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Minimum index for subfactors and entropy. II

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Introduction.

V. Jones [22] constructed his celebrated theory on index for type II_1 factors by using the notion of coupling constant. Kosaki [25] extended Jones' index theory to that for conditional expectations between arbitrary factors based on Connes' spatial theory [9] and Haagerup's theory on operator valued weights [17]. For von Neumann algebras $M\supseteqq N$, let $\mathcal{E}(M, N)$ denote the set of all faithful normal conditional expectations from M onto N, and $\mathcal{E}(M)$ the set of all faithful normal states on M. When $M\supseteqq N$ is a pair of factor and subfactor with $\mathcal{E}(M, N)\neq\emptyset$, Kosaki's index $\mathrm{Index}\,E$ varies depending on $E\in\mathcal{E}(M, N)$. But it was shown in [18] (independently by Longo [27]) that if $\mathrm{Index}\,E<\infty$ for some $E\in\mathcal{E}(M, N)$, then there exists a unique $E_0\in\mathcal{E}(M, N)$ which minimizes $\mathrm{Index}\,E$ for $E\in\mathcal{E}(M, N)$. So we can define the minimum index $[M:N]_0=\mathrm{Index}\,E_0$ for a pair $M\supseteqq N$.

Starting with the von Neumann entropy, we have at present several kinds of entropies in noncommutative probability theory (see [3, 4, 10, 11, 12, 29, 41, 43] for instance). Pimsner and Popa [33] exactly estimated the entropy $H(M|N)$ of a type II_1 factor M relative to its subfactor N in terms of Jones' index. This entropy extends the conditional entropy in commutative probability theory, and was first used by Connes and Størmer [12] to study the Kolmogorov-Sinai entropy of automorphisms of finite von Neumann algebras. As the natural generalization of $H(M|N)$ for finite von Neumann algebras, Connes [10] defined the entropy $H_\varphi(M|N)$ for general von Neumann algebras $M\supseteqq N$ and a normal state φ on M by using the notion of relative entropy. Here the relative entropy of normal positive functionals was first studied by Umegaki [41] in the semi-finite case, and was extended by Araki [3, 4] to the general case. On the other hand, taking account of Pimsner and Popa's estimate of $H(M|N)$, we introduced in [19] another entropy $K_\varphi(M|N)$ of a von Neumann algebra M relative to its subalgebra N and $\varphi\in\mathcal{E}(M)$ such that $E\in\mathcal{E}(M, N)$ with $\varphi\circ E=\varphi$ exists. For factors $M\supseteqq N$ and $E\in\mathcal{E}(M, N)$, we write $K_E(M|N)$ for $K_\varphi(M|N)$

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which is independent of φ with $\varphi\circ E=\varphi$. The relation between the minimum index $[M:N]_0$ and the entropy $K_E(M|N)$ was established in [19]. Also Kawakami [23] introduced the same entropy as $K_\varphi(M|N)$ in a restricted situation and obtained some related results.

In this paper, we continue to study the relation between index and entropy. This time, we mainly consider the entropies $H_\varphi(M|N)$ and $H_E(M|N)$, where the entropy $H_E(M|N)$ relative to $E\in\mathcal{E}(M, N)$ is defined as the supremum of $H_\varphi(M|N)$ for $\varphi\in\mathcal{E}(M)$ with $\varphi\circ E=\varphi$. § 1 contains definitions on index and entropy. In § 2, we present several basic properties of entropies $H_\varphi(M|N)$ and $H_E(M|N)$. In § 3, we further discuss $H_\varphi(M|N)$ related with the relative commutant $N'\cap M$. In §§ 4 and 5, the entropies $H_E(M|N)$ and $K_E(M|N)$ are investigated in connection with tensor products and crossed products. It is shown that $H_E(M|N)$ and $K_E(M|N)$ are well behaved under taking tensor products and crossed products in some cases. In § 6, when N is a factor, we estimate $H_E(M|N)$ compared with $K_E(M|N)$ and show that $H_E(M|N)\leqq K_E(M|N)$ for all $E\in\mathcal{E}(M, N)$. Moreover using the results in §§ 4 and 5 all together, we prove the equality $H_E(M|N)=K_E(M|N)$ when N is an infinite factor and $K_E(M|N)<\infty$. In § 7, let $M\supseteqq N$ be a pair of factor and subfactor with $[M:N]_0<\infty$. The results in [19] concerning the relation between $[M:N]_0$ and $K_E(M|N)$ are combined with the estimates in § 6. Consequently we have $H_E(M|N)\leqq\log[M:N]_0$ for all $E\in\mathcal{E}(M, N)$ and characterize $E\in\mathcal{E}(M, N)$ with $\operatorname{Index} E=[M:N]_0$ by means of the entropy $H_E(M|N)$. Finally in § 8, the formulas of entropies for basic constructions are given including some examples.

1. Definitions and preliminaries.

In this paper, von Neumann algebras are always assumed to be σ-finite. Let M be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and N a von Neumann subalgebra of M. We denote by $\mathfrak{S}(M)$ the set of all normal states on M and by $\mathcal{E}(M)$ the set of all faithful normal states on M. Let $\mathcal{E}(M, N)$ denote the set of all faithful normal conditional expectations from M onto N. For each $E\in\mathcal{E}(M, N)$, the operator valued weight E^{-1} from N' to M' is uniquely determined by the equation $d\varphi\circ E/d\phi=d\varphi/d\phi\circ E^{-1}$ of spatial derivatives where φ and ϕ are any faithful normal semifinite weights on N and M' respectively ([17, Theorem 6.13], [36, 12.11]). Now let $M\supseteqq N$ be a pair of factor and subfactor. Kosaki's index $\operatorname{Index} E$ of $E\in\mathcal{E}(M, N)$ is defined by $\operatorname{Index} E=E^{-1}(1)$ ([25]). This index value depends on the choice of $E\in\mathcal{E}(M, N)$. But when $\operatorname{Index} E<\infty$ for some $E\in\mathcal{E}(M, N)$ (hence $\operatorname{Index} E<\infty$ for all $E\in\mathcal{E}(M, N)$), we proved in [18] (also [27]) that there exists a unique $E_0\in\mathcal{E}(M, N)$ such that

$$\operatorname{Index} E_0=\min\{\operatorname{Index} E : E\in\mathcal{E}(M, N)\},$$

and E_0 is characterized by the condition

$$E_0^{-1}|N'\cap M=(\mathrm{Index}\,E_0)E_0|N'\cap M.$$

Furthermore $E_0|N'\cap M$ becomes a trace on $N'\cap M$. Thus we define the minimum index $[M:N]_0$ for a pair $M\supseteq N$ by $[M:N]_0=\mathrm{Index}\,E_0$. Also let $[M:N]_0=\infty$ if $\mathcal{E}(M,N)=\emptyset$ or $\mathrm{Index}\,E=\infty$ for all $E\in\mathcal{E}(M,N)$. Properties of the minimum index were presented in [**19**].

For a pair of general von Neumann algebras $M\supseteq N$ and $\varphi\in\mathfrak{S}(M)$, the entropy $H_\varphi(M|N)$ of M relative to N and φ introduced by Connes [**10**] is defined by

$$H_\varphi(M|N)=\sup_{(\varphi_i)}\sum_i\{S(\varphi,\varphi_i)-S(\varphi|N,\varphi_i|N)\},$$

where the supremum is taken over all finite families $(\varphi_1,\cdots,\varphi_n)$ of $\varphi_i\in M_*^+$ with $\sum\varphi_i=\varphi$. Here $S(\varphi,\psi)$ denotes the relative entropy of $\varphi,\psi\in M_*^+$ ([**3, 4**]). In particular, let M be a finite von Neumann algebra with a faithful normal trace τ, $\tau(1)=1$, and E_N be the conditional expectation $M\to N$ with respect to τ ([**40**]). Then $H(M|N)=H_\tau(M|N)$ is given by

$$H(M|N)=\sup_{(x_i)}\sum_i\{\tau(\eta E_N(x_i))-\tau(\eta(x_i))\},$$

where $\eta(t)=-t\log t$ on $[0,\infty)$ and the supremum is taken over all finite families $(x_1,\cdots,x_n)$ of $x_i\in M_+$ with $\sum x_i=1$. For a pair of type II_1 factors $M\supseteq N$, the entropy $H(M|N)$ was extensively developed by Pimsner and Popa [**33**] in connection with Jones' index $[M:N]$ $(=\mathrm{Index}\,E_N)$.

Let $M\supseteq N$ be general von Neumann algebras again. Given $E\in\mathcal{E}(M,N)$, we define the entropy $H_E(M|N)$ relative to E by

$$H_E(M|N)=\sup\{H_\varphi(M|N):\varphi\in\mathcal{E}(M),\ \varphi\circ E=\varphi\}.$$

For $\varphi\in\mathcal{E}(M)$ such that $E\in\mathcal{E}(M,N)$ with $\varphi\circ E=\varphi$ exists, we introduced in [**19**] another entropy $K_\varphi(M|N)$ relative to N and φ by

$$K_\varphi(M|N)=-S(\hat{\omega},\omega),$$

where $\omega=\varphi|N'\cap M$ and $\hat{\omega}=\varphi\circ(E^{-1}|N'\cap M)$. Here, since $E^{-1}|N'\cap M$ is not necessarily bounded, the relative entropy $S(\hat{\omega},\omega)$ is given by

$$S(\hat{\omega},\omega)=\inf\{S(\omega',\omega):\omega'\in(N'\cap M)_*^+,\ \omega'\leqq\hat{\omega}\}.$$

Also for $E\in\mathcal{E}(M,N)$, we define

$$K_E(M|N)=\sup\{K_\varphi(M|N):\varphi\in\mathcal{E}(M),\ \varphi\circ E=\varphi\}.$$

But when N is a factor, $K_\varphi(M|N)$ is independent of the choice of $\varphi\in\mathcal{E}(M)$ with $\varphi\circ E=\varphi$ and we can write

$$K_E(M|N)=-S(E\circ(E^{-1}|N'\cap M),\ E|N'\cap M),$$

because $E|N'\cap M$ and $E\circ(E^{-1}|N'\cap M)$ are scalar-valued.

Let $M \supseteqq N$ be factors and $E \in \mathcal{E}(M, N)$. Then $K_E(M|N)$ is exactly estimated in terms of Kosaki's index as follows ([19]): If $N' \cap M$ has a nonatomic part, then $K_E(M|N)=\infty$. If $N' \cap M$ is atomic and $\{e_k\}$ is a set of atoms in the centralizer $(N' \cap M)_E$ of E with $\sum e_k=1$, then

$$K_E(M|N) = \sum_k E(e_k) \log \frac{\operatorname{Index} E_{e_k}}{E(e_k)^2},$$

where $E_{e_k} \in \mathcal{E}(M_{e_k}, N_{e_k})$ is defined by $E_{e_k}(x)=E(e_k)^{-1}E(x)e_k$, $x \in M_{e_k}$. This estimate is completely analogous to that of $H(M|N)$ for type II_1 factors ([33]). Furthermore $K_E(M|N) \leqq \log[M:N]_0$, and the equality holds if and only if $\operatorname{Index} E=[M:N]_0$ (i.e. $E=E_0$).

However the entropy $K_E(M|N)$ is defined by using E^{-1} and restricting on the relative commutant $N' \cap M$, which is more closely connected with $\operatorname{Index} E$ than $H_\varphi(M|N)$ or $H_E(M|N)$. In fact, when $N' \cap M=\boldsymbol{C}$, it is clear by definition that $K_E(M|N)=\log \operatorname{Index} E$. But the same equality for $H_E(M|N)$ is not at all clear. The main purpose of this paper is to investigate the entropies $H_\varphi(M|N)$ and $H_E(M|N)$ related with the minimum index $[M:N]_0$ and compared with the entropy $K_E(M|N)$.

2. General properties of entropy.

In this section, let M be a von Neumann algebra and N a von Neumann subalgebra of M. We present general properties of entropies $H_\varphi(M|N)$ for $\varphi \in \mathfrak{S}(M)$ and $H_E(M|N)$ for $E \in \mathcal{E}(M, N)$.

PROPOSITION 2.1. *If α is an isomorphism of M onto $\alpha(M)$, then:*

(1) $H_\varphi(M|N) = H_{\varphi\circ\alpha^{-1}}(\alpha(M)|\alpha(N))$ *for all* $\varphi \in \mathfrak{S}(M)$,

(2) $H_E(M|N) = H_{\alpha\circ E\circ\alpha^{-1}}(\alpha(M)|\alpha(N))$ *for all* $E \in \mathcal{E}(M, N)$.

PROOF. (1) is immediate since $S(\varphi\circ\alpha^{-1}, \psi\circ\alpha^{-1})=S(\varphi, \psi)$ for $\varphi, \psi \in M_*^+$. (2) follows from (1). □

PROPOSITION 2.2. *Let L be a von Neumann algebra with $L \supseteqq M \supseteqq N$. If $\varphi \in \mathfrak{S}(L)$, then:*

(1) $H_\varphi(L|N) \leqq H_\varphi(L|M)+H_{\varphi|M}(M|N)$,

(2) $H_\varphi(L|N) \geqq H_\varphi(L|M)$,

(3) $H_\varphi(L|N) \geqq H_{\varphi|M}(M|N)$ *whenever* $F \in \mathcal{E}(L, M)$ *with* $\varphi\circ F=\varphi$ *exists.*

If $F \in \mathcal{E}(L, M)$ and $E \in \mathcal{E}(M, N)$, then:

(4) $H_{E\circ F}(L|N) \leqq H_F(L|M)+H_E(M|N)$,

(5) $H_{E\circ F}(L|N) \geqq H_E(M|N)$.

PROOF. (1) is obvious. (2) follows from the monotonicity of relative entropy

([**24, 39**]). To show (3), let $\psi_1, \cdots, \psi_n \in M_*^+$ and $\sum\psi_i = \varphi|M$. Taking $\varphi_i = \psi_i \circ F$, we have $\sum\varphi_i = \varphi$ so that

$$H_\varphi(L|N) \geqq \sum_i \{S(\varphi, \varphi_i) - S(\varphi|N, \varphi_i|N)\} \geqq \sum_i \{S(\varphi|M, \psi_i) - S(\varphi|N, \psi_i|N)\}$$

by monotonicity. Hence (3) holds. Moreover (4) and (5) are immediate from (1) and (3), respectively. □

PROPOSITION 2.3. *Let $M = \bigoplus_k M_k$ and $N = \bigoplus_k N_k$ for countable families $\{M_k\}$ and $\{N_k\}$ of von Neumann algebras with $M_k \supseteq N_k$. If $\varphi = \bigoplus_k \lambda_k \varphi_k$ where $\lambda_k \geqq 0$, $\sum\lambda_k = 1$, and $\varphi_k \in \mathfrak{S}(M_k)$, then*

$$H_\varphi(M|N) = \sum_k \lambda_k H_{\varphi_k}(M_k|N_k).$$

PROOF. Let $\psi_1, \cdots, \psi_n \in M_*^+$ and $\sum\psi_i = \varphi$. Writing $\psi_i = \bigoplus_k \lambda_k \psi_{ik}$ with $\psi_{ik} \in (M_k)_*^+$ and $\sum_i \psi_{ik} = \varphi_k$, we have

$$\sum_i \{S(\varphi, \psi_i) - S(\varphi|N, \psi_i|N)\} = \sum_k \lambda_k \sum_i \{S(\varphi_k, \psi_{ik}) - S(\varphi_k|N_k, \psi_{ik}|N_k)\}$$
$$\leqq \sum_k \lambda_k H_{\varphi_k}(M_k|N_k)$$

by the additivity of relative entropy for direct sums and by the scaling property of relative entropy ([**4**, Theorem 3.6]). Hence $H_\varphi(M|N) \leqq \sum\lambda_k H_{\varphi_k}(M_k|N_k)$. The reverse inequality is similarly shown. □

PROPOSITION 2.4. *If $\varphi \in \mathcal{E}(M)$ and $H_\varphi(M|N) = 0$, then $M = N$.*

PROOF. Let $\varphi \in \mathcal{E}(M)$ and suppose $H_\varphi(M|N) = 0$. If $\psi \in \mathcal{E}(M)$ satisfies $\psi \leqq c\varphi$ for some $c > 0$, then

$$S\left(\varphi|N, \frac{1}{c}\psi|N\right) = S\left(\varphi, \frac{1}{c}\psi\right),$$

so that by [**4**, Theorem 3.6]

$$S(\varphi|N, \psi|N) = S(\varphi, \psi) < \infty,$$

i.e. N is weakly sufficient for $\{\varphi, \psi\}$ in the sense of [**31**]. Hence due to [**31**, Theorem 4], we have $\psi \circ E_\varphi = \psi$ where $E_\varphi : M \to N$ is the Accardi-Cecchini generalized conditional expectation [**1**] with respect to φ. Because M_* is the closed linear span of $\{\psi \in \mathcal{E}(M) : \psi \leqq c\varphi$ for some $c > 0\}$, we get $E_\varphi = \mathrm{id}_M$. This shows $M = N$. □

PROPOSITION 2.5. *The function $\varphi \to H_\varphi(M|N)$ is lower semicontinuous in norm on $\mathfrak{S}(M)$.*

PROOF. Let $\varphi_n, \varphi \in \mathfrak{S}(M)$ and $\|\varphi_n - \varphi\| \to 0$. Taking the standard representation of M ([**2, 16**]), we have ξ_n and ξ in the natural positive cone such that $\varphi_n = (\cdot\xi_n|\xi_n)$ and $\varphi = (\cdot\xi|\xi)$. Then $\|\xi_n - \xi\| \leqq \|\varphi_n - \varphi\|^{1/2} \to 0$ by [**2**, Theorem 4]

([**16**, Lemma 2.10]). Let $\psi_1, \cdots, \psi_k \in M_*^+$ and $\sum\psi_i=\varphi$. For $1\leqq i\leqq k$, there exists a unique $a_i'\in M'$ such that $0\leqq a_i'\leqq e'$ and $\psi_i=(\cdot a_i'\xi|\xi)$, where e' is the projection onto $\overline{M\xi}$. Since $\sum\psi_i=\varphi$ implies $\sum a_i'=e'$, by replacing a_1' with $a_1'+(1-e')$, we can choose $a_1', \cdots, a_k'\in M_+'$ with $\sum a_i'=1$ such that $\psi_i=(\cdot a_i'\xi|\xi)$, $1\leqq i\leqq k$. Define $\psi_{n1}, \cdots, \psi_{nk}\in M_*^+$ by $\psi_{ni}=(\cdot a_i'\xi_n|\xi_n)$. Then $\sum_i\psi_{ni}=\varphi_n$ and $\|\psi_{ni}-\psi_i\|\to 0$ as $n\to\infty$, so that by [**4**, Theorem 3.7]

$$\sum_i\{S(\varphi, \psi_i)-S(\varphi|N, \psi_i|N)\} = \lim_{n\to\infty}\sum_i\{S(\varphi_n, \psi_{ni})-S(\varphi_n|N, \psi_{ni}|N)\}$$
$$\leqq \liminf_{n\to\infty} H_{\varphi_n}(M|N).$$

Therefore $H_\varphi(M|N)\leqq\liminf_{n\to\infty}H_{\varphi_n}(M|N)$. □

The next lemma will be very useful in the sequel.

Lemma 2.6. *If $\varphi\in\mathcal{E}(M)$ and there exists $E\in\mathcal{E}(M, N)$ with $\varphi\circ E=\varphi$, then*

$$H_\varphi(M|N) = \sup_{(\varphi_i)}\sum_i S(\varphi_i\circ E, \varphi_i),$$

where the supremum is taken over all $\varphi_1, \cdots, \varphi_n\in M_*^+$ with $\sum\varphi_i=\varphi$.

Proof. According to [**32**, Theorem 2] (extending [**30**, Theorem 5]), we get for $\psi\in M_*^+$

$$S(\varphi, \psi)=S(\varphi|N, \psi|N)+S(\psi\circ E, \psi).$$

This shows the desired formula. □

Proposition 2.7. *If $E\in\mathcal{E}(M, N)$, then $H_\varphi(M|N)$ is concave on $\{\varphi\in\mathfrak{S}(M): \varphi\circ E=\varphi\}$.*

Proof. Let $\varphi, \psi\in\mathfrak{S}(M)$, $\varphi\circ E=\varphi$, $\psi\circ E=\psi$, and $0<\lambda<1$. For $\varphi_1, \cdots, \varphi_m\in M_*^+$ with $\sum\varphi_i=\varphi$ and $\psi_1, \cdots, \psi_n\in M_*^+$ with $\sum\psi_j=\psi$, since $\sum_i\lambda\varphi_i+\sum_j(1-\lambda)\psi_j=\lambda\varphi+(1-\lambda)\psi$, we have by Lemma 2.6

$$H_{\lambda\varphi+(1-\lambda)\psi}(M|N) \geqq \sum_i S(\lambda\varphi_i\circ E, \lambda\varphi_i)+\sum_j S((1-\lambda)\psi_j\circ E, (1-\lambda)\psi_j)$$
$$= \lambda\sum_i S(\varphi_i\circ E, \varphi_i)+(1-\lambda)\sum_j S(\psi_j\circ E, \psi_j),$$

showing the conclusion required. □

Remark 2.8. If N is a finite dimensional factor with the normalized trace τ, then $H_E(M|N)=H_{\tau\circ E}(M|N)$ for all $E\in\mathcal{E}(M, N)$. Indeed for any $\psi\in\mathcal{E}(N)$, τ is in the convex hull of the unitary orbit of ψ, i.e. $\tau=\sum\lambda_k u_k\psi u_k^*$ with $\lambda_k>0$, $\sum\lambda_k=1$ and $u_k\in N$ unitary. Then $\tau\circ E=\sum\lambda_k u_k(\psi\circ E)u_k^*$, so that by Propositions 2.7 and 2.1(1)

$$H_{\tau\circ E}(M|N) \geqq \sum\lambda_k H_{u_k(\psi\circ E)u_k^*}(u_k M u_k^*|u_k N u_k^*) = H_{\psi\circ E}(M|N),$$

as desired.

PROPOSITION 2.9. *For every* $E\in\mathcal{E}(M, N)$

$$H_E(M|N)\leqq\log\|E^{-1}(1)\|,$$

where $\|E^{-1}(1)\|=\infty$ *if* $E^{-1}(1)$ *is unbounded.*

PROOF. Suppose $E^{-1}(1)$ is bounded and let $\lambda=\|E^{-1}(1)\|^{-1}$. Then it follows from [**19**, Proposition 1.9] that

$$E(x)\geqq E^{-1}(1)^{-1}x\geqq\lambda x,\qquad x\in M_+.$$

Let $\varphi\in\mathcal{E}(M)$ and $\varphi\circ E=\varphi$. For each $\varphi_1,\cdots,\varphi_n\in M_*^+$ with $\sum\varphi_i=\varphi$, since $\varphi_i\circ E\geqq\lambda\varphi_i$, we have by [**4**, Theorem 3.6]

$$\sum_i S(\varphi_i\circ E, \varphi_i)\leqq\sum_i S(\lambda\varphi_i, \varphi_i)=-\sum_i\varphi_i(1)\log\lambda=-\log\lambda.$$

Therefore $H_\varphi(M|N)\leqq-\log\lambda$ by Lemma 2.6, so that $H_E(M|N)\leqq-\log\lambda$. □

In fact, the above proposition was given at the end of [**10**] without proof. When $M\supseteqq N$ is a pair of factor and subfactor, this shows that $H_E(M|N)\leqq\log\operatorname{Index}E$ for all $E\in\mathcal{E}(M, N)$. But we shall establish a stronger inequality in Corollary 7.1.

The notion of commuting squares plays a fundamental role in index theory for type II_1 factors ([**15**]). In the next lemma, we show the monotonicity of entropies $H_\varphi(M|N)$ and $H_E(M|N)$ for a commuting square.

LEMMA 2.10. *Let* $M_1\supseteqq N_1$ *be von Neumann algebras with* $M_1\subseteqq M$ *and* $N_1\subseteqq N$. *Let* $\varphi\in\mathcal{E}(M)$, $E\in\mathcal{E}(M, N)$, $E_1\in\mathcal{E}(M, N_1)$ *and* $F_1\in\mathcal{E}(M, M_1)$ *be such that* $\varphi\circ E=\varphi\circ E_1=\varphi\circ F_1=\varphi$. *If the commuting square condition* $E\circ F_1=E_1$ *holds, then:*

(1) $H_\varphi(M|N)\geqq H_{\varphi|M_1}(M_1|N_1)$,

(2) $H_E(M|N)\geqq H_{E|M_1}(M_1|N_1)$.

PROOF. (1) Let $\psi_1,\cdots,\psi_n\in(M_1)_*^+$ and $\sum\psi_i=\varphi|M_1$. Taking $\varphi_1,\cdots,\varphi_n\in M_*^+$ with $\sum\varphi_i=\varphi$ by $\varphi_i=\psi_i\circ F_1$, we have

$$H_\varphi(M|N)\geqq\sum_i S(\varphi_i\circ E, \varphi_i)\geqq\sum_i S((\varphi_i\circ E)|M_1, \varphi_i|M_1)$$
$$=\sum_i S(\psi_i\circ(E|M_1), \psi_i)$$

by Lemma 2.6 and monotonicity. This implies $H_\varphi(M|N)\geqq H_{\varphi|M_1}(M_1|N_1)$.

(2) For each $\psi_1\in\mathcal{E}(M_1)$ with $\psi_1\circ(E|M_1)=\psi_1$, letting $\psi=\psi_1\circ E_1$, we have

$$\psi\circ E=\psi_1\circ E_1\circ E=\psi_1\circ E_1=\psi,$$
$$\psi\circ F_1=\psi\circ E\circ F_1=\psi\circ E_1=\psi.$$

Hence by (1) applied to ψ, we get

$$H_E(M|N)\geqq H_\psi(M|N)\geqq H_{\psi_1}(M_1|N_1),$$

implying $H_E(M|N)\geqq H_{E|M_1}(M_1|N_1)$. □

We end this section with martingale type convergence properties of entropies. Let $\{M_n\}$ and $\{N_n\}$ be increasing sequences of von Neumann subalgebras of M with $M_n \supseteqq N_n$ such that $M=\bigvee_n M_n$ and $N=\bigvee_n N_n$.

PROPOSITION 2.11. *For every* $\varphi \in \mathfrak{S}(M)$

$$H_\varphi(M|N) \leqq \liminf_{n\to\infty} H_{\varphi|M_n}(M_n|N_n).$$

PROOF. Given $\varphi_1, \cdots, \varphi_n \in M_*^+$ with $\sum \varphi_i = \varphi$, by the martingale convergence of relative entropy ([**4**, Theorem 3.9], [**24**, Theorem 4.1]), we have

$$\begin{aligned}\sum_i \{S(\varphi, \varphi_i) - S(\varphi|N, \varphi_i|N)\} &= \lim_{n\to\infty}\sum_i \{S(\varphi|M_n, \varphi_i|M_n) - S(\varphi|N_n, \varphi_i|N_n)\} \\ &\leqq \liminf_{n\to\infty} H_{\varphi|M_n}(M_n|N_n),\end{aligned}$$

implying the desired inequality. □

PROPOSITION 2.12. *Let* $\varphi \in \mathcal{E}(M)$ *and assume that there exist* $E_n \in \mathcal{E}(M, N_n)$ *with* $\varphi \circ E_n = \varphi$ *and* $F_n \in \mathcal{E}(M, M_n)$ *with* $\varphi \circ F_n = \varphi$. *If* $E_{n+1} \circ F_n = E_n$ *for all n, then*

(1) $H_\varphi(M|N) = \lim_{n\to\infty} H_{\varphi|M_n}(M_n|N_n)$ *increasingly,*

(2) $H_E(M|N) = \lim_{n\to\infty} H_{E_n|M_n}(M_n|N_n)$ *increasingly where* $E \in \mathcal{E}(M, N)$ *with* $\varphi \circ E = \varphi$.

PROOF. By [**37**] and [**7**, Lemma 2], there exists $E \in \mathcal{E}(M, N)$ such that $\varphi \circ E = \varphi$ and $E_n(x) \to E(x)$ strongly for all $x \in M$. We get $E_m \circ E_n = E_n$ for $m > n$ by induction, so that $E \circ F_n = E_n$ and particularly $E|M_n = E_n|M_n$. Hence (1) follows from Lemma 2.10(1) and Proposition 2.11. Also for each $\psi \in \mathcal{E}(M)$ with $\psi \circ E = \psi$, we get by Proposition 2.11

$$H_\psi(M|N) \leqq \liminf_{n\to\infty} H_{\psi|M_n}(M_n|N_n) \leqq \liminf_{n\to\infty} H_{E_n|M_n}(M_n|N_n),$$

since $(\psi|M_n)\circ(E_n|M_n) = \psi|M_n$. Hence

$$H_E(M|N) \leqq \liminf_{n\to\infty} H_{E_n|M_n}(M_n|N_n).$$

This together with Lemma 2.10(2) implies (2). □

3. Relative commutant and entropy.

Given $\varphi \in \mathcal{E}(M)$ such that $E \in \mathcal{E}(M, N)$ with $\varphi \circ E = \varphi$ exists, it was shown in [**19**, § 4] that if either $Z(M)$ or $Z(N)$ is atomic and if $K_\varphi(M|N) < \infty$, then $N' \cap M$ is atomic. In this section, we similarly discuss the relation between the entropy $H_\varphi(M|N)$ and the relative commutant $N' \cap M$ when N is a factor.

To begin with, we give the complete estimate of $H_\varphi(M|\boldsymbol{C})$ as follows:

PROPOSITION 3.1. (1) *If M has a nonatomic part, then* $H_\varphi(M|\boldsymbol{C}) = \infty$ *for* *every* $\varphi\in\mathcal{E}(M)$.

(2) *Assume that* M *is atomic. Let* tr *be the faithful normal semifinite trace on* M *such that* $\mathrm{tr}(q)=1$ *for every minimal projection* q *in* M. *If* $\varphi\in\mathcal{E}(M)$ *and* $a=d\varphi/d\,\mathrm{tr}$, *then* $H_{\varphi}(M|\boldsymbol{C})=\mathrm{tr}(\eta a)$.

PROOF. (1) Let M_{φ} be the centralizer of $\varphi\in\mathcal{E}(M)$. First let us prove that if M_{φ} has a nonatomic part with the support projection $e\neq 0$, then $H_{\varphi}(M|\boldsymbol{C})=\infty$. For each $n\geqq 1$, there are projections $e_1, \cdots, e_n$ in M_{φ} such that $\sum e_i=e$ and $\varphi(e_i)=\varphi(e)/n$. Taking $\varphi_1, \cdots, \varphi_n\in M_*^+$ with $\sum\varphi_i\leqq\varphi$ by $\varphi_i=\varphi(e_i\cdot)$, we have by Lemma 2.6 and monotonicity

$$H_{\varphi}(M|\boldsymbol{C})\geqq\sum_{i=1}^{n}S(\varphi_i(1)\varphi,\ \varphi_i)$$

$$\geqq\sum_{i=1}^{n}\varphi_i(e_i)\log\frac{\varphi_i(e_i)}{\varphi_i(1)\varphi(e_i)}=\varphi(e)\log\frac{n}{\varphi(e)},$$

showing $H_{\varphi}(M|\boldsymbol{C})=\infty$.

Now suppose $H_{\varphi}(M|\boldsymbol{C})<\infty$ for some $\varphi\in\mathcal{E}(M)$. Then M_{φ} and hence $Z(M)$ are atomic. So let $\{p_j\}$ be the set of all atoms in $Z(M)$ and $\varphi_j=\varphi(p_j)^{-1}|Mp_j$. Thanks to $M=\bigoplus_j Mp_j$ and $Z(M)=\bigoplus_j \boldsymbol{C}p_j$, we get by Propositions 2.2(2) and 2.3

$$\infty>H_{\varphi}(M|\boldsymbol{C})\geqq H_{\varphi}(M|Z(M))=\sum_j\varphi(p_j)H_{\varphi_j}(Mp_j|\boldsymbol{C}).$$

Thus to show the atomicness of M, we can assume that M is a factor. If M is a nonatomic semifinite factor, then M_{φ} includes a maximal abelian subalgebra A of M because σ^{φ} is inner. Also if M is a type III_{λ} factor with $0\leqq\lambda<1$, then it is known [**36**, 29.9] that M_{φ} includes a maximal abelian subalgebra A of M. In these cases, since A is nonatomic, we have $H_{\varphi}(M|\boldsymbol{C})\geqq H_{\varphi|A}(A|\boldsymbol{C})=\infty$ by Proposition 2.2(3) and the assertion proved first. Next let M be a type III_1 factor. Take projections $e_1, \cdots, e_n$ in M such that $\sum e_i=1$ and $\varphi(e_i)=1/n$, and define $\psi\in\mathcal{E}(M)$ by $\psi(x)=\sum\varphi(e_ixe_i)$, $x\in M$. Since $e_i\in M_{\psi}$, it follows as in the first argument that $H_{\psi}(M|\boldsymbol{C})\geqq\log n$. Furthermore, by [**13**] (the homogeneity of $\mathfrak{S}(M)$), there is a sequence $\{u_m\}$ of unitaries in M such that $\|\psi-u_m\varphi u_m^*\|\to 0$. Hence

$$H_{\psi}(M|\boldsymbol{C})\leqq\liminf_{m\to\infty}H_{u_m\varphi u_m^*}(M|\boldsymbol{C})=H_{\varphi}(M|\boldsymbol{C})$$

by Propositions 2.5 and 2.1(1), so that $H_{\varphi}(M|\boldsymbol{C})=\infty$. Thus we conclude that M is atomic.

(2) For $\varphi_1, \cdots, \varphi_n\in M_*^+$ with $\sum\varphi_i=\varphi$, letting $a_i=d\varphi_i/d\,\mathrm{tr}$, we get

$$\sum_i S(\varphi_i(1)\varphi,\ \varphi_i)=\mathrm{tr}(\eta a)+\sum_i\{\eta(\mathrm{tr}(a_i))-\mathrm{tr}(\eta a_i)\}\leqq\mathrm{tr}(\eta a),$$

because $\mathrm{tr}(\eta a_i)\geqq\eta(\mathrm{tr}(a_i))$. Now we write $a=\sum\lambda_kq_k$ with minimal projections q_k, $\sum q_k=1$, and take $\psi_k\in M_*^+$ such that $d\psi_k/d\,\mathrm{tr}=\lambda_kq_k$. Then since $\sum_{k=1}^{n}\psi_k\leqq\varphi$,

$$H_\varphi(M|\boldsymbol{C}) \geqq \sum_{k=1}^{n} S(\psi_k(1)\varphi, \psi_k) = \sum_{k=1}^{n} \eta(\lambda_k) \longrightarrow \mathrm{tr}(\eta a)$$

as $n\to\infty$. Hence $H_\varphi(M|\boldsymbol{C})=\mathrm{tr}(\eta a)$. □

The above (2) was given in [**10**, Théorème 5(D)] in the finite dimensional case. The entropy $H_\varphi(M|N)$ coincides with the von Neumann entropy particularly when $M=\boldsymbol{B}(\mathcal{H})$ and $N=\boldsymbol{C}$, so that the function $\varphi\mapsto H_\varphi(M|N)$ is not necessarily continuous (see [**43**]). As for the entropy $K_\varphi(M|\boldsymbol{C})$, we note [**19**, Example 4.6] that $K_\varphi(M|\boldsymbol{C})=2\,\mathrm{tr}(\eta a)$ if $M=\boldsymbol{B}(\mathcal{H})$ and $a=d\varphi/d\,\mathrm{tr}$. Hence the entropies $H_\varphi(M|N)$ and $K_\varphi(M|N)$ are not identical in general.

PROPOSITION 3.2. *Assume that N is a factor. If $N'\cap M$ has a nonatomic part, then $H_\varphi(M|N)=\infty$ for every $\varphi\in\mathcal{E}(M)$ such that $E\in\mathcal{E}(M, N)$ with $\varphi\circ E=\varphi$ exists.*

PROOF. Let $\varphi\in\mathcal{E}(M)$ and $E\in\mathcal{E}(M, N)$ be such that $\varphi\circ E=\varphi$. Because $\sigma_t^\varphi(N'\cap M)=N'\cap M$ as well as $\sigma_t^\varphi(N)=N$ for all $t\in\boldsymbol{R}$, there exists $F\in\mathcal{E}(M, N'\cap M)$ with $\varphi\circ F=\varphi$ ([**37**]). Then $E\circ F=\varphi$ since N is a factor. Hence we can apply Lemma 2.10(1) to $M_1=N'\cap M$ and $N_1=\boldsymbol{C}$, so that

$$H_\varphi(M|N) \geqq H_{\varphi|N'\cap M}(N'\cap M|\boldsymbol{C}).$$

Thus Proposition 3.1 shows the conclusion required. □

PROPOSITION 3.3. *Assume that N is a factor and $\mathcal{E}(M, N)\neq\emptyset$. If $N'\cap M$ is infinite dimensional, then $H_\varphi(M|N)=\infty$ for some $\varphi\in\mathcal{E}(M)$ such that $E\in\mathcal{E}(M, N)$ with $\varphi\circ E=\varphi$ exists.*

PROOF. By assumption, there is a sequence $\{e_k\}$ of nonzero projections in $N'\cap M$ with $\sum e_k=1$. Let $\{\lambda_k\}$ be a sequence of positive numbers such that $\sum\lambda_k=1$ and $\sum\eta(\lambda_k)=\infty$. Choosing an $\omega\in\mathcal{E}(N'\cap M)$, we define

$$\psi(x) = \sum_k \frac{\lambda_k}{\omega(e_k)}\omega(e_k x e_k), \qquad x\in N'\cap M.$$

Then $\psi\in\mathcal{E}(N'\cap M)$ and $e_k\in(N'\cap M)_\psi$. According to [**6**, Théorème 5.3], there exists $E\in\mathcal{E}(M, N)$ such that $E|N'\cap M=\psi$. Choose a $\varphi_0\in\mathcal{E}(N)$ and let $\varphi=\varphi_0\circ E$. Then $\varphi\in\mathcal{E}(M)$, $\varphi\circ E=\varphi$ and $\varphi|N'\cap M=\psi$. For each $n\geqq 1$, we can take $\varphi_1, \cdots, \varphi_n\in M_*^+$ with $\sum\varphi_k\leqq\varphi$ by $\varphi_k=\varphi(e_k\,\cdot)$ because $(N'\cap M)_\psi=(N'\cap M)_E \subseteqq M_\varphi$. Then

$$H_\varphi(M|N) \geqq \sum_{k=1}^{n} S(\varphi_k\circ E, \varphi_k) \geqq \sum_{k=1}^{n} \varphi_k(e_k)\log\frac{\varphi_k(e_k)}{\varphi_k(E(e_k))} = \sum_{k=1}^{n} \eta(\lambda_k),$$

showing $H_\varphi(M|N)=\infty$. □

4. Entropy for tensor products.

In this section, we investigate the entropies $H_\varphi(M|N)$ and $H_E(M|N)$ under taking tensor products. The addition formula of the entropy $K_\varphi(M|N)$ for tensor products was given in [**19**, Proposition 3.6]. In the following, let $M\supseteq N$ and $P\supseteq Q$ be two pairs of von Neumann algebras.

LEMMA 4.1. *Let $\varphi\in\mathfrak{S}(M)$ and $\psi\in\mathfrak{S}(P)$. For every $\varphi'\in M_*^+$ with $\varphi'\leqq\varphi$ and $\psi'\in P_*^+$ with $\psi'\leqq\psi$,*

$$S(\varphi\otimes\psi,\ \varphi'\otimes\psi')=\psi'(1)S(\varphi,\ \varphi')+\varphi'(1)S(\psi,\ \psi').$$

PROOF. By restricting φ on $M_{s(\varphi)}$ and ψ on $P_{s(\psi)}$ where $s(\varphi)$ is the support projection of φ, we can assume that $\varphi\in\mathcal{E}(M)$ and $\psi\in\mathcal{E}(P)$. For $0<\varepsilon<1$, let $\varphi'_\varepsilon=(1-\varepsilon)\varphi'+\varepsilon\varphi$ and $\psi'_\varepsilon=(1-\varepsilon)\psi'+\varepsilon\psi$. Since φ'_ε and ψ'_ε are faithful, the formula in [**30**, p. 70] implies that

$$S(\varphi\otimes\psi,\ \varphi'_\varepsilon\otimes\psi'_\varepsilon)=\psi'_\varepsilon(1)S(\varphi,\ \varphi'_\varepsilon)+\varphi'_\varepsilon(1)S(\psi,\ \psi'_\varepsilon).$$

Taking the limits as $\varepsilon\to0$ by [**4**, Theorem 3.7], we get the desired formula. □

PROPOSITION 4.2. (1) *For every $\varphi\in\mathfrak{S}(M)$ and $\psi\in\mathfrak{S}(P)$*

$$H_{\varphi\otimes\psi}(M\otimes P|N\otimes Q)\geqq H_\varphi(M|N)+H_\psi(P|Q).$$

(2) *For every $E\in\mathcal{E}(M,\ N)$ and $F\in\mathcal{E}(P,\ Q)$*

$$H_{E\otimes F}(M\otimes P|N\otimes Q)\geqq H_E(M|N)+H_F(P|Q).$$

PROOF. Given $\varphi_1,\ \cdots,\ \varphi_m\in M_*^+$ with $\sum\varphi_i=\varphi$ and $\psi_1,\ \cdots,\ \psi_n\in P_*^+$ with $\sum\psi_j=\psi$, since $\sum_{i,j}\varphi_i\otimes\psi_j=\varphi\otimes\psi$ and $\varphi_i\otimes\psi_j|N\otimes Q=(\varphi_i|N)\otimes(\psi_j|Q)$, we have by Lemma 4.1

$$H_{\varphi\otimes\psi}(M\otimes P|N\otimes Q)\geqq\sum_{i,j}\{S(\varphi\otimes\psi,\ \varphi_i\otimes\psi_j)-S(\varphi\otimes\psi|N\otimes Q,\ \varphi_i\otimes\psi_j|N\otimes Q)\}$$

$$=\sum_i\{S(\varphi,\ \varphi_i)-S(\varphi|N,\ \varphi_i|N)\}+\sum_j\{S(\psi,\ \psi_j)-S(\varphi|Q,\ \psi_j|Q)\},$$

implying (1). (2) is immediate from (1). □

The following is a particular case of Proposition 4.2(2):

$$H_{E\otimes\mathrm{id}_P}(M\otimes P|N\otimes P)\geqq H_E(N|N).$$

Now we establish the equality in the above when N is an infinite factor and P is an injective factor. To do this, we give the next lemma.

LEMMA 4.3. *Assume that N is an infinite factor.*

(1) *If e is an infinite projection in N, then*

$$H_{E|M_e}(M_e|N_e)=H_E(M|N).$$

(2) *If F is a type* I *factor, then*

$$H_{E\otimes \mathrm{id}_F}(M\otimes F|N\otimes F)=H_E(M|N).$$

PROOF. (1) By assumption, there is a $v\in N$ such that $v^*v=1$ and $vv^*=e$. Then $M_e=vMv^*$, $N_e=vNv^*$ and $E(x)=vE(v^*xv)v^*$, $x\in M_e$. Hence Proposition 2.1(2) shows the desired equality.

(2) Choose a minimal projection e in F. Then $M=(M\otimes F)_{1\otimes e}$, $N=(N\otimes F)_{1\otimes e}$ and $E=(E\otimes \mathrm{id}_F)|(M\otimes F)_{1\otimes e}$ under the obvious identification. Hence (2) follows from (1). □

PROPOSITION 4.4. *If N is an infinite factor and P is an injective factor with separable predual, then*

$$H_{E\otimes \mathrm{id}_P}(M\otimes P|N\otimes P)=H_E(M|N)$$

for every $E\in\mathcal{E}(M, N)$.

PROOF. Let us show that

$$H_{E\otimes \mathrm{id}_P}(M\otimes P|N\otimes P)\leqq H_E(M|N).$$

Because P is approximately finite dimensional ([8]), we can choose an increasing sequence $\{F_n\}$ of finite type I subfactors of P such that $P=\bigvee_n F_n$ (see [**14**]). Let $\varphi\in\mathcal{E}(M\otimes P)$ and $\varphi\circ(E\otimes\mathrm{id}_P)=\varphi$. Given $\varphi_1, \cdots, \varphi_k\in(M\otimes P)_*^+$ with $\sum\varphi_i=\varphi$, since

$$(\varphi|M\otimes F_n)\circ(E\otimes \mathrm{id}_{F_n})=\varphi|M\otimes F_n,$$
$$(\varphi_i|M\otimes F_n)\circ(E\otimes \mathrm{id}_{F_n})=\varphi_i\circ(E\otimes\mathrm{id}_P)|M\otimes F_n,$$

Lemmas 4.3(2) and 2.6 imply that

$$H_E(M|N)=H_{E\otimes\mathrm{id}_{F_n}}(M\otimes F_n|N\otimes F_n)\geqq \sum_i S(\varphi_i\circ(E\otimes\mathrm{id}_P)|M\otimes F_n,\ \varphi_i|M\otimes F_n).$$

Since $M\otimes P=\bigvee_n(M\otimes F_n)$, we get

$$H_E(M|N)\geqq\sum_i S(\varphi_i\circ(E\otimes\mathrm{id}_P),\ \varphi_i)$$

by the martingale convergence of relative entropy. This shows the conclusion required. □

REMARK 4.5. The above equality does not hold in the finite dimensional case. For instance, let $M_n=M_n(\boldsymbol{C})$ be the $n\times n$ matrix algebra with the normalized trace τ_n. As for the conditional expectation $\tau_m\otimes\mathrm{id}_{M_n}: M_m\otimes M_n\to \boldsymbol{C}\otimes M_n$ with respect to $\tau_{mn}=\tau_m\otimes\tau_n$, we have by Remark 2.8 and [**33**, Theorem 6.2]

$$H_{\tau_m\otimes\mathrm{id}_{M_n}}(M_m\otimes M_n|\boldsymbol{C}\otimes M_n)=\begin{cases}\log m^2, & m\leqq n,\\ \log mn, & m>n,\end{cases}$$

which is not equal to $H_{\tau_m}(M_m|\boldsymbol{C})=\log m$ whenever $m, n>1$.

Now suppose M is a factor on a Hilbert space $\mathscr{H}$. If $E\in\mathscr{E}(M, N)$ and $E^{-1}(1)<\infty$ ($E^{-1}(1)$ is a scalar), then $E'\in\mathscr{E}(N', M')$ is defined by $E'=E^{-1}(1)^{-1}E^{-1}$.

PROPOSITION 4.6. *Under the above situation, $H_{E'}(N'|M')$ is independent of the choice of a Hilbert space $\mathscr{H}$ where M' is infinite. (In particular, if M is a type* III *factor, then $H_{E'}(N'|M')$ is independent of any choice of $\mathscr{H}$.)*

PROOF. It suffices to show the following: If α is an isomorphism of M onto $\alpha(M)$ where M and $\alpha(M)$ are acting respectively on $\mathscr{H}$ and $\mathscr{H}_1$ such that M' and $\alpha(M)'$ are infinite, then

$$H_{(\alpha\circ E\circ\alpha^{-1})'}(\alpha(N)'|\alpha(M)')=H_{E'}(N'|M').$$

We may separately consider an amplication, an induction and a spatial isomorphism. Let $\alpha: x\in M\mapsto x\otimes 1\in M\otimes \boldsymbol{C}$ where $\mathscr{H}_1=\mathscr{H}\otimes\mathscr{K}$. Then $\alpha\circ E\circ\alpha^{-1}=E\otimes\mathrm{id}_{\boldsymbol{C}}$, so that $(\alpha\circ E\circ\alpha^{-1})'=E'\otimes\mathrm{id}_{B(\mathscr{K})}$ by [**19**, Proposition 1.7]. Hence we can apply Lemma 4.3(2). Let $\alpha: x\in M\mapsto xe\in Me$ where e is an infinite projection in M'. Then $(\alpha\circ E\circ\alpha^{-1})(xe)=E(x)e$ for $x\in M$, so that $(\alpha\circ E\circ\alpha^{-1})'=E'|N'_e$ by [**19**, Proposition 1.5]. Hence we can apply Lemma 4.3(1). For a spatial isomorphism, the desired equality is immediate from Proposition 2.1(2). $\square$

Based on the special properties of type III_1 factors, we have:

PROPOSITION 4.7. (1) *If N is a type III_1 factor and $E\in\mathscr{E}(M, N)$, then $H_E(M|N)=H_\varphi(M|N)$ for every $\varphi\in\mathfrak{S}(M)$ with $\varphi\circ E=\varphi$.*

(2) *Let N be an infinite factor and R_∞ the injective type III_1 factor with separable predual (i.e. the Araki-Woods factor). If $E\in\mathscr{E}(M, N)$, then $H_E(M|N)=H_{\varphi\otimes\omega}(M\otimes R_\infty|N\otimes R_\infty)$ for every $\omega\in\mathfrak{S}(R_\infty)$ and $\varphi\in\mathfrak{S}(M)$ with $\varphi\circ E=\varphi$.*

PROOF. (1) Let $\varphi, \psi\in\mathfrak{S}(M)$ be such that $\varphi\circ E=\varphi$ and $\psi\circ E=\psi$. Let $\varphi_0=\varphi|N$ and $\psi_0=\psi|N$. By [**13**], there is a sequence $\{u_n\}$ of unitaries in N such that $\|\psi_0-u_n\varphi_0u_n^*\|\to 0$. Then we get $\|\psi-u_n\varphi u_n^*\|\to 0$, so that $H_\psi(M|N)\leqq H_\varphi(M|N)$ by Propositions 2.5 and 2.1(1). Hence $H_\psi(M|N)=H_\varphi(M|N)$ by symmetry of φ, ψ.

(2) Because $N\otimes R_\infty$ is a type III_1 factor, Proposition 4.4 and the above (1) imply that

$$H_E(M|N)=H_{E\otimes\mathrm{id}_{R_\infty}}(M\otimes R_\infty|N\otimes R_\infty)=H_{\varphi\otimes\omega}(M\otimes R_\infty|N\otimes R_\infty)$$

for every $\omega\in\mathfrak{S}(R_\infty)$ and $\varphi\in\mathfrak{S}(M)$, $\varphi\circ E=\varphi$. $\square$

It is known [**20**, Lemma 4.4] that if N is a type III factor and $\varphi, \psi\in\mathfrak{S}(N)$, then ψ is in the closed convex hull of the unitary orbit of φ. So in view of Proposition 2.7, it seems possible that Proposition 4.7(1) is true even when N is any type III factor.

5. Entropy for crossed products.

In this section, we investigate the entropies $H_E(M|N)$ and $K_E(M|N)$ under taking crossed products. Let $M \supseteqq N$ be von Neumann algebras on a Hilbert space $\mathcal{H}$ and G a locally compact group. Let α be an action of G on M such that $\alpha_g(N)=N$ for all $g \in G$. The crossed products $\tilde{M}=M \rtimes_\alpha G$ of M by α and $\tilde{N}=N \rtimes_\alpha G$ of N by $\alpha|N$ are defined as follows:

$$\tilde{M}=\{\pi_\alpha(M) \cup (1 \otimes \lambda_G)\}'' \supseteqq \tilde{N}=\{\pi_\alpha(N) \cup (1 \otimes \lambda_G)\}'',$$

where π_α is the representation of $M \supseteqq N$ corresponding to α on $L^2(G, \mathcal{H})=\mathcal{H} \otimes L^2(G)$ and λ is the left regular representation of G on $L^2(G)$. See [28, 36] for duality theory of crossed products. In the following, we suppose the second axiom of countability for G, so that the von Neumann algebras appearing are all σ-finite.

Now let $E \in \mathcal{E}(M, N)$ and suppose E commutes with α, i.e. $E \circ \alpha_g=\alpha_g \circ E$, $g \in G$. Then since

$$(E \otimes \mathrm{id}_{B(L^2(G))})(\pi_\alpha(x))=\pi_\alpha(E(x)), \qquad x \in M,$$

we can define $\tilde{E} \in \mathcal{E}(\tilde{M}, \tilde{N})$ by $\tilde{E}=E \otimes \mathrm{id}_{B(L^2(G))}|\tilde{M}$, which satisfies $\tilde{E}(\pi_\alpha(x))=\pi_\alpha(E(x))$, $x \in M$. We call $\tilde{E}$ the canonical extension of E.

THEOREM 5.1. *If $E \in \mathcal{E}(M, N)$ commutes with α and $\tilde{E}$ is the canonical extension of E, then*

$$H_{\tilde{E}}(M \rtimes_\alpha G | N \rtimes_\alpha G) \geqq H_E(M|N).$$

Moreover if N is an infinite factor, then

$$H_{\tilde{E}}(M \rtimes_\alpha G | N \rtimes_\alpha G)=H_E(M|N).$$

PROOF. For each neighborhood V of the unit 1_G of G, we choose an $f_V \in L^1(G)_+$ whose support is included in V and such that $\|f_V\|_1=1$. Define $\psi_V \in \mathfrak{S}(\boldsymbol{B}(L^2(G)))$ by $\psi_V=(\cdot f_V^{1/2}|f_V^{1/2})$. Furthermore take a $\psi \in \mathcal{E}(\boldsymbol{B}(L^2(G)))$ and let $\psi_{V,\varepsilon}=(1-\varepsilon)\psi_V+\varepsilon\psi$ for $0<\varepsilon<1$. Now, for any $\varphi \in \mathcal{E}(M)$ with $\varphi \circ E=\varphi$, let $\varphi_1, \cdots, \varphi_n \in M_*^+$ and $\sum \varphi_i=\varphi$. Define $\tilde{\varphi}_{V,\varepsilon} \in \mathcal{E}(\tilde{M})$ and $\tilde{\varphi}_{V,\varepsilon,i} \in \tilde{M}_*^+$ by $\tilde{\varphi}_{V,\varepsilon}=\varphi \otimes \psi_{V,\varepsilon}|\tilde{M}$ and $\tilde{\varphi}_{V,\varepsilon,i}=\varphi_i \otimes \psi_{V,\varepsilon}|\tilde{M}$. Then $\tilde{\varphi}_{V,\varepsilon} \circ \tilde{E}=\tilde{\varphi}_{V,\varepsilon}$ and $\sum_i \tilde{\varphi}_{V,\varepsilon,i}=\tilde{\varphi}_{V,\varepsilon}$. Hence

$$H_{\tilde{E}}(\tilde{M}|\tilde{N}) \geqq \sum_i S(\tilde{\varphi}_{V,\varepsilon,i} \circ \tilde{E}, \tilde{\varphi}_{V,\varepsilon,i}) \geqq \sum_i S(\tilde{\varphi}_{V,\varepsilon,i} \circ \tilde{E} \circ \pi_\alpha, \tilde{\varphi}_{V,\varepsilon,i} \circ \pi_\alpha)$$

by monotonicity. We get for every $x \in M$

$$\tilde{\varphi}_{V,\varepsilon,i}(\pi_\alpha(x))=(1-\varepsilon)\int \varphi_i(\alpha_{g^{-1}}(x)) f_V(g) dg+\varepsilon(\varphi_i \otimes \psi)(\pi_\alpha(x)),$$

$$\tilde{\varphi}_{V,\varepsilon,i}(\tilde{E}(\pi_\alpha(x)))=(1-\varepsilon)\int \varphi_i(\alpha_{g^{-1}}(E(x))) f_V(g) dg+\varepsilon(\varphi_i \otimes \psi)(\pi_\alpha(E(x))),$$

so that $\tilde{\varphi}_{V,\varepsilon,i}\circ\pi_\alpha\to\varphi_i$ and $\tilde{\varphi}_{V,\varepsilon,i}\circ\tilde{E}\circ\pi_\alpha\to\varphi_i\circ E$ in the $\sigma(M_*, M)$-topology as $V\to\{1_G\}$ and $\varepsilon\to 0$. Therefore

$$H_{\tilde{E}}(\tilde{M}|\tilde{N}) \geqq \sum_i S(\varphi_i\circ E, \varphi_i)$$

by the joint $\sigma(M_*, M)$-lower semicontinuity of relative entropy ([**24**]). This implies $H_{\tilde{E}}(\tilde{M}|\tilde{N})\geqq H_E(M|N)$.

Next let us prove the second part. Let $\hat{\alpha}:\tilde{M}\to\tilde{M}\otimes\mathcal{L}(G)$ be the dual co-action of α where $\mathcal{L}(G)=\{\lambda_g : g\in G\}''$. Then $\hat{\alpha}|\tilde{N}$ is that of $\alpha|N$. So the crossed products $\tilde{\tilde{M}}=\tilde{M}\rtimes_{\hat{\alpha}}G$ of $\tilde{M}$ by $\hat{\alpha}$ and $\tilde{\tilde{N}}=\tilde{N}\rtimes_{\hat{\alpha}}G$ of $\tilde{N}$ by $\hat{\alpha}|\tilde{N}$ are defined as follows:

$$\tilde{\tilde{M}} = \{\hat{\alpha}(\tilde{M})\cup(1_{\tilde{M}}\otimes L^\infty(G))\}'' \supseteqq \tilde{\tilde{N}} = \{\hat{\alpha}(\tilde{N})\cup(1_{\tilde{M}}\otimes L^\infty(G))\}''.$$

If $x\in M$ and $g\in G$, then

$$\begin{aligned}(\tilde{E}\otimes\mathrm{id}_{\mathcal{L}(G)})&(\hat{\alpha}(\pi_\alpha(x)(1\otimes\lambda_g)))\\ &= (\tilde{E}\otimes\mathrm{id}_{\mathcal{L}(G)})(\pi_\alpha(x)(1\otimes\lambda_g)\otimes\lambda_g)=\pi_\alpha(E(x))(1\otimes\lambda_g)\otimes\lambda_g\\ &= \hat{\alpha}(\pi_\alpha(E(x))(1\otimes\lambda_g))=\hat{\alpha}(\tilde{E}(\pi_\alpha(x)(1\otimes\lambda_g))),\end{aligned}$$

so that $(\tilde{E}\otimes\mathrm{id}_{\mathcal{L}(G)})\circ\hat{\alpha}=\hat{\alpha}\circ\tilde{E}$ (i.e. $\tilde{E}$ is $\hat{\alpha}$-invariant). This shows that $\tilde{\tilde{E}}\in\mathcal{E}(\tilde{\tilde{M}}, \tilde{\tilde{N}})$ can be defined by $\tilde{\tilde{E}}=\tilde{E}\otimes\mathrm{id}_{B(L^2(G))}|\tilde{\tilde{M}}$, which satisfies $\tilde{\tilde{E}}\circ\hat{\alpha}=\hat{\alpha}\circ\tilde{E}$. For each compact subset K of G, letting $\xi_K=\mu(K)^{-1/2}\chi_K$ where $\mu(K)$ is the left Haar measure of K, we define $\psi_K\in\mathfrak{S}(\boldsymbol{B}(L^2(G)))$ by $\psi_K=(\cdot\xi_K|\xi_K)$. Then for every $g\in G$

$$\psi_K(\lambda_g) = \int\xi_K(g^{-1}h)\xi_K(h)dh=\frac{\mu(K\cap gK)}{\mu(K)} \longrightarrow 1$$

as $K\to G$. Take a $\psi\in\mathcal{E}(\boldsymbol{B}(L^2(G)))$ and let $\psi_{K,\varepsilon}=(1-\varepsilon)\psi_K+\varepsilon\psi$ for $0<\varepsilon<1$. Now, for any $\tilde{\varphi}\in\mathcal{E}(\tilde{M})$ with $\tilde{\varphi}\circ\tilde{E}=\tilde{\varphi}$, let $\tilde{\varphi}_1, \cdots, \tilde{\varphi}_n\in\tilde{M}_*^+$ and $\sum\tilde{\varphi}_i=\tilde{\varphi}$. Define $\tilde{\tilde{\varphi}}_{K,\varepsilon}\in\mathcal{E}(\tilde{\tilde{M}})$ and $\tilde{\tilde{\varphi}}_{K,\varepsilon,i}\in\tilde{\tilde{M}}_*^+$ by $\tilde{\tilde{\varphi}}_{K,\varepsilon}=\tilde{\varphi}\otimes\psi_{K,\varepsilon}|\tilde{\tilde{M}}$ and $\tilde{\tilde{\varphi}}_{K,\varepsilon,i}=\tilde{\varphi}_i\otimes\psi_{K,\varepsilon}|\tilde{\tilde{M}}$. Then $\tilde{\tilde{\varphi}}_{K,\varepsilon}\circ\tilde{\tilde{E}}=\tilde{\tilde{\varphi}}_{K,\varepsilon}$ and $\sum_i\tilde{\tilde{\varphi}}_{K,\varepsilon,i}=\tilde{\tilde{\varphi}}_{K,\varepsilon}$. Hence

$$H_{\tilde{\tilde{E}}}(\tilde{\tilde{M}}|\tilde{\tilde{N}}) \geqq \sum_i S(\tilde{\tilde{\varphi}}_{K,\varepsilon,i}\circ\tilde{\tilde{E}}, \varphi_{K,\varepsilon,i}) \geqq \sum_i S(\tilde{\tilde{\varphi}}_{K,\varepsilon,i}\circ\tilde{\tilde{E}}\circ\hat{\alpha}, \tilde{\tilde{\varphi}}_{K,\varepsilon,i}\circ\hat{\alpha}).$$

We get for every $x\in M$ and $g\in G$

$$\begin{aligned}&\tilde{\tilde{\varphi}}_{K,\varepsilon,i}(\hat{\alpha}(\pi_\alpha(x)(1\otimes\lambda_g)))\\ &\qquad= (1-\varepsilon)\tilde{\varphi}_i(\pi_\alpha(x)(1\otimes\lambda_g))\psi_K(\lambda_g)+\varepsilon\tilde{\varphi}_i(\pi_\alpha(x)(1\otimes\lambda_g))\psi(\lambda_g),\\ &\tilde{\tilde{\varphi}}_{K,\varepsilon,i}(\tilde{\tilde{E}}(\hat{\alpha}(\pi_\alpha(x)(1\otimes\lambda_g))))\\ &\qquad= (1-\varepsilon)\tilde{\varphi}_i(\tilde{E}(\pi_\alpha(x)(1\otimes\lambda_g)))\psi_K(\lambda_g)+\varepsilon\tilde{\varphi}_i(\tilde{E}(\pi_\alpha(x)(1\otimes\lambda_g)))\psi(\lambda_g),\end{aligned}$$

so that $\tilde{\tilde{\varphi}}_{K,\varepsilon,i}\circ\hat{\alpha}\to\tilde{\varphi}_i$ and $\tilde{\tilde{\varphi}}_{K,\varepsilon,i}\circ\tilde{\tilde{E}}\circ\hat{\alpha}\to\tilde{\varphi}_i\circ\tilde{E}$ in the $\sigma(\tilde{M}_*, \tilde{M})$-topology as $K\to G$ and $\varepsilon\to 0$. Therefore

$$H_{\tilde{\tilde{E}}}(\tilde{\tilde{M}}|\tilde{\tilde{N}}) \geqq \sum_i S(\tilde{\varphi}_i\circ\tilde{E}, \tilde{\varphi}_i),$$

implying $H_{\tilde{\tilde{E}}}(\tilde{\tilde{M}}|\tilde{\tilde{N}})\geqq H_{\tilde{E}}(\tilde{M}|\tilde{N})$.

Furthermore let $\gamma:\tilde{\tilde{M}}\to M\otimes \boldsymbol{B}(L^2(G))$ be the isomorphism giving the duality $\tilde{\tilde{M}}\simeq M\otimes \boldsymbol{B}(L^2(G))$. Then $\gamma|\tilde{\tilde{N}}$ gives the duality $\tilde{\tilde{N}}\simeq N\otimes \boldsymbol{B}(L^2(G))$. If $x\in M$, $g\in G$ and $f\in L^\infty(G)$, then

$$\begin{aligned}\gamma(\tilde{\tilde{E}}(\hat{\alpha}(\pi_\alpha(x)(1\otimes\lambda_g))(1_{\tilde{M}}\otimes f))) &= \gamma(\hat{\alpha}(\pi_\alpha(E(x))(1\otimes\lambda_g))(1_{\tilde{M}}\otimes f))\\ &= \pi_\alpha(E(x))(1\otimes\lambda_g)(1\otimes f) = (E\otimes \mathrm{id}_{B(L^2(G))})(\pi_\alpha(x)(1\otimes\lambda_g)(1\otimes f))\\ &= (E\otimes \mathrm{id}_{B(L^2(G))})(\gamma(\hat{\alpha}(\pi_\alpha(x)(1\otimes\lambda_g))(1_{\tilde{M}}\otimes f))).\end{aligned}$$

This shows $\gamma\circ\tilde{\tilde{E}}=(E\otimes \mathrm{id}_{B(L^2(G))})\circ\gamma$. Thus by Proposition 2.1(2)

$$H_{\tilde{\tilde{E}}}(\tilde{\tilde{M}}|\tilde{\tilde{N}}) = H_{E\otimes \mathrm{id}_{B(L^2(G))}}(M\otimes \boldsymbol{B}(L^2(G))|N\otimes \boldsymbol{B}(L^2(G))).$$

Since N is an infinite factor, Lemma 4.3(2) implies $H_{\tilde{\tilde{E}}}(\tilde{\tilde{M}}|\tilde{\tilde{N}})=H_E(M|N)$, so that $H_{\tilde{E}}(\tilde{M}|\tilde{N})=H_E(M|N)$. □

COROLLARY 5.2. *If N is an infinite factor and $E\in\mathcal{E}(M,N)$, then for $\varphi\in\mathcal{E}(M)$ with $\varphi\circ E=\varphi$*

$$H_{\tilde{E}}(M\rtimes_{\sigma^\varphi}\boldsymbol{R}|N\rtimes_{\sigma^\varphi}\boldsymbol{R}) = H_E(M|N).$$

PROOF. Thanks to $\varphi\circ E=\varphi$, it follows that $\sigma_t^\varphi(N)=N$ and $\sigma_t^\varphi\circ E=E\circ\sigma_t^\varphi$ for all $t\in\boldsymbol{R}$. Hence we can apply Theorem 5.1. □

REMARK 5.3. In the situation of the corollary, let τ be the canonical trace on $M\rtimes_{\sigma^\varphi}\boldsymbol{R}$ satisfying $\tau\circ\theta_t=e^{-t}\tau$, $t\in\boldsymbol{R}$, where θ is the dual action. An important fact shown in [**27**, §4] is that $\tilde{E}$ is the conditional expectation with respect to τ.

A finite family $\{a_1,\cdots,a_n\}$ in M is called a basis for $E\in\mathcal{E}(M,N)$ if $x=\sum_i a_i E(a_i^* x)$ for all $x\in M$ ([**33**, **42**]). The next lemma is a slight extension of [**42**, 2.5.3 and Remark].

LEMMA 5.4. (1) *If $\{a_1,\cdots,a_n\}$ in M is a basis for $E\in\mathcal{E}(M,N)$, then $E^{-1}(1)$ is bounded and $E^{-1}(x)=\sum_i a_i x a_i^*$ for $x\in N'$.*

(2) *Assume that N is a factor. If $E\in\mathcal{E}(M,N)$ and $E^{-1}(1)$ is bounded, then there exists a basis $\{a_1,\cdots,a_n\}$ in M for E.*

PROOF. (1) The conclusion was proved in [**42**] when $M\supseteqq N$ are acting on the standard Hilbert space determined by $\varphi_0\circ E$, $\varphi_0\in\mathcal{E}(N)$. Here note that the factorness assumption for $M\supseteqq N$ is unnecessary. Moreover we can show that the conclusion holds independently of the choice of $\mathcal{H}$ (see the proofs of [**19**, Proposition 3.2] and Proposition 4.6).

(2) The proof is the same as that of [**42**, 2.5.3] (based on [**25**, Corollary 3.4]) when we check the following: If N is finite, then so is N' in the standard representation of M. So let us prove this. The boundedness of $E^{-1}(1)$

means that E is of finite index in the sense of [5]. Hence $N'\cap M$ is finite dimensional by [5, Corollaire 3.19] (this is seen also from [25, Proposition 4.3]). The finiteness of N implies the semifiniteness of N'. Since $E'=E^{-1}(1)^{-1}E^{-1}$ belongs to $\mathcal{E}(N', M')$, M' and hence M are semifinite ([36, 10.21]). Thus applying [5, Lemma 3.23] to E and again to E' because $E'^{-1}(1)=E(E^{-1}(1))$ by [19, Proposition 1.2], we conclude that N' is finite. □

THEOREM 5.5. *Assume that N is a factor. If $E\in\mathcal{E}(M, N)$ commutes with α and if $(N'\cap M)_E\subseteq M^\alpha$, the fixed point algebra of α, and $E^{-1}(1)$ is bounded, then*

$$K_{\tilde{E}}(M\rtimes_\alpha G\,|\,N\rtimes_\alpha G)\leqq K_E(M|N).$$

PROOF. Since N is a factor and $E^{-1}(1)$ is bounded, $\omega=E|N'\cap M$ and $\hat{\omega}=E\circ(E^{-1}|N'\cap M)$ are faithful positive functionals on $N'\cap M$. It follows from [19, Theorem 3.3] that $(N'\cap M)_E$ is sufficient for $\{\omega, \hat{\omega}\}$ in the sense of [21]. Hence by [21, Theorem 4.1] (also [31, Theorem 4]), we obtain

$$K_E(M|N)=-S(\hat{\omega}|(N'\cap M)_E,\ \omega|(N'\cap M)_E).$$

If $x\in(N'\cap M)_E$, then for every $y\in M$

$$\tilde{E}(\pi_\alpha(x)\pi_\alpha(y))=\pi_\alpha(E(xy))=\pi_\alpha(E(yx))=\tilde{E}(\pi_\alpha(y)\pi_\alpha(x)),$$

and since $x\in M^\alpha$, for every $g\in G$

$$\tilde{E}(\pi_\alpha(x)(1\otimes\lambda_g))=\tilde{E}((1\otimes\lambda_g)\pi_\alpha(x)),$$

so that $\pi_\alpha(x)\in(\tilde{N}'\cap\tilde{M})_{\tilde{E}}$. Hence $\pi_\alpha((N'\cap M)_E)\subseteq(\tilde{N}'\cap\tilde{M})_{\tilde{E}}$. By Lemma 5.4(2), there exists a basis $\{a_1, \cdots, a_n\}$ in M for E. Then as shown in the proof of [19, Theorem 2.8], $\{\pi_\alpha(a_1), \cdots, \pi_\alpha(a_n)\}$ is a basis for $\tilde{E}$, so that by Lemma 5.4(1)

$$E^{-1}(x)=\sum_i a_i x a_i^*,\qquad x\in N',$$

$$\tilde{E}^{-1}(X)=\sum_i \pi_\alpha(a_i)X\pi_\alpha(a_i^*),\qquad X\in\tilde{N}'.$$

Given $\tilde{\varphi}\in\mathcal{E}(\tilde{M})$ with $\tilde{\varphi}\circ\tilde{E}=\tilde{\varphi}$, we get for every $x\in(N'\cap M)_E$

$$\tilde{\varphi}(\pi_\alpha(x))=\tilde{\varphi}(\tilde{E}(\pi_\alpha(x))=\tilde{\varphi}(\pi_\alpha(E(x)))=\omega(x),$$

$$\tilde{\varphi}(\tilde{E}^{-1}(\pi_\alpha(x)))=\tilde{\varphi}(\tilde{E}(\sum_i\pi_\alpha(a_i x a_i^*)))=\tilde{\varphi}(\pi_\alpha(E(E^{-1}(x))))=\hat{\omega}(x).$$

Therefore by monotonicity

$$\begin{aligned}K_{\tilde{\varphi}}(\tilde{M}|\tilde{N})&=-S(\tilde{\varphi}\circ\tilde{E}^{-1}|\tilde{N}'\cap\tilde{M},\ \tilde{\varphi}|\tilde{N}'\cap\tilde{M})\\&\leqq-S(\tilde{\varphi}\circ\tilde{E}^{-1}\circ\pi_\alpha|(N'\cap M)_E,\ \tilde{\varphi}\circ\pi_\alpha|(N'\cap M)_E)\\&=-S(\hat{\omega}|(N'\cap M)_E,\ \omega|(N'\cap M)_E)=K_E(M|N),\end{aligned}$$

implying $K_{\tilde{E}}(\tilde{M}|\tilde{N})\leqq K_E(M|N)$. □

Under additional assumptions, we obtain the following exact result which serves our purpose to connect $H_E(M|N)$ with $K_E(M|N)$.

THEOREM 5.6. *Assume that G is abelian and that M, N, $M\rtimes_{\alpha}G$ and $N\rtimes_{\alpha}G$ are all factors. If $E\in\mathcal{E}(M, N)$ commutes with α and if $(N'\cap M)_{E}\subseteqq M^{\alpha}$ and $K_{E}(M|N)<\infty$ (this is the case if* $\operatorname{Index}E<\infty$), *then*

$$K_{\tilde{E}}(M\rtimes_{\alpha}G|N\rtimes_{\alpha}G)=K_{E}(M|N).$$

PROOF. Given a nonzero projection e in $(N'\cap M)_{E}$, since $e\in M^{\alpha}$, the action α^{e} of G on M_{e} can be defined by $\alpha_{g}^{e}=\alpha_{g}|M_{e}$, so that $\alpha_{g}^{e}(N_{e})=N_{e}$, $g\in G$. Moreover let $\tilde{e}=\pi_{\alpha}(e)$, which belongs to $(\tilde{N}'\cap\tilde{M})_{\tilde{E}}$ as shown in the proof of Theorem 5.5. Then the following (1)-(4) hold:

(1) $\tilde{M}_{\tilde{e}}=M_{e}\rtimes_{\alpha^{e}}G$ and $\tilde{N}_{\tilde{e}}=N_{e}\rtimes_{\alpha^{e}}G$,

(2) E_{e} commutes with α^{e} and $\tilde{E}_{\tilde{e}}$ is the canonical extension of E_{e},

(3) $\operatorname{Index}\tilde{E}_{\tilde{e}}=\operatorname{Index}E_{e}$,

(4) $(\tilde{N}_{\tilde{e}}'\cap\tilde{M}_{\tilde{e}})_{\tilde{E}_{\tilde{e}}}=\tilde{e}(\tilde{N}'\cap\tilde{M})_{\tilde{E}}\tilde{e}$.

In fact, $\tilde{M}_{\tilde{e}}=M_{e}\rtimes_{\alpha^{e}}G$ is seen from

$$\tilde{e}(\pi_{\alpha}(x)(1\otimes\lambda_{g}))\tilde{e}=\pi_{\alpha^{e}}(exe)(e\otimes\lambda_{g}),\qquad x\in M,\ g\in G.$$

The other in (1) is analogous. Since

$$\begin{aligned}E_{e}(\alpha_{g}^{e}(x))&=E(e)^{-1}E(\alpha_{g}(x))e=E(e)^{-1}\alpha_{g}(E(x)e)\\&=\alpha_{g}^{e}(E_{e}(x)),\qquad x\in M_{e},\ g\in G,\end{aligned}$$

E_{e} commutes with α^{e}. So let $(E_{e})^{\sim}$ be the canonical extension of E_{e}. Since $\tilde{E}(\tilde{e})=\pi_{\alpha}(E(e))=E(e)$, we get for every $x\in M$ and $g\in G$

$$\begin{aligned}\tilde{E}_{\tilde{e}}(\tilde{e}(\pi_{\alpha}(x)(1\otimes\lambda_{g}))\tilde{e})&=E(e)^{-1}\tilde{E}(\pi_{\alpha}(exe)(1\otimes\lambda_{g}))\tilde{e}\\&=E(e)^{-1}\pi_{\alpha}(E(exe))(1\otimes\lambda_{g})\tilde{e}=E(e)^{-1}\pi_{\alpha^{e}}(E(exe)e)(e\otimes\lambda_{g})\\&=\pi_{\alpha^{e}}(E_{e}(exe))(e\otimes\lambda_{g})=(E_{e})^{\sim}(\pi_{\alpha^{e}}(exe)(e\otimes\lambda_{g})),\end{aligned}$$

so that $\tilde{E}_{\tilde{e}}=(E_{e})^{\sim}$, implying (2). Because $\operatorname{Index}(E_{e})^{\sim}=\operatorname{Index}E_{e}$ (see the proof of [**19**, Theorem 2.8]), (3) follows. For every $X\in\tilde{N}_{\tilde{e}}'\cap\tilde{M}_{\tilde{e}}$ and $Y\in\tilde{M}$, since

$$\begin{aligned}\tilde{E}(XY)\tilde{e}&=\tilde{E}(X\tilde{e}Y\tilde{e})\tilde{e}=\tilde{E}(\tilde{e})\tilde{E}_{\tilde{e}}(X\tilde{e}Y\tilde{e}),\\\tilde{E}(YX)\tilde{e}&=\tilde{E}(\tilde{e}Y\tilde{e}X)\tilde{e}=\tilde{E}(\tilde{e})\tilde{E}_{\tilde{e}}(\tilde{e}Y\tilde{e}X),\end{aligned}$$

it follows that $\tilde{E}(XY)=\tilde{E}(YX)$ if and only if $\tilde{E}_{\tilde{e}}(X\tilde{e}Y\tilde{e})=\tilde{E}_{\tilde{e}}(\tilde{e}Y\tilde{e}X)$. This shows (4) by [**6**, Corollaire 3.10].

Since $K_{E}(M|N)<\infty$, $N'\cap M$ and hence $(N'\cap M)_{E}$ are atomic ([**19**, Theorem 4.2]). So let $\{e_{k}\}$ be a set of atoms in $(N'\cap M)_{E}$ with $\sum e_{k}=1$. Then we have by [**19**, Theorem 4.2]

$$K_{E}(M|N)=\sum_{k}E(e_{k})\log\frac{\operatorname{Index}E_{e_{k}}}{E(e_{k})^{2}},$$

so that $\operatorname{Index}E_{e_{k}}<\infty$ for all k. Furthermore each e_{k} is an atom in $N'\cap M$ too, because σ^{E} is inner by a one parameter unitary group in $N'\cap M$ by [**6**, Proposition 3.11]. Hence $N'_{e_k}\cap M_{e_k}=\boldsymbol{C}$ for all k. This together with (3) above implies that

$$\operatorname{Index}\tilde{E}_{\tilde{e}_k}=\operatorname{Index}E_{e_k}=[M_{e_k}:N_{e_k}]_0$$

where $\tilde{e}_k=\pi_\alpha(e_k)$. But since G is abelian and $\tilde{M}_{\tilde{e}_k}\supseteqq\tilde{N}_{\tilde{e}_k}$ are factors, we have $[M_{e_k}:N_{e_k}]_0=[\tilde{M}_{\tilde{e}_k}:\tilde{N}_{\tilde{e}_k}]_0$ by [**19**, Theorem 2.8] and (1), so that $\operatorname{Index}\tilde{E}_{\tilde{e}_k}=[\tilde{M}_{\tilde{e}_k}:\tilde{N}_{\tilde{e}_k}]_0$ (i.e. $\tilde{E}_{\tilde{e}_k}$ gives the minimum index for $\tilde{M}_{\tilde{e}_k}\supseteqq\tilde{N}_{\tilde{e}_k}$). Now partition each $\tilde{e}_k$ into atoms $f_{k1},\cdots,f_{kn_k}$ in $(\tilde{N}'_{\tilde{e}_k}\cap\tilde{M}_{\tilde{e}_k})_{\tilde{E}_{\tilde{e}_k}}$. Then it follows from (4) that $\{f_{kj}:1\leqq k\leqq m,\ 1\leqq j\leqq n_k\}$ is a set of atoms in $(\tilde{N}'\cap\tilde{M})_{\tilde{E}}$ with $\sum_k\sum_{j=1}^{n_k}f_{kj}=1$. Noting $\tilde{E}_{f_{kj}}=(\tilde{E}_{\tilde{e}_k})_{f_{kj}}$, we get by [**25**, Proposition 4.2] and [**18**, Theorem 1]

$$\begin{aligned}\operatorname{Index}\tilde{E}_{f_{kj}}&=\tilde{E}_{\tilde{e}_k}(f_{kj})(\tilde{E}_{\tilde{e}_k})^{-1}(f_{kj})=(\operatorname{Index}\tilde{E}_{\tilde{e}_k})\tilde{E}_{\tilde{e}_k}(f_{kj})^2\\&=(\operatorname{Index}\tilde{E}_{\tilde{e}_k})\tilde{E}(\tilde{e}_k)^{-2}\tilde{E}(f_{kj})^2,\end{aligned}$$

so that

$$\frac{\operatorname{Index}\tilde{E}_{f_{kj}}}{\tilde{E}(f_{kj})^2}=\frac{\operatorname{Index}\tilde{E}_{\tilde{e}_k}}{\tilde{E}(\tilde{e}_k)^2}=\frac{\operatorname{Index}E_{e_k}}{E(e_k)^2}$$

for all k and $1\leqq j\leqq n_k$. Thus using [**19**, Theorem 4.2] again, we obtain

$$\begin{aligned}K_{\tilde{E}}(\tilde{M}\,|\,\tilde{N})&=\sum_k\sum_{j=1}^{n_k}\tilde{E}(f_{kj})\log\frac{\operatorname{Index}\tilde{E}_{f_{kj}}}{\tilde{E}(f_{kj})^2}\\&=\sum_k\sum_{j=1}^{n_k}\tilde{E}(f_{kj})\log\frac{\operatorname{Index}E_{e_k}}{E(e_k)^2}\\&=\sum_k E(e_k)\log\frac{\operatorname{Index}E_{e_k}}{E(e_k)^2}=K_E(M\,|\,N).\quad\square\end{aligned}$$

COROLLARY 5.7. *If $M\supseteqq N$ are type* III$_1$ *factors, $E\in\mathcal{E}(M,N)$ and $K_E(M|N)<\infty$, then for $\varphi\in\mathcal{E}(M)$ with $\varphi\circ E=\varphi$*

$$K_{\tilde{E}}(M\rtimes_{\sigma^\varphi}\boldsymbol{R}\,|\,N\rtimes_{\sigma^\varphi}\boldsymbol{R})=K_E(M\,|\,N).$$

PROOF. Since $(N'\cap M)_E\subseteqq M_\varphi\ (=M^{\sigma^\varphi})$ and $M\rtimes_{\sigma^\varphi}\boldsymbol{R}\supseteqq N\rtimes_{\sigma^\varphi}\boldsymbol{R}$ are II$_\infty$ factors ([38]), we apply Theorem 5.6. $\square$

6. Estimates of entropy.

The aim of this section is to estimate the entropies $H_\varphi(M|N)$ and $H_E(M|N)$ in comparison with $K_E(M|N)$ when N is a factor. For this sake, we first give some lemmas.

LEMMA 6.1. *Let $M\supseteqq N$ be von Neumann algebras and $\{f_n\}$ a sequence of projections in N with $f_n\uparrow 1$. If $\varphi\in\mathcal{E}(M)$ and $\varphi_n=\varphi(f_n)^{-1}\varphi|M_{f_n}$, then*

$$H_\varphi(M|N)\leqq\liminf_{n\to\infty}H_{\varphi_n}(M_{f_n}|N_{f_n}).$$

Proof. Given $\psi_1, \cdots, \psi_k \in M_*^+$ with $\sum\psi_i=\varphi$, let $\psi_{ni}=\varphi(f_n)^{-1}\psi_i|M_{f_n}$. Since $\|\varphi_n'-\varphi\|\to 0$ and $\|\psi_{ni}'-\psi_i\|\to 0$ as $n\to\infty$ where $\varphi_n'=\varphi_n(f_n\cdot f_n)$ and $\psi_{ni}'=\psi_{ni}(f_n\cdot f_n)$, we have

$$\sum_i\{S(\varphi, \psi_i)-S(\varphi|N, \psi_i|N)\} = \lim_{n\to\infty}\sum_i\{S(\varphi_n', \psi_{ni}')-S(\varphi_n'|N, \psi_{ni}'|N)\}$$

$$= \lim_{n\to\infty}\sum_i\{S(\varphi_n, \psi_{ni})-S(\varphi_n|N_{f_n}, \psi_{ni}|N_{f_n})\} \leqq \liminf_{n\to\infty} H_{\varphi_n}(M_{f_n}|N_{f_n}),$$

as desired. □

Lemma 6.2. *Let M be a semifinite von Neumann algebra and $\varphi\in\mathcal{E}(M)$. If $\{e_k\}$ is a set of projections in M_φ with $\sum e_k=1$, then*

$$H_\varphi(M|\bigoplus_k M_{e_k}) \leqq \sum_k \eta(\varphi(e_k)).$$

Proof. By Lemma 6.1, we can assume that $\{e_k\}$ is a finite set $\{e_1, \cdots, e_m\}$ and further that M is finite with a faithful normal trace τ, $\tau(1)=1$. Moreover it may be supposed by Proposition 2.5 that $a=d\varphi/d\tau$ is bounded. We get $ae_k=e_k a$ since $e_k\in M_\varphi$. Also $E\in\mathcal{E}(M, \bigoplus_k M_{e_k})$ with $\varphi\circ E=\varphi$ is given by $E(x)=\sum e_k x e_k$, $x\in M$. Thus it suffices to prove that

$$\sum_l S(\varphi_l\circ E, \varphi_l) \leqq \sum_k \eta(\varphi(e_k))$$

for each $\varphi_1, \cdots, \varphi_n\in M_*^+$ with $\sum\varphi_l=\varphi$. The proof in the following is a modification of that of [**33**, Lemma 4.3]. Let $b_l=a^{-1/2}(d\varphi_l/d\tau)a^{-1/2}$. For any $\varepsilon>0$, by spectral decomposition, we write $b_l=\sum_j\beta_{lj}f_{lj}+c_l$ for some $\beta_{lj}\geqq 0$, projections f_{lj} in M and $0\leqq c_l\leqq\varepsilon$. Define $\varphi_{lj}, \omega_l\in M_*^+$ by $d\varphi_{lj}/d\tau=\beta_{lj}a^{1/2}f_{lj}a^{1/2}$ and $d\omega_l/d\tau=a^{1/2}c_l a^{1/2}$. Since $\varphi_l=\sum_j\varphi_{lj}+\omega_l$, we have

$$S(\varphi_l\circ E, \varphi_l) \leqq \sum_j S(\varphi_{lj}\circ E, \varphi_{lj})+S(\omega_l\circ E, \omega_l).$$

Noting $\tau\circ E=\tau$, we get for $\varepsilon\leqq(\|a\|e)^{-1}$

$$S(\omega_l\circ E, \omega_l) = \tau(\eta E(a^{1/2}c_l a^{1/2}))-\tau(\eta(a^{1/2}c_l a^{1/2})) \leqq \eta(\|a\|\varepsilon).$$

Now write (ψ_i) for $(\varphi_{lj})_{l,j}$ and $d\psi_i/d\tau=\beta_i a^{1/2}f_i a^{1/2}$ with $\beta_i\geqq 0$ and projections f_i in M. Then

$$0 \leqq 1-\sum_i\beta_i f_i \leqq n\varepsilon,$$

$$\sum_l S(\varphi_l\circ E, \varphi_l) \leqq \sum_i S(\psi_i\circ E, \psi_i)+\delta_1(\varepsilon),$$

where $\delta_1(\varepsilon)\to 0$ as $\varepsilon\to 0$.

As in the proof of [**33**, Lemma 4.3], there exist projections g_{ij} in M and $\alpha_{kij}\geqq 0$ such that $\sum_j g_{ij}=f_i$ and

$$0 \leqq g_{ij}ae_k g_{ij}-\alpha_{kij}g_{ij} \leqq \varepsilon g_{ij}$$

for all k, i, j. Define $\psi_{ij}\in M_*^+$ by $d\psi_{ij}/d\tau=\beta_i a^{1/2}g_{ij}a^{1/2}$. Then

$$S(\psi_{ij}\circ E,\,\psi_{ij}) = \tau(\eta E(\beta_i a^{1/2}g_{ij}a^{1/2})) - \tau(\eta(\beta_i a^{1/2}g_{ij}a^{1/2}))$$
$$= \beta_i\{\tau(\eta(\sum_k e_k a^{1/2}g_{ij}a^{1/2}e_k)) - \tau(\eta(a^{1/2}g_{ij}a^{1/2}))\}$$
$$= \beta_i\{\sum_k \tau(\eta(g_{ij}ae_kg_{ij})) - \tau(\eta(g_{ij}ag_{ij}))\}\,.$$

We get for $\varepsilon \leqq (me)^{-1}$

$$\tau(\eta(g_{ij}ae_kg_{ij})) \leqq \tau(\eta(\alpha_{kij})) + \tau(\eta(\varepsilon g_{ij})) = \tau(g_{ij})\{\eta(\alpha_{kij}) + \eta(\varepsilon)\}\,,$$
$$\tau(\eta(g_{ij}ag_{ij})) \geqq \tau(\eta((\sum_k \alpha_{kij} + m\varepsilon)g_{ij})) - \tau(\eta(m\varepsilon g_{ij}))$$
$$= \tau(g_{ij})\{\eta(\sum_k \alpha_{kij} + m\varepsilon) - \eta(m\varepsilon)\}\,,$$

since

$$0 \leqq g_{ij}ag_{ij} - (\sum_k \alpha_{kij})g_{ij} \leqq m\varepsilon g_{ij}\,.$$

Therefore

$$\sum_{i,j} S(\psi_{ij}\circ E,\,\psi_{ij}) \leqq \sum_{i,j}\beta_i\tau(g_{ij})\{\sum_k \eta(\alpha_{kij}) + m\eta(\varepsilon) - \eta(\sum_k \alpha_{kij} + m\varepsilon) + \eta(m\varepsilon)\}$$
$$= \sum_{i,j}\beta_i\tau(g_{ij})\{\sum_k \eta(\alpha_{kij}) - \eta(\sum_k \alpha_{kij})\} + \delta_2(\varepsilon)\,,$$

where $\delta_2(\varepsilon)\to 0$ as $\varepsilon\to 0$ because $\sum_{ij}\beta_i\tau(g_{ij}) \leqq 1$ and $\sum_k \alpha_{kij} \leqq \|a\|$.

Now let

$$r_{kij} = \alpha_{kij}\beta_i\tau(g_{ij})\,,$$
$$s_{kij} = (\sum_{i',j'} \alpha_{ki'j'}\beta_{i'}\tau(g_{i'j'}))(\sum_{k'} \alpha_{k'ij}\beta_i\tau(g_{ij}))\,.$$

Then the direct calculation shows that

$$\sum_{i,j}\beta_i\tau(g_{ij})\{\sum_k \eta(\alpha_{kij}) - \eta(\sum_k \alpha_{kij})\} = \sum_k \eta(\sum_{i,j} r_{kij}) - \sum_{k,i,j} r_{kij}\log\frac{r_{kij}}{s_{kij}}\,.$$

Since

$$\varphi(e_k) - \sum_{i,j}\beta_i\tau(g_{ij}ae_kg_{ij}) = \tau(e_kae_k) - \sum_{i,j}\beta_i\tau(e_k a^{1/2}g_{ij}a^{1/2}e_k)$$
$$= \tau(e_k a^{1/2}(1 - \sum_i \beta_i f_i)a^{1/2}e_k)\,,$$

we get

$$0 \leqq \varphi(e_k) - \sum_{i,j}\beta_i\tau(g_{ij}ae_kg_{ij}) \leqq n\varepsilon\,.$$

Also

$$0 \leqq \sum_{i,j}\beta_i\tau(g_{ij}ae_kg_{ij}) - \sum_{i,j} r_{kij} \leqq \varepsilon\sum_{i,j}\beta_i\tau(g_{ij}) \leqq \varepsilon\,,$$

so that

$$0 \leqq \varphi(e_k) - \sum_{i,j} r_{kij} \leqq (n+1)\varepsilon\,.$$

This implies that

$$\sum_k \eta(\sum_{i,j} r_{kij}) \leqq \sum_k \eta(\varphi(e_k)) + \delta_3(\varepsilon),$$

where $\delta_3(\varepsilon) \to 0$ as $\varepsilon \to 0$. Furthermore since

$$\sum_{k,i,j} s_{kij} = (\sum_{k,i,j} r_{kij})^2 \leqq \sum_{k,i,j} r_{kij},$$

we get

$$\sum_{k,i,j} r_{kij} \log \frac{r_{kij}}{s_{kij}} \geqq 0,$$

because this is the relative entropy of (r_{kij}) and (s_{kij}). By the above estimates altogether, we have

$$\begin{aligned}\sum_l S(\varphi_l \circ E, \varphi_l) &\leqq \sum_{i,j} S(\psi_{ij} \circ E, \psi_{ij}) + \delta_1(\varepsilon) \\ &\leqq \sum_k \eta(\varphi(e_k)) + \delta_1(\varepsilon) + \delta_2(\varepsilon) + \delta_3(\varepsilon),\end{aligned}$$

implying the desired inequality as $\varepsilon \to 0$. □

LEMMA 6.3. *Let M be a general von Neumann algebra and $\varphi \in \mathcal{E}(M)$. If $\{e_k\}$ is a set of projections in M_φ with $\sum e_k = 1$, then*

$$H_\varphi(M | \bigoplus_k M_{e_k}) \leqq \sum_k \eta(\varphi(e_k)).$$

PROOF. We can assume by Lemma 6.1 that $\{e_k\}$ is a finite set. Let $N = \bigoplus_k M_{e_k}$. Since $\sigma_t^\varphi(N) = N$ for all $t \in \boldsymbol{R}$, we take the crossed products $\tilde{M} = M \rtimes_{\sigma^\varphi} \boldsymbol{R}$ and $\tilde{N} = N \rtimes_{\sigma^\varphi} \boldsymbol{R}$. Let $\tilde{e}_k = \pi_\sigma(e_k)$ where $\sigma = \sigma^\varphi$. Since

$$\pi_\sigma(\sum_k e_k x e_k)(1 \otimes \lambda_t) = \sum_k \tilde{e}_k (\pi_\sigma(x)(1 \otimes \lambda_t)) \tilde{e}_k, \qquad x \in M,\ t \in \boldsymbol{R},$$

it follows that $\tilde{N} = \bigoplus_k \tilde{M}_{\tilde{e}_k}$. Moreover $\tilde{E} \in \mathcal{E}(\tilde{M}, \tilde{N})$ given by $\tilde{E}(X) = \sum \tilde{e}_k X \tilde{e}_k$, $X \in \tilde{M}$, is the canonical extension of $E(x) = \sum e_k x e_k$, $x \in M$. For each $\varphi_1, \cdots, \varphi_n \in M_*^+$ with $\sum \varphi_i = \varphi$, define $\tilde{\varphi}_{V,\varepsilon} \in \mathcal{E}(\tilde{M})$ and $\tilde{\varphi}_{V,\varepsilon,i} \in \tilde{M}_*^+$ as in the first paragraph of the proof of Theorem 5.1. Then

$$\sum_i S(\tilde{\varphi}_{V,\varepsilon,i} \circ \tilde{E} \circ \pi_\sigma, \tilde{\varphi}_{V,\varepsilon,i} \circ \pi_\sigma) \leqq H_{\tilde{\varphi}_{V,\varepsilon}}(\tilde{M} | \tilde{N}) \leqq \sum_k \eta(\tilde{\varphi}_{V,\varepsilon}(\tilde{e}_k))$$

by Lemma 6.2 due to the semifiniteness of $\tilde{M}$. Letting $V \to \{1_G\}$ and $\varepsilon \to 0$, since $\tilde{\varphi}_{V,\varepsilon}(\tilde{e}_k) \to \varphi(e_k)$, we have as in the proof of Theorem 5.1

$$\sum_i S(\varphi_i \circ E, \varphi_i) \leqq \sum_k \eta(\varphi(e_k)),$$

as desired. □

In the sequel of this section, let $M \supseteqq N$ be von Neumann algebras and assume always that N is a factor. We are now in a position to prove the next theorem.

THEOREM 6.4. *Let $\varphi \in \mathcal{E}(M)$ and $E \in \mathcal{E}(M, N)$ with $\varphi \circ E = \varphi$. If $\{e_k\}$ is a* *set of projections in* $(N'\cap M)_E$ *with* $\sum e_k=1$ *and if* $\varphi_k=\varphi(e_k)^{-1}\varphi|M_{e_k}$, *then:*

(1) $H_\varphi(M|N)\geqq\sum_k\eta(\varphi(e_k))+\sum_k\varphi(e_k)H_{\varphi_k}(M_{e_k}|N_{e_k})$,

(2) $H_\varphi(M|N)\leqq 2\sum_k\eta(\varphi(e_k))+\sum_k\varphi(e_k)H_{\varphi_k}(M_{e_k}|N_{e_k})$.

PROOF. (1) For each k, let $\psi_{kj}\in(M_{e_k})_*^+$, $1\leqq j\leqq n_k$, be given with $\sum_j\psi_{kj}=\varphi_k$. Define $\varphi_{kj}\in M_*^+$ by $\varphi_{kj}=\varphi(e_k)\psi_{kj}(e_k\cdot e_k)$. Then $\sum_k\sum_{j=1}^{n_k}\varphi_{kj}=\varphi$ since

$$\sum_k\sum_{j=1}^{n_k}\varphi_{kj}(x)=\sum_k\varphi(e_kxe_k)=\varphi(x),\qquad x\in M.$$

Hence for each $m\geqq 1$, we get

$$H_\varphi(M|N)\geqq\sum_{k=1}^m\sum_{j=1}^{n_k}S(\varphi_{kj}\circ E,\varphi_{kj})\geqq\sum_{k=1}^m\sum_{j=1}^{n_k}S(\varphi_{kj}\circ E|M_{e_k},\varphi_{kj}|M_{e_k}).$$

But $\varphi_{kj}|M_{e_k}=\varphi(e_k)\psi_{kj}$ and $\varphi_{kj}\circ E|M_{e_k}=\varphi(e_k)^2\psi_{kj}\circ E_{e_k}$ since

$$\varphi_{kj}(E(x))=\varphi(e_k)\psi_{kj}(E(x)e_k)=\varphi(e_k)^2\psi_{kj}(E_{e_k}(x)),\qquad x\in M_{e_k}.$$

Therefore

$$\begin{aligned}H_\varphi(M|N)&\geqq\sum_{k=1}^m\sum_{j=1}^{n_k}S(\varphi(e_k)^2\psi_{kj}\circ E_{e_k},\varphi(e_k)\psi_{kj})\\&=\sum_{k=1}^m\eta(\varphi(e_k))+\sum_{k=1}^m\varphi(e_k)\sum_{j=1}^{n_k}S(\psi_{kj}\circ E_{e_k},\psi_{kj}).\end{aligned}$$

Thanks to $\varphi_k\circ E_{e_k}=\varphi_k$, taking the supremum over (ψ_{kj}) for each $1\leqq k\leqq m$, we obtain

$$H_\varphi(M|N)\geqq\sum_{k=1}^m\eta(\varphi(e_k))+\sum_{k=1}^m\varphi(e_k)H_{\varphi_k}(M_{e_k}|N_{e_k}),$$

implying the desired inequality as $m\to\infty$.

(2) Letting $\varphi'=\varphi|\oplus_kM_{e_k}$ and $\varphi''=\varphi|\oplus_kN_{e_k}$, we have by Proposition 2.2(1)

$$H_\varphi(M|N)\leqq H_\varphi(M|\bigoplus_kM_{e_k})+H_{\varphi'}(\bigoplus_kM_{e_k}|\bigoplus_kN_{e_k})+H_{\varphi''}(\bigoplus_kN_{e_k}|N).$$

Moreover we have by Lemma 6.3

$$H_\varphi(M|\bigoplus_kM_{e_k})=\sum_k\eta(\varphi(e_k)),$$

and by Proposition 2.3

$$H_{\varphi'}(\bigoplus_kM_{e_k}|N_{e_k})=\sum_k\varphi(e_k)H_{\varphi_k}(M_{e_k}|N_{e_k}).$$

Now let $\psi_1,\cdots,\psi_n\in(\oplus_kN_{e_k})_*^+$ and $\sum\psi_i=\varphi''$. Since

$$\psi_i(E(e_ky))=\varphi(e_k)\psi_i(y)\geqq\varphi(e_k)\psi_i(e_ky),\qquad y\in N,$$

we get $\psi_i\circ E|N_{e_k}\geqq\varphi(e_k)\psi_i|N_{e_k}$ for all k. Therefore

$$\sum_i S(\psi_i\circ(E\,|\bigoplus_k N_{e_k}),\ \psi_i)=\sum_i\sum_k S(\psi_i\circ E\,|\,N_{e_k},\ \psi_i\,|\,N_{e_k})$$
$$\leqq\sum_i\sum_k S(\varphi(e_k)\psi_i\,|\,N_{e_k},\ \psi_i\,|\,N_{e_k})=-\sum_i\sum_k\psi_i(e_k)\log\varphi(e_k)=\sum_k\eta(\varphi(e_k)),$$

so that

$$H_{\varphi''}(\bigoplus_k N_{e_k}\,|\,N)\leqq\sum_k\eta(\varphi(e_k)).$$

Thus we obtain the desired inequality. □

THEOREM 6.5. *Let $\{p_j\}$ be a set of projections in $Z(M)$ with $\sum p_j=1$.*

(1) *If $\varphi\in\mathcal{E}(M)$ and there exists $E\in\mathcal{E}(M,N)$ with $\varphi\circ E=\varphi$, then*

$$H_\varphi(M\,|\,N)=\sum_j\eta(\varphi(p_j))+\sum_j\varphi(p_j)H_{\varphi_j}(Mp_j\,|\,Np_j),$$

where $\varphi_j=\varphi(p_j)^{-1}\varphi\,|\,Mp_j$.

(2) *Assume that N is infinite. Then for every $E\in\mathcal{E}(M,N)$*

$$H_E(M\,|\,N)=\sum_j\eta(E(p_j))+\sum_j E(p_j)H_{E_{p_j}}(Mp_j\,|\,Np_j).$$

PROOF. (1) Since $Z(M)\subseteqq(N'\cap M)_E$, it suffices by Theorem 6.4(1) to show that

$$H_\varphi(M\,|\,N)\leqq\sum_j\eta(\varphi(p_j))+\sum_j\varphi(p_j)H_{\varphi_j}(Mp_j\,|\,Np_j).$$

But by Proposition 2.2(1)

$$H_\varphi(M\,|\,N)\leqq H_\varphi(M\,|\bigoplus_j Np_j)+H_{\varphi''}(\bigoplus_j Np_j\,|\,N),$$

where $\varphi''=\varphi\,|\oplus_j Np_j$. Thanks to $M=\oplus_j Mp_j$, we have by Proposition 2.3

$$H_\varphi(M\,|\bigoplus_j Np_j)=\sum_j\varphi(p_j)H_{\varphi_j}(Mp_j\,|\,Np_j),$$

and as in the proof of Theorem 6.4

$$H_{\varphi''}(\bigoplus_j Np_j\,|\,N)\leqq\sum_j\eta(\varphi(p_j)).$$

(2) Apply (1) above to $\{p_j\otimes 1\}$ in $Z(M\otimes R_\infty)$ and $\varphi\otimes\omega$ where $\omega\in\mathcal{E}(R_\infty)$ and $\varphi\in\mathcal{E}(M)$ with $\varphi\circ E=\varphi$. Then we have by Proposition 4.7(2)

$$\begin{aligned}H_E(M\,|\,N)&=H_{\varphi\otimes\omega}(M\otimes R_\infty\,|\,N\otimes R_\infty)\\&=\sum_j\eta(\varphi(p_j))+\sum_j\varphi(p_j)H_{\varphi_j\otimes\omega}(Mp_j\otimes R_\infty\,|\,Np_j\otimes R_\infty)\\&=\sum_j\eta(E(p_j))+\sum_j E(p_j)H_{E_{p_j}}(Mp_j\,|\,Np_j),\end{aligned}$$

because Np_j is infinite and $\varphi_j\circ E_{p_j}=\varphi_j$. □

THEOREM 6.6. *For every $E\in\mathcal{E}(M,N)$*

$$H_E(M\,|\,N)\leqq K_E(M\,|\,N).$$

PROOF. If $N'\cap M$ has a nonatomic part, then $H_E(M\,|\,N)=K_E(M\,|\,N)=\infty$ by

Proposition 3.2 and [19, Theorem 4.3]. So suppose $N'\cap M$ is atomic. If $\{p_j\}$ is the set of all atoms in $Z(M)$, then by Theorem 6.5(2)

$$H_E(M|N)=\sum_j \eta(E(p_j))+\sum_j E(p_j)H_{E_{p_j}}(Mp_j|Np_j),$$

and by [19, Theorem 4.1]

$$K_E(M|N)=\sum_j \eta(E(p_j))+\sum_j E(p_j)K_{E_{p_j}}(Mp_j|Np_j).$$

Thus we may assume that M as well as N is a factor. Choose a set $\{e_k\}$ of atoms in $(N'\cap M)_E$ with $\sum e_k=1$. Then for any $\varphi\in\mathcal{E}(M)$ with $\varphi\circ E=\varphi$, we deduce by Theorem 6.4(2), Proposition 2.9 and [19, Theorem 4.2] that

$$H_\varphi(M|N)\leqq\sum_k \varphi(e_k)\log\frac{\operatorname{Index}E_{e_k}}{\varphi(e_k)^2}=K_E(M|N).\quad\square$$

COROLLARY 6.7. *Assume that M is a finite von Neumann algebra with a faithful normal trace τ, $\tau(1)=1$. If $E_N: M\to N$ is the conditional expectation with respect to τ, then $H_{E_N}(M|N)=H_\tau(M|N)$. Moreover if M is of type* II_1, *then $H_{E_N}(M|N)=K_{E_N}(M|N)$.*

PROOF. It was shown in [19, Corollary 4.5] that if M is of type II_1, then $K_{E_N}(M|N)=H_\tau(M|N)$. When N is finite dimensional, $H_{E_N}(M|N)=H_\tau(M|N)$ follows from Remark 2.8. When N is of type II_1 (hence so is M), by Theorem 6.6

$$H_\tau(M|N)\leqq H_{E_N}(M|N)\leqq K_{E_N}(M|N)=H_\tau(M|N).\quad\square$$

THEOREM 6.8. *Assume that N is infinite. If $E\in\mathcal{E}(M,N)$ and $K_E(M|N)<\infty$, then*

$$H_E(M|N)=K_E(M|N).$$

PROOF. It suffices by Theorem 6.6 to prove $H_E(M|N)\geqq K_E(M|N)$. We can assume as in the proof of Theorem 6.6 that M as well as N is a factor. Let $\hat{M}=M\otimes R_\infty$, $\hat{N}=N\otimes R_\infty$ and $\hat{E}=E\otimes\mathrm{id}_{R_\infty}$. Then $\hat{M}\supseteqq\hat{N}$ are type III_1 factors. Associated with $\varphi\in\mathcal{E}(\hat{M})$ such that $\varphi\circ\hat{E}=\varphi$, we further take the crossed products $\tilde{M}=\hat{M}\rtimes_{\sigma^\varphi}\boldsymbol{R}$ and $\tilde{N}=\hat{N}\rtimes_{\sigma^\varphi}\boldsymbol{R}$ together with the canonical extension $\tilde{E}$ of $\hat{E}$. Since $\tilde{M}\supseteqq\tilde{N}$ are type II_∞ factors, we can write $\tilde{M}=\tilde{M}_e\otimes F$ and $\tilde{N}=\tilde{N}_e\otimes F$ where e is a finite projection in $\tilde{N}$ and F is a type I_∞ factor. Then the canonical trace τ on $\tilde{M}$ is written as $\tau=\tau_e\otimes\mathrm{tr}_F$ where τ_e is the normalized trace on $\tilde{M}_e$ and tr_F is the usual trace on F. Because $\tilde{E}$ is the conditional expectation with respect to τ (see Remark 5.3), we have $\tilde{E}=E_{\tilde{N}_e}\otimes\mathrm{id}_F$ where $E_{\tilde{N}_e}$ is the conditional expectation $\tilde{M}_e\to\tilde{N}_e$ with respect to τ_e. Now we deduce as follows:

$$\begin{aligned} H_E(M|N) &= H_{\hat{E}}(\hat{M}|\hat{N}) && \text{(Proposition 4.4)}\\ &= H_{\tilde{E}}(\tilde{M}|\tilde{N}) && \text{(Corollary 5.2)}\end{aligned}$$

$$\begin{aligned}
&\geqq H_{E_{\tilde{N}_e}}(\tilde{M}_e|\tilde{N}_e) && \text{(Proposition 4.2(2))}\\
&= K_{E_{\tilde{N}_e}}(\tilde{M}_e|\tilde{N}_e) && \text{(Corollary 6.7)}\\
&= K_{\tilde{\tilde{E}}}(\tilde{M}|\tilde{N}) && \text{([19, Proposition 3.6])}\\
&= K_{\hat{E}}(\hat{M}|\hat{N}) && \text{(Corollary 5.7)}\\
&= K_E(M|N) && \text{([19, Proposition 3.6]).} \quad \square
\end{aligned}$$

It is an open problem whether the conclusion of Theorem 6.8 holds when N is a type II_1 factor, also when $K_E(M|N)=\infty$.

7. Minimum index and entropy.

Throughout this section, let $M\supseteqq N$ be a pair of factor and subfactor such that $[M:N]_0=\mathrm{Index}\,E_0<\infty$ where $E_0\in\mathcal{E}(M, N)$. In [**19**, §6], we established the relation between the minimum index $[M:N]_0$ and the entropy $K_E(M|N)$. In this section, combining the results in [**19**] with the estimates in §6, we have the same relation between $[M:N]_0$ and $H_E(M|N)$.

Although the following is a corollary of Theorem 6.6 and [**19**, Proposition 6.1 and Theorem 6.3], we give a direct proof which may be interesting.

Corollary 7.1. *For every* $E\in\mathcal{E}(M, N)$

$$H_E(M|N)\leqq \log[M:N]_0\,,$$

and if $H_E(M|N)=\log[M:N]_0$ *then* $E=E_0$.

Proof. Let $\tau=E_0|N'\cap M$ which is a trace by [**18**, Theorem 1]. For each $E\in\mathcal{E}(M, N)$, let $a=d(E|N'\cap M)/d\tau$. For any $\varphi\in\mathcal{E}(M)$ with $\varphi\circ E=\varphi$, let $\varphi_1, \cdots, \varphi_n\in M_*^+$ be given with $\sum\varphi_k=\varphi$. Suppose for the moment that $\varphi_1, \cdots, \varphi_n$ are all faithful. Then according to [**30**, Theorem 4], we have

$$S(\varphi_k\circ E, \varphi_k)=i\lim_{t\to+0}\frac{1}{t}\varphi_k([D\varphi_k\circ E:D\varphi_k]_t-1)\,,$$

where $[D\varphi_k\circ E:D\varphi_k]$ is the Connes cocycle derivative of $\varphi_k\circ E$ and φ_k. Furthermore since by [**6**, Propositions 4.1 and 5.1]

$$\begin{aligned}
[D\varphi_k\circ E:D\varphi_k]_t &= [D\varphi_k\circ E:D\varphi_k\circ E_0]_t[D\varphi_k\circ E_0:D\varphi_k]_t\\
&= [DE:DE_0]_t[D\varphi_k\circ E_0:D\varphi_k]_t = a^{it}[D\varphi_k\circ E_0:D\varphi_k]_t\,,
\end{aligned}$$

we get

$$S(\varphi_k\circ E, \varphi_k)=S(\varphi_k\circ E_0, \varphi_k)-\varphi_k(\log a)\,,$$

so that

$$\sum_k S(\varphi_k\circ E, \varphi_k)=\sum_k S(\varphi_k\circ E_0, \varphi_k)-\varphi(\log a)\,.$$

Therefore by the proof of Proposition 2.9

$$\sum_k S(\varphi_k \circ E, \varphi_k) \leqq \log [M:N]_0 + \tau(\eta a).$$

When $\varphi_1, \cdots, \varphi_n$ are not necessarily faithful, taking $(1-\varepsilon)\varphi_k + \varepsilon\varphi$ where $0<\varepsilon<1$ and letting $\varepsilon \to 0$, we have the above inequality by the lower semicontinuity of relative entoropy. This implies that

$$H_E(M|N) \leqq \log [M:N]_0 + \tau(\eta a).$$

Hence $H_E(M|N) \leqq \log [M:N]_0$ because $\tau(\eta a) \leqq \eta(\tau(a)) = 0$. Moreover if $H_E(M|N) = \log [M:N]_0$, then $\tau(\eta a) = 0$ so that $a=1$, implying $E=E_0$ by [**6**, Théorème 5.3]. □

In addition to several characterizations in [**18**, **19**] for $E \in \mathcal{E}(M, N)$ having the minimum index, Theorem 6.8 and [**19**, Theorem 6.3] show the following:

COROLLARY 7.2. *Assume that N is infinite. Then the following conditions for $E \in \mathcal{E}(M, N)$ are equivalent:*

(i) $\mathrm{Index}\, E = [M:N]_0$, *i.e.* $E=E_0$;

(ii) $H_E(M|N) = \log [M:N]_0$;

(iii) $H_E(M|N) = \log \mathrm{Index}\, E$.

REMARK 7.3. Let N be not necessarily infinite. Then Theorem 7.2 holds when (ii) and (iii) are replaced by the following (ii)′ and (iii)′ where P is any infinite factor:

(ii)′ $H_{E \otimes \mathrm{id}_P}(M \otimes P | N \otimes P) = \log [M:N]_0$;

(iii)′ $H_{E \otimes \mathrm{id}_P}(M \otimes P | N \otimes P) = \log \mathrm{Index}\, E$.

COROLLARY 7.4. *If $E \in \mathcal{E}(M, N)$ and $N' \cap M = \boldsymbol{C}$ (this is the case if* $\mathrm{Index}\, E < 4$*), then*

$$H_E(M|N) = K_E(M|N) = \log \mathrm{Index}\, E.$$

PROOF. By assumption, $\mathcal{E}(M, N)$ consists of one element E. Hence $K_E(M|N) = \log \mathrm{Index}\, E$ by [**19**, Theorem 6.3]. When N is infinite, Theorem 6.8 implies $H_E(M|N) = K_E(M|N)$. When N is of type II_1, so is M (see the proof of Lemma 5.4) and E is the conditional expectation with respect to the normalized trace τ on M, so that Corollary 6.7 implies $H_E(M|N) = K_E(M|N)$. When N is finite dimensional, the conclusion is trivial since $N' \cap M = \boldsymbol{C}$ forces $M=N$. □

All other results in [**19**, §6] can be translated by replacing $K_E(M|N)$ with $H_E(M|N)$ due to Theorem 6.6 or 6.8, while we omit the details.

8. Basic constructions and entropy.

As in §7, let $M \supseteqq N$ be a pair of factor and subfactor with $[M:N]_0 < \infty$. Given $E \in \mathcal{E}(M, N)$, repeating the basic constructions [**25**] started from E, we obtain the tower of factors:

$$N \subsetneqq M_0 = M \subsetneqq M_1 \subsetneqq M_2 \subsetneqq \cdots$$

with $E_n \in \mathcal{E}(M_n, M_{n-1})$, $n \geqq 1$, satisfying $\mathrm{Index}\, E_n = \mathrm{Index}\, E$. Concerning the entropies $H_{E_n}(M_n | M_{n-1})$ and $K_{E_n}(M_n | M_{n-1})$, we have:

PROPOSITION 8.1. *For every* $n \geqq 1$,

(1) $H_{E_{2n}}(M_{2n} | M_{2n-1}) = H_E(M | N)$,

(2) $H_{E_{2n+1}}(M_{2n+1} | M_{2n}) = H_{E_1}(M_1 | M)$,

(3) $K_{E_{2n}}(M_{2n} | M_{2n-1}) = K_E(M | N)$,

(4) $K_{E_{2n+1}}(M_{2n+1} | M_{2n}) = K_{E_1}(M_1 | M)$.

Moreover if $e_1, \cdots, e_m$ *are atoms in* $(N' \cap M)_E$ *with* $\sum e_k = 1$, *then*

$$K_{E_1}(M_1 | M) = \sum_k \frac{E^{-1}(e_k)}{\mathrm{Index}\, E} \log \frac{(\mathrm{Index}\, E)^2 (\mathrm{Index}\, E_{e_k})}{E^{-1}(e_k)^2}.$$

PROOF. It suffices for (1)-(4) to prove the case $n=1$. Let J and J_1 be the modular conjugations determined respectively by $\varphi_0 \circ E$ and $\varphi_0 \circ E \circ E_1$ where $\varphi_0 \in \mathcal{E}(N)$. Then we have by the method of basic construction

$$JMJ = M' = J_1 M_2 J_1, \qquad JNJ = M_1' = J_1 M_1 J_1,$$

$$JE(J \cdot J)J = (\mathrm{Index}\, E)^{-1} E_1^{-1} = J_1 E_2(J_1 \cdot J_1) J_1.$$

Hence Proposition 2.1(2) implies that

$$H_{E_2}(M_2 | M_1) = H_{(\mathrm{Index}\, E)^{-1} E_1^{-1}}(M' | M_1') = H_E(M | N).$$

The proof of $H_{E_3}(M_3 | M_2) = H_{E_1}(M_1 | M)$ is analogous. Next since

$$J(N' \cap M)J = M' \cap M_1 = J_1(M_1' \cap M_2)J_1,$$

$$E(J \cdot J) | M' \cap M_1 = (\mathrm{Index}\, E)^{-1} E_1^{-1} | M' \cap M_1 = E_2(J_1 \cdot J_1) | M' \cap M_1,$$

$$E^{-1}(J \cdot J) | M' \cap M_1 = (\mathrm{Index}\, E) E_1 | M' \cap M_1 = E_2^{-1}(J_1 \cdot J_1) | M' \cap M_1,$$

we get

$$K_{E_2}(M_2 | M_1) = -S(E_2^{-1} | M_1' \cap M_2, E_2 | M_1' \cap M_2)$$
$$= -S(E^{-1} | N' \cap M, E | N' \cap M) = K_E(M | N),$$

and analogously $K_{E_3}(M_3 | M_2) = K_{E_1}(M_1 | M)$.

Now let us show the formula of $K_{E_1}(M_1 | M)$ required. Because $\sigma_t^E = \sigma_{-t}^{E^{-1}}$ for $t \in \boldsymbol{R}$ by [**17**, Theorem 6.13], it follows that

$$J(N' \cap M)_E J = J(N' \cap M)_{E^{-1}} J = (M' \cap M_1)_{E_1}.$$

Hence $Je_1 J, \cdots, Je_m J$ are atoms in $(M' \cap M_1)_{E_1}$. Furthermore we have by [**25**, Proposition 4.2]

$$\mathrm{Index}\, (E_1)_{Je_k J} = E_1(Je_k J) E_1^{-1}(Je_k J)$$
$$= (\mathrm{Index}\, E)^{-1} E^{-1}(e_k)(\mathrm{Index}\, E) E(e_k) = \mathrm{Index}\, E_{e_k}.$$

Thus by [**19**, Theorem 4.2]

$$K_{E_1}(M_1|M)=\sum_k E_1(Je_kJ)\log\frac{\operatorname{Index}(E_1)_{Je_kJ}}{E_1(Je_kJ)^2}$$

$$=\sum_k\frac{E^{-1}(e_k)}{\operatorname{Index}E}\log\frac{(\operatorname{Index}E)^2(\operatorname{Index}E_{e_k})}{E^{-1}(e_k)^2}. \quad \square$$

The following examples show that $H_{E_1}(M_1|M)=H_E(M|N)$ does not generally hold.

EXAMPLE 8.2. Let $M=R$ be the hyperfinite type II_1 factor with the normalized trace τ. For $m\geqq 2$, choose nonzero projections $e_1,\cdots,e_m$ in M with $\sum e_k=1$ such that $\alpha_k=\tau(e_k)$, $1\leqq k\leqq m$, are all different. Taking isomorphisms $\theta_k: M_{e_1}\to M_{e_k}$ for $2\leqq k\leqq m$, we define a subfactor N of M by

$$N=\{x\oplus\theta_2(x)\oplus\cdots\oplus\theta_m(x): x\in M_{e_1}\}.$$

Then it is easy to check that

$$N'\cap M=\boldsymbol{C}e_1+\cdots+\boldsymbol{C}e_m.$$

Let $E_N: M\to N$ be the conditional expectation with respect to τ, and $E_M: M_1\to M$ be that with respect to the normalized trace τ_1 on M_1. Note [**25**] that E_M coincides with the conditional expectation obtained by the basic construction from E_N. Then since $M_{e_k}=N_{e_k}$, we get by [**25**, Theorem 4.4]

$$[M:N]=\operatorname{Index}E_N=\sum_k\alpha_k^{-1},$$

$$[M:N]_0=\operatorname{Index}E_0=m^2,$$

and by Corollary 6.7

$$H_{E_N}(M|N)=K_{E_N}(M|N)=H_\tau(M|N)=2\sum_k\eta(\alpha_k).$$

Here $E_0\in\mathcal{E}(M,N)$ having the minimum index m^2 is given by $E_0(e_k)=1/m$, $1\leqq k\leqq m$. Hence $E_N=a^{1/2}E_0a^{1/2}$ $(=E_0(a^{1/2}\cdot a^{1/2}))$ where $a=\sum m\alpha_ke_k$. Because $E_N^{-1}=a^{-1/2}E_0^{-1}a^{-1/2}$ by [**19**, Proposition 1.2] and $E_0^{-1}(e_k)=m$ by [**18**, Theorem 1], we get

$$E_N^{-1}(e_k)=E_0(m^{-1}\alpha_k^{-1}e_k)=\alpha_k^{-1},\qquad 1\leqq k\leqq m.$$

Therefore by Proposition 8.1

$$H_{E_M}(M_1|M)=K_{E_M}(M_1|M)=2\sum_k\eta\Big(\frac{\alpha_k^{-1}}{\sum_i\alpha_i^{-1}}\Big).$$

For instance, if we take $m=3$ and $(\alpha_1,\alpha_2,\alpha_3)=(1/6,1/3,1/2)$, then $H_{E_M}(M_1|M)\neq H_{E_N}(M|N)$. According to [**33**], Jones' subfactor R_λ of R in [**22**] where $\lambda=[R:R_\lambda]^{-1}<1/4$ is an example of the case $m=2$. In [**35**], a subfactor of a type II_1 factor is called a locally trivial subfactor if it has the above form of the case $m=2$. Also choosing a sequence $e_1,e_2,\cdots$ of projections in M with

$\sum e_k=1$ and $\alpha_k=\tau(e_k)$, $\sum\eta(\alpha_k)<\infty$, and defining N as above, we obtain a subfactor N of M such that $[M:N]=\infty$ but $H_{E_N}(M|N)<\infty$.

EXAMPLE 8.3. Let M be an infinite factor. For $m\geqq 2$, choose nonzero projections $e_1, \cdots, e_m$ in M with $\sum e_k=1$. Taking $v_k\in M$, $2\leqq k\leqq m$, such that $v_k^*v_k=e_1$ and $v_kv_k^*=e_k$, we define a subfactor N of M by

$$N=\{x\oplus v_2xv_2^*\oplus\cdots\oplus v_mxv_m^*: x\in M_{e_1}\}.$$

Then the following are easily shown:

$$N'\cap M=\sum_{i,j=1}^{m}\boldsymbol{C}v_iv_j^*\simeq M_m(\boldsymbol{C})$$

where $v_1=e_1$. For $\alpha_1, \cdots, \alpha_m>0$ with $\sum\alpha_k=1$, define

$$E(x)=\sum_{i,j=1}^{m}\alpha_jv_iv_j^*xv_jv_i^*, \qquad x\in M.$$

Then $E\in\mathcal{E}(M, N)$ is directly checked. Since $e_k\in(N'\cap M)_E$ and $M_{e_k}=N_{e_k}$, we get

$$\text{Index } E=\sum_k\alpha_k^{-1},$$

$$[M:N]_0=\text{Index } E_0=m^2,$$

and by Theorem 6.8

$$H_E(M|N)=K_E(M|N)=2\sum_k\eta(\alpha_k).$$

Here $E_0\in\mathcal{E}(M, N)$ is defined as E above with $\alpha_k=1/m$, $1\leqq k\leqq m$. Since $E=a^{1/2}E_0a^{1/2}$ where $a=\sum m\alpha_ke_k$, we get $E^{-1}(e_k)=\alpha_k^{-1}$, $1\leqq k\leqq m$, so that $H_{E_1}(M_1|M)=K_{E_1}(M_1|M)$ is given by the same formula as in Example 8.2.

PROPOSITION 8.4. *If there exist a projection e in M and an isomorphism $\theta: M_e\to M_{1-e}$ such that $N=\{x\oplus\theta(x): x\in M_e\}$, then $[M:N]_0=4$ and $K_{E_n}(M_n|M_{n-1})=K_E(M|N)$ for every $E\in\mathcal{E}(M, N)$ and $n\geqq 1$.*

PROOF. By assumption, either $N'\cap M=\boldsymbol{C}e+\boldsymbol{C}(1-e)$ or $N'\cap M\simeq M_2(\boldsymbol{C})$ occurs. In either case, $[M:N]_0=\text{Index } E_0=4$ as in Examples 8.2 and 8.3, where $E_0|N'\cap M$ is a trace with $E_0(e)=E_0(1-e)=1/2$. For the second assertion, it suffices by Proposition 8.1 to show that $K_{E_1}(M_1|M)=K_E(M|N)$ for every $E\in\mathcal{E}(M, N)$. Replacing E by uEu^* with a unitary u in $N'\cap M$, we may assume that e and $1-e$ are atoms in $(N'\cap M)_E$. Then $E=a^{1/2}E_0a^{1/2}$ for some $a=2\alpha_1e+2\alpha_2(1-e)$ with $\alpha_1, \alpha_2>0$, $\alpha_1+\alpha_2=1$. Hence the calculation in Examples 8.2 and 8.3 implies that

$$\begin{aligned}K_{E_1}(M_1|M)&=2\eta\Big(\frac{\alpha_1^{-1}}{\alpha_1^{-1}+\alpha_2^{-1}}\Big)+2\eta\Big(\frac{\alpha_2^{-1}}{\alpha_1^{-1}+\alpha_2^{-1}}\Big)\\&=2\eta(\alpha_2)+2\eta(\alpha_1)=K_E(M|N). \quad\square\end{aligned}$$

Finally let $M_{-1}=N\subseteqq M_0=M\subseteqq M_1\subseteqq M_2\subseteqq \cdots$ be the basic constructions with $E_n\in\mathcal{E}(M_n, M_{n-1})$ started with $E=E_0$ having the minimum index $[M:N]_0<\infty$. It is proved in [**26**] that $[M_n:N]_0=[M:N]_0^{n+1}$ for all $n\geqq 1$, or equivalently $E_0\circ E_1\circ\cdots\circ E_n$ gives the minimum index for $M_n\supseteqq N$. This result enables us to prove the next theorem.

THEOREM 8.5. *Under the above situation, the following assertions hold:*

(1) *For every* n, $k\geqq 0$

$$K_{E_n\circ\cdots\circ E_{n+k}}(M_{n+k}\,|\,M_{n-1})=(k+1)\log[M:N]_0.$$

(2) *If* N *is infinite or if* $N'\cap M=\boldsymbol{C}$, *then for every* n, $k\geqq 0$

$$H_{E_n\circ\cdots E_{n+k}}(M_{n+k}\,|\,M_{n-1})=(k+1)\log[M:N]_0.$$

PROOF. (1) For each n, $k\geqq 0$, we see by the result stated above that $E_n\circ\cdots\circ E_{n+k}$ gives the minimum index for $M_{n+k}\supseteqq M_{n-1}$. Hence by [**19**, Theorem 6.3]

$$K_{E_n\circ\cdots\circ E_{n+k}}(M_{n+k}\,|\,M_{n-1})=\log[M_{n+k}:M_{n-1}]_0=(k+1)\log[M:N]_0.$$

(2) When N is infinite, the desired equality follows from Theorem 6.8 and the above (1). When N is of type II_1 (hence so is M) and $N'\cap M=\boldsymbol{C}$, each E_n is the conditional expectation with respect to the normalized trace on M_n. Therefore by corollary 6.7 and [**34**, Theorem 3.1]

$$H_{E_n\circ\cdots\circ E_{n+k}}(M_{n+k}\,|\,M_{n-1})=\log[M_{n+k}:M_{n-1}]=\log[M_{n+k}:M_{n-1}]_0.$$

Finally the case of N being finite dimensional is trivial, because $N'\cap M=\boldsymbol{C}$ forces $M=N$. □

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