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On hypoellipticity for a certain operator with double characteristic

Dedicated to Professor Mutsuhide Matsumura on his 60th birthday

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§ 1. Introduction and result.

In this paper, we consider C^{∞}-hypoellipticity for the operator

$$P=D_1^2+D_2^2+D_3^2+x_3^2D_4^2+(f(x_1)-1)D_4, \tag{1.1}$$

$$\left(D_j=-i\frac{\partial}{\partial x_j} \qquad j=1, 2, 3, 4\right)$$

in neighborhoods ($\subset \boldsymbol{R}^4$) of the hypersurface $x_1=0$. Here we assume that the function $f(x_1)$ has the following properties:

(A.1) (i) $f(0)=0$, $\quad f(x_1)>0 \quad$ *if* $x_1\neq 0$.

(ii) $f(x_1)$ *is monotone in the intervals* $[0, \delta)$ *and* $(-\delta, 0]$ *for some* $\delta>0$.

Notice that the above operator (1.1) is a degenerate elliptic operator with double characteristic $\Sigma=\{(x, \xi)\in T^*\boldsymbol{R}^4\setminus 0\,;\ \xi_1=\xi_2=\xi_3=x_3=0\}$. Also notice that the canonical symplectic form $\sigma=\sum_j dx_j\wedge d\xi_j$ is of constant rank $(=2)$ on $T_\rho\Sigma$ for any point $\rho\in\Sigma$. A. Grigis [**3**] treated a class of such operators after the important work of L. Boutet de Monvel [**1**]. He has given a condition which is necessary and sufficient for them to be hypoelliptic with loss of one derivative. Roughly speaking, his condition is that Melin's invariant (=subprincipal symbol +positive trace/2) does not take non-positive (real) values on the characteristic manifold Σ. For the operator (1.1), it becomes $0<f(x_1)<2$ if $\mathcal{I}m\ f(x_1)=0$ (*cf.* the condition (b) of théorème 0.1 in [**3**]). So, under the assumption (i) of (A.1), the operator (1.1) does not satisfy the condition on the hypersurface $x_1=0$. Nevertheless, it has a possibility to be hypoelliptic with loss of more than one derivatives.

First, let us give a condition of non-hypoellipticity for the operator (1.1):

THEOREM 1. *In addition to the hypothesis* (A.1), *we assume that*

(A.2) *there exist positive numbers* δ_1 *and* ε *such that*

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$$|x_1 \log f(x_1)| \geqq \varepsilon \qquad \text{for} \quad 0 < x_1 < \delta_1 .$$

Then, the operator (1.1) *is not hypoelliptic in any neighborhood of the hypersurface* $x_1=0$.

We remark that there are the same conditions as (A.1) and (A.2) in the works of Y. Morimoto [**12**] and T. Hoshiro [**9**] where the hypoellipticity for infinitely degenerate elliptic operators is treated. The purpose of the present paper is to point out that the operator (1.1) has a similar structure as in them. Next we give a sufficient condition for the operator (1.1) to be hypoelliptic.

THEOREM 2. *In addition to the hypothesis* (A.1), *we assume that*

$$\text{(A.3)} \qquad \lim_{x_1 \to 0} |x_1 \log f(x_1)| = 0 .$$

Then, the operator (1.1) *is hypoelliptic in some neighborhood of the hypersurface* $x_1=0$.

As in the works [**12**], [**8**] and [**9**] the most significant example is the operator (1.1) with $f(x_1)=\exp(-1/|x_1|^{\sigma})$ $(\sigma>0)$. It is hypoelliptic on $x_1=0$ if and only if $\sigma<1$. Also we remark here that, in view of our proof of Theorem 2, it would be obvious that the operator

$$P_1 = D_1^2+D_2^2+x_2^2D_3^2+(f(x_1)-1)D_3$$

is hypoelliptic without the assumption (A.3). It is analogous to the fact that Fediĭ's operator is also hypoelliptic without the assumption (A.3) (see [**9**]). Such a difference of the conditions for hypoellipticity can be understood from propagation of singularities along double characteristic manifolds. Generally, the singularities can propagate along the leaves of foliations of $T_\rho\Sigma\cap T_\rho\Sigma^\sigma$ for $\rho\in\Sigma$ (where $T_\rho\Sigma^\sigma$ is the orthogonal space of $T_\rho\Sigma$ with respect to the symplectic form σ). For the operator (1.1), Melin's invariant vanishes on $\Lambda=\Sigma\cap\{x_1=0, \xi_4>0\}$(where it may not be hypoelliptic microlocally) and, for any $\rho\in\Lambda$, there is a vector $(=\partial_{x_2})\in T_\rho\Sigma\cap T_\rho\Sigma^\sigma$ which is tangent to Λ. Thus on the operator (1.1), it is possible for singularities to propagate along Λ and so, such an assumption as (A.3) is necessary for the operator (1.1) to be hypoelliptic (it can be regarded as a condition for preventing the propagation of singularities). On the other hand, it can be easily observed that, for P_1, there is no vector playing the role as ∂_{x_2} above.

There have been several works in the cases where the above mentioned L. Boutet de Monvel-A. Grigis' condition is violated. See for example, V.V. Grušin [**5**], K. Taira [**14**], B. Helffer [**6**] E.M. Stein [**13**], A. Grigis-L.P. Rothschild [**4**] and K.H. Kwon [**11**]. We do not explain here their works. However we note that their situations and ours are different to each other. Also our result seems to be extended to a certain class of operators characterized geometrically. The author wants to consider it in a future paper.

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§ 2. Preliminaries.

In the present section, we recall some techniques due to L. Boutet de Monvel [**1**]. They are necessary for a reduction in our proof of Theorem 2.

First let us denote by $h_j(t)$ the j-th Hermite function, i.e.,

$$h_j(t)=\pi^{-1/4}(2^j j!)^{-1/2}\Big(\frac{d}{dt}-t\Big)^j \exp\Big(-\frac{t^2}{2}\Big).$$

Choosing a function $\phi(t)\in C^\infty(\boldsymbol{R})$ so that (i) $0\leqq\phi(t)\leqq 1$, (ii) $\phi(t)\equiv 0$ for $|t|\leqq 1$, (iii) $\phi(t)\equiv 1$ for $|t|\geqq 2$, we introduce a sequence of operators H_j, $j=0, 1, 2, \cdots$ in such a way that

$$\begin{aligned}(2.1)\qquad H_j:\ v(x_1, x_2, x_4) &\longrightarrow (H_j v)(x_1, x_2, x_3, x_4)\\ &=(2\pi)^{-1}\int e^{ix_4\xi_4}\phi(\xi_4)|\xi_4|^{1/4}h_j(x_3|\xi_4|^{1/2})\hat{v}(x_1, x_2, \xi_4)d\xi_4,\end{aligned}$$

where $\hat{v}$ denotes the partial Fourier transform of v w.r.t. x_4. Notice that the adjoint of H_j is defined by

$$\begin{aligned}(2.2)\qquad H_j^*:\ u(x_1, x_2, x_3, x_4) &\longrightarrow (H_j^* u)(x_1, x_2, x_4)\\ &=(2\pi)^{-1}\iint e^{ix_4\xi_4}\phi(\xi_4)|\xi_4|^{1/4}h_j(y_3|\xi_4|^{1/2})\hat{u}(x_1, x_2, y_3, \xi_4)dy_3 d\xi_4.\end{aligned}$$

We set now $\Pi_j=H_jH_j^*$ and, in addition we denote by Π_{00} a pseudodifferential operator with symbol $1-\phi(\xi_4)^2$ which is of class $OPS^0_{\lambda,1,0}$ with $\lambda=(1+\xi_4{}^2)^{1/2}$. Here we adopt the notation from H. Kumano-go [**10**] Chapter 7.

We consider here some properties of these operators related to our operator (1.1), (*cf.* A. Grigis [**3**] Section III.3).

PROPOSITION 1. (i) *Π_j, $j=0, 1, 2, \cdots$ are pseudodifferential operators of class $OPS^0_{\lambda,1/2,1/2}$ with $\lambda=(1+\xi_4{}^2)^{1/2}$.*

(ii) *For any $u\in L^2(\boldsymbol{R}^4)$,*

$$u=\Pi_{00}u+\sum_{j=0}^{\infty}\Pi_j u,$$

with the right hand side being convergent in $L^2(\boldsymbol{R}^4)$.

Hereafter we denote $\Pi_*=I-\Pi_{00}-\Pi_0(=\sum_{j=1}^\infty \Pi_j)$.

(iii) *For any $j, k=0, 1, 2, \cdots$*

$$H_j^*H_k=\delta_{jk}\cdot\phi(D_4)^2.$$

(iv) *For any* $j=0, 1, 2, \cdots$ *it holds*

$$PH_j = H_jP_j \quad and \quad H_j^*P = P_jH_j^*,$$

where $P_j = D_1{}^2+D_2{}^2+(2j+1)|D_4|+(f(x_1)-1)D_4$.

(v) *Let us set* $y=(x_1, x_2, x_4)$ *and* $\eta=(\xi_1, \xi_2, \xi_4)$. *Then the following inclusions hold:*

$$WF(H_0{}^*u) \subset \{(y, \eta)\in T^*\boldsymbol{R}^3\setminus 0;\ (x_1, x_2, 0, x_4;\ \xi_1, \xi_2, 0, \xi_4)\in WF(u)\}$$

and

$$WF(H_0v)\subset\{(x_1, x_2, 0, x_4;\ \xi_1, \xi_2, 0, \xi_4)\in T^*\boldsymbol{R}^4\setminus 0;\ (y, \eta)\in WF(v)\}.$$

REMARK. In view of (iv), it is obvious that the operators Π_j, $j=0, 1, 2, \cdots$ commute with our operator P.

PROOF. (i) First observe that the integral operator with kernel $K(t, s)=h(t)\overline{h(s)}$ $(h\in\mathcal{S}(\boldsymbol{R}))$ can be regarded as a pseudodifferential operator with symbol $e^{-it\tau}h(t)\overline{\hat{h}(\tau)}$. Indeed, from Plancherel's formula, it follows that

$$h(t)\int\overline{h(s)}f(s)ds=(2\pi)^{-1}h(t)\int\overline{\hat{h}(\tau)}\hat{f}(\tau)d\tau.$$

So, with aid of the property $\hat{h}_j(\tau)=(-i)^jh_j(\tau)$, Π_j can be regarded as a pseudodifferential operator with symbol

$$i^j\cdot\phi(\xi_4)^2\cdot e^{-ix_3\xi_3}h_j(x_3|\xi_4|^{1/2})h_j(\xi_3/|\xi_4|^{1/2}). \tag{2.3}$$

This immediately yields the assertion (i).

The properties (ii) and (iii) are direct consequences of the fact that the sequence of the Hermite functions $\{h_j\}_{j=0}^{\infty}$ is an orthonormal basis in $L^2(\boldsymbol{R})$.

The properties in (iv) immediately follow from the fact that $h_j(t)$, $j=0, 1, 2, \cdots$ are the eigenfunctions of the Hermite operator, i.e.,

$$\left(-\frac{d^2}{dt^2}+t^2\right)h_j(t) = (2j+1)h_j(t). \tag{2.4}$$

The assertions in (v) are consequences of the fact that the distribution kernels of the operators H_0 and H_0^* have respectively the following integral expressions:

$$(2\pi)^{-1}\pi^{-1/4}\int\exp\{i(x_4-y_4)\xi_4-x_3{}^2|\xi_4|/2\}\phi(\xi_4)|\xi_4|^{1/4}d\xi_4,$$

and

$$(2\pi)^{-1}\pi^{-1/4}\int\exp\{i(x_4-y_4)\xi_4-y_3{}^2|\xi_4|/2\}\phi(\xi_4)|\xi_4|^{1/4}d\xi_4.$$

The stationary phase method enables us to compute the wave front set of these kernels (see L. Hörmander [5] Theorem 8.1.9), and this yields (v) (see Theorem 8.2.13 of [5]). ■

§3. Proof of non-hypoellipticity.

The proof of Theorem 1 is almost identical to that of Theorem 1 in [9]. We start by considering the following eigenvalue problem (with real parameter ξ):

$$\tag{3.1} \begin{cases} \left(-\dfrac{d^2}{dt^2}+f(t)|\xi|\right)v(t)=\lambda v(t), & -a<t<a \\ v(a)=v(-a)=0. \end{cases}$$

Denote by $\lambda_1(\xi)$ the smallest eigenvalue and by $v(t;\xi)$ the corresponding eigenfunction normalized so that $\int_{a}^{a}|v(t;\xi)|^2dt=1$. Let us now recall (see Section 2 of [9]) that our assumptions (A.1) and (A.2) imply that

(3.2) there exists a positive constant C such that

$$\lambda_1(\xi)\leqq(C\log|\xi|)^2 \qquad \text{for } |\xi| \text{ sufficiently large,}$$

and

(3.3) for any positive number a' satisfying $0<a'<a$,

$$\int_{-a'}^{a'}|v(t;\xi)|^2dt \longrightarrow 1 \quad \text{as} \quad |\xi|\to\infty.$$

Set now

$$\tag{3.4} u_\xi(x)=\exp(\sqrt{\lambda_1(\xi)}\cdot x_2+i|\xi|x_4)v(x_1;\xi)h_0(x_3|\xi|^{1/2}).$$

We are going to show that the one parameter family of the functions (3.4) with properties (3.2) and (3.3) contradicts hypoellipticity for the operator (1.1).

First observe that, if the operator P is hypoelliptic, then it holds the following inequality: For any integer $k>0$ and for any open sets $\omega'\Subset\omega$, there exist an integer k' and a constant C_1 such that

$$\tag{3.5} \|D_4{}^k u\|_{L^2(\omega')}\leqq C_1\{\sum_{|\alpha|\leqq k'}\|D^\alpha Pu\|_{L^2(\omega)}+\|u\|_{L^2(\omega)}\},$$
$$\text{for any} \quad u\in C^\infty(\bar\omega).$$

In the above inequality, (3.5), let us set $\omega=\{x\in\boldsymbol{R}^4;\ |x_1|<a,\ 0<x_2<a,\ |x_3|<a,\ 0<x_4<a\}$ and $\omega'=\{x\in\boldsymbol{R}^4;\ |x_1|<a',\ a'/2<x_2<a',\ |x_3|<a',\ a'/2<x_4<a'\}$ with sufficiently small constants a and a' satisfying $0<a'<a$ (recall that P does not depend on the variables (x_2, x_4)).

Now, notice that $u_\xi(x)$ is a solution of the equation $Pu_\xi(x)\equiv0$ in ω for arbitrary $\xi>0$. This easily follows from the property (2.4) and the definition of $v(t;\xi)$. So, if one substitutes $u_\xi(x)$ to (3.5), then the first term of the right hand side vanishes.

On the other hand, it could be seen that the property (3.2) yields the estimate

(3.6) $\|u_\xi\|_{L^2(\omega)} \leqq C_2 \cdot |\xi|^C$ for $|\xi|$ sufficiently large,

with some constant C_2, and that the property (3.3) guarantees the inequality

(3.7) $\|D_4^k u_\xi\|_{L^2(\omega')} \geqq C_3 \cdot |\xi|^k$ for $|\xi|$ sufficiently large,

with another positive constant C_3.

Finally we can easily see that there is a contradiction among (3.5), (3.6) and (3.7) taking a positive integer k so that $k>C$. This finishes the proof.

§4. Proof of hypoellipticity.

Since we consider the hypoellipticity in a small neighborhood of hypersurface $x_1=0$, we can modify $f(x_1)$ outside some neighborhood of $x_1=0$. So we assume that $f(x_1) \in C^\infty(\boldsymbol{R})$ satisfies $0 \leqq f(x_1) < 1$ preserving the properties (A.1) and (A.3). At first, let us explain the plan of our proof. In order to show hypoellipticity of P, one can consider it, by dividing P into three parts: $P\Pi_{00}$, $P\Pi_0$ and $P\Pi_*$. More precisely, since $u=\Pi_{00}u+\Pi_0 u+\Pi_* u$, the smoothness of u comes from those of all terms in the right hand side. Also notice that the operators Π_{00}, Π_0 and Π_* commute with P. So, if we show the hypoellipticity of the equations $P\Pi_{00}u=\Pi_{00}f$, $P\Pi_0 u=\Pi_0 f$ and $P\Pi_* u=\Pi_* f$ (i.e., the smoothness of f implies those of $\Pi_{00}u$, $\Pi_0 u$ and $\Pi_* u$), then our proof would be completed.

In addition, let us remark that it suffices to show the smoothness of the solution u with respect to the variable x_4, since P is non-characteristic with respect to the other variables. To be more precise, we now introduce the following Sobolev space:

DEFINITION. We denote by $H^{k,l}(k, l \in \boldsymbol{R})$ the space of all distributions $u \in \mathcal{S}'(\boldsymbol{R}^4)$ satisfying

$$\int |\hat{u}(\xi_1, \xi_2, \xi_3, \xi_4)|^2 (1+\xi_1^2+\xi_2^2+\xi_3^2)^k (1+\xi_4^2)^l d\xi < \infty .$$

In the present section, we are going to prove that, if f is C^∞ (w.r.t. all variables) in a neighborhood of a certain point on $x_1=0$, then the solution $u \in H^{0,-\infty}(=\bigcup_l H^{0,l})$ belongs to $H^{0,\infty}(=\bigcap_l H^{0,l})$ there. It may be seen that one can easily show that the smoothness of the solution u w.r.t. the variables (x_1, x_2, x_3), by writing the equation $Pu=f$ as

$$(D_1^2+D_2^2+D_3^2)u = -\{x_3^2 D_4^2+(f(x_1)-1)D_4\}u+f$$

and observing recursively that the right hand side belongs to $H^{2k,\infty}$, $k=0, 1, 2, \cdots$. For the precise discussion, *cf.* the first part of Section 4 of [**9**].

I. It would be quite obvious that $\Pi_{00}u\in H^{0,\infty}$ since Π_{00} is a pseudodifferential operator with symbol $1-\phi(\xi_4)^2$.

II. Next we consider the equation $P\Pi_*u=\Pi_*f$. We shall show that for any positive integer k, one can construct a parametrix Q so that $QP\Pi_*\equiv\Pi_*$ mod $OPS_{\lambda,1/2,1/2}^{-k}$ with $\lambda=(1+\xi_4^2)^{1/2}$. (Note that $K\in OPS_{\lambda,1/2,1/2}^{-k}$ is a *regularizer* of order k with respect to the variable x_4, *cf.* Theorem 1.6 in Chap. 7 of [**10**].) The argument below is essentially due to L. Boutet de Monvel [**1**] and A. Grigis [**3**]. Roughly speaking, their idea is that, since P_j is semi-elliptic for $j\geqq 1$, one can build the parametrix Q. In order to make this section readable, we shall show this explicitly.

(**1**). Let us choose now functions $\psi_j\in C_0^\infty(\boldsymbol{R})$, $j=1, 2, 3$ so that $\psi_1(t)\equiv 1$ for $|t|\leqq 1/8$, $\psi_3(t)\equiv 0$ for $|t|\geqq 1/4$ and $\psi_1\Subset\psi_2\Subset\psi_3$. (Here $\psi_1\Subset\psi_2$ means that, in the support of ψ_1, ψ_2 is identically equal to 1.) Further we choose $\phi_j\in C^\infty(\boldsymbol{R})$, $j=1, 2, 3$ so that $\phi\Subset\phi_1\Subset\phi_2\Subset\phi_3$ (where ϕ is the same one in Section 2) and $\phi_3(t)\equiv 0$ for $|t|\leqq 1/2$. Denote by φ_j, $j=1, 2, 3$ pseudodifferential operators with symbols $\varphi_j(\xi_1, \xi_2, \xi_4)=\psi_j(|\xi_4|/(\xi_1^2+\xi_2^2))\phi_j(\xi_4)$, $j=1, 2, 3$, respectively.

Now notice that, in the support of $\psi_3(|\xi_4|/(\xi_1^2+\xi_2^2))$, it holds

$$\xi_1^2+\xi_2^2+\xi_3^2+x_3^2\xi_4^2+(f(x_1)-1)\xi_4\geqq(\xi_1^2+\xi_2^2+\xi_3^2+|\xi_4|)/4.$$

Denote by Q_1 a pseudodifferential operator with symbol

$$\sigma(Q_1)=\{\xi_1^2+\xi_2^2+\xi_3^2+x_3^2\xi_4^2+(f(x_1)-1)\xi_4\}^{-1}\varphi_3(\xi_1, \xi_2, \xi_4).$$

Then the symbol calculus of class $S_{\lambda,1/2,0}$ gives that

$$Q_1P=\varphi_3-K \quad \text{with} \quad K\in OPS_{\lambda,1/2,0}^{-1/2}.$$

This immediately implies that

$$Q_1P\varphi_2=(I-K)\varphi_2.$$

Moreover we now use the Neumann series expansion. Set

$$Q_2=(I+K+K^2+\cdots+K^{2k-1})Q_1.$$

Then it is clear that

$$\begin{aligned}\varphi_1Q_2P&\equiv\varphi_1Q_2P\varphi_2 \mod OPS_\lambda^{-\infty}\\&=\varphi_1+K_1 \quad \text{with} \quad K_1\in OPS_{\lambda,1/2,0}^{-k}.\end{aligned}\tag{4.1}$$

(**2**). In the region complimentary to the one considered in (1), we shall construct the parametrix in the following way. First let us write the symbol $\sigma(P_j^N)$(where N will be chosen later sufficiently large) by the sum of semi-homogenous parts:

$$\sigma(P_j^N)=p_{2N,j}+p_{2N-1,j}+\cdots+p_{0,j},$$

with each $p_{k,j}$ having the property that

$$p_{k,j}(x_1;\lambda\xi_1,\lambda\xi_2,\lambda^2\xi_4)=\lambda^k p_{k,j}(x_1;\xi_1,\xi_2,\xi_4),\qquad for\quad \lambda>0.$$

In order to make $(r_{-2N,j}+r_{-2N-1,j}+\cdots)\circ\sigma(P_j^N)\sim 1$, we collect the terms by the degree of the semi-homogeneity:

$$\begin{cases} r_{-2N,j}\cdot p_{2N,j}=1, \\ r_{-2N-\nu,j}\cdot p_{2N,j}+\sum\limits_{\substack{l<\nu \\ l+\alpha_1+m=\nu}}\partial_{\xi_1}^{\alpha_1}r_{-2N-l,j}\cdot D_{x_1}^{\alpha_1}p_{2N-m,j}/\alpha_1!=0, \\ \text{for}\quad \nu=1,2,\cdots. \end{cases}$$

(Here notice that $p_{2N,j}=\{\xi_1^2+\xi_2^2+(2j+1)|\xi_4|+(f(x_1)-1)\xi_4\}^N$ is semi-elliptic for $j\geqq 1$.)

Choose functions $\psi_j\in C_0^\infty(\boldsymbol{R})$, $j=4,5$, so that $\psi_5(t)\equiv 1$ for $|t|\leqq 1/10$, $1-\psi_5\supseteq 1-\psi_4\supseteq 1-\psi_1$ and set

$$\begin{aligned} q_{3,j}=&(r_{-2N,j}+r_{-2N-1,j}+\cdots+r_{-2N-2k+1,j}) \\ &\times\{1-\psi_5(j|\xi_4|/(\xi_1^2+\xi_2^2))\}\phi_3(\xi_4). \end{aligned}$$

Also, we denote by $Q_{3,j}$ and $(1-\psi_4)\phi_2$ pseudodifferential operators with symbols $q_{3,j}$ and $\{1-\psi_4(|\xi_4|/(\xi_1^2+\xi_2^2))\}\phi_2(\xi_4)$, respectively.

Observe now that, in the support of $1-\psi_5(j|\xi_4|/(\xi_1^2+\xi_2^2))$, it holds

$$|\partial_\xi^\alpha\partial_{x_1}^{\beta_1}p_{k,j}|\leqq C_1(j+1)^{k/2}(1+|\xi_4|)^{(k-\alpha_1-\alpha_2)/2-\alpha_4},$$

with a positive constant C_1 independent of j. Hence, by induction, we can obtain the following inequalities: In the support of $\{1-\psi_5(j|\xi_4|/(\xi_1^2+\xi_2^2))\}\phi_3(\xi_4)$,

$$|\partial_\xi^\alpha\partial_{x_1}^{\beta_1}r_{-2N-\nu,j}|\leqq C_2(j+1)^{-N-(\nu+\alpha_1+\alpha_2)/2}(1+|\xi_4|)^{-N-(\nu+\alpha_1+\alpha_2)/2-\alpha_4}$$

and

$$|\partial_\xi^\alpha\partial_{x_1}^{\beta_1}q_{3,j}|\leqq C_3(j+1)^{-N-(\alpha_1+\alpha_2)/2}(1+|\xi_4|)^{-N-(\alpha_1+\alpha_2)/2-\alpha_4}$$

with some positive constants C_2 and C_3 independent of j.

We now remark that the symbol $\sigma(H_jQ_{3,j}H_j^*)$ is equal to

$$q_{3,j}(x_1;\xi_1,\xi_2,\xi_4)h_j(x_3|\xi_4|^{1/2})h_j(\xi_3/|\xi_4|^{1/2})\cdot i^j\cdot e^{-ix_3\xi_3}\cdot\phi(\xi_4)^2,$$

and that the Hermite functions have the property:

$$\left|t^\alpha\frac{d^\beta}{dt^\beta}h_j(t)\right|\leqq C_{\alpha\beta}(j+1)^{(1+\alpha+\beta)/2},$$

with some positive constant $C_{\alpha\beta}$ independent of j. (See G. Folland [2] page 54.)

Finally we obtain the following inequality:

$$(4.2) \qquad |\partial_\xi^\alpha \partial_x^\beta \sigma(H_j Q_{3,j} H_j^*)| \leqq C'_{\alpha\beta}(j+1)^{1-N+|\alpha|+|\beta|}(1+|\xi_4|)^{-N+(\beta_3-\alpha_1-\alpha_2-\alpha_3)/2-\alpha_4},$$

where $C'_{\alpha\beta}$ is a positive constant independent of j. This immediately implies that the series $Q_3=\sum_{j=1}^{\infty} H_j Q_{3,j} H_j^*$ converges with respect to the semi-norms in $S_{\lambda,1/2,1/2}^{-N}$ up to degree $N-3$. Moreover, since $1-\varphi_1 \in \{1-\psi_4(|\xi_4|/(\xi_1^2+\xi_2^2))\}\phi_2(\xi_4) \in \{1-\psi_5(j|\xi_4|/(\xi_1^2+\xi_2^2))\}\phi_3(\xi_4)$ in the support of $\phi(\xi_4)$, we have

$$\begin{aligned}(4.3) \qquad & (1-\varphi_1)Q_3 P^N \Pi_* \\ & \equiv (1-\varphi_1)\sum_{j=1}^{\infty} H_j Q_{3,j} H_j^* P^N (1-\psi_4)\phi_2 \Pi_* \mod OPS_{\lambda}^{-\infty} \\ & = (1-\varphi_1)\sum_{j=1}^{\infty} H_j Q_{3,j} P_j^N (1-\psi_4)\phi_2 H_j^* \cdot \phi(D_4)^2 \\ & = (1-\varphi_1)\sum_{j=1}^{\infty} H_j (I+K_{3,j}) H_j^* \cdot \phi(D_4)^2 \\ & = (1-\varphi_1)\Pi_* + K_3,\end{aligned}$$

with K_3 being of class $OPS_{\lambda,1/2,1/2}^{-k}$ (note that the series $\sum_{j=1}^{\infty} H_j K_{3,j} H_j^*$ converges w.r.t. the semi-norms in $S_{\lambda,1/2,1/2}^{-k}$ up to degree $k-3$).

So, from (4.1) and (4.3), we can conclude that the operator $Q=\varphi_1 Q_2 + (1-\varphi_1)Q_3 P^{N-1}$ has the property mentioned above (notice that $S_{\lambda,1/2,0}^{-k} \subset S_{\lambda,1/2,1/2}^{-k}$).

REMARK. In the above construction of the parametrix, we have to choose N sufficiently large depending the order of the regularity (i.e., the exponent l of $H^{0,l}$). The reason is that one needs the information of the semi-norms of $\sigma(Q_3)$ more and more as one considers the smoothness (w.r.t. x_4) of the solution u of higher order.

III. Finally let us consider the equation $P\Pi_0 u = \Pi_0 f$. We are going to show that, if f is smooth in a neighborhood of a certain point, then $\Pi_0 u$ is also smooth there. First recall (v) of Proposition 1. In order to prove $\Pi_0 u = H_0 H_0^* u$ is smooth, it suffices to show that $WF(H_0^* u)=\emptyset$. Next let us multiply the operator H_0^* from the left to the both sides of the equation $P\Pi_0 u=\Pi_0 f$. Then, from (iii) and (iv) of Proposition 1, it follows

$$P_0 H_0^* u = H_0^* f.$$

Therefore one can easily conclude that it suffices to show the micro-local hypoellipticity of P_0 (in $\boldsymbol{R}^3$), since it is known that $WF(H_0^* f)=\emptyset$ (recall (v) of Proposition 1). Also this would be shown by the method of the previous papers [8] and [9]. In fact, the assumptions (A.1) and (A.3) imply the following inequalities:

$$(4.4) \qquad \|D_1 v\|^2 + \|D_2 v\|^2 \leqq (P_0 v, v), \qquad \text{for any} \quad v \in C_0^\infty(\boldsymbol{R}^3),$$

and

(4.5) given any $\varepsilon>0$, there exists a positive constant C_ε such that

$$\|\log\langle D_4\rangle v\|^2 \leqq \varepsilon(P_0 v, v)+C_\varepsilon\|v\|^2, \qquad \text{for any} \quad v\in C_0^\infty(\boldsymbol{R}^3).$$

To obtain these estimates, we use partial Fourier transform w.r.t. x_4. Let

$$\begin{aligned}\hat{P}_0 &= D_1^2+D_2^2+|\xi_4|+(f(x_1)-1)\xi_4\\ &= D_1^2+D_2^2+F(x_1;\ \xi_4).\end{aligned}$$

Then, it is clear that

$$F(x_1;\ \xi_4)\begin{cases} = f(x_1)|\xi_4| & if \quad \xi_4>0,\\ \geqq |\xi_4| & if \quad \xi_4<0.\end{cases} \tag{4.6}$$

Thus, the inequality (4.4) is trivial. Moreover, let us recall that, by "sew together argument" one can prove from (A.1) and (A.3) the following inequality:

(4.7) Given any $\varepsilon>0$, there exists a constant C'_ε such that

$$\begin{aligned}\int|\log\langle\xi_4\rangle w(x_1)|^2dx_1 \leqq \varepsilon\int\{|D_1w(x_1)|^2+f(x_1)|\xi_4||w(x_1)|^2\}dx_1\\ +C'_\varepsilon\int|w(x_1)|^2dx_1,\end{aligned}$$

$$\text{for any} \quad w\in C_0^\infty(\boldsymbol{R}).$$

(For detail, *cf.* Section 3 of [**9**].) Thus the inequalities (4.4), (4.6) and (4.7) yield (4.5). Finally, it could be obvious that our assertion follows from the estimates (4.4) and (4.5). (*Cf.* Theorem 1 and its corollary in T. Hoshiro [8] or Theorem 1 in Y. Morimoto [**12**].)

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