadlag processes following an analytic perturbation method. Thus we will be concerned with the case where L is written as L_1+L_2, L_1 is a well known operator, for example, a generator of a diffusion process, or of a stable process, and L_2 is a perturbation of L_1. Then we can get easily regularities of solutions of equations associated with L. In order to study homogenization of cadlag processes, we will also use that sample paths of cadlag processes are represented as a solution of a stochastic differential equation of jump type. Therefore we will start with a class of Lévy measure as in (A.1)-(3) below, which contains the following measure as a typical example.

$$(1.4)\quad \nu(x,dy)=|y|^{-d-\alpha_0}dy + \{1_{\{0<\rho\le e\}}(\rho)e^{-1-\alpha(x)}+1_{\{\rho>e\}}(\rho)\rho^{-1-\alpha(x)}(\log\rho)^{\beta(x)}\} \times d\rho\{\sigma(d\omega)+\delta_{\{p(x)\}}(d\omega)\},$$

where $1<\alpha_0<2$, $\rho=|y|$, $\omega=y/|y|\in S^{d-1}$, σ is a finite measure on S^{d-1}, $\alpha(x)$, $\beta(x)$, $p(x)$ are periodic continuous functions with period 1, $1<\alpha(x)<2$, $\beta(x)\in\boldsymbol{R}$, and $p(x)\in S^{d-1}$.

In Section 3 we will study homogenization of $\{X^L(t)\}$ under the assumptions (A.1)-(A.4) below. The essential assumption is that there exists the limit Lévy measure $\nu^*(\cdot)=\lim_{\varepsilon\downarrow0}\int_{T^d}\nu(x,\cdot/\varepsilon)\mu(dx)/\varepsilon^\alpha K(1/\varepsilon)$ for some $\alpha\in(1,2)$ and slowly varying function K, where μ is the invariant probability measure of the cadlag process on $\boldsymbol{T}^d$ governed by L. The scaled cadlag process $\{\varepsilon X^L(t/\varepsilon^\alpha K(1/\varepsilon))\}$ is identical in law with the cadlag process $\{X^{L^\varepsilon}(t)\}$ governed by L^ε of the form (3.2) with ν^ε given by (3.1). The above essential assumption leads us to the conclusion that $\{X^{L^\varepsilon}(t)\}$ converges, as $\varepsilon\downarrow 0$, to the cadlag process $\{X^{L^*}(t)\}$ governed by L^* of the form (3.6). We will show this main result (Theorem 3.1) by the same method as in [3].

Section 4 is devoted to some examples. We can derive from the examples there that, in the case Lévy measure is given by (1.4), if $\alpha^-\equiv\min_x\alpha(x)<\alpha_0$, then the process $\{\varepsilon X^L(t/\varepsilon^{\alpha^-}|\log\varepsilon|^{\beta^+})\}$ converges to the process $\{X^{L^*}(t)\}$ as $\varepsilon\downarrow 0$, where $\beta^+=\max_x\beta(x)$, and L^* is given by

$$L^*f(x)=\int_{y=\rho\omega\in\boldsymbol{R}^d}\{f(x+y)-f(x)-y\cdot\nabla f(x)\}\rho^{-1-\alpha^-}d\rho\sigma^*(d\omega),$$

with $\sigma^*(\Theta)=\mu(\{x\in\boldsymbol{T}^d:\ \alpha(x)=\alpha^-,\ \beta(x)=\beta^+\})\sigma(\Theta)+\mu(\{x\in\boldsymbol{T}^d:\ \alpha(x)=\alpha^-,\ \beta(x)=\beta^+\}\cap p^{-1}(\Theta))$, $\Theta\in\mathscr{B}(S^{d-1})$.

2. Preliminaries.

Let $C(E)$ be the set of all real valued continuous functions on E and $C_b(E)$ the subset of $C(E)$ consisting of those bounded functions. Let $C^n(E)$ be the set of all real valued n times continuously differentiable functions on E and $C_b^n(E)$ the subspace of $C^n(E)$ consisting of those functions with bounded derivatives up to order n. $B(E)$ stands for the set of all real valued bounded Borel mesurable functions on E. $C_0(E)$ is the space of real valued continuous functions on E vanishing at infinity, and $C_0^n(E)$ is the subspace of $C^n(E)$ consisting of those functions with derivatives belonging to $C_0(E)$ up to order n. For a real valued function f we use the following notations: $\nabla_x f(x, y)=(\partial_{x_i}f(x, y))$, $\nabla_x^2 f(x, y)=(\partial_{x_i}\partial_{x_j}f(x, y))$, $\nabla_x\nabla_y f(x, y)=(\partial_{x_i}\partial_{y_j}f(x, y))$, etc. We also use the notation $\|f\|=\sup_{x\in E}|f(x)|$ for a real or vector valued function f on E. For real numbers c_1 and c_2, $c_1\wedge c_2$ and $c_1\vee c_2$ stand for $\min\{c_1, c_2\}$ and $\max\{c_1, c_2\}$, respectively.

For a, b and ν appeared in a Lévy type operator L defined by (1.1), we now assume the following:

(A.1)

(1) Case A: The matrix a vanishes, or
Case B: a is positive definite, and each component a_{ij} belongs to $C_b^2(\boldsymbol{R}^d)$.

(2) For every i, $b_i\in C_b(\boldsymbol{R}^d)$ in Case A, or $b_i\in C_b^1(\boldsymbol{R}^d)$ in Case B.

(3) $\nu(x, dy)$ is represented as

$$\nu(x, \Gamma) = \int_\Gamma c(x, y) n(y) dy + \int_0^\infty \int_U 1_\Gamma(\rho p(x, u)) g(x, \rho, u) d\rho m(du),$$
$$\Gamma \in \mathcal{B}(\boldsymbol{R}^d \setminus \{0\}).$$

(i) $c \geqq 0, \in C_b(\boldsymbol{R}^{2d})$, and $\inf_x c(x, 0) > 0$. There exist positive numbers M, γ_0, h_0 such that $\|c(\cdot, y) - c(\cdot, 0)\| \leqq M|y|^{\gamma_0}$ for $|y| \leqq h_0$ in Case A. $c(\cdot, y) \in C_b^1(\boldsymbol{R}^d)$ for fixed y with $\|\nabla_x c\| < \infty$ in Case B.

(ii) $n(y) = n(\rho\omega) = n_0(\omega)\rho^{-d-\alpha_0}$, $\rho = |y|$, $\omega = y/|y| \in S^{d-1}$, for some $\alpha_0 \in (1, 2)$ and $n_0 \geqq 0, \not\equiv 0$ and either $n_0 \in C_b^d(S^{d-1})$ in Case A, or $n_0 \in C_b(S^{d-1})$ in Case B.

(iii) $(U, \mathcal{B}(U), m)$ is a finite measure space.

(iv) $p: \boldsymbol{R}^d \times U \to S^{d-1}$ is Borel measurable, and $p(\cdot, u) \in C_b(\boldsymbol{R}^d)$ for fixed u in Case A, or $p(\cdot, u) \in C_b^1(\boldsymbol{R}^d)$ for fixed u and $\|\nabla_x p\| < \infty$ in Case B.

(v) $g: \boldsymbol{R}^d \times (0, \infty) \times U \to [0, \infty)$ is Borel measurable, $g(\cdot, \cdot, u) \in C(\boldsymbol{R}^d \times (0, \infty))$ for each u, and either there exists a $\beta \in (1, \alpha_0)$ such that

$$\int_0^\infty (\rho^\beta \wedge \rho) \|g(\cdot, \rho, \cdot)\| d\rho < \infty$$

in Case A, or $g(\cdot, \rho, u) \in C_b^1(\boldsymbol{R}^d)$ for fixed ρ, u and there exists a $\beta \in (1, 2)$ such that

$$\int_0^\infty (\rho^\beta \wedge \rho)(\|g(\cdot, \rho, \cdot)\| + \|\nabla_x g(\cdot, \rho, \cdot)\|) d\rho < \infty$$

in Case B.

(A.2) $a_{ij}(x)$, $b_i(x)$, $i, j = 1, 2, \cdots, d$, $c(x, y)$, $p(x, u)$, $g(x, \rho, u)$ are periodic in x with period 1 for fixed y, ρ, u.

Then we have the following theorem.

THEOREM 2.1. *Assume* (A.1) *and* (A.2). (i) *There exists a cadlag process* $\{X^L(t)\}$ *on* $\boldsymbol{R}^d$ *governed by L.* (ii) *The cadlag process* $\{\mathfrak{X}^L(t)\}$ *on the d-dimensional torus* $\boldsymbol{T}^d$ *induced by* $\{X^L(t)\}$ *has a unique invariant probability measure* μ *on* $\boldsymbol{T}^d$. (iii) *Let* $\{T_t^L\}$ *be the semigroup associated with* $\{X^L(t)\}$. *Let f be a function of* $C_b(\boldsymbol{R}^{2d})$ *such that* $f(x, y)$ *is periodic in x with period* 1 *for each y;* $\int_{\boldsymbol{T}^d} f(x, y)\mu(dx) = 0$, $y \in \boldsymbol{R}^d$; $f(x, \cdot) \in C_b^3(\boldsymbol{R}^d)$ *for fixed x with* $\|\nabla_y f\| + \|\nabla_y^2 f\| + \|\nabla_y^3 f\| < \infty$. *Moreover, in Case B, assume that* $f \in C_b^1(\boldsymbol{R}^{2d})$; $\partial_{x_i} f(x, \cdot) \in C_b^1(\boldsymbol{R}^d)$ *for each x and i; and* $\|\nabla_x \nabla_y f\| < \infty$. *Then the integral* $u(x, y) \equiv \int_0^\infty T_t^L f(\cdot, y)(x) dt$ *converges absolutely. u belongs to* $C_b^1(\boldsymbol{R}^{2d})$, $\partial_{x_i} u(x, y)$ *is uniformly continuous on* $\boldsymbol{R}^d$ *in x for fixed y,* $\partial_{y_i} u \in C_b^1(\boldsymbol{R}^{2d})$, $i = 1, 2, \cdots, d$, *and*

$$\|u\| + \|\nabla_x u\| + \|\nabla_y u\| + \|\nabla_x \nabla_y u\| + \|\nabla_y^2 u\|$$
$$\leqq c(\|f\| + \|\nabla_y f\| + \|\nabla_y^2 f\|),$$

for some positive constant c independent of f. Particularly, $u \in C_b^2(\boldsymbol{R}^{2d})$ *in Case*

B. Moreover it holds, in both Cases A and B, that $-Lu(x, y)=f(x, y)$, x, $y\in \boldsymbol{R}^d$, *where* L *is applied to the variable* x.

REMARK 2.2. If $U=S^{d-1}$ and $p(x, u)=u$, then, by virtue of [**6**], we get the assertion (i) in Case A. In [**14**] Stroock pointed out the existence of a *strong Feller continuous* cadlag process governed by L in Case B. Therefore the assertions (i) and (ii) corresponding to that case follow from his results.

Now we sketch the proof in the same way as in [**3**]. We assume (A.1) and (A.2) throughout this section. Following a routine method, we set

$$L_1 f(x)=\begin{cases}\displaystyle\int_{\boldsymbol{R}^d}\{f(x+y)-f(x)-y\cdot\nabla f(x)\}n(y)dy, & \text{in Case A,}\\[2mm] \displaystyle\frac{1}{2}\sum_{i,j=1}^d \partial_{x_i}(a_{ij}(x)\partial_{x_j}f(x)), & \text{in Case B,}\end{cases}$$

and $L_2=L_0-L_1$, where L_0 is given by (1.1) with $b_i(x)/c(x, 0)$ and $\nu(x, dy)/c(x, 0)$ in place of $b_i(x)$ and $\nu(x, dy)$, respectively, in Case A, or $L_0=L$ in Case B. Let $p^{L_1}(t, x, y)$ be the transition function of the α_0-stable process in Case A, or of the diffusion process in Case B, governed by L_1. Let $\{T_t^{L_1}\}$ and $\{G_\lambda^{L_1}\}$ be the associated semigroup and resolvent, that is, for $f\in B(\boldsymbol{R}^d)$,

$$T_t^{L_1}f(x)=\int_{\boldsymbol{R}^d}p^{L_1}(t, x, y)f(y)dy,$$

$$G_\lambda^{L_1}f(x)=\int_0^\infty e^{-\lambda t}T_t^{L_1}f(x)dt.$$

First we note the following properties from [**5**] in Case A, and from [**7**] in Case B. Put $a_0=\alpha_0$ in Case A, or $=2$ in Case B. We denote by c_i $(i=1, 2, \cdots)$ positive constants independent of λ, f, y, t etc. throughout this section. Let us fix a sufficiently large λ_0. Then it holds that

$$G_\lambda^{L_1}:\ C_0(\boldsymbol{R}^d)\longrightarrow C_0^1(\boldsymbol{R}^d), \tag{2.1}$$

$$G_\lambda^{L_1}:\ B(\boldsymbol{R}^d)\longrightarrow C_b^1(\boldsymbol{R}^d), \tag{2.2}$$

$$\|\nabla G_\lambda^{L_1}f\|\leqq c_1\lambda^{-(a_0-1)/a_0}\|f\|, \tag{2.3}$$

$$\|\nabla(G_\lambda^{L_1}f(\cdot+y)-G_\lambda^{L_1}f(\cdot))\|\leqq c_2\lambda^{-(a_0-1-r)/a_0}\|f\|\,|y|^r, \tag{2.4}$$

for $\lambda\geqq\lambda_0$, $f\in B(\boldsymbol{R}^d)$, $y\in\boldsymbol{R}^d$, where an $r\in(0, a_0-1)$ is fixed arbitrarily. Furthermore, in Case B we have

$$G_\lambda^{L_1}:\ C_b^1(\boldsymbol{R}^d)\longrightarrow C_b^2(\boldsymbol{R}^d), \tag{2.5}$$

$$\|\nabla^2 G_\lambda^{L_1}f\|\leqq c_3\lambda^{-1/2}(\|f\|+\|\nabla f\|), \tag{2.6}$$

$$\|\nabla^2(G_\lambda^{L_1}f(\cdot+y)-G_\lambda^{L_1}f(\cdot))\|\leqq c_4\lambda^{-(1-r)/2}(\|f\|+\|\nabla f\|)|y|^r, \tag{2.7}$$

for $\lambda\geqq\lambda_0$, $f\in C_b^1(\boldsymbol{R}^d)$, $y\in\boldsymbol{R}^d$, where an $r\in(0, 1)$ is fixed arbitrarily.

By using these facts we show the following.

LEMMA 2.3. *Fix an* $r\in(0\vee(\alpha_0-1-\gamma_0), \alpha_0-1)$ *in Case A, or an* $r\in(\alpha_0-1, 1)$ *in Case B. Then*

$$L_2G_\lambda^{L_1}: C_0(\boldsymbol{R}^d)\longrightarrow C_0(\boldsymbol{R}^d), \tag{2.8}$$

$$L_2G_\lambda^{L_1}: B(\boldsymbol{R}^d)\longrightarrow C_b(\boldsymbol{R}^d), \tag{2.9}$$

$$\|L_2G_\lambda^{L_1}f\| \leqq c_6(\lambda^{-(\alpha_0-1-r)/\alpha_0}\vee\lambda^{-(\alpha_0-\beta)/\alpha_0})\|f\|, \tag{2.10}$$

for $\lambda\geqq\lambda_0$ *and* $f\in B(\boldsymbol{R}^d)$. *Moreover in Case B,*

$$L_2G_\lambda^{L_1}: C_b^1(\boldsymbol{R}^d)\longrightarrow C_b^1(\boldsymbol{R}^d), \tag{2.11}$$

$$\|\nabla L_2G_\lambda^{L_1}f\| \leqq c_6(\lambda^{-(1-r)/2}\vee\lambda^{-(2-\beta)/2})(\|f\|+\|\nabla f\|), \tag{2.12}$$

for $\lambda\geqq\lambda_0$ *and* $f\in C_b^1(\boldsymbol{R}^d)$.

PROOF. Let $\lambda\geqq\lambda_0$ and put $Hf(x, y)=G_\lambda^{L_1}f(x+y)-G_\lambda^{L_1}f(x)-y\cdot\nabla G_\lambda^{L_1}f(x)$, and

$$\begin{aligned}L_2G_\lambda^{L_1}f(x) &= \frac{b(x)}{c(x, 0)}\cdot\nabla G_\lambda^{L_1}f(x)+\int_{\boldsymbol{R}^d}Hf(x, y)\Big(\frac{c(x, y)}{c(x, 0)}-1\Big)n(y)dy \\ &\quad +\int_0^\infty\int_U Hf(x, \rho p(x, u))\frac{g(x, \rho, u)}{c(x, 0)}d\rho n(du) \\ &\equiv J_1f(x)+J_2f(x)+J_3f(x), \qquad \text{in Case A,}\end{aligned}$$

$$\begin{aligned}L_2G_\lambda^{L_1}f(x) &= \sum_{i=1}^d\Big(b_i(x)-\frac{1}{2}\sum_{j=1}^d\partial_{x_j}a_{ij}(x)\Big)\partial_{x_i}G_\lambda^{L_1}f(x) \\ &\quad +\int_{\boldsymbol{R}^d}Hf(x, y)c(x, y)n(y)dy \\ &\quad +\int_0^\infty\int_U Hf(x, \rho p(x, u))g(x, \rho, u)d\rho n(du) \\ &\equiv J_1f(x)+J_2f(x)+J_3f(x), \qquad \text{in Case B.}\end{aligned}$$

By means of (A.1) and (2.1)-(2.4), we see that $H: B(\boldsymbol{R}^d)\to C(\boldsymbol{R}^{2d})$, $f\in C_0(\boldsymbol{R}^d)\mapsto Hf(\cdot, y)\in C_0(\boldsymbol{R}^d)$, $f\in B(\boldsymbol{R}^d)\mapsto Hf(\cdot, y)\in C_b(\boldsymbol{R}^d)$, and

$$\|Hf(\cdot, y)\| \leqq c_7\lambda^{-(\alpha_0-1-r)/\alpha_0}\|f\|(|y|^{r+1}\wedge|y|), \tag{2.13}$$

for $f\in B(\boldsymbol{R}^d)$, $y\in\boldsymbol{R}^d$, where an $r\in(0, \alpha_0-1)$ is fixed arbitrarily. Also J_1: $C_0(\boldsymbol{R}^d)\to C_0(\boldsymbol{R}^d)$, J_1: $B(\boldsymbol{R}^d)\to C_b(\boldsymbol{R}^d)$, and

$$\|J_1f\| \leqq c_8\lambda^{-(\alpha_0-1)/\alpha_0}\|f\|, \qquad f\in B(\boldsymbol{R}^d).$$

In view of (A.1), $\|c(\cdot, y)/c(\cdot, 0)-1\|\leqq c_9(|y|^{\gamma_0}\wedge 1)$, $y\in\boldsymbol{R}^d$, in Case A, and $\|c\|\leqq c_{10}$ in Case B. Taking an r as in the lemma and using the dominated convergence theorem, we find that J_2: $C_0(\boldsymbol{R}^d)\to C_0(\boldsymbol{R}^d)$, J_2: $B(\boldsymbol{R}^d)\to C_b(\boldsymbol{R}^d)$, and

$$\|J_2 f\| \leqq c_{11}\lambda^{-(\alpha_0-1-r)/\alpha_0}\|f\|\times\begin{cases}\displaystyle\int_{R^d}(|y|^{r+1+\gamma_0}\wedge|y|)n(y)dy, & \text{in Case A,}\\ \displaystyle\int_{R^d}(|y|^{r+1}\wedge|y|)n(y)dy, & \text{in Case B,}\end{cases}$$

$$= c_{12}\lambda^{-(\alpha_0-1-r)/\alpha_0}\|f\|, \qquad f \in B(\boldsymbol{R}^d).$$

Putting $r=\beta-1$ in (2.13), we have, by the same reason as above, that J_3: $C_0(\boldsymbol{R}^d)\to C_0(\boldsymbol{R}^d)$, J_3: $B(\boldsymbol{R}^d)\to C_b(\boldsymbol{R}^d)$, and

$$\|J_3 f\| \leqq c_{13}\lambda^{-(\alpha_0-\beta)/\alpha_0}\|f\|\int_0^\infty(\rho^\beta\wedge\rho)\|g(\cdot,\rho,\cdot)\|d\rho$$

$$= c_{14}\lambda^{-(\alpha_0-\beta)/\alpha_0}\|f\|, \qquad f\in B(\boldsymbol{R}^d).$$

Thus we obtain (2.8)-(2.10).

We are concentrated on Case B in the rest of the proof. Fix an $f\in C_b^1(\boldsymbol{R}^d)$ arbitrarily. By virtue of (A.1) and (2.3)-(2.7),

$$\|\nabla_x Hf(\cdot, y)\| \leqq c_{15}\lambda^{-(1-r)/2}(\|f\|+\|\nabla f\|)(|y|^{r+1}\wedge|y|), \tag{2.14}$$

for $y\in\boldsymbol{R}^d$ with a fixed $r\in(0, 1)$, and

$$J_1 f \in C_b^1(\boldsymbol{R}^d), \qquad \|\nabla J_1 f\| \leqq c_{16}\lambda^{-1/2}(\|f\|+\|\nabla f\|).$$

(A.1) and the dominated convergence theorem imply that

$$J_2 f \in C_b^1(\boldsymbol{R}^d), \qquad \|\nabla J_2 f\|\leqq c_{17}\lambda^{-(1-r)/2}(\|f\|+\|\nabla f\|),$$

where an r is arbitrarily fixed within $(\alpha_0-1, 1)$. Noting that $\|\nabla_y Hf(\cdot, y)\|\leqq c_{18}\lambda^{-1/2}(\|f\|+\|\nabla f\|)(|y|\wedge 1)$, $y\in\boldsymbol{R}^d$, and setting $r=\beta-1$ in (2.13) and (2.14), we get similarly that

$$J_3 f \in C_b^1(\boldsymbol{R}^d), \qquad \|\nabla J_3 f\| \leqq c_{19}\lambda^{-(2-\beta)/2}(\|f\|+\|\nabla f\|).$$

Thus (2.11) and (2.12) follow. ■

We now denote by $\tilde{L}_1$ the generator of the strongly continuous semigroup $\{T_t^{\tilde{L}_1}\}$ with $C_0(\boldsymbol{R}^d)$ as the domain. Define the operator $\tilde{L}_0$ by $\tilde{L}_0=\tilde{L}_1+L_2$ with the domain $D(\tilde{L}_0)=D(\tilde{L}_1)(\supset C_0^2(\boldsymbol{R}^d))$. Then $\tilde{L}_0$: $D(\tilde{L}_1)\to C_0(\boldsymbol{R}^d)$ because of (2.8). We see that $\tilde{L}_0$ is the smallest closed extension of the operator L_0 restricted to $C_0^2(\boldsymbol{R}^d)$ and $\tilde{L}_0$ has the strong negative property, that is, $f\in D(\tilde{L}_0)$ and $f(x_0)=\max_x f(x)$ imply $\tilde{L}_0 f(x_0)\leqq 0$. Therefore there exists a unique strongly continuous Markovian semigroup $\{T_t^{\tilde{L}_0}\}$ on $C_0(\boldsymbol{R}^d)$ with the generator $\tilde{L}_0$. Let $\{X^{L_0}(t)\}$ be a cadlag process on $\boldsymbol{R}^d$ associated with $\{T_t^{\tilde{L}_0}\}$ and $P^{L_0}(t, x, \cdot)$ the transition probability. $\{T_t^{\tilde{L}_0}\}$ and the resolvent $\{G_\lambda^{\tilde{L}_0}\}$ are naturally extended to the operators on $B(\boldsymbol{R}^d)$ in the following way.

$$T_t^{L_0} f(x) = \int_{\boldsymbol{R}^d} f(y) P^{L_0}(t, x, dy),$$

$$G_\lambda^{L_0} f(x) = \int_0^\infty e^{-\lambda t} T_t^{L_0} f(x) dt,$$

for $f \in B(\boldsymbol{R}^d)$. Then, in view of (2.9) and (2.10),

$$G_\lambda^{L_0} f = G_\lambda^{L_1}(I - L_2 G_\lambda^{L_1})^{-1} f,$$

for $f \in B(\boldsymbol{R}^d)$ and sufficiently large λ. Combining this with $G_\lambda^{L_1} 1 = 1/\lambda$, (2.2)-(2.7) and (2.9)-(2.12), we have the following.

LEMMA 2.4. *Let $r \in (0, a_0 - 1)$. Then it holds that*

$$G_\lambda^{L_0} 1 = 1/\lambda, \tag{2.15}$$

$$G_\lambda^{L_0}: B(\boldsymbol{R}^d) \longrightarrow C_b^1(\boldsymbol{R}^d), \tag{2.16}$$

$$\|\nabla G_\lambda^{L_0} f\| \leqq c_{20} \lambda^{-(a_0-1)/a_0} \|f\|, \tag{2.17}$$

$$\|\nabla(G_\lambda^{L_0} f(\cdot + y) - G_\lambda^{L_0} f(\cdot))\| \leqq c_{21} \lambda^{-(a_0-1-r)/a_0} \|f\| |y|^r, \tag{2.18}$$

for sufficiently large λ, $f \in B(\boldsymbol{R}^d)$, and $y \in \boldsymbol{R}^d$. Especially, in Case B,

$$G_\lambda^{L_0}: C_b^1(\boldsymbol{R}^d) \longrightarrow C_b^2(\boldsymbol{R}^d), \tag{2.19}$$

$$\|\nabla^2 G_\lambda^{L_0} f\| \leqq c_{22} \lambda^{-1/2}(\|f\| + \|\nabla f\|), \tag{2.20}$$

$$\|\nabla^2(G_\lambda^{L_0} f(\cdot + y) - G_\lambda^{L_0} f(\cdot))\| \leqq c_{23} \lambda^{-(1-r)/2}(\|f\| + \|\nabla f\|) |y|^r, \tag{2.21}$$

for sufficiently large λ, $f \in C_b^1(\boldsymbol{R}^d)$ and $y \in \boldsymbol{R}^d$.

We next show that the semigroup $\{T_t^{L_0}\}$ has the strong Feller property. Since $L = L_0$ in Case B, the associated cadlag process $\{X^{L_0}(t)\}$ is nothing but the one governed by L. Hence this property is already obtained in Case B as noted in Remark 2.2. We thus only consider Case A in the following lemma, whose proof is also available for Case B.

LEMMA 2.5.

$$T_t^{L_0}: B(\boldsymbol{R}^d) \longrightarrow C(\boldsymbol{R}^d), \qquad t > 0.$$

PROOF. We use an idea in [**15**]. Let us repeat above argument for the space time semigroup $\{\hat{T}_t^{L_1}\}$ and resolvent $\{\hat{G}_\lambda^{L_1}\}$, where

$$\hat{T}_t^{L_1} \hat{f}(s, x) = \int_{\boldsymbol{R}^d} \hat{f}(s+t, y) p(t, x, y) dy,$$

$$\hat{G}_\lambda^{L_1} \hat{f}(s, x) = \int_0^\infty e^{-\lambda t} \hat{T}_t^{L_1} \hat{f}(s, x) dt,$$

for $\hat{f} \in B(\boldsymbol{R}^{d+1})$ and $(s, x) \in \boldsymbol{R} \times \boldsymbol{R}^d$. Then there exists a unique strongly continuous Markovian semigroup $\{\hat{T}_t^{L_0}\}$ on $C_0(\boldsymbol{R}^{d+1})$ with the generator $\hat{L}_0$ which is the smallest closed extention of $\partial+L_0$ restricted to $C_0^2(\boldsymbol{R}^{d+1})$, where $(\partial+L_0)\hat{f}(s, x)=\partial_s\hat{f}(s, x)+L_0\hat{f}(s, x)$, L_0 being applied to the variable x. $\{\hat{T}_t^{\hat{L}_0}\}$ and the resolvent $\{\hat{G}_\lambda^{\hat{L}_0}\}$ are extended to the operators on $B(\boldsymbol{R}^{d+1})$, and it holds that

$$(2.22)\qquad \hat{G}_\lambda^{\hat{L}_0}\hat{f}(s, \cdot)=\hat{G}_\lambda^{\hat{L}_1}(I-L_2\hat{G}_\lambda^{\hat{L}_1})^{-1}\hat{f}(s, \cdot)\in C_b^1(\boldsymbol{R}^d),$$

for sufficiently large λ, $\hat{f}\in B(\boldsymbol{R}^{d+1})$ and $s\in\boldsymbol{R}$.

Now let us fix sufficiently large λ, $f\in B(\boldsymbol{R}^d)$ and $t>0$. Put

$$\hat{f}_{t,\lambda}(s, x)=\frac{1}{t}1_{[0,t]}(s)e^{\lambda s}T_{t-s}^{L_0}f(x).$$

We then have $T_t^{L_0}f(\cdot)=\hat{G}_\lambda^{L_0}\hat{f}_{t,\lambda}(0, \cdot)$, where $\{\hat{G}_\lambda^{L_0}\}$ is the space time resolvent induced by $\{T_t^{L_0}\}$. Since $\{\hat{G}_\lambda^{L_0}\}=\{\hat{G}_\lambda^{\hat{L}_0}\}$, the assertion of the lemma follows from (2.22). ∎

We denote by $\{\mathfrak{X}^{L_0}(t)\}$ the cadlag process on $\boldsymbol{T}^d$ induced by $\{X^{L_0}(t)\}$. Let $\{\mathfrak{T}_t^{L_0}\}$ and $\{\mathfrak{G}_\lambda^{L_0}\}$ be the associated semigroup and resolvent, respectively. We should notice that Lemmas 2.4 and 2.5 are also valid for functions on $\boldsymbol{T}^d$, $\{\mathfrak{G}_\lambda^{L_0}\}$ and $\{\mathfrak{T}_t^{L_0}\}$.

LEMMA 2.6. *There exists a unique invariant probability measure μ_0 on $\boldsymbol{T}^d$ such that*

$$(2.23)\qquad \left\|\mathfrak{T}_t^{L_0}\mathfrak{f}(\cdot)-\int_{\boldsymbol{T}^d}\mathfrak{f}d\mu_0\right\|\leqq c_{24}e^{-c_{25}t}\|\mathfrak{f}\|,\qquad t>0,\ \mathfrak{f}\in B(\boldsymbol{T}^d).$$

PROOF. First note that $\{\mathfrak{T}_t^{L_0}\}$ satisfies the strong Feller property in the strict sense ([8]). In the same way as in [3], we can show that the transition probability $\mathfrak{P}^{L_0}$ of $\{\mathfrak{X}^{L_0}(t)\}$ satisfies $\mathfrak{P}^{L_0}(t, \mathfrak{x}, \mathfrak{B})>0$ for $t>0$, $\mathfrak{x}\in\boldsymbol{T}^d$, and nonempty open sets $\mathfrak{B}\subset\boldsymbol{T}^d$. In view of (2.15), $\{\mathfrak{X}^{L_0}(t)\}$ is conservative. Hence Theorem 1.1 in [17] leads us to the conclusion of the lemma. ∎

LEMMA 2.7. *Let f be an element of $C_b(\boldsymbol{T}^d\times\boldsymbol{R}^d)$ such that $f(\mathfrak{x}, \cdot)\in C_b^3(\boldsymbol{R}^d)$ for fixed $\mathfrak{x}$ with $\|\nabla_y f\|+\|\nabla_y^2 f\|+\|\nabla_y^3 f\|<\infty$, and $\int_{\boldsymbol{T}^d}f(\mathfrak{x}, y)\mu_0(d\mathfrak{x})=0$, $y\in\boldsymbol{R}^d$. Moreover in Case B assume that $f\in C_b^1(\boldsymbol{T}^d\times\boldsymbol{R}^d)$, $\partial_{\mathfrak{x}_i}f(\mathfrak{x}, \cdot)\in C_b^1(\boldsymbol{R}^d)$ for each $\mathfrak{x}$ and i, and $\|\nabla_{\mathfrak{x}}\nabla_y f\|<\infty$. Then* (i) *the integral $\mathfrak{R}f(\mathfrak{x}, y)\equiv\int_0^\infty\mathfrak{T}_t^{L_0}f(\cdot, y)(\mathfrak{x})dt$ is absolutely convergent*; (ii) *$\mathfrak{R}f\in C_b^1(\boldsymbol{T}^d\times\boldsymbol{R}^d)$, $\partial_{y_i}\mathfrak{R}f\in C_b^1(\boldsymbol{T}^d\times\boldsymbol{R}^d)$, $i=1, 2, \cdots, d$, and*

$$\begin{aligned}\|\mathfrak{R}f\|+\|\nabla_{\mathfrak{x}}\mathfrak{R}f\|+\|\nabla_y\mathfrak{R}f\|+\|\nabla_{\mathfrak{x}}\nabla_y\mathfrak{R}f\|+\|\nabla_y^2\mathfrak{R}f\|\\ \leqq c_{26}(\|f\|+\|\nabla_y f\|+\|\nabla_y^2 f\|);\end{aligned}$$

(iii) *$\mathfrak{R}f\in C_b^2(\boldsymbol{T}^d\times\boldsymbol{R}^d)$ in Case B*; (iv) *$-\mathfrak{L}_0\mathfrak{R}f(\cdot, y)=f(\cdot, y)$, $y\in\boldsymbol{R}^d$, where $\mathfrak{L}_0$ means the operator L_0 acting on functions on $\boldsymbol{T}^d$.*

PROOF. Let $\widetilde{\mathfrak{L}}_0$ be the generator of $\{\mathfrak{T}_t^{L_0}\}$ restricted to $C(\boldsymbol{T}^d)$. Let us arbitrarily fix an f satisfying all of the conditions of the lemma. By means of (2.23),

$$\|\mathfrak{R}f\| \leqq \int_0^\infty \sup_{\mathfrak{x}, y} |\mathfrak{T}_t^{L_0} f(\cdot, y)(\mathfrak{x})| dt \leqq c_{27}\|f\|,$$

which implies the assertion (i).

With the aid of the resolvent equation,

$$\mathfrak{R}f(\mathfrak{x}, y) = \mathfrak{G}_\lambda^{L_0}(f(\cdot, y)+\lambda\mathfrak{R}f(\cdot, y))(\mathfrak{x}), \qquad \lambda>0,\ \mathfrak{x}\in\boldsymbol{T}^d,\ y\in\boldsymbol{R}^d. \tag{2.24}$$

From now on we fix a sufficiently large λ and set $\Lambda f(\mathfrak{x}, y)=f(\mathfrak{x}, y)+\lambda\mathfrak{R}f(\mathfrak{x}, y)$. Obviously,

$$\|\Lambda f\| \leqq c_{28}\|f\|.$$

This with (2.24) and (2.16) leads us to the fact $\mathfrak{R}f(\cdot, y)\in C_b^1(\boldsymbol{T}^d)$, whence $\Lambda f(\cdot, y)\in C(\boldsymbol{T}^d)$. By using (2.24) again, we see that $\mathfrak{R}f(\cdot, y)\in D(\widetilde{\mathfrak{L}}_0)$ and $-\widetilde{\mathfrak{L}}_0\mathfrak{R}f(\cdot, y)=f(\cdot, y)$. Since $\widetilde{\mathfrak{L}}_0=\mathfrak{L}_0$ on $C^1(\boldsymbol{T}^d)$ in Case A, or on $C^2(\boldsymbol{T}^d)$ in Case B, the assertion (iv) follows from the assertions (ii) and (iii).

Since $\int_{\boldsymbol{T}^d}\partial_{y_i}f(\mathfrak{x}, y)\mu_0(d\mathfrak{x})=0$ for every y and i,

$$\partial_{y_i}\mathfrak{R}f(\mathfrak{x}, y) = \mathfrak{R}(\partial_{y_i}f)(\mathfrak{x}, y) = \mathfrak{G}_\lambda^{L_0}(\Lambda(\partial_{y_i}f)(\cdot, y))(\mathfrak{x}).$$

Similarly,

$$\partial_{y_i}\partial_{y_j}\mathfrak{R}f(\mathfrak{x}, y) = \mathfrak{R}(\partial_{y_i}\partial_{y_j}f)(\mathfrak{x}, y) = \mathfrak{G}_\lambda^{L_0}(\Lambda(\partial_{y_i}\partial_{y_j}f)(\cdot, y))(\mathfrak{x}).$$

Combining (2.24) and above two formulas with Lemma 2.4, we see that $\mathfrak{R}f$ belongs to $C_b^1(\boldsymbol{T}^d\times\boldsymbol{R}^d)$, $\partial_{y_i}\mathfrak{R}f\in C_b^1(\boldsymbol{T}^d\times\boldsymbol{R}^d)$, $i=1, 2, \cdots, d$, and

$$\begin{aligned}
&\|\nabla_{\mathfrak{x}}\mathfrak{R}f\| = \sup_y\|\nabla\mathfrak{G}_\lambda^{L_0}(\Lambda f(\cdot, y))\| \leqq c_{29}\|f\|,\\
&\|\nabla_{\mathfrak{x}}(\mathfrak{R}f(\cdot+\mathfrak{z}, \cdot)-\mathfrak{R}f(\cdot, \cdot))\|\\
&\qquad = \sup_y\|\nabla(\mathfrak{G}_\lambda^{L_0}(\Lambda f(\cdot, y))(\cdot+\mathfrak{z})-\mathfrak{G}_\lambda^{L_0}(\Lambda f(\cdot, y))(\cdot))\|\\
&\qquad \leqq c_{30}\|f\|\,|\mathfrak{z}|^r,\\
&\|\nabla_y\mathfrak{R}f\|+\|\nabla_y^2\mathfrak{R}f\|+\|\nabla_{\mathfrak{x}}\nabla_y\mathfrak{R}f\| \leqq c_{27}(\|\nabla_y f\|+\|\nabla_y^2 f\|)+c_{29}\|\nabla_y f\|,\\
&\|\nabla_{\mathfrak{x}}\nabla_y(\mathfrak{R}f(\cdot+\mathfrak{z}, \cdot)-\mathfrak{R}f(\cdot, \cdot))\| \leqq c_{30}\|\nabla_y f\|\,|\mathfrak{z}|^r,\\
&\|\nabla_{\mathfrak{x}}\nabla_y(\mathfrak{R}f(\cdot, \cdot+z)-\mathfrak{R}f(\cdot, \cdot))\|+\|\nabla_y^2(\mathfrak{R}f(\cdot+\mathfrak{z}, \cdot)-\mathfrak{R}f(\cdot, \cdot))\|\\
&\qquad \leqq c_{29}(|\mathfrak{z}|+|z|)(\|\nabla_y f\|+\|\nabla_y^2 f\|),\\
&\|\nabla_y^2(\mathfrak{R}f(\cdot, \cdot+z)-\mathfrak{R}f(\cdot, \cdot))\| \leqq c_{27}\|\nabla_y^3 f\|\,|z|.
\end{aligned}$$

for $\mathfrak{z}\in\boldsymbol{T}^d$ and $z\in\boldsymbol{R}^d$, where r is fixed arbitrarily within $(0, a_0-1)$. Thus assertion (ii) follows.

For the assertion (iii) it is enough to notice the following. By virtue of

Lemma 2.4,

$$\|\nabla_{\mathfrak{x}}^2 \mathfrak{R} f\| \leqq c_{31}(\|f\|+\|\nabla_{\mathfrak{x}} f\|),$$
$$\|\nabla_{\mathfrak{x}}^2(\mathfrak{R} f(\cdot+\mathfrak{z}, \cdot)-\mathfrak{R} f(\cdot, \cdot))\| \leqq c_{32}(\|f\|+\|\nabla_{\mathfrak{x}} f\|)|\mathfrak{z}|^r,$$
$$\|\nabla_{\mathfrak{x}}^2(\mathfrak{R} f(\cdot, \cdot+z)-\mathfrak{R} f(\cdot, \cdot))\| \leqq c_{31}(\|\nabla_y f\|+\|\nabla_{\mathfrak{x}}\nabla_y f\|)|z|,$$

for $\mathfrak{z}\in \boldsymbol{T}^d$ and $z\in \boldsymbol{R}^d$, where r is fixed arbitrarily within (0, 1). ■

We are now in the position to give.

PROOF OF THEOREM 2.1. Since $L_0=L$ in Case B, the assertions of the theorem corresponding to that case have been already verified in above argument. We only consider Case A. The cadlag process $\{X^L(t)\}$ governed by L is given as the time changed process $\{X^{L_0}(\varphi(t))\}$, where $\varphi(t)$ is the inverse function of $t\mapsto\int_0^t c(X^{L_0}(s), 0)^{-1}ds$. Then $\mu(d\mathfrak{x})\equiv\left(\int_{T^d} c(\mathfrak{x}, 0)^{-1}\mu_0(d\mathfrak{x})\right)^{-1}c(\mathfrak{x}, 0)^{-1}\mu_0(d\mathfrak{x})$ is the unique invariant probability measure of $\{\mathfrak{X}^L(t)\}$. Set $\bar{f}(x, y)=f(x, y)/c(x, 0)$ for $f\in B(\boldsymbol{R}^{2d})$ such that $\int_{T^d} f(x, y)\mu(dx)=0$, $y\in \boldsymbol{R}^d$. Obviously

$$\int_0^\infty T_t^L f(\cdot, y)(x)dt = \int_0^\infty T_t^{L_0}\bar{f}(\cdot, y)(x)dt, \qquad x, y \in \boldsymbol{R}^d,$$

which is absolutely convergent. If f satisfies the conditions in the part (iii) of the theorem, then the function on $\boldsymbol{T}^d\times\boldsymbol{R}^d$ induced by $\bar{f}$ satisfies the conditions of Lemma 2.7, and hence we get the assertion (iii) of the theorem. ■

3. Main theorem.

For each $\varepsilon>0$ and Lévy measure ν, we set

$$\nu^\varepsilon(x, \Gamma) = \frac{\nu(x, \Gamma/\varepsilon)}{\varepsilon^\alpha K(1/\varepsilon)}, \qquad x \in \boldsymbol{R}^d, \Gamma \in \mathcal{B}(\boldsymbol{R}^d), \tag{3.1}$$

where $\alpha>0$ and K is a slowly varying function, that is, K is a positive continuous function on $[0, \infty)$ such that $\lim_{\rho\to\infty} K(c\rho)/K(\rho)=1$, $c>0$. We define the following operator.

$$\begin{aligned} L^\varepsilon f(x) = &\frac{1}{2}\frac{\varepsilon^{2-\alpha}}{K(1/\varepsilon)}\sum_{i,j=1}^d a_{ij}\left(\frac{x}{\varepsilon}\right)\partial_{x_i}\partial_{x_j} f(x) \\ &+\frac{\varepsilon^{1-\alpha}}{K(1/\varepsilon)}\sum_{i=1}^d b_i\left(\frac{x}{\varepsilon}\right)\partial_{x_i} f(x) \\ &+\int_{R^d}\left\{f(x+y)-f(x)-\sum_{i=1}^d y_i\partial_{x_i} f(x)\right\}\nu^\varepsilon\left(\frac{x}{\varepsilon}, dy\right). \end{aligned} \tag{3.2}$$

Under the assumptions (A.1) and (A.2), there exist cadlag processes $\{X^L(t)\}$

and $\{X^{L^\varepsilon}(t)\}$ on $\boldsymbol{R}^d$ governed by L and L^ε, respectively. Note that the scaled process $\{\varepsilon X^L(t/\varepsilon^\alpha K(1/\varepsilon))\}$ is equivalent to $\{X^{L^\varepsilon}(t)\}$ in the sense of law.

Let μ be the invariant probability measure of the cadlag process $\{\mathfrak{X}^L(t)\}$ on $\boldsymbol{T}^d$ induced by $\{X^L(t)\}$ as stated in Theorem 2.1. We impose the following assumptions.

(A.3) $$\int_{T^d} b_i d\mu = 0, \qquad i = 1, 2, \cdots, d.$$

(A.4) There exist real numbers $\alpha \in (1, 2)$, $\rho_0 > 0$, a slowly varying function K and a finite measure n^* on S^{d-1} such that

(3.3) $$\sup_{x,\omega} c(x, \rho\omega)\rho^{-1-\alpha_0} + \sup_{x,u} g(x, \rho, u) \leqq \rho^{-1-\alpha} K(\rho), \quad \rho \geqq \rho_0,$$

(3.4) $$\lim_{r\to\infty} \frac{1}{r}\int_{\rho_0}^r \frac{\bar{n}(\rho, \cdot)}{\rho^{-1-\alpha}K(\rho)} d\rho = n^*(\cdot),$$

where σ_0 is the area element of S^{d-1} and $\bar{n}$ is given as

$$\bar{n}(\rho, \Theta) = \int_{T^d} \mu(dx)\Big(\int_\Theta c(x, \rho\omega)\rho^{-1-\alpha_0} n_0(\omega)\sigma_0(d\omega)$$
$$+ \int_U 1_\Theta(p(x, u)) g(x, \rho, u) m(du)\Big), \qquad \rho > 0, \Theta \in \mathcal{B}(S^{d-1}).$$

Setting

(3.5) $$\nu^*(\Gamma) = \int_{\rho\omega\in\Gamma} \rho^{-1-\alpha} d\rho\, n^*(d\omega), \qquad \Gamma \in \mathcal{B}(\boldsymbol{R}^d),$$

and we define

(3.6) $$L^* f(x) = \int_{R^d} \{f(x+y) - f(x) - y\cdot\nabla f(x)\}\nu^*(dy).$$

Let P_x^ε and P_x^* be the probability measures on $W \equiv D([0, \infty) \to \boldsymbol{R}^d)$ induced by the cadlag processes $\{X^{L^\varepsilon}(t)\}$ and $\{X^{L^*}(t)\}$ on $\boldsymbol{R}^d$ governed by L^ε and L^* starting at x, respectively.

THEOREM 3.1. *Assume* (A.1)-(A.4). *Then P_x^ε converges to P_x^* as $\varepsilon \downarrow 0$.*

In order to prove Theorem 3.1, we will first note that the path functions of the cadlag process $\{X^{L^\varepsilon}(t)\}$ starting at x are given as a solution of a stochastic differential equation of jump type. By using it, we will then show the tightness of $\{P_x^\varepsilon\}_{0<\varepsilon\leq 1}$ and the characterization of the limit process in Lemmas 3.6 and 3.7, respectively.

We assume (A.1)-(A.4) throughout this section. We may also assume that $\rho_0 > 1$ and $K(\rho) = K(\rho_0)$ for $0 \leqq \rho \leqq \rho_0$ without loss of generality.

First of all, we recall some properties of slowly varying functions from [13].

(3.7) $$\lim_{\rho\to\infty} \rho^{-c} K(\rho) = \lim_{\rho\to\infty} \rho^{-c}/K(\rho) = 0, \qquad c > 0.$$

For $c>0$, put

$$K_{1,c}(\rho)=\rho^{-c}\sup_{0\leq r\leq\rho} r^{c}K(r),\qquad K_{2,c}(\rho)=\rho^{c}\sup_{\rho\leq r<\infty} r^{-c}K(r),$$
$$K_{3,c}(\rho)=\rho^{c}\inf_{0\leq r\leq\rho} r^{-c}K(r),\qquad K_{4,c}(\rho)=\rho^{-c}\inf_{\rho\leq r<\infty} r^{c}K(r).$$

Then it holds that

(3.8) $$\lim_{\rho\to\infty}K_{i,c}(\rho)/K(\rho)=1,\qquad i=1,2,3,4.$$

For $\varepsilon>0$, put

(3.9) $$\bar{\nu}^{\varepsilon}(\Gamma)=\int_{T^d}\nu^{\varepsilon}(x,\Gamma)\mu(dx),\qquad \Gamma\in\mathcal{B}(\boldsymbol{R}^d).$$

LEMMA 3.2. *$\bar{\nu}^{\varepsilon}$ converges to ν^* vaguely on $\boldsymbol{R}^d\setminus\{0\}$ as $\varepsilon\downarrow 0$.*

PROOF. Fix $0<r<R<\infty$ and $\Theta\in\mathcal{B}(S^{d-1})$ with $\nu^*(\partial\Theta)=0$, arbitrarily. It is enough to show

$$\lim_{\varepsilon\downarrow 0}\bar{\nu}^{\varepsilon}((r,R]\times\Theta)=\nu^*((r,R]\times\Theta).$$

Note that

$$\bar{\nu}^{\varepsilon}((r,R]\times\Theta)=\frac{1}{\varepsilon^{\alpha}K(1/\varepsilon)}\int_{T^d}\nu(x,(r/\varepsilon,R/\varepsilon]\times\Theta)\mu(dx)$$
$$=\frac{1}{\varepsilon^{1+\alpha}K(1/\varepsilon)}\int_r^R\bar{n}(\rho/\varepsilon,\Theta)d\rho.$$

Put

$$A(\rho,\Theta)=\int_{\rho_0}^{\rho}\frac{\bar{n}(u,\Theta)}{u^{-1-\alpha}K(u)}du.$$

Then

$$\bar{\nu}^{\varepsilon}((r,R]\times\Theta)=\frac{\varepsilon}{K(1/\varepsilon)}\int_r^R\rho^{-1-\alpha}K(\rho/\varepsilon)\frac{d}{d\rho}A(\rho/\varepsilon,\Theta)d\rho$$
$$\leqq\frac{\varepsilon}{K(1/\varepsilon)}R^cK_{1,c}(R/\varepsilon)\int_r^R\rho^{-1-\alpha-c}\frac{d}{d\rho}A(\rho/\varepsilon,\Theta)d\rho$$
$$=\frac{K_{1,c}(R/\varepsilon)}{K(1/\varepsilon)}R^c\Big\{\varepsilon A(R/\varepsilon,\Theta)R^{-1-\alpha-c}-\varepsilon A(r/\varepsilon,\Theta)r^{-1-\alpha-c}$$
$$+(1+\alpha+c)\varepsilon\int_r^R A(\rho/\varepsilon,\Theta)\rho^{-2-\alpha-c}d\rho\Big\},$$

for every $c>0$. (3.4) tells us that $\lim_{\varepsilon\downarrow 0}(\rho/\varepsilon)^{-1}A(\rho/\varepsilon,\Theta)=n^*(\Theta)$ for each $\rho>0$. Since $\{A(\rho/\varepsilon,\Theta):0<\varepsilon\leqq 1, r\leqq\rho\leqq R\}$ is bounded, we find, by (3.8), that

$$\overline{\lim_{\varepsilon\downarrow 0}}\,\bar{\nu}^{\varepsilon}((r,R]\times\Theta)\leqq\frac{R^c}{\alpha+c}n^*(\Theta)(r^{-\alpha-c}-R^{-\alpha-c}),\qquad c>0,$$

and hence, letting $c\downarrow 0$,

$$\overline{\lim_{\varepsilon\downarrow 0}}\,\bar{\nu}^{\varepsilon}((r, R]\times\Theta) \leqq \nu^{*}((r, R]\times\Theta).$$

By using $K_{4,c}$, we get, in the same way as above,

$$\underline{\lim}_{\varepsilon\downarrow 0}\,\bar{\nu}^{\varepsilon}((r, R]\times\Theta) \geqq \nu^{*}((r, R]\times\Theta). \quad \blacksquare$$

We next rewrite the Lévy measure ν. Fix $\omega_0\in S^{d-1}$ and $u_0\in U$ with $m(\{u_0\})=0$, arbitrarily. For $v=(\omega, u)\in V\equiv S^{d-1}\times U$, we set

$$m_0(dv) = \delta_{\{\omega_0\}}(d\omega)m(du)+n_0(\omega)\sigma_0(d\omega)\delta_{\{u_0\}}(du),$$

$$p_0(x, v) = \begin{cases} \omega, & \text{if } \omega\neq\omega_0,\ u=u_0, \\ p(x, u), & \text{otherwise,} \end{cases}$$

$$g_0(x, \rho, v) = \begin{cases} c(x, \rho\omega)\rho^{-1-\alpha_0}, & \text{if } \omega\neq\omega_0,\ u=u_0, \\ g(x, \rho, u), & \text{otherwise.} \end{cases}$$

Then

$$\nu(x, \Gamma) = \int_0^{\infty}\int_V 1_{\Gamma}(\rho p_0(x, v))g_0(x, \rho, v)d\rho m_0(dv), \qquad \Gamma\in\mathcal{B}(\boldsymbol{R}^d\setminus\{0\}).$$

We also note the following representation due to Tsuchiya [**16**].

(3.10) $$\nu(x, \Gamma) = \int_0^{\infty}\int_V 1_{\Gamma}(\eta(x, \rho, v))\rho^{-1-\alpha}K(\rho)d\rho m_0(dv), \qquad \Gamma \in \mathcal{B}(\boldsymbol{R}^d\setminus\{0\}).$$

Here η is given as follows. We set $G_0(x, \rho, v)=\int_{\rho}^{\infty}g_0(x, r, v)dr(\in[0, \infty))$. For each x and v, let $H_0(x, \cdot, v)$ be the right continuous inverse function of $\rho\mapsto G_0(x, \rho, v)$, that is, $H_0(x, \rho, v)=\sup\{r>0:\ G_0(x, r, v)>\rho\}$, where $\sup\emptyset=0$. Put $\eta(x, \rho, v)=H_0(x, k(\rho), v)p_0(x, v)$, with $k(\rho)=\int_{\rho}^{\infty}r^{-1-\alpha}K(r)dr$.

We observe the following estimate.

LEMMA 3.3. *There is a positive constant C_1 such that*

(3.11) $$|\eta(x, \rho, v)| = H_0(x, k(\rho), v) \leqq C_1 y(\rho), \qquad x \in \boldsymbol{R}^d,\ \rho>0,\ v\in V,$$

where $\beta_0=\alpha_0\vee\beta$, and $y(\rho)=\rho^{\alpha/\beta_0}$ $(0\leqq\rho\leqq 1)$, $=\rho$ $(\rho>1)$.

PROOF. If $\rho\geqq\rho_0$, then (3.3) implies that $G_0(x, \rho, v)\leqq k(\rho)$, and hence

(3.12) $$H_0(x, k(\rho), v) \leqq \rho, \qquad x \in \boldsymbol{R}^d,\ v \in V.$$

In the case where $\rho\leqq\rho_0$, by means of (A.1),

$$G_0(x, \rho, (\omega, u)) \leqq \begin{cases} (\|c\|/\alpha_0)\rho^{-\alpha_0}+k(\rho_0), & \text{if } \omega\neq\omega_0,\ u=u_0, \\ \rho^{-\beta}\int_0^{\rho_0}r^{\beta}\|g(\cdot, r, \cdot)\|dr+k(\rho_0), & \text{otherwise,} \end{cases}$$
$$\leqq c_1\rho^{-\beta_0},$$

where c_1 is a positive constant independent of ρ. From this, if $k(\rho)\geqq c_1\rho_0^{-\beta_0}$, then

$$H_0(x, k(\rho), v) \leqq c_1^{1/\beta_0} k(\rho)^{-1/\beta_0}, \qquad x \in \boldsymbol{R}^d, v \in V.$$

Since $\lim_{\rho\downarrow 0}\rho^\alpha k(\rho) \in (0, \infty)$, we find that

$$H_0(x, k(\rho), v) \leqq c_2 \rho^{\alpha/\beta_0}, \qquad x \in \boldsymbol{R}^d,\ v \in V,\ \rho \leqq \rho_0, \tag{3.13}$$

with some positive c_2 independent of ρ. (3.12) and (3.13) complete the proof. ■

For each $\varepsilon>0$ we define the function n^ε by

$$n^\varepsilon(\rho) = \rho^{-1-\alpha} K(\rho/\varepsilon)/K(1/\varepsilon), \qquad \rho>0. \tag{3.14}$$

LEMMA 3.4. *For every* $\varepsilon \in (0, 1]$,

$$\int_0^1 (\varepsilon y(\rho/\varepsilon))^\gamma n^\varepsilon(\rho) d\rho \leqq C_2 \kappa_\gamma^+(\varepsilon), \qquad \gamma > \alpha \vee \beta_0, \tag{3.15}$$

$$\int_1^\infty (\varepsilon y(\rho/\varepsilon))^\gamma n^\varepsilon(\rho) d\rho \leqq C_2 \kappa_\gamma^-(\varepsilon), \qquad 0 \leqq \gamma < \alpha \wedge \beta_0, \tag{3.16}$$

where C_2 *is a positive constant depending only on* α, β_0, γ *and* $K(\rho_0)$, *and*

$$\kappa_\gamma^+(\varepsilon) = \frac{\varepsilon^{\gamma-\alpha}}{K(1/\varepsilon)} + \frac{K_{1,(\gamma-\alpha)/2}(1/\varepsilon)}{K(1/\varepsilon)},$$

$$\kappa_\gamma^-(\varepsilon) = \frac{K_{2,(\alpha-\gamma)/2}(1/\varepsilon)}{K(1/\varepsilon)}.$$

PROOF. Set $c=(\gamma-\alpha)/2$. Then

$$\begin{aligned}
&\int_0^1 (\varepsilon y(\rho/\varepsilon))^\gamma \rho^{-1-\alpha} K(\rho/\varepsilon) d\rho \\
&= \varepsilon^{(1-\alpha/\beta_0)\gamma} K(\rho_0) \int_0^\varepsilon \rho^{\alpha\gamma/\beta_0 - 1 - \alpha}\, d\rho + \int_\varepsilon^1 \rho^{\gamma-1-\alpha} K(\rho/\varepsilon) d\rho \\
&\leqq \varepsilon^{\gamma-\alpha} K(\rho_0) + \varepsilon^c \sup_{1 \leq u \leq 1/\varepsilon} u^c K(u) \int_0^1 \rho^{\gamma-1-\alpha-c} d\rho \\
&\leqq \varepsilon^{\gamma-\alpha} K(\rho_0) + K_{1,c}(1/\varepsilon)/(\gamma-\alpha-c).
\end{aligned}$$

Thus we get (3.15). (3.16) is also obtained in the same way. ■

It follows (3.7) and (3.8) that

$$\sup_{0<\varepsilon\leq 1} \kappa_\gamma^+(\varepsilon) < \infty, \qquad \gamma > \alpha \vee \beta_0, \tag{3.17}$$

$$\sup_{0<\varepsilon\leq 1} \kappa_\gamma^-(\varepsilon) < \infty, \qquad 0 \leqq \gamma < \alpha \wedge \beta_0. \tag{3.18}$$

Now the path functions of the cadlag process $\{X^{L^\varepsilon}(t)\}$ starting at x are given as a solution of a stochastic differential equation of jump type. Namely, for each $\varepsilon>0$ and $x \in \boldsymbol{R}^d$, we have a cadlag process $X^\varepsilon = (X^\varepsilon(t))_{t\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ with a reference family $(\mathcal{F}_t)_{t\geq 0}$ such that there are

(i) a d-dimensional $(\mathcal{F}_t)$-Brownian motion $(B(t))_{t\geq 0}$ with $B(0)=0$ a.s.,

(ii) an $(\mathcal{F}_t)$-stationary Poisson point process p^ε on $[0,\infty)\times V$ with characteristic measure $n^\varepsilon(\rho)d\rho m_0(dv)$,

(iii) a d-dimensional cadlag process $X^\varepsilon=(X^\varepsilon(t))_{t\geq 0}$ adapted to $(\mathcal{F}_t)_{t\geq 0}$, and

(iv) with probability one, $X^\varepsilon(t)=(X_1^\varepsilon(t),\cdots,X_d^\varepsilon(t))$, $B(t)=(B_1(t),\cdots,B_d(t))$ and the Poisson random measure N^ε induced by p^ε satisfy

$$
\begin{aligned}
X_i^\varepsilon(t) &= x_i+\frac{\varepsilon^{1-\alpha/2}}{\sqrt{K(1/\varepsilon)}}\sum_{j=1}^{d}\int_0^t\sigma_{ij}\Big(\frac{X^\varepsilon(s)}{\varepsilon}\Big)dB_j(s)\\
&\quad+\frac{\varepsilon^{1-\alpha}}{K(1/\varepsilon)}\int_0^t b_i\Big(\frac{X^\varepsilon(s)}{\varepsilon}\Big)ds\\
&\quad+\int_0^{t+}\int_0^\infty\int_V\varepsilon\eta_i\Big(\frac{X^\varepsilon(s-)}{\varepsilon},\frac{\rho}{\varepsilon},v\Big)M^\varepsilon(dsd\rho dv),\\
&\qquad\qquad i=1,2,\cdots,d,
\end{aligned}
\tag{3.19}
$$

where $\sigma=(\sigma_{ij})$ is the square root of a, $\eta=(\eta_i)$, and $M^\varepsilon(dsd\rho dv)=N^\varepsilon(dsd\rho dv)-dsn^\varepsilon(\rho)d\rho m_0(dv)$.

Note that the above statement (i) and the second term of the right hand side of (3.19) are ignored in Case A. Also note that $(\Omega,\mathcal{F},P)$, $(\mathcal{F}_t)_{t\geq 0}$, $(B(t))_{t\geq 0}$ may depend on ε.

In view of Theorem 2.1, the function $\varphi_i(\cdot)\equiv\int_0^\infty T_t^\dagger b_i(\cdot)dt$ belongs to $C_b^1(\boldsymbol{R}^d)$ with uniformly continuous derivatives in Case A, or belongs to $C_b^2(\boldsymbol{R}^d)$ in Case B, and satisfies $-L\varphi_i=b_i$, $i=1,2,\cdots,d$. We set

$$
Y_i^\varepsilon(t)=X_i^\varepsilon(t)+\varepsilon\varphi_i(X^\varepsilon(t)/\varepsilon),\qquad i=1,2,\cdots,d.
\tag{3.20}
$$

Then, with the aid of Itô's formula,

$$
\begin{aligned}
Y_i^\varepsilon(t) &= x_i+\varepsilon\varphi_i(x/\varepsilon)+\frac{\varepsilon^{1-\alpha/2}}{\sqrt{K(1/\varepsilon)}}\sum_{j=1}^{d}\int_0^t\sigma_{ij}\Big(\frac{X^\varepsilon(s)}{\varepsilon}\Big)dB_j(\varepsilon)\\
&\quad+\frac{\varepsilon^{1-\alpha/2}}{\sqrt{K(1/\varepsilon)}}\sum_{j,k=1}^{d}\int_0^t\partial_{x_j}\varphi_i\Big(\frac{X^\varepsilon(s)}{\varepsilon}\Big)\sigma_{jk}\Big(\frac{X^\varepsilon(s)}{\varepsilon}\Big)dB_k(s)\\
&\quad+\int_0^{t+}\int_0^1\int_V\Phi_i^\varepsilon(s-,\rho,v)M^\varepsilon(dsd\rho dv)\\
&\quad+\int_0^{t+}\int_1^\infty\int_V\Phi_i^\varepsilon(s-,\rho,v)M^\varepsilon(dsd\rho dv)\\
&\equiv x_i+\varepsilon\varphi_i(x/\varepsilon)+F_{1i}^\varepsilon(t)+F_{2i}^\varepsilon(t)+I_{1i}^\varepsilon(t)+I_{2i}^\varepsilon(t),\qquad i=1,2,\cdots,d,
\end{aligned}
\tag{3.21}
$$

where

$$
\Phi_i^\varepsilon(s,\rho,v)=\varepsilon\eta_i^\varepsilon(s,\rho,v)+\varepsilon\Big\{\varphi_i\Big(\frac{X^\varepsilon(s)}{\varepsilon}+\eta^\varepsilon(s,\rho,v)\Big)-\varphi_i\Big(\frac{X^\varepsilon(s)}{\varepsilon}\Big)\Big\},
\tag{3.22}
$$

$$\eta_1^{\varepsilon}(s, \rho, v) = \eta_i\Big(\frac{X^{\varepsilon}(s)}{\varepsilon}, \frac{\rho}{\varepsilon}, v\Big). \tag{3.23}$$

(3.21) is sometimes simply written as

$$Y^{\varepsilon}(t) = x + \varepsilon\varphi(x/\varepsilon) + F_1^{\varepsilon}(t) + F_2^{\varepsilon}(t) + I_1^{\varepsilon}(t) + I_2^{\varepsilon}(t).$$

We note the following

LEMMA 3.5. *There exists a positive constant* C_3 *such that*

$$E(|I_1^{\varepsilon}(\tau+\delta) - I_1^{\varepsilon}(\tau)|^2) \leqq C_3\delta\kappa_2^{+}(\varepsilon), \tag{3.24}$$

$$E(|I_2^{\varepsilon}(\tau+\delta) - I_2^{\varepsilon}(\tau)|) \leqq C_3\delta\kappa_1^{-}(\varepsilon), \tag{3.25}$$

for $0<\varepsilon\leqq 1$, $\delta>0$, *and* $(\mathcal{F}_t)$*-stopping time* τ.

PROOF. By virtue of (3.11) and (3.22),

$$\begin{aligned} &E(|I_1^{\varepsilon}(\tau+\delta) - I_1^{\varepsilon}(\tau)|^2) \\ &\quad = E\Big[\int_{\tau}^{\tau+\delta}\int_0^1\int_V |\Phi^{\varepsilon}(s, \rho, v)|^2 ds\, n^{\varepsilon}(\rho) d\rho m_0(dv)\Big] \\ &\quad \leqq c_1 E\Big[\int_{\tau}^{\tau+\delta}\int_0^1\int_V |\varepsilon\eta^{\varepsilon}(s, \rho, v)|^2 ds\, n^{\varepsilon}(\rho) d\rho m_0(dv)\Big] \\ &\quad \leqq c_2\delta\int_0^1 \varepsilon^2 y(\rho/\varepsilon)^2 n^{\varepsilon}(\rho) d\rho, \end{aligned}$$

where c_1 and c_2 are positive constants independent of $\varepsilon, \delta, \tau$. Combining this with (3.15), we get (3.24). In the same way, we also get (3.25). ■

Now we will show the tightness of $\{P_x^{\varepsilon}\}_{0<\varepsilon\leqq 1}$. Following a criteria due to Aldous [**1**; Theorem 1], it suffices to show the following.

LEMMA 3.6. *Let* $T>0$. *Then*

$$\lim_{R\to\infty}\sup_{0<\varepsilon\leqq 1} P(\sup_{0\leqq t\leqq T} |X^{\varepsilon}(t)|>R) = 0, \tag{3.26}$$

$$\lim_{\varepsilon\downarrow 0} P(|X^{\varepsilon}(\tau+\delta^{\varepsilon}) - X^{\varepsilon}(\tau)|>h) = 0, \tag{3.27}$$

for every $h>0$, $(\mathcal{F}_t)$*-stopping time* τ *not greater than* T, *and nonnegative numbers* δ^{ε} *with* $\lim_{\varepsilon\downarrow 0}\delta^{\varepsilon}=0$.

PROOF. Let R be sufficiently large so that $R\geqq 2|x|+4\|\varphi\|$. By using (3.20), (3.21) and Lemma 3.5,

$$\begin{aligned} &P(\sup_{0\leqq t\leqq T} |X^{\varepsilon}(t)|>R) \\ &\quad \leqq (8/R)^2 E(|F_1^{\varepsilon}(T)|^2 + |F_2^{\varepsilon}(T)|^2 + |I_1^{\varepsilon}(T)|^2) + (8/R)E(|I_2^{\varepsilon}(T)|) \\ &\quad \leqq (c_1/R^2)T\{\varepsilon^{2-\alpha}/K(1/\varepsilon) + \kappa_2^{+}(\varepsilon)\} + (c_1/R)T\kappa_1^{-}(\varepsilon), \end{aligned}$$

with a positive constant c_1 independent of ε, R, T. (3.26) follows from (3.7), (3.17) and (3.18).

Fix h, τ, δ^{ε} arbitrarily as in the lemma. Choose a sufficiently smalll $\varepsilon_0>0$ such that $4\varepsilon_0\|\varphi\|<h$. By means of (3.20) and (3.21),

$$\begin{aligned}|X^{\varepsilon}(\tau+\delta^{\varepsilon})-X^{\varepsilon}(\tau)| &\leqq |Y^{\varepsilon}(\tau+\delta^{\varepsilon})-Y^{\varepsilon}(\tau)|+2\varepsilon\|\varphi\| \\ &\leqq \sum_{i=1,2}|F_i^{\varepsilon}(\tau+\delta^{\varepsilon})-F_i^{\varepsilon}(\tau)|+\sum_{i=1,2}|I_i^{\varepsilon}(\tau+\delta^{\varepsilon})-I_i^{\varepsilon}(\tau)|+h/2, \\ &\qquad 0<\varepsilon\leqq\varepsilon_0.\end{aligned}$$

Therefore, in view of Lemma 3.5,

$$\begin{aligned}&P(|X^{\varepsilon}(\tau+\delta^{\varepsilon})-X^{\varepsilon}(\tau)|>h) \\ &\quad\leqq \sum_{i=1,2}(8/h)^2E(|F_i^{\varepsilon}(\tau+\delta^{\varepsilon})-F_i^{\varepsilon}(\tau)|^2) \\ &\qquad +(8/h)^2E(|I_1^{\varepsilon}(\tau+\delta^{\varepsilon})-I_1^{\varepsilon}(\tau)|^2)+(8/h)E(|I_2^{\varepsilon}(\tau+\delta^{\varepsilon})-I_2^{\varepsilon}(\tau)|) \\ &\quad\leqq (c_2/h^2)\delta^{\varepsilon}\{\varepsilon^{2-\alpha}/K(1/\varepsilon)+\kappa_2^{+}(\varepsilon)\}+(c_2/h)\delta^{\varepsilon}\kappa_1^{-}(\varepsilon),\end{aligned}$$

for some positive c_2 independent of ε, h, τ, δ^{ε}. (3.27) follows from (3.7), (3.17) and (3.18). ∎

The following lemma tells us the characterization of the limit process.

LEMMA 3.7. *Let f be a real valued infinitely continuously differentiable function with compact support. Then it holds that*

$$E[f(X^{\varepsilon}(t))|\mathcal{F}_s]-f(X^{\varepsilon}(s))-E\left[\int_s^t L^*f(X^{\varepsilon}(u))du\,|\,\mathcal{F}_s\right]\longrightarrow 0 \quad \text{as} \quad \varepsilon\downarrow 0,$$

uniformly in s and t ($s<t$) of each compact set of $[0, \infty)$.

PROOF. In the following, $0\leqq s<t$ and $o(1)$ means a random variable whose expectation converges to 0, as $\varepsilon\downarrow 0$, uniformly in s and t of each compact set of $[0, \infty)$. We put

$$F(x, y)=f(x+y)-f(x)-y\cdot\nabla f(x).$$

By means of (3.20),

$$E[f(X^{\varepsilon}(t))|\mathcal{F}_s]-f(X^{\varepsilon}(s))=E[f(Y^{\varepsilon}(t))|\mathcal{F}_s]-f(Y^{\varepsilon}(s))+o(1). \tag{3.28}$$

Applying Itô's formula to $f(Y^{\varepsilon}(t))$ and noting (3.7), we see that the right hand side of (3.28) is equal to

$$E\left[\int_s^t\int_0^{\infty}\int_V F(Y^{\varepsilon}(u),\ \Phi^{\varepsilon}(u, \rho, v))du\, n^{\varepsilon}(\rho)d\rho m_0(dv)\,|\,\mathcal{F}_s\right]+o(1). \tag{3.29}$$

At this point we divide our argument into three steps.

Step 1. (3.29) is equal to

$$E\left[\int_s^t\int_0^{\infty}\int_V F(X^{\varepsilon}(u),\ \varepsilon\eta^{\varepsilon}(u, \rho, v))du\, n^{\varepsilon}(\rho)d\rho m_0(dv)\,|\,\mathcal{F}_s\right]+o(1). \tag{3.30}$$

In fact,

$$|F(y,\xi)-F(z,\zeta)| \leqq c_1\{|y-z|(|\xi|\wedge|\xi|^2)+|\xi-\zeta|(1\wedge(|\xi|+|\zeta|))\},$$

for $y, z, \xi, \zeta \in \boldsymbol{R}^d$, where c_1 only depends on d and $\|\nabla^k f\|$, $k=1, 2, 3$. Hence, by virtue of (3.20), (3.22) and Lemma 3.3, the expectation of the difference between (3.29) and (3.30) except $o(1)$-terms is dominated by

$$\begin{aligned}
& c_2 E\Big[\int_s^t\int_0^\infty\int_V\Big\{\varepsilon(|\Phi^\varepsilon(u,\rho,v)|\wedge|\Phi^\varepsilon(u,\rho,v)|^2) \\
& \quad +\varepsilon\Big|\varphi\Big(\frac{X^\varepsilon(u)}{\varepsilon}+\eta^\varepsilon(u,\rho,v)\Big)-\varphi\Big(\frac{X^\varepsilon(u)}{\varepsilon}\Big)\Big| \\
& \quad \times\{1\wedge(|\Phi^\varepsilon(u,\rho,v)|+\varepsilon|\eta^\varepsilon(u,\rho,v)|)\}\Big\}du\, n^\varepsilon(\rho)d\rho m_0(dv)|\mathcal{F}_s\Big] \\
& \leqq c_3 E\Big[\int_s^t\int_0^\infty\int_V \varepsilon\{|\varepsilon\eta^\varepsilon(u,\rho,v))|\wedge|\varepsilon\eta^\varepsilon(u,\rho,v)|^2 \\
& \quad +(1\wedge|\eta^\varepsilon(u,\rho,v)|)(1\wedge|\varepsilon\eta^\varepsilon(u,\rho,v)|)\}du\, n^\varepsilon(\rho)d\rho m_0(dv)|\mathcal{F}_s\Big] \\
& \leqq c_4|t-s|\Big[\int_0^1\{\varepsilon(\varepsilon y(\rho/\varepsilon))^2+\varepsilon^\gamma(\varepsilon y(\rho/\varepsilon))^{2-\gamma}\}n^\varepsilon(\rho)d\rho \\
& \quad +\varepsilon\int_1^\infty(\varepsilon y(\rho/\varepsilon)+1)n^\varepsilon(\rho)d\rho\Big] \\
& \leqq c_5|t-s|\{\varepsilon\kappa_2^+(\varepsilon)+\varepsilon^\gamma\kappa_{2-\gamma}^+(\varepsilon)+\varepsilon\kappa_1^-(\varepsilon)+\varepsilon\kappa_0^-(\varepsilon)\} \\
& \leqq c_5|t-s|\varepsilon^\gamma\sup_{0<\varepsilon\leqq 1}\{\kappa_2^+(\varepsilon)+\kappa_{2-\gamma}^+(\varepsilon)+\kappa_1^-(\varepsilon)+\kappa_0^-(\varepsilon)\} \\
& = o(1),
\end{aligned}$$

where $0<\gamma<2-\alpha\vee\beta_0$, and $c_i\,(i=2,\cdots,5)$ are positive constants independent of ε, t and s.

Step 2. (3.30) is equal to

$$\begin{aligned}
(3.31)\qquad & E\Big[\int_s^t\int_{\boldsymbol{R}^d}F(X^\varepsilon(u),z)du\,\nu^\varepsilon\Big(\frac{X^\varepsilon(u)}{\varepsilon},dz\Big)|\mathcal{F}_s\Big]+o(1) \\
& = E\Big[\int_s^t\int_{\boldsymbol{R}^d}F(X^\varepsilon(u),z)du\,\bar{\nu}^\varepsilon(dz)|\mathcal{F}_s\Big]+o(1),
\end{aligned}$$

where ν^ε and $\bar{\nu}^\varepsilon$ are defined by (3.1) and (3.9), respectively. The left hand side of (3.31) follows directly from (3.1) and (3.10). In order to get the right hand side, we put

$$g^\varepsilon(x,y)=\int_{\boldsymbol{R}^d}F(y,z)\{\nu^\varepsilon(x,dz)-\bar{\nu}^\varepsilon(dz)\}.$$

It is enough to show

$$(3.32)\qquad E\Big[\int_s^t g^\varepsilon\Big(\frac{X^\varepsilon(u)}{\varepsilon},X^\varepsilon(u)\Big)du\,|\mathcal{F}_s\Big]=o(1).$$

Note that $g^\varepsilon \in C_b(\boldsymbol{R}^{2d})$, $g^\varepsilon(\cdot, y)$ is periodic with period 1 for each y, $\int_{T^d} g^\varepsilon(x, y)\mu(dx)=0$, $y\in\boldsymbol{R}^d$, $g^\varepsilon(x, y)$ is infinitely continuously differentiable in y for fixed x. Moreover,

$$\begin{aligned}(3.33)\qquad \|\nabla_y^k g^\varepsilon\| &\leqq \frac{c_6}{\varepsilon^\alpha K(1/\varepsilon)}\sup_x\int_{R^d}|z|^2\wedge|z|\nu(x, d_z(z/\varepsilon))\\ &\leqq c_7\int_0^\infty(\varepsilon y(\rho/\varepsilon))^2\wedge(\varepsilon y(\rho/\varepsilon))n^\varepsilon(\rho)d\rho\\ &\leqq c_8\sup_{0<\varepsilon\leqq 1}\{\kappa_2^+(\varepsilon)+\kappa_1^-(\varepsilon)\}<\infty,\end{aligned}$$

for $k=0, 1, 2, \cdots$, and positive constants c_i $(i=6, 7, 8)$ independent of ε. In particular, in Case B, $g^\varepsilon\in C_b^1(\boldsymbol{R}^{2d})$, $\partial_{x_i}g^\varepsilon(x, y)$ is infinitely continuously differentiable in y for each x and i, and $\|\nabla_x\nabla_y^k g^\varepsilon\|<\infty$, $k=0, 1, 2\cdots$. Hence g^ε satisfies all of the conditions in (iii) of Theorem 2.1. Therefore the integral $\phi^\varepsilon(x, y)\equiv\int_0^\infty T_t^L g^\varepsilon(\cdot, y)(x)dt$ converges absolutely, and either $\phi^\varepsilon\in C_b^1(\boldsymbol{R}^{2d})$ with uniformly continuous derivatives in Case A, or $\phi^\varepsilon\in C_b^2(\boldsymbol{R}^{2d})$ in Case B. Also,

$$\begin{aligned}(3.34)\qquad &\|\phi^\varepsilon\|+\|\nabla_y\phi^\varepsilon\|+\|\nabla_x\nabla_y\phi^\varepsilon\|+\|\nabla_y^2\phi^\varepsilon\|\\ &\quad\leqq c_9(\|g^\varepsilon\|+\|\nabla_y g^\varepsilon\|+\|\nabla_y^2 g^\varepsilon\|)\leqq c_{10},\end{aligned}$$

with positive constants c_9 and c_{10} independent of ε. We now apply Itô's formula to $\phi^\varepsilon(X^\varepsilon(t)/\varepsilon, X^\varepsilon(t))$. Then

$$\begin{aligned}&\varepsilon^\alpha K(1/\varepsilon)\Big\{E\Big[\phi^\varepsilon\Big(\frac{X^\varepsilon(t)}{\varepsilon}, X^\varepsilon(t)\Big)\,|\,\mathcal{F}_s\Big]-\phi^\varepsilon\Big(\frac{X^\varepsilon(s)}{\varepsilon}, X^\varepsilon(s)\Big)\Big\}\\ &=E\Big[\int_0^t\Big\{\frac{1}{2}\sum_{i,j}a_{ij}\Big(\frac{X^\varepsilon(u)}{\varepsilon}\Big)\partial_{x_i}\partial_{x_j}\phi^\varepsilon\Big(\frac{X^\varepsilon(u)}{\varepsilon}, X^\varepsilon(u)\Big)\\ &\qquad+b\Big(\frac{X^\varepsilon(u)}{\varepsilon}\Big)\cdot\nabla_x\phi^\varepsilon\Big(\frac{X^\varepsilon(u)}{\varepsilon}, X^\varepsilon(u)\Big)\Big\}du\,|\,\mathcal{F}_s\Big]\\ &\quad+\varepsilon E\Big[\int_s^t\Big\{\frac{1}{2}\sum_{i,j}a_{ij}\Big(\frac{X^\varepsilon(u)}{\varepsilon}\Big)\Big\{\partial_{x_i}\partial_{y_j}\phi^\varepsilon\Big(\frac{X^\varepsilon(u)}{\varepsilon}, X^\varepsilon(u)\Big)\\ &\qquad+\partial_{x_j}\partial_{y_i}\phi^\varepsilon\Big(\frac{X^\varepsilon(u)}{\varepsilon}, X^\varepsilon(u)\Big)+\varepsilon\partial_{y_i}\partial_{y_j}\phi^\varepsilon\Big(\frac{X^\varepsilon(u)}{\varepsilon}, X^\varepsilon(u)\Big)\Big\}\\ &\qquad+b\Big(\frac{X^\varepsilon(u)}{\varepsilon}\Big)\cdot\nabla_y\phi^\varepsilon\Big(\frac{X^\varepsilon(u)}{\varepsilon}, X^\varepsilon(u)\Big)\Big\}du\,|\,\mathcal{F}_s\Big]\\ &\quad+E\Big[\int_s^t\int_{R^d}\Big\{\phi^\varepsilon\Big(\frac{X^\varepsilon(u)}{\varepsilon}+z, X^\varepsilon(u)+\varepsilon z\Big)-\phi^\varepsilon\Big(\frac{X^\varepsilon(u)}{\varepsilon}, X^\varepsilon(u)\Big)\\ &\qquad-z\cdot\nabla_x\phi^\varepsilon\Big(\frac{X^\varepsilon(u)}{\varepsilon}, X^\varepsilon(u)\Big)\end{aligned}$$

$$-\varepsilon z\cdot\nabla_y\phi^\varepsilon\Big(\frac{X^\varepsilon(u)}{\varepsilon},\ X^\varepsilon(u)\Big)\Big\}du\,\nu\Big(\frac{X^\varepsilon(u)}{\varepsilon},\ dz\Big)\,|\,\mathcal{F}_s\Big].$$

Since $-L\phi^\varepsilon(\cdot,\ y)(x)=g^\varepsilon(x,\ y)$, and a_{ij}, b_i are bounded, by means of (3.34) we find that

$$\begin{aligned}
&E\Big[\int_s^t g^\varepsilon\Big(\frac{X^\varepsilon(u)}{\varepsilon},\ X^\varepsilon(u)\Big)du\,|\,\mathcal{F}_s\Big]\\
&=E\Big[\int_s^t\int_{R^d}\Big\{\phi^\varepsilon\Big(\frac{X^\varepsilon(u)}{\varepsilon}+z,\ X^\varepsilon(u)+\varepsilon z\Big)-\phi^\varepsilon\Big(\frac{X^\varepsilon(u)}{\varepsilon}+z,\ X^\varepsilon(u)\Big)\\
&\qquad-\varepsilon z\cdot\nabla_y\phi^\varepsilon\Big(\frac{X^\varepsilon(u)}{\varepsilon},\ X^\varepsilon(u)\Big)\Big\}du\,\nu\Big(\frac{X^\varepsilon(u)}{\varepsilon},\ dz\Big)\,|\,\mathcal{F}_s\Big]+o(1).
\end{aligned}$$

By using (3.34) again,

$$\begin{aligned}
&\Big|\int_{R^d}\{\phi^\varepsilon(x/\varepsilon+z,\ x+\varepsilon z)-\phi^\varepsilon(x/\varepsilon+z,\ x)-\varepsilon z\cdot\nabla_y\phi^\varepsilon(x/\varepsilon,\ x)\}\,\nu(x/\varepsilon,\ dz)\Big|\\
&\leqq c_{11}\varepsilon\{\|\nabla_y\phi^\varepsilon\|+\|\nabla_x\nabla_y\phi^\varepsilon\|+\varepsilon\|\nabla_y^2\phi\|\}\int_{R^d}|z|^2\wedge|z|\,\nu(x/\varepsilon,\ dz)\\
&\leqq c_{12}\varepsilon,
\end{aligned}$$

with positive c_{11} and c_{12} independent of x and ε. Thus (3.32) follows.

Step 3. Now the assertion of the lemma is obtained as follows. By the same argument as for (3.33), for any $\delta>0$, there exist $0<\rho_1<\rho_2<\infty$ such that

$$\overline{\lim_{\varepsilon\downarrow 0}}\ \sup_{x\in R^d}\int_{\{|z|\leq\rho_1\}\cup\{|z|\geq\rho_2\}}|F(x,\ z)|(\bar{\nu}^\varepsilon(dz)+\nu^*(dz))<\delta.$$

Since $F(x,\ z)$ is uniformly continuous and has a compact support on $\boldsymbol{R}^d\times\{\rho_1\leqq|z|\leqq\rho_2\}$, in view of Lemma 3.2,

$$\lim_{\varepsilon\downarrow 0}\sup_{x\in R^d}\Big|\int_{\rho_1\leq|z|\leq\rho_2}F(x,\ z)(\bar{\nu}^\varepsilon(dz)-\bar{\nu}^*(dz))\Big|=0.$$

Thus we arrive at the conclusion of the lemma. ■

4. Examples.

Throughout this section we assume that $a(x)\equiv 0$ and $b(x)\equiv 0$. Set

$$\nu_0(x,\ dy)=c(x,\ y)n(y)dy,$$

where $c(x,\ y)$ and $n(y)$ fulfill the conditions (A.1)-(3)-(i), (ii) and $c(x,\ y)$ is periodic in x with period 1 for each y.

1. We will start with the simplest case such that

$$\nu(x,\ dy)=\nu_0(x,\ dy).$$

Note that there is the unique invariant probability measure μ of the cadlag process on $\boldsymbol{T}^d$ governed by L given as (1.1) with $\nu=\nu_0$. Suppose that $c(x, y)$ has the following asymptotic representation

$$c(x, y)=c_0(x, y)|y|^{-\delta_0}K_0(|y|), \qquad x\in\boldsymbol{R}^d,\ |y|\geqq\rho_0, \tag{4.1}$$

for a sufficiently large ρ_0. Here c_0 is a nonnegative bounded continuous function on $\boldsymbol{R}^{2d}$, $\delta_0\geqq 0$, and K_0 is a slowly varying function, where K_0 is bounded if $\delta_0=0$. The scaled cadlag process $\{\varepsilon X^L(t/\varepsilon^{\alpha_0+\delta_0}K_0(1/\varepsilon))\}$ is equivalent to the cadlag process $\{X^{L^\varepsilon}(t)\}$ governed by the following L^ε.

$$L^\varepsilon f(x)=\int_{\boldsymbol{R}^d}\{f(x+y)-f(x)-y\cdot\nabla f(x)\}\frac{c(x/\varepsilon, y/\varepsilon)}{\varepsilon^{\delta_0}K_0(1/\varepsilon)}n(y)dy.$$

If there exists the limit function

$$c_0^*(\omega)\equiv\lim_{r\to\infty}\frac{1}{r}\int_{\rho_0}^r d\rho\int_{\boldsymbol{T}^d}c_0(x, \rho\omega)\mu(dx), \qquad \omega\in S^{d-1}, \tag{4.2}$$

then $\{X^{L^\varepsilon}(t)\}$ converges to the stable process governed by L^* as $\varepsilon\downarrow 0$, where $n_0^*(d\omega)=c_0^*(\omega)n_0(\omega)\sigma_0(d\omega)$, and

$$L^*f(x)=\int_{y=\rho\omega\in\boldsymbol{R}^d}\{f(x+y)-f(x)-y\cdot\nabla f(x)\}\rho^{-1-\alpha_0-\delta_0}d\rho n_0^*(d\omega). \tag{4.3}$$

The case where $d=1$, $\delta_0=0$ and $K_0=constant$ is reduced to [**3**].

2. We next consider the following case.

$$\nu(x, dy)=\nu_0(x, dy)+\nu_1(x, dy),$$

where ν_1 is given as

$$\nu_1(x, \Gamma)=\int_0^\infty\int_{S^{d-1}}1_\Gamma(\rho\omega)g_1(x, \rho, \omega)d\rho\sigma_1(d\omega),$$

σ_1 is a finite measure on S^{d-1}, and g_1 satisfies the condition (A.1)-(3)-(v) corresponding to Case A, and is periodic in x with period 1. Let μ be the invariant measure of the cadlag process on $\boldsymbol{T}^d$ governed by L given by (1.1) with $\nu=\nu_0+\nu_1$. Suppose the following asymptotic behavior

$$g_1(x, \rho, \omega)=c_1(x, \rho, \omega)\rho^{-1-\alpha_1(x)}K_1(\rho)^{\beta_1(x)}, \qquad x\in\boldsymbol{R}^d,\ \omega\in S^{d-1},\ \rho\geqq\rho_1,$$

for a sufficiently large ρ_1, where c_1 is nonnegative, bounded on $\boldsymbol{R}^d\times(0, \infty)\times S^{d-1}$, continuous in (x, ρ), periodic in x with period 1; α_1 is continuous, periodic with period 1, and $1<\alpha_1^-\equiv\min_x\alpha_1(x)\leqq\max_x\alpha_1(x)<2$; K_1 is a slowly varying function; and β_1 is continuous and periodic with period 1. We assume (4.1). Put $\beta_1^+=\max_x\beta_1(x)$, $\alpha=(\alpha_0+\delta_0)\wedge\alpha_1^-$, and $K(\rho)=K_0(\rho)$ if $\alpha_0+\delta_0\leqq\alpha_1^-$, $=K_1(\rho)^{\beta_1^+}$ otherwise. The scaled cadlag process $\{\varepsilon X^L(t/\varepsilon^\alpha K(1/\varepsilon))\}$ is identical with the cadlag process $\{X^{L^\varepsilon}(t)\}$ governed by the following

$$L^{\varepsilon}f(x)=\int_{y=\rho\omega\in R^{d}}\{f(x+y)-f(x)-y\cdot\nabla f(x)\}$$
$$\times\left\{\frac{c(x/\varepsilon,\ y/\varepsilon)}{\varepsilon^{\alpha-\alpha_0}K(1/\varepsilon)}n(y)dy+\frac{g_1(x/\varepsilon,\ \rho/\varepsilon,\ \omega)}{\varepsilon^{1+\alpha}K(1/\varepsilon)}d\rho\sigma_1(d\omega)\right\}.$$

We will observe to what process $\{X^{L^{\varepsilon}}(t)\}$ converges as $\varepsilon\downarrow 0$.

(Case 1) $\alpha_0+\delta_0<\alpha_1^-$, or $\alpha_0+\delta_0=\alpha_1^-$ and $\varliminf_{\rho\to\infty}K_0(\rho)/K_1(\rho)^{\beta_1^+}>1$.

In this case we assume (4.2). Then the limit process is the stable process governed by L^* given by (4.3).

(Case 2) $\alpha_0+\delta_0=\alpha_1^-$ and $\lim_{\rho\to\infty}K_0(\rho)/K_1(\rho)^{\beta_1^+}=1$.

In this case we assume, besides (4.2), that there exists the limit

$$c_1^*(\omega)\equiv\lim_{r\to\infty}\frac{1}{r}\int_{\rho_1}^{r}d\rho\int_{\{x\in T^d;\ \alpha_1(x)=\alpha_1^-,\ \beta_1(x)=\beta_1^+\}}c_1(x,\ \rho,\ \omega)\mu(dx),\tag{4.4}$$

for $\omega\in S^{d-1}$. Then the limit process is governed by the following L^*.

$$L^*f(x)=\int_{y=\rho\omega\in R^d}\{f(x+y)-f(x)-y\cdot\nabla f(x)\}\times\rho^{-1-\alpha}d\rho\{n_0^*(d\omega)+n_1^*(d\omega)\},\tag{4.5}$$

where

$$n_1^*(d\omega)=c_1^*(\omega)\sigma_1(d\omega).$$

(Case 3) $\alpha_0+\delta_0>\alpha_1^-$, or $\alpha_0+\delta_0=\alpha_1^-$ and $\varlimsup_{\rho\to\infty}K_0(\rho)/K_1(\rho)^{\beta_1^+}<1$.

In this case we only assume (4.4). Then the limit process is the stable process governed by L^* given by (4.5) with $n_0^*\equiv 0$.

3. Finally we consider the case that

$$\nu(x,\ dy)=\nu_0(x,\ dy)+\nu_2(x,\ dy),$$

where

$$\nu_2(x,\ dy)=g_2(x,\ \rho)d\rho\delta_{\{p(x)\}}(d\omega),$$

p is an S^{d-1}-valued continuous periodic function, and g_2 satisfies the condition (A.1)-(3)-(v) corresponding to Case A, is periodic in x with period 1. Note that the assumption (A.1)-(3)-(iii), (iv) hold with $U=\{1\}$, $m(du)=\delta_{\{1\}}(du)$, $p(x,u)=p(x)$. We denote by μ the invariant measure of the cadlag process on $\boldsymbol{T}^d$ governed by L defined by (1.1) with $\nu=\nu_0+\nu_2$. Suppose

$$g_2(x,\ \rho)=c_2(x,\ \rho)\rho^{-1-\alpha_2(x)}K_2(\rho)^{\beta_2(x)},\qquad x\in\boldsymbol{R}^d,\ \rho\geqq\rho_2,$$

for a sufficiently large ρ_2, where c_2 is nonnegative, bounded, continuous on $\boldsymbol{R}^d\times(0,\ \infty)$, periodic in x with period 1; α_2 is continuous, periodic with period 1, and $1<\alpha_2^- \equiv \min_x \alpha_2(x) \leqq \max_x \alpha_2(x) < 2$; K_2 is a slowly varying function; and β_2 is continuous and periodic with period 1. We also assume (4.1). Set $\beta_2^+ = \max_x \beta_2(x)$, $\alpha = (\alpha_0+\delta_0) \wedge \alpha_2^-$, and $K(\rho) = K_0(\rho)$ if $\alpha_0+\delta_0 \leqq \alpha_2^-$, $= K_2(\rho)^{\beta_2^+}$ otherwise. The scaled cadlag process $\{\varepsilon X^L(t/\varepsilon^\alpha K(1/\varepsilon))\}$ is equivalent to the cadlag process $\{X^{L^\varepsilon}(t)\}$ governed by

$$L^\varepsilon f(x) = \int_{y=\rho\omega \in R^d} \{f(x+y) - f(x) - y \cdot \nabla f(x)\}$$
$$\times \left\{ \frac{c(x/\varepsilon, y/\varepsilon)}{\varepsilon^{\alpha-\alpha_0} K(1/\varepsilon)} n(y) dy + \frac{g_2(x/\varepsilon, \rho/\varepsilon)}{\varepsilon^{1+\alpha} K(1/\varepsilon)} d\rho \delta_{\{p(x/\varepsilon)\}}(d\omega) \right\}.$$

Dividing into three cases as above, we observe the limit process of $\{X^{L^\varepsilon}(t)\}$.

(Case 1) $\alpha_0+\delta_0 < \alpha_2^-$, or $\alpha_0+\delta_0 = \alpha_2^-$ and $\varliminf_{\rho\to\infty} K_0(\rho)/K_2(\rho)^{\beta_2^+} > 1$.

Assume (4.2). Then the limit process is governed by L^* of the form (4.3).

(Case 2) $\alpha_0+\delta_0 = \alpha_2^-$ and $\lim_{\rho\to\infty} K_0(\rho)/K_2(\rho)^{\beta_2^+} = 1$.

In this case we assume, besides (4.2), that there exists the limit measure

$$n_2^*(\Theta) \equiv \lim_{r\to\infty} \frac{1}{r} \int_{\rho_2}^r d\rho \int_{\{x \in T^d:\ \alpha_2(x)=\alpha_2^-,\ \beta_2(x)=\beta_2^+\} \cap p^{-1}(\Theta)} c_2(x, \rho) \mu(dx), \tag{4.6}$$

for $\Theta \in \mathscr{B}(S^{d-1})$. Then the limit process is governed by the following L^*.

$$L^* f(x) = \int_{y=\rho\omega \in R^d} \{f(x+y) - f(x) - y \cdot \nabla f(x)\} \times \rho^{-1-\alpha} d\rho \{n_0^*(d\omega) + n_2^*(d\omega)\}. \tag{4.7}$$

(Case 3) $\alpha_0+\delta_0 > \alpha_2^-$, or $\alpha_0+\delta_0 = \alpha_2^-$ and $\varlimsup_{\rho\to\infty} K_0(\rho)/K_2(\rho)^{\beta_2^+} < 1$.

In this case we only assume (4.6). Then the limit process is the stable process governed by L^* given by (4.7) with $n_0^* \equiv 0$.

4. Let $\nu = \sum_{i=1}^j \nu_{0i} + \sum_{i=1}^k \nu_{1i} + \sum_{i=1}^l \nu_{2i}$, where ν_{0i}, ν_{1i} and ν_{2i} are Lévy measures of the type of ν_0, ν_1 and ν_2 mentioned above, respectively, $i=1, 2, \cdots$. Then it is easy to see that Theorem 3.1 holds for this ν. Especially, in the case where ν is given as (1.4), we get the assertion mentioned in the last paragraph of Section 1.

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References

[1] D. Aldous, Stopping times and tightness, Ann. Probab., **6** (1978), 335-340.

[2] A. Bensoussan, J.L. Lions and G.C. Papanicolaou, Sur quelques phénomènes asymptotiques stationnaires, C.R. Acad. Sc. Paris, Série A, **281** (1975), 89-94.

[3] M. Horie, T. Inuzuka and H. Tanaka, Homogenization of certain one-dimensional discontinuous Markov processes, Hiroshima Math. J., **7** (1977), 629-641.

[4] T. Komatsu, Markov processes associated with certain integro-differential operators, Osaka J. Math., **10** (1973), 271-303.

[5] T. Komatsu, On the martingale problem for generators of stable processes with perturbations, Osaka J. Math., **21** (1984), 113-132.

[6] T. Komatsu, Pseudo-differential operators and Markov processes, J. Math. Soc. Japan, **36** (1984), 387-418.

[7] O.A. Ladyzěnskaja, V.A. Solonnikov and N.N. Ural'ceva, Linear and Quasilinear Equations of Parabolic Type, Transl. Math. Monographs, **23**, Amer. Math. Soc., Providence, 1968.

[8] P.A. Meyer, Les résolvantes fortement fellériennes d'après Mokobodzki, Séminaire de Probabilités II, Université de Strasbourg, Lecture Notes in Math., **51**, Springer, 1968, pp. 171-174.

[9] A. Negoro and M. Tsuchiya, Markov semigroups associated with one-dimensional Lévy operators-regularity and convergence-, Stochastic Methods in Biology (eds. M. Kimura et al.), Lecture Notes in Biomath., **70**, Springer, 1987, pp. 185-193.

[10] A. Negoro and M. Tsuchiya, Convergence and uniqueness theorems for Markov processes associated with Lévy operators, Probability Theory and Mathematical Statistics (eds. S. Watanabe and Yu. V. Prokhorov), Lecture Notes in Math., **1299**, Springer, 1988, pp. 348-356.

[11] A. Negoro and M. Tsuchiya, Stochastic processes and semigroups associated with degenerate Lévy generating operating operators, Stochastics and Stochastics Reports, **26** (1989), 29-61.

[12] G.C. Papanicolaou, D.W. Stroock and S.R.S. Varadhan, Martingale approach to some limit theorems, Statistical Mechanics and Dynamical Systems, Univ. Conf. Turbulence (ed. D. Ruelle), Duke Univ. Math. Ser. III, 1977.

[13] E. Seneta, Regular Varying Functions, Lecture Notes in Math., **508**, Springer, 1976.

[14] D.W. Stroock, Diffusion processes associated with Lévy generators, Z. Wahrsch. verw. Gebiete, **32** (1975). 209-244.

[15] D.W. Stroock and S.R.S. Varadhan, Diffusion processes with continuous coefficients, I and II, Comm. Pure Appl. Math., **22** (1969), 345-400 and 479-530.

[16] M. Tsuchiya, The Lipshits condition for coefficients of jump term of stochastic differential equations associated with decomposed Lévy measures (preprint).

[17] H. Watanabe, Potential operator of a recurrent strong Feller process in the strict sense and boundary value problem, J. Math. Soc. Japan, **16** (1964), 83-95.

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