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# Indivisibility of class numbers of totally imaginary quadratic extensions and their Iwasawa invariants

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## § 0. Introduction.

We denote by $l$ an odd prime number. Hartung [2] proved that there exist infinitely many imaginary quadratic fields whose class numbers are not divisible by $l$. In this paper, we generalize this result to the case of totally imaginary quadratic extensions over a totally real algebraic number field. Moreover we generalize the result due to Horie [3] on Iwasawa invariants of basic $\boldsymbol{Z}_l$-extensions.

We denote by $F$ a totally real algebraic number field and by $m$ its degree over the field $\boldsymbol{Q}$ of rational numbers. We denote by $n(p)$ for a prime $p$ the maximum value of $n$ such that the primitive $p^n$-th roots $\zeta_{p^n}$ of unity are at most of degree 2 over $F$. If $F$ is fixed we have $n(p)=0$ for almost all $p$. So we put $w_F=2^{n(2)+1}\prod_{p\neq 2}p^{n(p)}$. We denote by $\zeta_F(s)$ the Dedekind zeta function of $F$. We know by Serre [9] that $w_F\zeta_F(-1)$ is a rational integer. We denote by $h_K$ the class number of an algebraic number field $K$. The relative class number $h_{K/F}=h_K/h_F$ is an integer when $K$ is a totally imaginary quadratic extension over a totally real algebraic number field $F$. The main result of this paper is the following:

THEOREM. *Let $F$ be a totally real algebraic number field of finite degree. Let $l$ be an odd prime which does not divide $w_F\zeta_F(-1)$. Then there exist infinitely many quadratic extensions $K/F$ with the following properties:*

(i) *$K$ is totally imaginary,*

(ii) *the relative class number $h_{K/F}$ of $K/F$ is not divisible by $l$,*

(iii) *each prime ideal of $F$ over $l$ does not split in $K$.*

If $F=\boldsymbol{Q}$, this is the result due to Hartung [2], since $w_{\boldsymbol{Q}}\zeta_{\boldsymbol{Q}}(-1)=-2$. In order to get Theorem, we use trace formulas and $l$-adic representations related to automorphic forms obtained from division quaternion algebras over $F$.

Let $K/F$ be a totally imaginary quadratic extension. We denote by $\mu_K^-$

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(resp. $\lambda_K^-$) the minus $\mu$-invariant (resp. $\lambda$-invariant) of the basic $\boldsymbol{Z}_l$-extension of $K$. We get:

COROLLARY. *Let $F$ be a totally real algebraic number field. Let $l$ be an odd prime which does not divide $w_F\zeta_F(-1)$. Then there exist infinitely many totally imaginary quadratic extensions $K/F$ such that $\mu_K^-=\lambda_K^-=0$.*

In §1, we summarize the result of Ohta [4] about the $l$-adic representations of the absolute Galois group of $F$ related to automorphic forms. In §2, we summarize the trace formulas of Hecke operators obtained by Shimizu [6], [7] and [8]. In §3, we prove our Theorem by using the results summarized in the previous sections. Moreover we prove our Corollary by using the criterion in Friedman [1]. In §4, we discuss the case that $l$ divides $w_F\zeta_F(-1)$ for real quadratic fields $F/\boldsymbol{Q}$. The author does not know wheather the condition on $w_F\zeta_F(-1)$ is indispensable or not. We have never gotten any counterexamples of the Theorem under the case that $l$ divides $w_F\zeta_F(-1)$.

NOTATION. We denote by $\boldsymbol{C}$ (resp. $\boldsymbol{R}$, $\boldsymbol{Q}_l$) the field of complex numbers (resp. real numbers, $l$-adic numbers). We denote by $R^\times$ the group of invertible elements of a ring $R$ with unity. We denote by $\bar{F}$ the algebraic closure of $F$ and $\mathrm{Gal}(\bar{F}/F)$ the absolute Galois group over $F$. We denote by $N_{K/k}$ the norm map from $K$ to $k$. We denote by $\mathrm{M}_n(K)$ the ring of matrices of degree $n$ with coefficients in $K$. We put $\mathrm{GL}_n(K)=(\mathrm{M}_n(K))^\times$.

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## §1. $l$-adic representations.

Let $B$ be a division quaternion algebra over $F$ such that $B\otimes_{\boldsymbol{Q}}\boldsymbol{R}\cong\mathrm{M}_2(\boldsymbol{R})\times\boldsymbol{H}^{m-1}$, where $\boldsymbol{H}$ denotes the Hamilton quaternion algebra over $\boldsymbol{R}$. We denote by $\iota$ an involution of $B$ and by $\nu=\nu_{B/F}$ the reduced norm of $B/F$. We denote by $\boldsymbol{B}_A^\times$ the idele group of $B$. Let $S$ be an open subgroup of $\boldsymbol{B}_A^\times$ such that $S^\iota=S$ and $S=B_{\infty+}^\times\times S_0$, where $B_{\infty+}^\times=\{(t_1,\cdots,t_m)\in(B\otimes_{\boldsymbol{Q}}\boldsymbol{R})^\times : \nu(t_1)>0\}$ and $S_0$ is an open compact subgroup of the finite part $\boldsymbol{B}_f^\times$ of $\boldsymbol{B}_A^\times$. Let $\rho$ be a representation of $B^\times$, which will be constructed later as in Ohta [4]. Let $\mathfrak{S}(S,\rho)$ be the space of automorphic forms introduced later. We denote by $\mathfrak{T}(\mathfrak{p})$ and $\mathfrak{T}(\mathfrak{p},\mathfrak{p})$ the Hecke operators acting on $\mathfrak{S}(S,\rho)$ for a prime ideal $\mathfrak{p}$ of $F$. Ohta [4] got:

THEOREM. *There exists an $l$-adic representation*

$$\phi_{S,\rho}:\mathrm{Gal}(\bar{F}/F)\longrightarrow\mathrm{GL}_{2\dim_{\boldsymbol{C}}\mathfrak{S}(S,\rho)}(\boldsymbol{Q}_l),$$

*which has the following properties:*

(i) *If a prime* $\mathfrak{p}$ *of* $F$ *divides neither* $l$ *nor the discriminant* $D(B/F)$ *of* $B/F$ *and* $S$ *contains the group of units in a maximal order of the completion* $B_{\mathfrak{p}}$ *of* $B$ *at* $\mathfrak{p}$, *then* $\psi_{S,\rho}$ *is unramified at* $\mathfrak{p}$,

(ii) $\det(1-\psi_{S,\rho}(\sigma_{\mathfrak{p}})T)=\det(1-\mathfrak{T}(\mathfrak{p})T+N_{F/Q}(\mathfrak{p})\mathfrak{T}(\mathfrak{p},\mathfrak{p})T^2|_{\mathfrak{S}(S,\rho)})$, *where* $\sigma_{\mathfrak{p}}$ *is a Frobenius element at* $\mathfrak{p}$ *in* $\mathrm{Gal}(\bar{F}/F)$.

Next we take $S$ and $\rho$. We put $S_0=\prod_{\mathfrak{p}}\mathfrak{o}_{\mathfrak{p}}^{\times}$, where $\mathfrak{o}_{\mathfrak{p}}$ is a closure of a maximal order $\mathfrak{o}$ in $B$. We put $n=2$ (resp. $n=4$) for $l\geqq 5$ (resp. $l=3$). We get the representation $\rho$ as in p. 41 of Ohta [4], putting by $n_1=\cdots=n_m=n$ and $w=0$. Next we define the space $\mathfrak{S}(S,\rho)$ of automorphic forms. Let $B_+^{\times}$ be a subgroup of $B^{\times}$ consisting of elements $x\in B^{\times}$ whose reduced norm $\nu(x)$ is totally positive. We decompose $\boldsymbol{B}_{A+}^{\times}=\boldsymbol{B}_{\infty+}^{\times}\times\boldsymbol{B}_f^{\times}$ into $\boldsymbol{B}_{A+}^{\times}=\bigcup_{i=1}^{h}Sx_iB_+^{\times}$, where $h=h_B$ is the class number of $B$. We put $\Gamma_{S_i}=x_i^{-1}Sx_i\cap B_+^{\times}$. Thus $\Gamma_{S_i}$ is a Fuchsian group of the first kind in $\mathrm{SL}_2(\boldsymbol{R})$. We introduce the representation $\Psi$ of $\mathrm{GL}_2^+(\boldsymbol{R})\times(\boldsymbol{H}^{\times})^{m-1}$ by

$$\Psi((t_1,\cdots,t_m))=\prod_{i=2}^{m}\nu(t_i)^{-n/2}\rho_n(t_2)\otimes\cdots\otimes\rho_n(t_m),$$

where $\rho_n$ is the symmetric tensor representation of degree $n$ of $\mathrm{GL}_2(\boldsymbol{C})$. We see that the degree of $\Psi$ is $(n+1)^{m-1}$. We denote by $p$ the composite of the natural embedding $B^{\times}\to(B\otimes_{\boldsymbol{Q}}\boldsymbol{R})^{\times}=\mathrm{GL}_2(\boldsymbol{R})\times(\boldsymbol{H}^{\times})^{m-1}$ and the projection to $\mathrm{GL}_2(\boldsymbol{R})$. For a $\boldsymbol{C}^{(n+1)^{m-1}}$-valued function $f(z)$ on the complex upper half plane $\mathfrak{H}$, we put $f|_{[\gamma]}(z)=\Psi(\gamma)^{-1}j(p(\gamma),z)^{-(n+2)}f(z)$, where $j(p(\gamma),z)=cz+d$ for $p(\gamma)=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{GL}_2(\boldsymbol{R})$. We denote by $\mathfrak{S}(\Gamma_{S_i},\Psi)$ the space of $\boldsymbol{C}^{(n+1)^{m-1}}$-valued holomorphic functions $f(z)$ on $\mathfrak{H}$ such that $f|_{[\gamma]}(z)=f(z)$ for every $\gamma\in\Gamma_{S_i}$. We put $\mathfrak{S}(S,\rho)=\bigoplus_{i=1}^{h}\mathfrak{S}(\Gamma_{S_i},\Psi)$. The Theorem in this section is valid for $\mathfrak{S}(S,\rho)$.

## § 2. Trace formulas.

Many people calculated the traces of Hecke operators. In this section we compute the traces of Hecke operators referring to Shimizu [6], [7] and [8].

Let $\Gamma$ be a Fuchsian group of the first kind such that $\Gamma\backslash\mathfrak{H}$ is compact. So far as we use in this paper, we quote the formula, for example, from Shimizu [7] as follows:

$$\mathrm{tr.}\,\mathfrak{T}(\Gamma\alpha\Gamma)=v(\Gamma\backslash\mathfrak{H})\,\mathrm{tr.}\,\Psi(g_0)(\mathrm{sgn}\,g_0)^{k+2}\frac{k-1}{4\pi}$$
$$-\sum_{g\in\mathfrak{G}_1}\frac{\mathrm{tr.}\,\Psi(g)}{[\Gamma(g):Z(\Gamma)]}\frac{\zeta(g)^{k-1}-\eta(g)^{k-1}}{\zeta(g)-\eta(g)}(\det g)^{1-k/2}.$$

Notations are the same as in Theorem 1 of Shimizu [7]. $\mathfrak{G}_1$ is a complete system of inequivalent elliptic elements.

Next we compute traces of Hecke operators acting on $\mathfrak{S}(\Gamma_{S_i}, \Psi)$. We set $k=n+2$, where $n$ is as in § 1. We put $\mathfrak{T}(\mathfrak{q}, \mathfrak{D}_i)=\sum_{(\nu(\alpha))=\mathfrak{q}}\Gamma_{S_i}\alpha\Gamma_{S_i}$ for an integral ideal $\mathfrak{q}$ of $F$ as in Shimizu [**6**], where $\mathfrak{O}_i=x_i^{-1}Sx_i\cap B$. For investigating $\mathfrak{G}_1$ we put $\Omega_0$ the set of isomorphism classes of orders $\mathfrak{o}$ of totally imaginary quadratic extensions over $F$ in $B$ satisfying the following properties:

(i) no prime factor of $D(B/F)$ splits in $F(\mathfrak{o})$,

(ii) the conductor of $\mathfrak{o}$ is prime to $D(B/F)$.

We put $\mathfrak{o}=F(\alpha)\cap\mathfrak{D}_i$ for an elliptic element $\alpha\in\mathfrak{D}_i$ such that $(\nu(\alpha))=\mathfrak{q}$. It follows that $\mathfrak{o}$ is in $\Omega_0$. We know that there exists $\gamma\in\Gamma_{S_i}$ such that $\alpha'=\pm\gamma^{-1}\alpha\gamma$ if and only if $\mathfrak{o}=F(\alpha)\cap\mathfrak{D}_i$ and $\mathfrak{o}'=F(\alpha')\cap\mathfrak{D}_i$ are $\Gamma_{S_i}$-conjugate to each other. Therefore the number of $\Gamma_{S_i}$-equivalence classes of elliptic elements $\alpha$ such that $F(\alpha)\cap\mathfrak{D}_i=\mathfrak{o}$ is equal to the number of $\Gamma_{S_i}$-conjugacy classes of $\mathfrak{o}$. Let $\Gamma_0$ be the group of all units in $\mathfrak{D}_i$. By Shimizu [**7**] the number of $\Gamma_0$-conjugacy classes of $\mathfrak{o}$ is equal to $(h(\mathfrak{o})/2h_B)\prod_{\mathfrak{p}|D(B/F)}(1-(\mathfrak{o}/\mathfrak{p}))$, where $h(\mathfrak{o})$ is the class number of $\mathfrak{o}$ and $(\mathfrak{o}/\mathfrak{p})=1$ (resp. $-1, 0$) when $\mathfrak{p}$ splits completely (resp. remains prime, is ramified) in $F(\mathfrak{o})/F$. We denote by $E$ (resp. $E^+$) the group of units (resp. totally positive units) of the ring of integers of $F$. We get by $[\Gamma_0:\Gamma_{S_i}]=[E:E^+]$ that the number of $\Gamma_{S_i}$-conjugacy classes of $\mathfrak{o}$ is equal to

$$\frac{h(\mathfrak{o})}{h_B}\frac{[E:E^+]}{2}\prod_{\mathfrak{p}|D(B/F)}\left(1-\left(\frac{\mathfrak{o}}{\mathfrak{p}}\right)\right).$$

We see that $\Gamma\alpha\Gamma\cap Z(\mathrm{GL}_2(\boldsymbol{R}))\neq\emptyset$ if and only if there exists $q_0\in F$ such that $\mathfrak{q}=(q_0{}^2)$. We notice that $\Gamma(\alpha)$ is the group $E(\mathfrak{o})$ of units of $\mathfrak{o}=F(\alpha)\cap\mathfrak{D}_i$ and $Z(\Gamma)=E$ (cf. Shimizu [**6**]). Thus we get

$$\mathrm{tr.}\,\mathfrak{T}(\mathfrak{q}, \mathfrak{D}_i)=\delta(\mathfrak{q})\,\mathrm{tr.}\,\Psi(g_0)v(\Gamma_{S_i}\backslash\mathfrak{H})\frac{n+1}{4\pi}$$

$$-\frac{[E:E^+]}{2}\sum_{\mathfrak{o}\in\Omega_0}\frac{h(\mathfrak{o})}{h_B}\frac{\prod_{\mathfrak{p}|D(B/F)}\left(1-\left(\frac{\mathfrak{o}}{\mathfrak{p}}\right)\right)}{[E(\mathfrak{o}):E]}\sum_{\substack{\alpha\in J(\mathfrak{o})\\ \alpha \bmod E}}\mathrm{tr.}\,\Psi(\alpha)\frac{\zeta_\alpha^{n+1}-\eta_\alpha^{n+1}}{\zeta_\alpha-\eta_\alpha}(\det\alpha)^{-n/2},$$

where $J(\mathfrak{o})=\{\alpha\in\mathfrak{o}:\alpha\notin F, (\nu(\alpha))=\mathfrak{q}\}$, $\zeta_\alpha$ and $\eta_\alpha$ are eigenvalues of $\alpha$, and $\delta(\mathfrak{q})=1$ in the case of $\mathfrak{q}=(q_0{}^2)$ for some integer $q_0$ of $F$ and otherwise $\delta(\mathfrak{q})=0$.

We know that the Hecke operators $\mathfrak{T}(\mathfrak{q})$ of Ohta [**4**] act on $\mathfrak{S}(S, \rho)=\bigoplus_{i=1}^h\mathfrak{S}(\Gamma_{S_i}, \Psi)$. Hence we have $\mathrm{tr.}\,\mathfrak{T}(\mathfrak{q})=\sum_{i=1}^h\mathrm{tr.}\,\mathfrak{T}(\mathfrak{q}, \mathfrak{D}_i)$. As we see that the formula of $\mathrm{tr.}\,\mathfrak{T}(\mathfrak{q}, \mathfrak{D}_i)$ is independent of $i$ and that $\delta(\mathfrak{q})=0$ for a prime ideal $\mathfrak{q}$ of $F$, we get:

(1) $\mathrm{tr.}\,\mathfrak{T}(\mathfrak{q})=$

$$-\frac{[E:E^+]}{2}\sum_{\mathfrak{c}\in\Omega_0}h(\mathfrak{o})\frac{\prod_{\mathfrak{p}|D(B/F)}\left(1-\left(\frac{\mathfrak{o}}{\mathfrak{p}}\right)\right)}{[E(\mathfrak{o}):E]}\sum_{\substack{\alpha\in J(\cdot)\\ \alpha \bmod E}}\mathrm{tr.}\,\Psi(\alpha)\frac{\zeta_\alpha^{n+1}-\eta_\alpha^{n+1}}{\zeta_\alpha-\eta_\alpha}(\det\alpha)^{-n/2}.$$

We see $\mathrm{tr.}\,\mathfrak{T}((1))=\dim_C\mathfrak{S}(S, \rho)$ and $\delta((1))=1$. In this case, we can take $g_0=1$. By using $v(\Gamma_{S_i}\backslash\mathfrak{H})=v(\Gamma_{S_1}\backslash\mathfrak{H})$, we get:

(2) $$\dim_C \mathfrak{S}(S,\rho)=h_B(n+1)^m v(\Gamma_{S_1}\backslash\mathfrak{H})/4\pi$$
$$-\frac{[E:E^+]}{2}\sum_{\mathfrak{o}\in\Omega_0}h(\mathfrak{o})\frac{\prod_{\mathfrak{p}\mid D(B/F)}\left(1-\left(\frac{\mathfrak{o}}{\mathfrak{p}}\right)\right)}{[E(\mathfrak{o}):E]}\sum_{\substack{\alpha\in J(\mathfrak{o})\\ \alpha \bmod E}}\mathrm{tr}.\,\Psi(\alpha)\frac{\zeta_\alpha^{n+1}-\eta_\alpha^{n+1}}{\zeta_\alpha-\eta_\alpha}(\det\alpha)^{-n/2}.$$

By Shimizu [6] (See also [8].), we have

$$v(\Gamma_0\backslash\mathfrak{H})=\frac{D_F^{3/2}2^{2-m}\zeta_F(2)}{\pi^{2m-1}[E:E^+]}\frac{h_F}{h_B}\prod_{\mathfrak{p}\mid D(B/F)}(N_{F/Q}\mathfrak{p}-1),$$

where $D_F$ is the discriminant of $F$. From (2) we get by the functional equation of $\zeta_F(s)$ and the equality $v(\Gamma_{S_i}\backslash\mathfrak{H})=[E:E^+]v(\Gamma_0\backslash\mathfrak{H})$,

(3) $$\dim_C\mathfrak{S}(S,\rho)=(-1)^m(n+1)^m h_F\zeta_F(-1)\prod_{\mathfrak{p}\mid D(B/F)}(N_{F/Q}\mathfrak{p}-1)$$
$$-\frac{[E:E^+]}{2}\sum_{\mathfrak{o}\in\Omega_0}h(\mathfrak{o})\frac{\prod_{\mathfrak{p}\mid D(B/F)}\left(1-\left(\frac{\mathfrak{o}}{\mathfrak{p}}\right)\right)}{[E(\mathfrak{o}):E]}\sum_{\substack{\alpha\in J(\mathfrak{o})\\ \alpha \bmod E}}\mathrm{tr}.\,\Psi(\alpha)\frac{\zeta_\alpha^{n+1}-\eta_\alpha^{n+1}}{\zeta_\alpha-\eta_\alpha}(\det\alpha)^{-n/2}.$$

## § 3. Proof of Theorem.

Let $l$ be an odd prime which does not divide $w_F\zeta_F(-1)$.

First we prove that there exists at least one totally imaginary quadratic extension $K/F$ whose relative class number $h_{K/F}$ is not divisible by $l$.

For the case of $n(l)>0$, we take a prime ideal $\mathfrak{p}_l$ of $F$ as follows. Because $k_l=F(\zeta_l)$ is a totally imaginary quadratic extension over $F$, we take a prime ideal $\mathfrak{p}_l$ of $F$ which is unramified and of degree 1 over $\boldsymbol{Q}$ and splits completely in $k_l/\boldsymbol{Q}$ but not in $k_l(\zeta_{l^{n(l)+1}})/\boldsymbol{Q}$. We see $N_{F/Q}\mathfrak{p}_l\equiv 1 \bmod l^{n(l)}$ and $N_{F/Q}\mathfrak{p}_l\not\equiv 1 \bmod l^{n(l)+1}$.

We can determine the quaternion algebra $B/F$ by giving even number of prime spots which are ramified in $B/F$ (e.g. Weil [10] Chap. XIII). We take $B/F$ as follows:

(i) the only one real prime is unramified in $B/F$ and other real primes are ramified,

(ii) $\mathfrak{p}_l$ is ramified in $B/F$, if $n(l)>0$,

(iii) each prime ideal $\mathfrak{l}$ over $l$ is ramified in $B/F$,

(iv) the other prime ideals $\mathfrak{p}$ which are ramified in $B/F$ satisfy $N_{F/Q}\mathfrak{p}\not\equiv 1 \bmod l$.

Then $\prod_{\mathfrak{p}\mid D(B/F)}(N_{F/Q}\mathfrak{p}-1)$ is divisible by $l^{n(l)}$ not by $l^{n(l)+1}$. Thus the $l$-adic order of $\prod_{\mathfrak{p}\mid D(B/F)}(N_{F/Q}\mathfrak{p}-1)$ is equal to that of $w_F$. By the assumption that $l$ does not divide $w_F\zeta_F(-1)$, we see that the first term of the formula (3) is divisible by $l^{e_F}$ but not by $l^{e_F+1}$, where $e_F$ stands for the exponent of $l$ in $h_F$. We take $S$ and $\rho$ as in § 1.

Now we assume that every totally imaginary quadratic extension $K$ over $F$

has the relative class number $h_{K/F}$ which is divisible by $l$. Because $\mathfrak{p}_l$ divides $D(B/F)$ for the case of $n(l)>0$, $\Omega_0$ does not contain any order $\mathfrak{o}$ containing the primitive $l$-th roots of unity. Thus we see that the second term of (2) is divisible by $l^{e_F+1}$, because $h(\mathfrak{o})$ is a multiple of $h_{F(\mathfrak{o})}=h_{F(\mathfrak{o})/F}h_F$ and an $l$-adic integer (e. g. Prestel [5] p. 188). Thus we get:

$$\dim_C \mathfrak{S}(S, \rho)\equiv 0 \bmod l^{e_F} \quad \text{and} \quad \dim_C \mathfrak{S}(S, \rho)\not\equiv 0 \bmod l^{e_F+1}. \tag{4}$$

We put $H_l=\{g\in \mathrm{GL}_{2\dim_C \mathfrak{S}(S,\rho)}(\boldsymbol{Q}_l): g\equiv 1 \bmod l^{e_F+1}\}$. Let $M_l$ be the fixed field by $\psi_{S,\rho}^{-1}(H_l)$, where $\psi_{S,\rho}$ is the $l$-adic representation as in § 1. Let $\mathfrak{q}$ be a prime ideal of $F$ such that $\mathfrak{q}$ splits completely in $M_l/F$ and does not divide $lD(B/F)$. Therefore we get $\psi_{S,\rho}(\sigma_\mathfrak{q})\equiv 1 \bmod l^{e_F+1}$, and by (4)

$$\mathrm{tr}.\, \psi_{S,\rho}(\sigma_\mathfrak{q})\equiv 2\dim_C \mathfrak{S}(S, \rho)\not\equiv 0 \bmod l^{e_F+1},$$

where $\sigma_\mathfrak{q}$ is a Frobenius element at $\mathfrak{q}$ in $\mathrm{Gal}(\bar{F}/F)$. By (1), we get:

(5) $\quad \mathrm{tr}.\, \mathfrak{T}(\mathfrak{q}) =$

$$-\frac{[E:E^+]}{2}\sum_{\mathfrak{o}\in\Omega_0} h(\mathfrak{o})\frac{\prod_{\mathfrak{p}|D(B/F)}\left(1-\left(\frac{\mathfrak{o}}{\mathfrak{p}}\right)\right)}{[E(\mathfrak{o}):E]}\sum_{\substack{\alpha\in J(\mathfrak{o})\\ \alpha \bmod E}} \mathrm{tr}.\, \Psi(\alpha)\frac{\zeta_\alpha^{n+1}-\eta_\alpha^{n+1}}{\zeta_\alpha-\eta_\alpha}(\det\alpha)^{-n/2}.$$

We see by $(\nu(\alpha))=\mathfrak{q}$ that $(\det\alpha)^{-n/2}$ is an $l$-adic integer in $\bar{\boldsymbol{Q}}_l$. We see that $[E(\mathfrak{o}):E]$ is prime to $l$, because of the assumption that $\mathfrak{p}_l$ divides $D(B/F)$ for the case of $n(l)>0$. Because $h_{F(\mathfrak{o})/F}$ is divisible by $l$ and $h(\mathfrak{o})$ is a multiple of $h_{F(\mathfrak{o})/F}h_F$ and an $l$-adic integer, we get $\mathrm{tr}.\, \mathfrak{T}(\mathfrak{q})\equiv 0 \bmod l^{e_F+1}$. This contradicts the equality $\mathrm{tr}.\, \mathfrak{T}(\mathfrak{q})=\mathrm{tr}.\, \psi_{S,\rho}(\sigma_\mathfrak{q})$, which is contained in Theorem of Ohta [4] cited in § 1. Thus we see that there exists a totally imaginary quadratic extension $K$ over $F$ whose relative class number $h_{K/F}$ is not divisible by $l$. We see by (iii) that each prime ideal of $F$ over $l$ does not split in $K$. Moreover we can take $K$ which contains no primitive $l$-th root of unity, because $\mathfrak{p}_l$ divides $D(B/F)$, if $n(l)>0$.

Finally we prove that there exist infinitely many totally imaginary quadratic extensions over $F$ whose relative class numbers are not divisible by $l$.

Let $K_1, \cdots, K_s$ be totally imaginary quadratic extensions over $F$ whose relative class numbers are not divisible by $l$. Here we can assume that $K_i$ contains no primitive $l$-th root of unity. We take a prime ideal $\mathfrak{q}_i$ of $F$ such that $\mathfrak{q}_i$ splits completely in $K_i/F$ and $N_{F/\boldsymbol{Q}}\mathfrak{q}_i\not\equiv 1 \bmod l$ for each $1\leqq i\leqq s$. We take a division quaternion algebra $B/F$ as follows:

(i) the only one real prime is unramified in $B/F$ and other real primes are ramified,

(ii) $\mathfrak{p}_l$ is ramified in $B/F$, if $n(l)>0$,

(iii) $\mathfrak{q}_1, \cdots, \mathfrak{q}_s$ are ramified in $B/F$,

(iv) each prime ideal $\mathfrak{l}$ over $l$ is ramified in $B/F$,

(v) the other prime ideals $\mathfrak{p}$ which are ramified in $B/F$ satisfy $N_{F/Q}\mathfrak{p}\not\equiv 1 \bmod l$.
We take $S$ and $\rho$ as in § 1. We get as before $\dim_C \mathfrak{S}(S, \rho)\equiv 0 \bmod l^{e_F}$ and $\dim_C \mathfrak{S}(S, \rho)\not\equiv 0 \bmod l^{e_F+1}$. No order of $K_1, \cdots, K_s$ is contained in $\Omega_0$, because $\mathfrak{q}_1, \cdots, \mathfrak{q}_s$ divide $D(B/F)$. We get a contradiction by a similar argument as before, if we assume there exists no other totally imaginary quadratic extension over $F$ other than $K_1, \cdots, K_s$ whose relative class number is not divisible by $l$. We see by (iv) that each prime ideal of $F$ over $l$ does not split in these totally imaginary quadratic extensions. The proof of Theorem is complete.

Next we prove Corollary.

We take totally imaginary quadratic extensions $K/F$ such as in Theorem. We see that the relative class numbers $h_{K/F}$ are not divisible by $l$ and each prime ideal $\mathfrak{l}$ of $F$ over $l$ does not split completely in $K/F$. Therefore we get $\mu_K^- = \lambda_K^- = 0$ by the criterion in Friedman [**1**]. By the same discussion in the proof of Theorem, we also see that there exist infinitely many totally imaginary quadratic extensions $K/F$ such that $\mu_K^- = \lambda_K^- = 0$.

## § 4. The case that $l$ divides $w_F\zeta_F(-1)$.

We now take a prime number $l$ which divides $w_F\zeta_F(-1)$. To construct numerical examples for this case we first prove the following:

PROPOSITION. *We assume that there exists at least one totally imaginary quadratic extension $k/F$ with the following properties:*

(i) *the roots of unity of $k$ are $\pm 1$,*

(ii) *at least one prime ideal $\mathfrak{p}$ of $F$ which does not divide $2l$ is ramified in $k/F$,*

(iii) *the relative class number $h_{k/F}$ is not divisible by $l$.*

*Then there exist infinitely many totally imaginary quadratic extensions $K/F$ whose relative class numbers $h_{K/F}$ are not divisible by $l$.*

First we prove that there exists at least one totally imaginary quadratic extension $K/F$ other than $k$ whose relative class number $h_{K/F}$ is not divisible by $l$.

We denote by $\mathfrak{o}_k$ the ring of integers of $k$. Let $\alpha$ be an imaginary element of $\mathfrak{o}_k$ such that $N_{k/F}(\alpha)$ is prime to $l$. We put $\psi_n(\alpha)=(\alpha^{n+1}-(\alpha^\sigma)^{n+1})/(\alpha-\alpha^\sigma)$, where $\alpha^\sigma$ is the conjugate of $\alpha$ over $F$. We put $T(\alpha)=\alpha+\alpha^\sigma$ and $N(\alpha)=\alpha\alpha^\sigma$. We show that we may assume, by changing $\alpha$ if necessary, $\psi_n(\alpha)\not\equiv 0 \bmod \mathfrak{l}$ and $N(\alpha)\not\equiv 0 \bmod \mathfrak{l}$ for each prime ideal $\mathfrak{l}$ of $F$ over $l$. For $l\geqq 5$, we put $n=2$ in § 1. If $\psi_2(\alpha)\equiv 0 \bmod \mathfrak{l}$, we change $\alpha$ to $\alpha+x$ for an integer $x$ of $F$. We get $\psi_2(\alpha+x)\equiv 3x(x+T(\alpha))$ and $N(\alpha+x)\equiv x^2+T(\alpha)x+N(\alpha) \bmod \mathfrak{l}$. We see that there exists $x$

such that $\phi_2(\alpha+x)\not\equiv 0$ and $N(\alpha+x)\not\equiv 0 \bmod \mathfrak{l}$, because there exist at least 5 residue classes modulo $\mathfrak{l}$. For $l=3$, we put $n=4$ in §1. If $\phi_4(\alpha)\equiv 0 \bmod \mathfrak{l}$, we get $\phi_4(\alpha+x) \equiv -x^4+T(\alpha)x^3+(T(\alpha)^2-N(\alpha))x^2-T(\alpha)(T(\alpha)^2+N(\alpha))x$ and $N(\alpha+x) \equiv x^2+T(\alpha)x+N(\alpha) \bmod \mathfrak{l}$ for an integer $x$ of $F$. If the degree of $\mathfrak{l}$ is at least 2, there exists $x$ such that $\phi_4(\alpha+x)\not\equiv 0$ and $N(\alpha+x)\not\equiv 0 \bmod \mathfrak{l}$, because there exist at least 9 residue classes modulo $\mathfrak{l}$. If the degree of $\mathfrak{l}$ is 1, we see $N(\alpha)^2\equiv 1 \bmod \mathfrak{l}$. We get $T(\alpha)^4+1\equiv 0 \bmod \mathfrak{l}$ by $\phi_4(\alpha)\equiv T(\alpha)^4+N(\alpha)^2\equiv 0 \bmod \mathfrak{l}$. This is a contradiction. Moreover we can simultaneously take such $x$ for any $\mathfrak{l}$ over $l$.

Moreover we add a congruence condition modulo $\mathfrak{p}$ to $\alpha$. There exists $y\in\mathfrak{o}_k$ such that $y\not\equiv y^\sigma \bmod \mathfrak{P}^2$ for the prime ideal $\mathfrak{P}$ of $k$ over $\mathfrak{p}$ and the generator $\sigma$ of the Galois group of $k/F$, because $\mathfrak{p}$ is tamely ramified in $k/F$. We take $\alpha$ satisfying $\alpha\equiv y \bmod \mathfrak{P}^2$. Therefore we get $\alpha-\alpha^\sigma\equiv 0 \bmod \mathfrak{P}$ and $\alpha-\alpha^\sigma \not\equiv 0 \bmod \mathfrak{P}^2$. Thus we get $(\alpha-\alpha^\sigma)^2\equiv 0 \bmod \mathfrak{p}$ and $(\alpha-\alpha^\sigma)^2\not\equiv 0 \bmod \mathfrak{p}^2$. Considering Satz 4 and Lemma 10 of Prestel [**5**] and $(\alpha-\alpha^\sigma)^2=T(\alpha)^2-4N(\alpha)$, we see that the conductor of an order $\mathfrak{o}$ in $\mathfrak{o}_k$ containing $\alpha$ is not divisible by $\mathfrak{p}$. Since there exist finitely many orders $\mathfrak{o}$ in $\mathfrak{o}_k$ containing $\alpha$, we denote by $\mathfrak{r}_1, \cdots, \mathfrak{r}_s$ the prime divisors of the conductors of these orders. We have $\mathfrak{r}_i\neq\mathfrak{p}$ $(1\leqq i\leqq s)$. By class field theory we can take $\alpha$ such that $(\alpha)$ is a prime ideal of $k$ which splits completely in $k/\boldsymbol{Q}$. We put $N=N_{k/F}(\alpha)$. There exist finitely many algebraic integer $x+y\sqrt{-\delta'}$ with $x, y, \delta'\in F$ such that $x^2+\delta' y^2=N$ and $\delta'$ is totally positive. We denote by $\delta, \delta_1, \cdots, \delta_t$ these $\delta'$. At this time we take $\delta, \delta_1, \cdots, \delta_t$ such that $k=F(\sqrt{-\delta})$, $k_1=F(\sqrt{-\delta_1})$, $\cdots$, $k_t=F(\sqrt{-\delta_t})$ are different extensions. Let $\mathfrak{p}_i'$ be a prime ideal of $F$ which splits completely in $k_i/F$ and remains prime in $k/F$. We take a division quaternion algebra $B/F$ as follows:

(i) the only one real prime is unramified in $B/F$ and other real primes are ramified,

(ii) $\mathfrak{p}_1', \cdots, \mathfrak{p}_t'$ and $\mathfrak{r}_1, \cdots, \mathfrak{r}_s$ are ramified in $B/F$,

(iii) $\mathfrak{p}$ is unramified in $B/F$,

(iv) $\mathfrak{p}_l$ is ramified in $B/F$, if $n(l)>0$,

(v) the other prime ideals $\mathfrak{q}$ which are ramified in $B/F$ remain prime in $k/F$.

We take $S$ and $\rho$ as in §1, and consider $\mathfrak{S}(S, \rho)$. We consider the trace formula (1) in §2. We see that $k, k_1, \cdots, k_t$ are only totally imaginary quadratic extensions which contain algebraic integers whose norm to $F$ are equal to $N$. But no order in $k_1, \cdots, k_t$ appears in $\Omega_0$, because $\mathfrak{p}_1', \cdots, \mathfrak{p}_t'$ are ramified in $B/F$. Let $\mathfrak{o}_k$ be the ring of integers of $k$. Thus it is sufficient to consider the orders of $k$ for calculating tr. $\mathfrak{T}((N))$. Let $\mathfrak{o}_k$ be the ring of integers of $k$. No order of $k$ other than $\mathfrak{o}_k$ appears in $\Omega_0$ because $\mathfrak{r}_1, \cdots, \mathfrak{r}_s$ are ramified in $B/F$. By the assumption (i) and (ii) of Proposition, we get $E_k=E$. Therefore we get $J(\mathfrak{o}_k) =\{\alpha, \alpha^\sigma\}$. Thus we get:

$$\mathrm{tr.}\,\mathfrak{T}((N))=-\frac{[E:E^{+}]}{2}h_{k}\prod_{\mathfrak{p}\mid D(B/F)}\left(1-\left(\frac{\mathfrak{o}_{k}}{\mathfrak{p}}\right)\right)\sum_{\beta=\alpha,\alpha^{\sigma}}\mathrm{tr.}\,\Psi(\beta)\frac{\zeta_{\beta}^{n+1}-\eta_{\beta}^{n+1}}{\zeta_{\beta}-\eta_{\beta}}N^{-n/2}.$$

Let $\alpha=\alpha^{(1)}, \overline{\alpha^{(1)}}, \cdots, \alpha^{(m)}, \overline{\alpha^{(m)}}$ be the conjugates of $\alpha$ such that $\overline{\alpha^{(i)}}$ is the complex conjugation of $\alpha^{(i)}$ over a conjugate field $F^{(i)}$ of $F$. By the definition of the symmetric tensor representation, we get

$$\mathrm{tr.}\,\Psi(\alpha)\frac{\zeta_{\alpha}^{n+1}-\eta_{\alpha}^{n+1}}{\zeta_{\alpha}-\eta_{\alpha}}=\prod_{i=1}^{m}\frac{(\alpha^{(i)})^{n+1}-(\overline{\alpha^{(i)}})^{n+1}}{\alpha^{(i)}-\overline{\alpha^{(i)}}}\times\prod_{i=2}^{m}(N^{(i)})^{-n/2}$$

because of $\zeta_{\alpha}=\alpha^{(1)}$ and $\eta_{\alpha}=\overline{\alpha^{(1)}}$. So we see

$$\mathrm{tr.}\,\mathfrak{T}((N))=-[E:E^{+}]h_{k}\prod_{\mathfrak{p}\mid D(B/F)}\left(1-\left(\frac{\mathfrak{o}_{k}}{\mathfrak{p}}\right)\right)\mathrm{tr.}\,\Psi(\alpha)\frac{\zeta_{\alpha}^{n+1}-\eta_{\alpha}^{n+1}}{\zeta_{\alpha}-\eta_{\alpha}}N^{-n/2}.$$

By the congruence condition modulo $\mathfrak{l}$ of $\alpha$, we see that

$$\mathrm{tr.}\,\Psi(\alpha)\frac{\zeta_{\alpha}^{n+1}-\eta_{\alpha}^{n+1}}{\zeta_{\alpha}-\eta_{\alpha}}$$

is prime to $l$. By the assumption (iii), we get:

$$\mathrm{tr.}\,\mathfrak{T}((N))\equiv 0 \bmod l^{e_F} \quad\text{and}\quad \mathrm{tr.}\,\mathfrak{T}((N))\not\equiv 0 \bmod l^{e_F+1}.$$

We take $H_l$ and $M_l$ as in § 3. We see that $\mathfrak{p}$ is unramified in $M_l/F$, because $\mathfrak{p}$ does not divide $lD(B/F)$. We get $M_l\cap k=F$ by the assumption (ii) of Proposition. We take a prime ideal $\mathfrak{Q}$ of $F$ which decomposes in $M_l/F$ in the same manner as $(N)$, and remains prime in $k/F$. Thus there is no element $\beta$ of $k$ such that $(N_{k/F}(\beta))=\mathfrak{Q}$. So there appears no order of $k$ in the formula of $\mathrm{tr.}\,\mathfrak{T}(\mathfrak{Q})$. Because of $\mathrm{tr.}\,\mathfrak{T}(\mathfrak{Q})\equiv\mathrm{tr.}\mathfrak{T}((N))\not\equiv 0 \bmod l^{e_F+1}$, we see that there exists other totally imaginary quadratic extension $K$ whose relative class number $h_{K/F}$ is not divisible by $l$. Moreover $K$ contains no primitive $l$-th root of unity, because $\mathfrak{p}_l$ divides $D(B/F)$.

Next we prove that there exist infinitely many totally imaginary quadratic extensions over $F$ whose relative class numbers are not divisible by $l$. Let $k$, $K_1, \cdots, K_u$ be such quadratic extensions over $F$ which contain no primitive $l$-th root of unity. We take prime ideal $\mathfrak{q}_i$ of $F$ which splits completely in $K_i/F$ and remains prime in $k/F$ for each $1\leqq i\leqq u$. We take a division quaternion algebra $B/F$ satisfying (i)$\sim$(v) in which $\mathfrak{q}_i$ is ramified. Then we see by similar argument that there exist anothor totally imaginary quadratic extension over $F$ whose relative class number is not divisible by $l$. The proof of Proposition is complete.

Next we consider the numerical examples. We take $F=\boldsymbol{Q}(\sqrt{p})$, where $p\equiv 1 \bmod 4$ is a prime. We denote by $h(-q)$ (resp. $h(-pq)$) the class number of $\boldsymbol{Q}(\sqrt{-q})$ (resp. $\boldsymbol{Q}(\sqrt{-pq})$), where $q$ is a prime number. For $L=F(\sqrt{-q})=\boldsymbol{Q}(\sqrt{p}, \sqrt{-q})$, we see $h_{L/F}=h(-q)h(-pq)/2$. If $h(-q)$ and $h(-pq)$ are prime to $l$ and $q$ is not equal to 2, 3, $p$ nor $l$, then $L$ satisfies the assumption of Proposition. Using an electric computer we can find $q$ such that $h(-q)$ and $h(-pq)$ are prime to $l$ for $3\leqq l\leqq 47$ and $p\leqq 17389$, even if $w_F\zeta_F(-1)$ is divisible by $l$. There are 986 $p$'s. We write the number of $p$'s such that $l$ divides $w_{Q(\sqrt{p})}\zeta_{Q(\sqrt{p})}(-1)$ in the following table.

Table.

| $l$ | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 | 37 | 41 | 43 | 47 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| number of $p$ | 365 | 205 | 141 | 91 | 75 | 62 | 50 | 48 | 33 | 30 | 18 | 23 | 20 | 25 |

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