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A characterization of the association schemes of Hermitian forms

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Let $Y=(X, \{R_i\}_{0\leq i\leq d})$ be an association scheme whose parameters coincide with those of the association scheme $\mathrm{Her}(d, q)$ of Hermitian forms in d-dimensional space over the field $GF(q^2)$. Suppose that every edge of the distance-regular graph $\Gamma=(X, R_1)$ is contained in a clique of size q. It is shown that if $d\geqq 3$ then Y is isomorphic to $\mathrm{Her}(d, q)$. In the case $d=2$ a generalized quadrangle with the parameters (q, q^2) is reconstructed from Y.

1. Introduction.

The present paper is a continuation of [IS1] where the particular case $q=2$ was treated completely and some results concerning the general situation were proved. A detailed discussion of the schemes of Hermitian forms is contained in [BI], [BCN] and [IS1]. Here we give only the necessary definitions.

Let X be the set of all Hermitian forms (singular or nonsingular) in the space of dimension d over $GF(q^2)$ and $R_0, R_1, \cdots, R_d$ be the relations on X defined as follows

$$(x, y)\in R_i \quad \text{if and only if } \mathrm{rank}(x-y)=i,\ 0\leqq i\leqq d .$$

Then $Y=(X, \{R_i\}_{0\leq i\leq d})$ is a (P and Q)-polynomial association scheme known as the scheme of Hermitian forms $\mathrm{Her}(d, q)$. The distance-regular graph $\Gamma=(X, R_1)$ related to the scheme $\mathrm{Her}(d, q)$ has the following parameters:

$$\begin{aligned} b_i&=(q^{2d}-q^{2i})/(q+1), \\ c_i&=(q^{i-1}(q^i-(-1)^i))/(q+1), \\ a_i&=(q^{2i}-q^{i-1}(q^i-(-1)^i)-1)/(q+1). \end{aligned} \tag{1}$$

Apparently for the first time these facts were proved in [**Wan**].

The main result of the paper is the following.

THEOREM A. *Let Γ be a distance-regular graph of diameter $d\geqq 2$, whose* *parameters* b_i, c_i, a_i, $0 \leqq i \leqq d$ *satisfy the relations* (1) *for some integer* $q \geqq 2$. *Suppose that every edge of* Γ *is contained in a clique of size* q.

(i) *If* $d=2$ *then* Γ *is isomorphic to the subgraph induced by the vertices which are at distance* 2 *from a fixed vertex in the point graph of a generallized quadrangle with parameters* (q, q^2).

(ii) *If* $d \geqq 3$ *then* q *is a prime power and* Γ *is isomorphic to the graph related to the scheme* $Her(d, q)$.

REMARKS. Since $a_1=q-2$, the hypothesis of the theorem implies that every edge of Γ is in a unique clique of size q. It is not assumed in the theorem that q is a prime power. We obtain this fact as a corollary only in the case $d \geqq 3$. Besides the classical generalized quadrangles (corresponding to the graphs related to Her$(2, q)$), a number of other series with parameters (q, q^2) are known, see [**PT**]. In all examples q is a prime power, but up to our knowledge no proof exists that q must be so in all generalized quadrangles with parameters (q, q^2).

In what follows it is assumed that Γ is a distance-regular graph satisfying the hypothesis of Theorem A. In view of [**IS1**] we will assume that $q \geqq 3$, but the most part of our arguments are valid in the case $q=2$ as well. In order to simplify the notation we will not use a special symbol for the vertex set of Γ (as well as for all other graphs in the paper). So $x \in \Gamma$ means that x is a vertex of Γ.

It follows from the general theory [**BI**] that Γ can be considered as a set $\{x^* \mid x \in \Gamma\}$ of vectors in a k-dimensional vector space over $\boldsymbol{R}$ such that

$$\langle x^*, y^* \rangle = q_1(i) \qquad \text{if } d(x, y) = i\,.$$

Here $k=b_0$ is the valency of the graph Γ, $\langle\, ,\rangle$ is the usual inner product,

$$q_1(i)=((-q)^{2d-i}-1)/(q+1)\,, \qquad 0 \leqq i \leqq d\,,$$

and d is the usual distance function on Γ.

In Section 3 of [**IS1**] a number of propositions concerning Γ have been proved. Here we formulate those results in the following two lemmas. Recall that if $x \in \Gamma$ then $\Gamma_i(x)=\{z \mid z \in \Gamma,\ d(x, z)=i\}$. Let Δ, $\Sigma \subseteq \Gamma$ then $d(\Delta, \Sigma)= \min\{d(u, v) \mid u \in \Delta,\ v \in \Sigma\}$. Instead of $d(\{x\}, \Delta)$ we will write $d(x, \Delta)$. In addition $\Gamma_i(\Delta)=\{y \mid y \in \Gamma,\ d(y, \Delta)=i\}$ and Δ^* is the sum of all vectors u^* for $u \in \Delta$.

LEMMA 1.1. *Let* $x \in \Gamma$ *and* $\Gamma_1(x)=\{y_1, y_2, \cdots, y_k\}$ *be the set of all neighbours of* x *in* Γ. *Then the Gram matrix* $\|\langle y_i^*, y_j^* \rangle\|_{k \times k}$ *is nonsingular, i.e. the vectors* $y_1^*, y_2^*, \cdots, y_k^*$ *form a basis of* V. □

LEMMA 1.2. *Let* $x, y \in \Gamma$, $d(x, y)=i$, $1 \leqq i \leqq d$. *Then* x *and* y *are contained in a uniquely determined subgraph* $\Delta(x, y)$ *of* Γ. *Moreover the following asser-*

tions hold:

(i) $\Delta(x, y)$ *is distance-regular with parameters of the graph related to* $Her(i, q)$*;*

(ii) *if vertices* u, v *are contained in* $\Delta(x, y)$ *and* $d(u, v)=s$ *then each path of length* $t \leqq s+1$ *between* u *and* v *is contained in* $\Delta(x, y)$*; in particular* $\Delta(u, v) \subseteqq \Delta(x, y)$ *and* $\Delta(x, y)$ *is geodetically closed;*

(iii) *for a vertex* $x \in \Gamma$ *the set* $\pi(x)$ *of all subgraphs* $\Delta(x, y)$*, for* $y \in \Gamma_i(x)$*,* $1 \leqq i \leqq d-1$*, with the incidence relation defined by inclusion, form a projective space* $PG(d-1, q^2)$. □

In the graph related to Her(d, q) the subgraphs $\Delta(x, y)$ has the following interpretation. Let x be the null form and $y \in \Gamma_i(x)$. So y is a form of rank i and the radical rad(y) of the form has dimension $d-i$. Then $\Delta(x, y) = \{z \mid z \in \Gamma, \text{rad}(y) \subseteqq \text{rad}(z)\}$. In the general case these subgraphs can be defined in terms of the space V as follows. For $y \in \Gamma_i(x)$ let $V(x, y)$ be the subspace of V generated by the vectors from the set $\{x^*\} \cup \{z^* \mid d(z, x)=1, d(z, y) \leqq i\}$. Then $\Delta(x, y)$ is induced by the set of all vertices v such that $v^* \in V(x, y)$. Notice that if $d(x, y)=1$ then $\Delta(x, y)$ is the unique clique containing the edge $\{x, y\}$.

It is known ([**BCN**], [**IS1**]) that the distance-regular graph related to Her(d, q) is isomorphic to the subgraph induced by the vertices which are at the maximal distance from a fixed vertex in the dual polar space graph of ${}^2A_{2d-1}(q)$. If $d \geqq 3$ then the latter is characterized by its parameters, see [**BCN**], [**IS2**]. A graph with the parameters of the ${}^2A_3(q)$-graph is the point graph of type a generalized quadrangle with parameters (q, q^2) [**CGS**]. In view of this observation an approach to the characterization of Her(d, q) was proposed in [**IS1**]. This approach implies a reconstruction of the dual polar space graph from Γ. In the present paper we realize this approach. Namely, we reconstruct a generalized quadrangle in the case $d=2$ and the dual polar space of type ${}^2A_5(q)$ in the case $d=3$. It turns out that for $d>3$ it is possible to use some inductive arguments.

If we consider the representation of the graph of type ${}^2A_{2d-1}(q)$ in the eigenspace related to the exceptional Q-polynomial structure and restrict it on the graph related to Her(d, q) we obtain another representation of this graph. This representation exists also in the general situation and can be produced as follows.

Let $W = V \oplus V_0$ where V_0 is a 1-dimensional space generated by a vector e which is orthogonal to V, with $\langle e, e \rangle = 1$. For $x \in \Gamma$ let $\hat{x} = \alpha x^* + \beta e$ where $\alpha = (q+1)^{1/2}/q^d$ and $\beta = 1/q^d$. If $d(x, y)=i$ then $\langle \hat{x}, \hat{y} \rangle = (-q)^{-i}$. For a subgraph Δ of Γ let $\hat{\Delta}$ be the sum of the vectors $\hat{x}$ for all $x \in \Delta$.

2. The case $d=2$.

It was proved in [**Bo**] that in the point graph of a generalized quadrangle with parameters (q, q^2) the subgraph induced by the vertices which are at distance 2 from a fixed vertex is distance-regular with the parameters of the graph related to $\mathrm{Her}(2, q)$. The purpose of this section is to prove Theorem A (i), i.e. to reconstruct a generalized quadrangle with parameters (q, q^2) from a distance-regular graph with the parameters of $\mathrm{Her}(2, q)$.

2a. Some equalities.

Let $x\in\Gamma$ and $\{y_1, y_2, \cdots, y_k\}=\Gamma_1(x)$. It follows from our assumptions that the graph induced by $\Gamma_1(x)$ is a disjoint union of cliques. So we can assume that $y_{s(q-1)+i}$ is adjacent to $y_{s(q-1)+j}$ for $0\leqq s\leqq q^2$ and $1\leqq i<j\leqq q-1$.

Let us fix an orthonormal basis $\{e_1, \cdots, e_k\}$ in V. For $v\in V$ let v^i denotes the i-th coordinate of v in the basis $\{e_1, \cdots, e_k\}$. By symmetry we can assume that there are a, b and c such that

$$(y_i^*)^j=\begin{cases} a & \text{if } i=j\,, \\ b & \text{if } d(y_i, y_j)=1\,, \\ c & \text{otherwise.}\end{cases}$$

LEMMA 2.1.

$$(a-b)^2=q^3\,,$$
$$(a+(q-2)b-(q-1)c)^2=q^2\,.$$

PROOF. The inner products $\langle y_1^*, y_i^*\rangle$ for $i=1, 2$ and q are equal to $q_1(0)=q^3-q^2+q-1$, $q_1(1)=-q^2+q-1$ and $q_1(2)=q-1$ respectively. Evaluating these equalities in terms of a, b and c we obtain

$$a^2+(q-2)b^2+(q-1)q^2c^2=q_1(0)\,,$$
$$2ab+(q-3)b^2+(q-1)q^2c^2=q_1(1)\,,$$
$$2ac+2(q-2)bc+(q-1)(q^2-1)c^2=q_1(2)\,.$$

The equalities stated in the lemma are linear combinations of these equalities with coefficients $(1, -1, 0)$ and $(1, q-2, -q+1)$ respectively. □

Let $w\in\Gamma_2(x)$. Since every edge of Γ is contained in a unique clique there are no adjacent vertices in $\Gamma_1(x)\cap\Gamma_1(w)$. So by symmetry we can assume, that there are α, β and γ with

$$(w^*)^i=\begin{cases} \alpha & \text{if } d(w, y_i)=1\,, \\ \beta & \text{if } d(w, y_i)\neq 1 \text{ and } d(\Delta(x, y_i), w)=1\,, \\ \gamma & \text{otherwise.}\end{cases}$$

LEMMA 2.2.

$$(\alpha-\beta)(a-b) = -q^2,$$
$$(\alpha+(q-2)\beta-(q-1)\gamma)(a+(q-2)b-(q-1)c) = -q^2.$$

PROOF. Without loss of generality we can assume that $y_1 \in \Gamma_1(x) \cap \Gamma_1(w)$ and $y_i \notin \Gamma_1(x) \cap \Gamma_1(w)$ for $q \leqq i \leqq 2(q-1)$. Then w^* has the form

$$(\underbrace{\alpha, \beta, \cdots, \beta}_{q-1}, \underbrace{\gamma, \gamma, \cdots, \gamma}_{q-1}, \cdots).$$

The inner products $\langle w^*, y_i^*\rangle$ for $i=1$, 2 and q are equal to $q_1(1)$, $q_1(2)$ and $q_1(2)$ respectively. Evaluating these equalities and considering linear combinations of them with the same coefficients as in Lemma 2.1, we obtain the equalities stated in the present lemma. $\square$

COROLLARY 2.3.

$$(\alpha-\beta)^2 = q,$$
$$(\alpha+(q-2)\beta-(q-1)\gamma)^2 = q^2.$$

PROOF follows from Lemmas 2.1 and 2.2. $\square$

Now let $w, v \in \Gamma_2(x)$. The set of maximal cliques passing through a vertex y will be denoted by $Q(y)$. We introduce the following parameters. Set $n = \#\{y \in \Gamma_1(x) \mid d(y, v) = d(y, w) = 1\}$ and $m = \#\{\Sigma \in Q(x) \mid d(\Sigma, v) = d(\Sigma, w) = 1\}$. Corollary 2.3 enables us to express the inner product $\langle v^*, w^*\rangle$ in terms of m and n.

PROPOSITION 2.4. $\langle w^*, v^*\rangle = (n+m)q-(q-1)^2(q+1)$.

PROOF. By the parameters of Γ, among the cliques containing x there are exactly $m-q^2+2q+1$ cliques which contain no vertices adjacent to w or v and exactly $2(q^2-q-m)$ cliques which contain adjacent vertices only for one of the vertices w and v. So we obtain the equality

$$\langle w^*, v^*\rangle = (m-q^2+2q+1)(q-1)\gamma^2+2(q^2-q-m)\gamma(\alpha+(q-2)\beta)$$
$$+(m-n)(2\alpha\beta+(q-3)\beta^2)+n(\alpha^2+(q-2)\beta^2).$$

Expand the right side and separate the monoms depending on m, depending on n and depending neither on m nor on n. Then we come to the expression

$$\langle w^*, v^*\rangle = (m/(q-1))((\alpha+(q-2)\beta-(q-1)\gamma)^2-(\alpha-\beta)^2)+n(\alpha-\beta)^2+S,$$

where S stands for the part not depending on m and n. By Corrollary 2.3 this is the same as

$$\langle w^*, v^*\rangle = (m+n)q+S.$$

In order to calculate S apply this formula to the case $w=v$. In this case $m=n=q(q-1)$ and $\langle w^*, v^*\rangle=q_1(0)$. So $S=-(q-1)^2(q+1)$. □

Let us apply Proposition 2.4 to the case $d(v, w)=1$. If a vertex $y\in\Gamma_1(x)$ is adjacent to both v and w then $y\in\Delta(v, w)$. But the graph induced by $\Gamma_1(x)$ is a disjoint union of cliques. So such a vertex y is unique. Hence $n\leqq 1$ if $d(v, w)=1$.

We reformulate this fact in the following way. For $y\in\Gamma_2(x)$ let $\lambda(y)$ (respectively $\mu(y)$) denote the set of cliques $\Sigma\in Q(x)$ such that $d(\Sigma, y)=1$ (respectively $d(\Sigma, y)=2$). Since $\Gamma_1(x)\cap\Gamma_1(y)$ contains no adjacent vertices, we have $|\lambda(y)|=q(q-1)$ and $|\mu(y)|=q+1$. Finally $|\lambda(v)\cap\lambda(w)|=m$.

COROLLARY 2.5. *Let $d(v, w)=1$. Then either*

i) $|\Gamma_1(x)\cap\Gamma_1(v)\cap\Gamma_1(w)|=1$, $|\lambda(v)\cap\lambda(w)|=q^2-2q-1$ *and* $|\mu(v)\cap\mu(w)|=0$;

or

ii) $|\Gamma_1(x)\cap\Gamma_1(v)\cap\Gamma_1(w)|=0$, $|\lambda(v)\cap\lambda(w)|=q^2-2q$ *and* $|\mu(v)\cap\mu(w)|=1$.

PROOF. Since $d(v, w)=1$ we have $\langle v^*, w^*\rangle=q_1(1)$. So by Proposition 2.4 $m+n=q^2-2q$. □

2b. Classes of cliques.

Let Σ be a clique. If $y\in\Gamma_1(\Sigma)$ then Σ contains exactly one vertex adjacent to y. Hence we can calculate the cardinalities of $\Gamma_1(\Sigma)$ and $\Gamma_2(\Sigma)$. We have $|\Gamma_1(\Sigma)|=|\Sigma|\cdot(q-1)q^2=q^3(q-1)$ and $|\Gamma_2(\Sigma)|=|\Gamma|-|\Sigma|-|\Gamma_1(\Sigma)|=q(q^2-1)$.

LEMMA 2.6. *$\Gamma_2(\Sigma)$ is a disjoint union of cliques having size q and contains no other edges.*

PROOF. For $x\in\Gamma_2(\Sigma)$ we shall prove that in $Q(x)$ there is exactly one clique which lies in $\Gamma_2(\Sigma)$ and that any other clique in $Q(x)$ intersects $\Gamma_2(\Sigma)$ exactly in x.

If a clique $\Theta\in Q(x)$ belongs to all the sets $\lambda(v)$, $v\in\Sigma$, then each vertex of Σ is adjacent to some vertex from $\Theta-\{x\}$. But $|\Sigma|>|\Theta-\{x\}|$, so at least one vertex of Σ is adjacent to two vertices of Θ. This is impossible due to our assumption on cliques in Γ. So the sets $\mu(v)$, $v\in\Sigma$ cover $Q(x)$. If $v, w\in\Sigma$ then $\Delta(v, w)=\Sigma\subseteqq\Gamma_2(x)$ and hence $|\Gamma_1(x)\cap\Gamma_1(v)\cap\Gamma_1(w)|=0$. By Corollary 2.5 $|\mu(v)\cap\mu(w)|=1$. So we have a set $Q(x)$ of size q^2+1 covered by a family $\{\mu(v)\mid v\in\Sigma\}$ of q subsets, each of which has size $q+1$ and any two of which intersect exactly in one point. Now it is easy to see that all the sets $\mu(v)$, $v\in\Sigma$ have a clique Θ in common. By the definition $\Theta\subseteqq\Gamma_2(\Sigma)$. Each other clique $\Xi\in Q(x)$ lies in $\mu(v)$ for exactly one vertex $v\in\Xi$. Hence each vertex from $\Sigma-\{v\}$ is adjacent to a unique vertex from $\Xi-\{x\}$, i.e. $\Sigma-\{v\}$ and $\Xi-\{x\}$ are joined by a matching. □

A clique Θ is said to be *congruent* to a clique Σ if either $\Theta=\Sigma$ or $d(\Theta, \Sigma)=2$. By Lemma 2.6 the congruency is an equivalence relation. A clique Θ is said to be *adjacent* to a clique Σ if they are joined by a matching.

LEMMA 2.7. *Let Σ be a clique and $x\in\Gamma_1(\Sigma)$. Then there is exactly one clique in $Q(x)$ which lies in $\Gamma_1(\Sigma)$.*

PROOF. Let $\{y\}=\Sigma\cap\Gamma_1(x)$. If $v, w\in\Sigma-\{y\}$ then $\{y\}=\Gamma_1(x)\cap\Gamma_1(v)\cap\Gamma_1(w)$. By Corollary 2.5 $\mu(v)\cap\mu(w)=\emptyset$ and hence the union of $\mu(v)$ over all $v\in\Sigma-\{y\}$ covers q^2-1 cliques in $Q(x)$. Since $|Q(x)|=q^2+1$ there is just one clique $\Xi\in Q(x)-\{\Delta(x, y)\}$ which belongs to all the sets $\lambda(v)$, $v\in\Sigma-\{y\}$. $\square$

It is easy to see that the cliques Ξ and Σ in the above proof are adjacent. A set $\mathcal{S}$ of cliques in Γ will be called a *spread* if the following conditions hold

a) for each vertex $v\in\Gamma$ there is exactly one clique $\Sigma=\Sigma(v)\in\mathcal{S}$ such that $v\in\Sigma$,

b) if x, y are adjacent vertices of Γ then either $\Sigma(x)=\Sigma(y)$ or $\Sigma(x)$ is adjacent to $\Sigma(y)$.

For a clique Σ let $\mathcal{S}(\Sigma)$ denote the set of all cliques in Γ which lie in $\Gamma_i(\Sigma)$ for some $i=0, 1$ or 2.

PROPOSITION 2.8. *The set $\mathcal{S}(\Sigma)$ forms a spread. Moreover if $\mathcal{S}$ is a spread and $\Sigma\in\mathcal{S}$ then $\mathcal{S}=\mathcal{S}(\Sigma)$.*

PROOF. By Lemmas 2.6 and 2.7 cliques in $\mathcal{S}(\Sigma)$ are disjoint and they cover all vertices of Γ. For $i=1, 2$, let $v_i\in\Theta_i\in\mathcal{S}(\Sigma)$ and let v_1 and v_2 be adjacent. If $v_1\in\Sigma\cup\Gamma_2(\Sigma)$ then $\Gamma_1(\Theta_1)=\Gamma_1(\Sigma)$. So either $\Theta_2=\Theta_1$ or Θ_2 is adjacent to Θ_1. Now let $v_1, v_2\in\Gamma_1(\Sigma)$. By the definition $\Theta_1\cap\Theta_2=\emptyset$. If Θ_2 intersects $\Gamma_2(\Theta_1)$ then there exists a clique Ξ congruent to Θ_1 such that $\Xi\cap\Theta_2\neq\emptyset$. But $\Gamma_1(\Theta_1)=\Gamma_1(\Xi)$. Hence Ξ is adjacent to Σ. Now Lemma 2.7 implies that $\Xi=\Theta_2$ The contradiction proves the first claim of the lemma.

Let $\mathcal{S}$ be a spread. By Lemma 2.7 if a clique $\Sigma\in\mathcal{S}$ and clique Ξ is adjacent to Σ then $\Xi\in\mathcal{S}$. Hence if $\Sigma\in S$ then $\mathcal{S}(\Sigma)\subseteq\mathcal{S}$. Since cliques from $\mathcal{S}(\Sigma)$ cover Γ we have $\mathcal{S}=\mathcal{S}(\Sigma)$. $\square$

2c. Generalized quadrangle.

Now we start with the reconstruction of a generalized quadrangle from Γ. Let $C=\{c_1, c_2, \cdots, c_s\}$ be the set of all classes of congruent cliques. Let us construct a graph $\tilde{\Gamma}$ with the vertex set $\{g\}\cup C\cup\Gamma$ where g is an additional vertex. The adjacency is defined as follows:

(a) $x, y\in\Gamma$ are adjacent if and only if they are adjacent in Γ;

(b) a vertex c_i is adjacent to a vertex $x\in\Gamma$ if and only if x is contained in a clique from c_i;

(c) $c_i, c_j \in C$ for $i \neq j$ are adjacent if and only if c_i and c_j are contained in the same spread;

(d) $\tilde{\Gamma}_1(g)=C$.

By Proposition 2.8 the subgraph in $\tilde{\Gamma}$ induced by C is a disjoint union of cliques. The size of any clique in C is equal to the number of classes of congruent cliques in the corresponding spread. If Σ is a clique in Γ then its congruency class covers $|\Sigma \cup \Gamma_2(\Sigma)|=q^3$ vertices. So each spread contains exactly q classes. Spreads in Γ are in a correspondence with cliques in Γ containing a fixed vertex. Now it is easy to see that any vertex of $\tilde{\Gamma}$ is adjacent to exactly $q(q^2+1)$ vertices of $\tilde{\Gamma}$. Let us introduce two sets of cliques of size $q+1$ in $\tilde{\Gamma}$. Let P_1 (respectively P_2) be the set of all cliques in $\tilde{\Gamma}$ having the form $\Sigma \cup \{x\}$ where Σ is a clique in Γ (respectively in C) and x is the congruency class containing Σ (respectively $x=g$). Let $P=P_1 \cup P_2$.

LEMMA 2.9. *If x is a vertex of $\tilde{\Gamma}$ and $\Sigma \in P$ then $d(\Sigma, x) \leqq 1$. Moreover, there is exactly one vertex $y \in \Sigma$ such that $d(y, x)=d(\Sigma, x)$.*

PROOF. Let $\Sigma \in P_1$ and $c \in \Sigma \cap C$. If $x \in \Gamma$ then $d(\Sigma, x) \leqq 1$ since all vertices from $\Gamma_2(\Sigma-\{c\})$ are covered by cliques from c. If $c \neq c' \in C$ then there is a clique in c' which intersects $\Sigma-\{c\}$. So $d(\Sigma, c')=1$. Finally $d(g, c)=1$.

Now let $\Sigma \in P_2$. Since classes from $\Sigma-\{g\}$ form a spread, any vertex of Γ is at distance 1 from Σ. Any other vertex of $\tilde{\Gamma}$ is at distance 1 or 0 from $g \in \Sigma$. So in any case if $\Sigma \in P$ and x is a vertex of $\tilde{\Gamma}$ then $d(\Sigma, x) \leqq 1$. Now the equality $|\tilde{\Gamma}|=1+q(q^2+1)+q^4=(q+1)+(q+1)\cdot q\cdot q^2$ shows that any vertex in $\Gamma_1(\Sigma)$ is adjacent to only one vertex of Σ. □

PROPOSITION 2.10. *$\tilde{\Gamma}$ is the point graph of a generalized quadrangle with the parameters (q, q^2).*

PROOF. If is easy to see that any edge of $\tilde{\Gamma}$ is contained in some clique from P. So by Lemma 2.9 P is the set of all cliques of $\tilde{\Gamma}$.

Now let the vertices of $\tilde{\Gamma}$ and the cliques from P be the elements of a rank 2 geometry $\mathcal{G}$ and let the incidence on $\mathcal{G}$ be defined by inclusion. Since any cycle of length 3 in $\tilde{\Gamma}$ is contained in a clique from P, the girth of $\mathcal{G}$ is at least 8. By Lemma 2.9 the diameter of $\mathcal{G}$ is four. So $\mathcal{G}$ is a generalized quadrangle. □

It is easy to see that Proposition 2.10 implies Theorem A (i).

2d. Other possibilities.

Let us construct a code from Γ. The codewords will be in a correspondence with the vertices of Γ while places in the codewords will correspond to spreads.

Let us mark the congruency classes from each spread by the integers from 1 to q. The i-th place of the codeword corresponding to a vertex $v\in\Gamma$ contains the number of the class covering $v\in\Gamma$ in the spread number i. For $x\in\Gamma$ let $\tilde{x}$ be the codeword corresponding to x, and for $x, y\in\Gamma$ let $\delta(\tilde{x}, \tilde{y})$ denote the Hamming distance between the codewords, i.e. the number of nonequal coordinates. Then the following lemma holds.

LEMMA 2.11. *Let $x, y\in\Gamma$. Then*

$$\delta(\tilde{x}, \tilde{y}) = \begin{cases} 0, & \text{if } x=y, \\ q^2, & \text{if } d(x, y)=1, \\ q^2-q, & \text{if } d(x, y)=2. \quad \square \end{cases}$$

It follows from Lemma 2.11 that the weight enumerator of the constructed code is the following:

$$|\Gamma|^{-1}\sum_{x,y\in\Gamma} z^{\delta(\tilde{x},\tilde{y})} = 1+(q^2-q)(q^2+1)z^{q^2-q}+(q-1)(q^2+1)z^{q^2}.$$

By remark 8.2 in [**CGS**] a code with this enumerator is an orthogonal array of strength 3, index q, length of codewords q^2+1 and nonzero distances q^2 and q^2-q. In the same paper it is shown that existence of such an array is equivalent to existence of a generalized quadrangle with parameters (q, q^2).

At the end of the section let us show how the generalized quadrangle $\mathcal{Q}$ can be reconstructed just in its natural representation as a set of vectors in an eigenspace of the corresponding association scheme.

LEMMA 2.12. *Let Σ and Ξ be cliques of size q in Γ. Then $\Sigma^*=\Xi^*$ (equivalently $\hat{\Sigma}=\hat{\Xi}$) if and only if Σ and Ξ are congruent cliques.*

PROOF. From the structure of cliques in Γ it follows that Σ and Ξ have at most one vertex in common. Keeping this fact in mind one can check the lemma by direct calculation of $\langle\Sigma^*, \Sigma^*\rangle$ and $\langle\Sigma^*, \Xi^*\rangle$. $\square$

REMARK. Let Γ satisfy the hypothesis of Theorem A for some $d\geqq 2$. Since the inner product $\langle\hat{x}, \hat{y}\rangle$ does not depend on d, the claim analogous to Lemma 2.12 is valid for cliques in a subgraph $\Delta(x, y)$ of Γ for $d(x, y)=2$.

Let us define the following set of vectors in the space W. For a clique Σ in Γ put $w_\Sigma=-\hat{\Sigma}$. Due to Lemma 2.12 we can write w_c instead of w_Σ where c is the congruency class containing Σ. Put $w_g=e$. Notice that $w_g=\hat{\Gamma}/q^2$. Finally for $x\in\Gamma$ put $w_x=\hat{x}$.

LEMMA 2.13. *Let x, y be any vertices of $\tilde{\Gamma}$ and $i=d(x, y)$. Then $\langle w_x, w_y\rangle =(-q)^{-i}$.*

PROOF. Direct calculation. □

It is easy to see that we have obtained the desired representation of $\tilde{\Gamma}$.

3. The case $d=3$.

Now let us turn to the case $d=3$.

By Lemma 1.2 (i) and (ii) any pair of vertices $x, y \in \Gamma$ which are at distance 2 in Γ lie in a unique subgraph $\Delta(x, y)$. This subgraph is distance-regular and its parameters coincide with that of Her $(2, q)$. Such a subgraph will be called a *prequad* since by Proposition 2.10 it is isomorphic to the graph induced by the set of vertices which are at the maximal distance from a fixed vertex in the point graph of some generalized quadrangle. By Lemma 1.2 (iii) the set $Q(x)$ of cliques and the set $P(x)$ of prequads containing a fixed vertex x in Γ form a projective plane $\pi(x)$ of order q^2.

If Δ, Ξ are distinct prequads having a vertex in common, then by Lemma 1.2 (ii) $\Delta \cap \Xi$ is a connected graph. Since $\pi(x)$ is a projective plane for $x \in \Delta \cap \Xi$, we have that $\Delta \cap \Xi$ is a clique.

3a. Classes of prequads.

In this subsection Δ denotes a fixed prequad in Γ. Let us consider the decomposition of Γ with respect to Δ. The following lemma is very useful.

LEMMA 3.1. *Let a prequad Ξ intersect Δ and $x \in \Xi$. Then $d(\Delta, x) = d(\Delta \cap \Xi, x)$.*

PROOF. Put $\Sigma = \Delta \cap \Xi$. If $d(\Delta, x) < d(\Sigma, x)$ then $d(\Delta, x) = 1$ and $d(\Sigma, x) = 2$. Let $y \in \Sigma$, $z \in \Delta$ such that $d(z, x) = 1$. There is a path of length at most 3 passing from x to y through z. Since $d(x, y) = 2$, Lemma 1.2 (ii) implies $z \in \Sigma$. □

The prequad Δ contains q^4 vertices and its valency is $(q-1)(q^2+1)$. By Lemma 1.2 (ii) each vertex from $\Gamma_1(\Delta)$ is adjacent to exactly one vertex in Δ. So $|\Gamma_1(\Delta)| = q^4 \cdot (q-1)q^4 = q^9 - q^8$.

LEMMA 3.2. *Let $x \in \Delta$, $y \in \Gamma_1(\Delta) \cap \Gamma_1(x)$. Then there is a bijection φ between the cliques in $\Gamma_1(\Delta)$ containing y and the cliques in Δ containing x. If $\Sigma = \varphi(\Theta)$ then Σ and Θ are joined by a matching.*

PROOF. Let Ξ be a prequad containing x and y. Then $\Xi \cap \Delta$ is a clique. By Lemmas 3.1 and 2.7 there is just one clique in $\Xi \cap \Gamma_1(\Delta)$ containing y. So we have the desired bijection. □

By the above lemma if $y \in \Gamma_1(\Delta)$ then $Q(y)$ contains exactly one clique Θ intersecting Δ and exactly q^2+1 cliques from $\Gamma_1(\Delta)$. Let Σ be any other clique in $Q(y)$. Then Σ intersects $\Gamma_2(\Delta)$. Let Ξ be the prequad passing through Σ and Θ. Then by Lemma 1.2 (iii) $\Xi\cap\Delta$ is a clique. On the other hand $y\in\Sigma$ and $d(y, \Xi\cap\Delta)=1$. By Lemma 2.6 this implies that $|\Sigma\cap\Gamma_2(\Xi\cap\Delta)|=1$. Hence $|\Sigma\cap\Gamma_2(\Delta)|=1$ and $|\Gamma_1(y)\cap\Gamma_2(\Delta)|=q^4-1$.

Now to calculate the cardinality of $\Gamma_2(\Delta)$ we should determine the number of vertices from $\Gamma_1(\Delta)$ adjacent to a fixed vertex $y\in\Gamma_2(\Delta)$. As it was proved above, if $\Sigma\in Q(y)$ and $\Sigma\cap\Gamma_1(\Delta)\neq\emptyset$ then $\Sigma\cap\Gamma_2(\Delta)=\{y\}$. Let $\lambda(y)$ (respectively $\mu(y)$) denote the set of all cliques $\Sigma\in Q(y)$ having nonempty (respectively empty) intersection with $\Gamma_1(\Delta)$. Finally, let $\nu(y)$ denote the set of all prequads from $P(y)$ intersecting Δ.

We start with two trivial remarks concerning these sets.

(A) *If $\Sigma\in\lambda(y)$ then Σ lies in exactly one prequad from $\nu(y)$.* Surely, by Lemma 3.1 such a prequad contains $\Delta\cap\Gamma_1(\Sigma)$.

(B) *If $\Xi\in\nu(y)$ then Ξ contains exactly q^2 cliques from $\lambda(y)$.* This fact is due to Lemmas 3.1 and 2.6.

Since $\pi(y)$ is a projective plane of order q^2 then by (A) and (B) all prequads from $\nu(y)$ have a clique $\Theta_y\in\mu(y)$ in common. If $\Xi\in\nu(y)$ then by Lemma 2.6 Θ_y is congruent in Ξ to the clique $\Sigma=\Xi\cap\Delta$. By the remark after Lemma 2.12 $\hat{\Theta}_y=\hat{\Sigma}$. So if Ξ_1 is another prequad from $\nu(y)$ and $\Sigma_1=\Xi_1\cap\Delta$ then $\hat{\Sigma}_1=\hat{\Sigma}$. Now the remark after Lemma 2.12 implies that Σ and Σ_1 are congruent in Δ. Each congruency class in a prequad consists of q^2 cliques. Hence $|\nu(y)|\leqq q^2$ and by (A) and (B) y is adjacent to at most $(q-1)q^4$ vertices from $\Gamma_1(\Delta)$.

Now we are in a position to calculate the cardinality of $\Gamma_2(\Delta)$. On one hand $|\Gamma_2(\Delta)|\geqq|\Gamma_1(\Delta)|\cdot(q^4-1)/((q-1)q^4)=q^8-q^4$. On the other hand $|\Gamma_2(\Delta)|\leqq |\Gamma|-|\Delta|-|\Gamma_1(\Delta)|=q^9-q^4-(q^9-q^8)=q^8-q^4$. So $|\Gamma_2(\Delta)|=q^8-q^4$. In particular for each vertex y of Γ the inequality $d(\Delta, y)\leqq 2$ holds. In particular each clique from $\mu(y)$ is contained in $\Gamma_2(\Delta)$. Another consequence is the following.

LEMMA 3.3. *If $y\in\Gamma_2(\Delta)$ then $|\nu(y)|=q^2$. Moreover, the subgraph induced by $\Delta\cap\Gamma_2(y)$ is a disjoint union of cliques which form a congruency class in Δ.* □

REMARK. Lemma 3.3 enables us to calculate $\langle\Delta^*, z^*\rangle$ for $z\in\Gamma_2(\Delta)$. A direct calculation shows that $\langle\Delta^*, z^*\rangle$ coincides with $\langle\Delta^*, u^*\rangle$ for $u\in\Delta$. On the other hand if $z\in\Gamma_1(\Delta)$ and $\{t\}=\Delta\cap\Gamma_1(z)$ then by Lemma 1.2 (ii) for any $s\in\Delta$ we have $d(z, s)=d(t, s)+1$. Now it is straightforward to check that $\langle\Delta^*, z^*\rangle=-q^4\neq\langle\Delta^*, t^*\rangle=q^4(q-1)$.

By Lemma 3.3 $|\lambda(y)|=q^4$ and $|\mu(y)|=q^2+1$. The point Θ_y of the projective plane $\pi(y)$ is contained in exactly q^2+1 lines. Since $|\nu(y)|=q^2$, cliques from $\mu(y)$ form a line in $\pi(y)$. So there is a prequad Δ_y such that for a clique Σ we have $\Sigma\in\mu(y)$ if and only if $y\in\Sigma\subseteqq\Delta_y$.

PROPOSITION 3.4. *If y, z are two adjacent vertices in $\Gamma_2(\Delta)$ then $\Delta_y=\Delta_z$. In other words, $\Gamma_2(\Delta)$ is a disjoint union of prequads.*

PROOF. Let Ξ be a prequad and t be a vertex of Ξ. Put $A=\Gamma_1(t)$ and $B=\Xi_1(t)$. By Lemma 1.1 $\{x^* \mid x\in A\}$ is a basis of V. So the vector Ξ^* is a linear combination of vectors in the basis. Since for $x\in A$ the inner product $\langle\Xi^*, x^*\rangle$ depends only on whether $x\in B$ or $x\notin B$, one can see that $\Xi^*=\alpha A^*+\beta B^*$ for some α and β. By the analogous reason $t^*=\gamma A^*$ for some γ. Hence

$$\Xi^*=\delta t^*+\beta B^*$$

for appropriate δ and β. Notice that α, β, γ and δ do not depend on the particular choice of t and Ξ. In addition it is easy to see that $\beta\neq 0$.

Let us apply this formula to Δ_y with respect to vertices y and z and then calculate $\langle\Delta^*, \Delta_y^*\rangle$. By the remark after Lemma 3.3 the value $\langle\Delta^*, s^*\rangle$ depends on the parity of $i=d(\Delta, s)$. Hence any vertex adjacent to z in Δ_y lies in $\Gamma_2(\Delta)$ and $\Delta_y=\Delta_z$. So we have proved that Δ_y is the unique prequad that is contained in $\Gamma_2(x)$ and contains $\{y\}$. □

A prequad Ξ is said to be *congruent* to a prequad Δ if either $\Xi=\Delta$ or $d(\Xi, \Delta)=2$. By Proposition 3.4 congruency of prequads is an equivalence relation. It follows from a direct calculation that for prequads Ξ, Θ we have $\Xi^*=\Theta^*$ if and only if Ξ and Θ are congruent. So the equality of vectors is another way to define the notion of congruency of prequads.

LEMMA 3.5. *If $y\in\Gamma_1(\Delta)$ then there is exactly one prequad Ξ which is contained in $\Gamma_1(\Delta)$ and contains y. Moreover there is a matching between Δ and Ξ which determines an isomorphism of Δ and Ξ.*

PROOF. Let us calculate the number of prequads intersecting $D=\Delta\cup\Gamma_2(\Delta)$. By Lemma 3.1 if Ξ is such a prequad then either $\Xi\subseteq D$ or $\Xi\cap D$ is a disjoint union of q^2 cliques. Any prequad contains exactly $q^3(q^2+1)$ cliques. Any clique from D is contained in exactly q^2 prequads intersecting $\Gamma_1(\Delta)$. So the number of prequads intersecting D is $q^4+q^4\cdot q^3(q^2+1)\cdot q^2/q^2=q^9+q^7+q^4$. The total number of prequads in Γ is $q^9\cdot(q^4+q^2+1)/q^4=q^5(q^4+q^2+1)$. So there are exactly q^5-q^4 prequads in $\Gamma_1(\Delta)$. By Lemma 3.2 for any vertex $y\in\Gamma_1(\Delta)$ there is at most one prequad Ξ in $\Gamma_1(\Delta)$ which contains y. Since $|\Gamma_1(\Delta)|=q^9-q^8=(q^5-q^4)q^4$, such a prequad Ξ exists. □

If prequads Δ and Ξ are joined by a matching, then these prequads will be called *adjacent*.

Now let us generalize the notion of spread introduced in Section 2. A set $\mathcal{S}$ of cliques (respectively, prequads) in Γ will be called 1-*spread* (respectively, 2-*spread*) if the following conditions hold:

a) for each vertex $v \in \Gamma$ there is exactly one clique (prequad) $\Sigma = \Sigma(v) \in \mathcal{S}$ such that $v \in \Sigma$;

b) if x, y are adjacent vertices of Γ then either $\Sigma(x) = \Sigma(y)$ or $\Sigma(x)$ is adjacent to $\Sigma(y)$.

For a prequad Ξ, let $\mathcal{S}_2(\Xi)$ denote the set of all prequads in Γ which lie in $\Gamma_i(\Xi)$ for $i=0, 1$ or 2.

PROPOSITION 3.6. *The set $\mathcal{S}_2(\Xi)$ forms a 2-spread. Moreover, if $\mathcal{S}$ is a 2-spread and $\Xi \in \mathcal{S}$ then $\mathcal{S} = \mathcal{S}_2(\Xi)$.*

PROOF. Repeat that of Proposition 2.8. □

By the above definition and Proposition 3.6 2-spreads are in a bijection with the prequads containing a fixed vertex $x \in \Gamma$. So there are exactly q^4+q^2+1 2-spreads.

3b. Classes of cliques.

Using Proposition 3.6 it is easy to describe all 1-spreads in Γ. For a clique Σ let $\mathcal{S} = \mathcal{S}_1(\Sigma)$ be the minimal set of cliques in Γ such that

a) $\Sigma \in \mathcal{S}$, and

b) if Θ_1, Θ_2 are adjacent cliques and $\Theta_1 \in \mathcal{S}$ then $\Theta_2 \in \mathcal{S}$.

PROPOSITION 3.7. *$\mathcal{S}_1(\Sigma)$ is a 1-spread and each 1-spread in Γ is of this type.*

PROOF. We should show only that for any vertex $x \in \Gamma$ there is exactly one clique in $\mathcal{S}_1(\Sigma)$ containing x. For a clique Θ let $\mathcal{T}(\Theta)$ be the set of 2-spreads containing prequads which contain Θ. If Θ_1 and Θ_2 are adjacent cliques and Δ is a prequad containing Θ_1 then either $\Theta_2 \subset \Delta$ or $\Theta_2 \subseteq \Gamma_1(\Delta)$. By Lemmas 3.2 and 3.5 in either case there is a prequad Ξ adjacent to Δ which contains Θ_2. Since $\mathcal{S}_2(\Xi) = \mathcal{S}_2(\Delta)$ we have proved that $\mathcal{T}(\Theta_1) = \mathcal{T}(\Theta_2)$. Now by transitivity if Θ_2 is a clique from $\mathcal{S}(\Theta_1)$ then $\mathcal{T}(\Theta_1) = \mathcal{T}(\Theta_2)$ too.

On the other hand it is easy to see that if Θ_1 and Θ_2 are intersecting cliques then either $\Theta_1 = \Theta_2$ or $\mathcal{T}(\Theta_1) \neq \mathcal{T}(\Theta_2)$. So for any vertex $x \in \Gamma$ there is exactly one clique in $\mathcal{S}(\Sigma)$ containing x. □

By Proposition 3.7 1-spreads are in a bijection with the cliques containing a fixed vertex of Γ. So there are exactly q^4+q^2+1 1-spreads.

Let us discuss the notion of congruency of cliques in Γ. Till now we have used this notion only in the sense of "congruency in a prequad". Let us prove that this congruency is an equivalence relation on the set of all cliques in Γ.

LEMMA 3.8. *If Θ_1, Θ_2 are congruent in a prequad Δ and Θ_2, Θ_3 are congruent in a prequad Ξ then there is a prequad containing Θ_1, Θ_3 and these cliques* *are congruent in the prequad.*

PROOF. Without loss of generality we assume that $\Xi \neq \Delta$. If $y \in \Theta_3$ it follows from Lemma 3.1 that $\Theta_3 = \Theta_y$, i.e. Θ_3 is the clique which is common for all the prequads from $\nu(y)$ (see definitions before Lemma 3.3). Since Θ_1 and Θ_2 are congruent in Δ, by Lemma 3.3 there is a prequad in $\nu(y)$ which contains Θ_1 and Θ_3. □

Now let us study the relations between 1- and 2-spreads. Let $\mathcal{S}_i$ be an i-spread, $i=1, 2$. Let Σ be a clique from $\mathcal{S}_1$.

LEMMA 3.9. *If $\Sigma \subset \Xi$ for some prequad Ξ from $\mathcal{S}_2$ then any clique from $\mathcal{S}_1$ is contained in a prequad from $\mathcal{S}_2$. Moreover, if Σ_1, $\Sigma_2 \in \mathcal{S}_1$ are adjacent (congruent) then the prequads from $\mathcal{S}_2$ which contain them are also adjacent (congruent).*

PROOF. Let $\Sigma \subset \Xi$ and Θ be a clique adjacent to Σ. By Lemma 3.1 either $\Theta \subset \Xi$ or $\Theta \subset \Gamma_1(\Xi)$. In the latter case by Lemmas 3.2 and 3.5 Θ is contained in a prequad adjacent to Ξ. By connectivity any clique from $\mathcal{S}_1$ is contained in a prequad from $\mathcal{S}_2$.

If Θ is congruent to Σ then by Lemma 3.1 either $\Theta \subset \Xi$ or $\Theta \subset \Gamma_2(\Xi)$. If the latter holds then by Proposition 3.4 Θ is contained in a prequad which is congruent to Ξ. □

It follows in particular from this lemma that if Σ is not contained in a prequad from $\mathcal{S}_2$ then this is true for any other clique from $\mathcal{S}_1$.

LEMMA 3.10. *Let Σ be a clique intersecting a prequad $\Xi \in \mathcal{S}_2$ and Σ do not lie in Ξ. Let $\mathcal{A}$ be the set of all cliques from $\mathcal{S}_1 = \mathcal{S}_1(\Sigma)$ intersecting Ξ. Then*

(a) *no cliques from $\mathcal{A}$ are congruent,*

(b) *there is mapping φ from $\mathcal{S}_1$ onto $\mathcal{A}$ such that if Θ_1, Θ_2 are adjacent cliques from $\mathcal{S}_1$ then Θ_1 is congruent to $\varphi(\Theta_1)$ and $\varphi(\Theta_1)$ is adjacent to $\varphi(\Theta_2)$.*

PROOF. Suppose that cliques Σ_1 and Σ_2 from $\mathcal{A}$ are congruent. Then by Lemma 3.8 these cliques lie in a prequad Δ. Now since $d(\Sigma_1, \Sigma_2)=2$ it is easy to see that $\Xi \cap \Delta$ contains a pair of vertices at distance 2; a contradiction.

We claim that a clique from $\mathcal{S}_1$ is congruent to a unique clique from $\mathcal{A}$. The total number of cliques in $\mathcal{S}_1$ is $|\Gamma|/q = q^8$. A clique $\Theta \in \mathcal{S}_1$ is contained in q^2+1 prequads; in such a prequad, Θ is congruent to q^2-1 cliques distinct from Θ and by Proposition 3.7 all these cliques are contained in $\mathcal{S}_1$. Since any two distinct prequads have at most one clique in common, we conclude that Θ is congruent to q^4 cliques from $\mathcal{S}_1$. On the other hand $|\mathcal{A}| = |\Xi| = q^4$, so the claim follows by (a). For a clique $\Theta \in \mathcal{S}_1$ let $\varphi(\Theta)$ denote the unique clique from $\mathcal{A}$ which is congruent to Θ.

Now we should show that if Θ_1 and Θ_2 are adjacent cliques from $\mathcal{S}_1$ then $\varphi(\Theta_1)$ and $\varphi(\Theta_2)$ are adjacent. By Lemma 3.8 there is a prequad Ξ_1 which contains Θ_1 and $\varphi(\Theta_1)$. By Lemma 3.9 Θ_2 is contained either in Ξ_1 or in a prequad Ξ_2 adjacent to Ξ_1. In either case there is a clique Φ which is congruent to Θ_2 and adjacent to $\varphi(\Theta_1)$. Let Δ be a prequad which contains $\varphi(\Theta_1)$ and Φ. Since Φ and $\Xi \cap \Delta$ lie in distinct spreads in Δ there is a clique Θ_3 which is congruent to Φ (so Θ_3 is also adjacent to $\varphi(\Theta_1)$) and intersects $\Xi \cap \Delta$. By the previous paragraph $\Theta_3 = \varphi(\Theta_2)$. □

3c. The dual polar space graph.

Let us construct a graph $\tilde{\Gamma}$ in the following way. The set of vertices of $\tilde{\Gamma}$ is $\{g\} \cup P \cup Q \cup \Gamma$, where P (respectively, Q) is the set of all congruency classes of prequads (respectively, cliques) and g is an additional vertex. The adjacency in $\tilde{\Gamma}$ is defined by the following:

a) $\tilde{\Gamma}_1(g) = P$,

b) two classes of prequads (respectively, cliques) are adjacent if and only if they contain two adjacent prequads (respectively, cliques),

c) a class of cliques is adjacent to a class of prequads if there is a clique Σ and a prequad Ξ in these classes such that $\Sigma \subset \Xi$,

d) a class C of cliques is adjacent to a vertex $x \in \Gamma$ if and only if there is a clique in C which contains x,

e) the adjacency on Γ is the same as above.

It follows directly from the definition that $\tilde{\Gamma}_i(g)$ for $i=1, 2, 3$ coincides with P, Q, Γ respectively.

Let us study the structure of the subgraphs of $\tilde{\Gamma}$ induced by P and Q. It is easy to see that each 2-spread determines a clique of size q in $\tilde{\Gamma}_1(g)$ and all these q^4+q^2+1 cliques are disjoint. In particular $|\tilde{\Gamma}_1(g)| = q(q^4+q^2+1)$. By Lemma 3.10 a connected component of $\tilde{\Gamma}_2(g)$ corresponds to some 1-spread and is isomorphic to a prequad of Γ. Notice that this implies in particular that all prequads in Γ are isomorphic but we will not make use of this fact. Since there are q^4+q^2+1 1-spreads in Γ we have $|\tilde{\Gamma}_2(g)| = q^4(q^4+q^2+1)$. Hence $|\tilde{\Gamma}| = (q+1)(q^3+1)(q^5+1)$.

LEMMA 3.11. *Any edge in $\tilde{\Gamma}$ is contained in a unique clique. Any clique in $\tilde{\Gamma}$ has size $q+1$ and it is the union of a clique from $\tilde{\Gamma}_i(g)$ for some $i=1, 2$ or 3 and of a vertex from $\tilde{\Gamma}_{i-1}(g)$.*

PROOF. Let $\{x, y\}$ be an edge of $\tilde{\Gamma}$. We should prove that x, y and the set A of all the vertices which are adjacent to both x and y form a clique in

$\tilde{\Gamma}$ and that this clique has the shape stated in the lemma. Let $x \in \tilde{\Gamma}_s(g)$ and $y \in \tilde{\Gamma}_t(g)$. We may assume that $s \leqq t$. The proof is divided into consideration of six cases depending on the pair (s, t).

$(s, t)=(0, 1)$. In this case the conclusion is obvious.

$(s, t)=(1, 1)$. We should only prove that there is no vertex $z \in \tilde{\Gamma}_2(g)$ adjacent to x and y.

Since the classes x and y are adjacent, they are subclasses of a 2-spread $\mathcal{S}$. By Lemma 3.9 any class of cliques can be adjacent to at most one congruency class from $\mathcal{S}$.

$(s, t)=(1, 2)$. By the same reason as above $A \cap \tilde{\Gamma}_1(g)=\emptyset$. Let Ξ be a prequad from the class x and Σ be a clique from the class y such that $\Sigma \subset \Xi$. If z is a class of cliques which is adjacent to y then by Lemma 3.9 and 3.5 there is a clique Θ in z which is adjacent to Σ. If in addition z is adjacent to x then by Lemma 3.9 Θ is contained in a prequad from x. Hence $\Theta \subseteqq \Xi$. Now it is easy to see that $A \cup \{y\}$ is a clique in $\tilde{\Gamma}_2(g)$.

$(s, t)=(2, 2)$. First of all since x, y lie in the same 1-spread there are no vertices in $\Gamma=\tilde{\Gamma}_3(g)$ which are adjacent to both x and y.

Now suppose that u is a congruency class of prequads which is adjacent to x and y. Let $\Xi \in u$ and $\Sigma \in x$ such that $\Sigma \subset \Xi$. If Θ is a clique from y which is adjacent to Σ then by Lemma 3.9 Θ lies in Ξ. So u is uniquely determined by x and y. Finally, the classes of cliques from Ξ which define in Ξ the same spread as Σ, form the unique clique from $\tilde{\Gamma}_2(g)$ containing the edge $\{x, y\}$. Since $\Xi \in u$, any vertex from this clique is adjacent to u.

$(s, t)=(2, 3)$. If a class z of cliques is adjacent to x then it determines the same 1-spread. So it is not adjacent to the vertex y. If Σ is a clique from x which contains y then it is clear that $A=\Sigma-\{y\}$.

$(s, t)=(3, 3)$. It is easy to see that in this case A consists of the class of the clique Σ which contains x and y, and all the vertices from $\Sigma-\{x, y\}$. $\square$

Let us calculate the valency of $\tilde{\Gamma}$. If $x=g$ or $x \in \tilde{\Gamma}_3(g)$ then x is contained in exactly q^4+q^2+1 cliques. Let $x \in \tilde{\Gamma}_1(g)$. The class x consists of q^4 prequads. Each prequad contains $q^3(q^2+1)$ cliques. We have already calculated in the proof of Lemma 3.10 that each congruency class of cliques consists of $1+(q^2+1)(q^2-1)=q^4$ cliques. So each class of prequads is adjacent to exactly $q^3(q^2+1)$ classes of cliques. Now we have that the vertex x is contained in exactly $1+q^3(q^2+1)/q=q^4+q^2+1$ cliques in $\tilde{\Gamma}$. Finally let $x \in \tilde{\Gamma}_2(g)$. Since any connected component of $\tilde{\Gamma}_2(g)$ is isomorphic to a prequad, x lies in exactly q^2+1 cliques intersecting $\tilde{\Gamma}_1(g)$. On the other hand the class x consists of q^4 cliques. So x is contained in exactly q^4 cliques intersecting $\tilde{\Gamma}_3(g)$. Hence in either case the number of cliques in $\tilde{\Gamma}$ containing a vertex x is equal to q^4+q^2+1.

By Lemma 3.11 if Σ is a clique in $\tilde{\Gamma}_i(g)$ for some $i>0$ then there is exactly one vertex $v(\Sigma)$ in $\tilde{\Gamma}_{i-1}$ which is adjacent to all vertices from Σ. Let us study in detail the case $i=2$. Let Ξ be a connected component of the graph induced by $\tilde{\Gamma}_2(g)$. Since Ξ is isomorphic to a prequad, we can use the notions of congruency and adjacency on the set of cliques in Ξ.

LEMMA 3.12. *If Σ, Θ are adjacent (respectively, congruent) cliques in Ξ then $v(\Sigma)$ and $v(\Theta)$ are adjacent (respectively, coincide). If Σ and Θ determine distinct spreads in Ξ then $v(\Sigma)\neq v(\Theta)$.*

PROOF. At first let Σ and Θ define the same spread in Ξ. Let $\mathcal{S}$ be the 1-spread in Γ corresponding to Ξ and Δ be a prequad which does not contain a clique from $\mathcal{S}$. Then Lemma 3.10 provides us with a bijection φ from Δ to Ξ. Put $\Sigma_1=\varphi^{-1}(\Sigma)$ and $\Theta_1=\varphi^{-1}(\Theta)$.

Let Π be a prequad containing Σ_1 and a clique from $\mathcal{S}$. Then by Lemma 3.9 Π contains all cliques from $\mathcal{S}$ which intersect Σ_1. Since the classes of these cliques form Σ we have that the class of Π coincides with $v(\Sigma)$. Now by Lemma 3.9 there is a prequad $\Phi\in\mathcal{S}(\Pi)$ such that $\Theta_1\subseteq\Phi$. Again by Lemma 3.9 Φ contains all the cliques from $\mathcal{S}$ which intersect Θ_1. Hence the class of Φ coincides with $v(\Theta)$. Since Π and Φ are from the same 2-spread and they contain Σ_1 and Θ_1, by Lemma 3.9 $v(\Sigma)$ and $v(\Theta)$ are adjacent or coincide depending on adjacency or congruency of Σ and Θ.

Finally let Σ and Θ determine distinct spreads in Ξ. Then up to congruency Σ and Θ have nontrivial intersection. By Lemma 3.11 $v(\Sigma)\neq v(\Theta)$. □

REMARK. As we have proved above a vertex $x\in\tilde{\Gamma}_1(g)$ is contained in exactly q^4+q^2 cliques intersecting $\tilde{\Gamma}_2(g)$. Hence Lemma 3.12 implies in particular that x is adjacent to vertices of exactly q^2+1 connected components of $\tilde{\Gamma}_2(g)$.

3d. Quads in $\tilde{\Gamma}$.

Now let us define a family of special subgraphs (quads) of the graph $\tilde{\Gamma}$.

For a prequad Ξ from Γ let $A=A_1\cup A_2\cup A_3$ where A_1 is the vertex set of Ξ, A_2 is the set of classes of cliques from Ξ and A_3 is the one-element set consisting of the class of the prequad Ξ. Then by Proposition 2.10 the subgraph in $\tilde{\Gamma}$ induced by A is isomorphic to the point graph of a generalized quadrangle. Let $\mathcal{P}_1$ denote the set of all subgraphs in $\tilde{\Gamma}$ which can be obtained in this way.

For a connected component Ξ of the graph $\tilde{\Gamma}_2(g)$ let $B=B_1\cup B_2\cup B_3$ where B_1 is the vertex set of Ξ, B_2 is a set of all vertices $v(\Sigma)$ where Σ is a clique from Ξ and $B_3=\{g\}$. By Lemma 3.12 and Proposition 2.10 the subgraph of $\tilde{\Gamma}$ induced by B is also isomorphic to the point graph of a generalized quadrangle.

Let $\mathscr{P}_2$ denotes the set of all subgraphs in $\tilde{\Gamma}$ of this shape and let $\mathscr{P}=\mathscr{P}_1\cup\mathscr{P}_2$. The elements of $\mathscr{P}$ will be called *quads*.

Notice that in a quad any edge is contained in a clique of size $q+1$. Hence if a quad contains an edge $\{a, b\}$ of $\tilde{\Gamma}$ then it contains the unique clique of $\tilde{\Gamma}$ containing $\{a, b\}$.

LEMMA 3.13. *Each pair of cliques in $\tilde{\Gamma}$ having a nontrivial intersection is contained in a quad.*

PROOF. Let Σ and Θ be two intersecting cliques from $\tilde{\Gamma}$ and $x\in\Sigma\cap\Theta$. The proof of the lemma depends on the position of cliques Σ and Θ. Below the triple (i, α, β) marks the case when $x\in\tilde{\Gamma}_i(g)$ and Σ (respectively, Θ) intersects $\tilde{\Gamma}_{i+\alpha}(g)$ (respectively, $\tilde{\Gamma}_{i+\beta}(g)$).

CASE $(3, -1, -1)$: Let Ξ be a prequad in Γ which contain $\Sigma\cap\Gamma$ and $\Theta\cap\Gamma$. Then the quad defined by Ξ, contains Σ and Θ.

CASE $(2, +1, +1)$: The cliques $\Sigma\cap\Gamma$ and $\Theta\cap\Gamma$ are congruent. By Lemma 3.8 there is a prequad Ξ in Γ which contains them. The quad corresponding to Ξ, contains Σ and Θ.

CASE $(2, +1, -1)$: Let $c\in\Theta\cap\tilde{\Gamma}_1(g)$ and $f\in\Sigma\cap\tilde{\Gamma}_2(g)$. By Lemma 3.9 the clique $\Sigma\cap\Gamma$ is contained in a prequad Ξ from the class c. The quad defined by Ξ, contains Σ. Moreover, it contains the edge $\{c, f\}$. Hence it contains the whole clique Θ.

CASE $(2, -1, -1)$: Let Ξ be the connected component of the graph $\tilde{\Gamma}_2(g)$ which contains x. By definition the quad defined by Ξ, contains Σ and Θ.

CASE $(1, +1, +1)$: First of all if Σ and Θ intersect the same connected component of $\tilde{\Gamma}_2(g)$ then the quad defined by this component contains Σ and Θ.

Now let Σ and Θ intersect distinct components (say Ξ and Δ) of $\tilde{\Gamma}_2(g)$. A vertex of a quad is contained in exactly q^2+1 cliques in the quad. On the other hand if Φ is a quad from $\mathscr{P}_1$ containing x then the cliques in Φ passing through x correspond to distinct spreads in the prequad $\Phi\cap\Gamma$. Hence these cliques intersect distinct connected components of $\tilde{\Gamma}_2(g)$. Now the remark after Lemma 3.12 implies that each quad Φ from $\mathscr{P}_1$ which contains x, intersects Ξ and Δ and the cliques $\Phi\cap\Xi$ and $\Phi\cap\Delta$ are congruent to $\Sigma\cap\tilde{\Gamma}_2(g)$ and $\Theta\cap\tilde{\Gamma}_2(g)$ respectively by Lemma 3.12.

By Lemma 3.3 if Π and Λ are congruent prequads in Γ then the cliques from Π which are congruent to some clique from Λ form a spread in Π. So a pair of quads from $\mathscr{P}_1$ passing through x has exactly one clique in common. Since the total number of quads from $\mathscr{P}_1$ passing through x is q^4 and each congruency class in Ξ or in Δ has cardinality q^2, for arbitrary Σ and Θ there is exactly one quad in $\mathscr{P}_1$ containing both of them.

CASE $(1, +1, -1)$: Let Ξ be the connected component of $\tilde{\Gamma}_2(g)$ which contains $\Sigma \cap \tilde{\Gamma}_2(g)$. The corresponding quad contains Σ. Moreover, it contains $\{x, g\}$. Hence it contains Θ.

CASE $(1, -1, -1)$: This case does not occur.

CASE $(0, +1, +1)$: Let $\mathcal{S}_2$ and $\mathcal{S}_2'$ be the 2-spreads corresponding to $\Sigma - \{g\}$ and $\Theta - \{g\}$. Let $y \in \Gamma$ and Ξ_1, Ξ_2 be the prequads from $\mathcal{S}_2$ and $\mathcal{S}_2'$ respectively which pass through y. Then the 1-spread $\mathcal{S}_1(\Xi_1 \cap \Xi_2)$ defines a connected component in $\tilde{\Gamma}_1(g)$ and hence the quad from $\mathcal{P}_2$. It is easy to see that this quad contains both Σ and Θ. □

COROLLARY 3.14. *For each vertex x of $\tilde{\Gamma}$ the cliques and the quads passing through x form a projective plane of order q^2.*

PROOF. There are exactly $a = (q^4+q^2+1)+q^9(q^4+q^2+1)/q^4 = (q^4+q^2+1)(q^5+1)$ quads and exactly $b = (q+1)(q^3+1)(q^5+1)\cdot(q^4+q^2+1)(q^4+q^2)/2$ pairs of intersecting cliques in $\tilde{\Gamma}$. Since each quad (having $(q+1)(q^3+1)$ vertices) contains exactly $c = (q+1)(q^3+1)\cdot(q^2+1)q^2/2$ pairs of intersecting cliques, the equality $b = ac$ implies that each pair of intersecting cliques is contained in exactly one quad. Notice that one can see this fact just from the proof of Lemma 3.13.

Let A (respectively B) be the set of all cliques (respectively quads) in $\tilde{\Gamma}$ passing through x. It was proved above that $|A| = q^4+q^2+1$. On the other hand a quad from B contains exactly q^2+1 cliques from A. Hence each clique from A is contained in $((q^4+q^2+1)-1)/((q^2+1)-1) = q^2+1$ quads. In particular, the cardinality of B is also q^4+q^2+1. The number of pairs of quads intersecting in a fixed clique from A is $(q^2+1)q^2/2$. Hence the total number of pairs of intersecting quads from B is $(q^4+q^2+1)(q^4+q^2)/2$ and it is equal to the number of all pairs of quads from B. □

REMARK. Corollary 3.14 means that the diagram geometry whose elements are the vertices, the cliques and the quads from $\tilde{\Gamma}$, is a (connected) geometry with diagram C_3.

Let us now prove another important lemma concerning quads.

LEMMA 3.15. *Let x be a vertex and Ξ be a quad from $\tilde{\Gamma}$. Then $d(\Xi, x) \leqq 1$ and there is exactly one vertex y of Ξ such that $d(\Xi, x) = d(y, x)$.*

PROOF. Suppose to the contrary that $d(\Xi, x) = n$, $n \geqq 2$. Let $x_0 = x, x_1, \cdots, x_n$ be the shortest path joining x with Ξ. Let Δ be the quad containing x_{n-2}, x_{n-1} and x_n. By Corollary 3.14 the intersection $\Xi \cap \Delta$ contains a clique. Now by Lemma 2.9 $d(\Xi \cap \Delta, x_{n-2}) \leqq 1$; a contradiction. Hence $d(\Xi, x) \leqq 1$.

Any vertex of Ξ is adjacent to exactly $q \cdot q^4$ vertices from $\tilde{\Gamma} - \Xi$. Since $|\tilde{\Gamma}| = (q+1)(q^3+1)(q^5+1)$ and $|\Xi| = (q+1)(q^3+1)$, there is exactly one vertex $y \in \Xi$

such that $d(\Xi, x)=d(y, x)$. □

COROLLARY 3.16. *Let Ξ be a quad, $x, y \in \Xi$ and s be the distance between x and y in Ξ. Then any path of length at most $s+1$ joining x and y, is contained in Ξ.*

PROOF. If $s=1$ then the conclusion follows from Lemma 3.11. Let $s=2$. If we have a path (x, a, y) then by Lemma 3.15 $a \in \Xi$. Suppose that we have a path (x, a, b, y). Let Δ be a quad passing through x, a and b. Then by Lemma 2.9 either $b \in \Delta \cap \Xi$ or $d(\Delta \cap \Xi, b)=1$. In the latter case $\Xi=\Delta$ as they contain x, y which are at distance 2 apart. □

Now we are in a position to prove the main result of the section which implies Theorem A (ii) in the case $d=3$.

PROPOSITION 3.17. *$\tilde{\Gamma}$ is a distance-regular graph with the parameters of the dual polar space graph of type ${}^2A_5(q)$. In particular, $\tilde{\Gamma}$ is isomorphic to that graph.*

PROOF. Let Ω denote the dual polar space graph of type ${}^2A_5(q)$. Let x be a vertex of $\tilde{\Gamma}$. If a vertex y is at distance 1 or 2 from x then there exists a quad Ξ which contains both x and y. By Corollary 3.16 a path of length at most $d(x, y)+1$ lies in Ξ. Hence the parameters c_i and a_i for $i=1, 2$ exist and coincide with those of Ω. Since the valency of $\tilde{\Gamma}$ is the same as that of Ω the parameters b_0, b_1 and b_2 also exist and are as stated.

Now let $y \in \tilde{\Gamma}_3(x)$. By Lemma 3.15 for any quad Ξ passing through x there is just one vertex in Ξ adjacent to y. Hence $c_i=q^4+q^2+1$.

Thus $\tilde{\Gamma}$ is distance-regular and its parameters coincide with the parameters of Ω and by [**BCN**], [**IS2**] $\tilde{\Gamma} \cong \Omega$. □

COROLLARY 3.18. *The graph Γ is isomorphic to the graph of Hermitian forms in 3-dimensional space over $GF(q^2)$.* □

3e. The representation of $\tilde{\Gamma}$ as a set of vectors.

Let us introduce an additional set of vectors in W. For a vertex $x \in \Gamma$ put $w_x=\hat{x}$. For a clique Σ in Γ put $w_\Sigma=-\hat{\Sigma}$. For a prequad Δ put $w_\Delta=\hat{\Delta}/q^2$. Finally put $w_g=w_\Gamma=-\hat{\Gamma}/q^6$. As it was mentioned before if two cliques (or prequads) are congruent then the corresponding vectors coincide. Hence for each vertex x of $\tilde{\Gamma}$ there is a well-defined vector w_x.

LEMMA 3.19. *If $x, y \in \tilde{\Gamma}$ and $d(x, y)=i$ then $\langle w_x, w_y\rangle=(-q)^{-i}$.*

PROOF. The claim can be checked by direct calculations, but we propose another kind of arguments. The dual polar space graph Ω has a representation as a system of norm one vectors in the eigenspace U of dimension $q(q^5+1)/(q+1)$ (which coincides with $\dim(W)$). In this representation if v, w are vectors corresponding to some vertices of Ω at distance i then $\langle v, w\rangle=(-q)^{-i}$. By Proposition 3.17 for a fixed vertex x the system of vectors $\Omega_3(x)$ is isomorphic to Γ. If Σ is a clique in Ω then the sum of vectors over Σ is zero. Now it is easy to verify that the vectors w_Σ for Σ being a clique or a prequad in Γ correspond to vectors of Ω. □

4. The case $d\geqq 4$.

In order to deal with the case $d\geqq 4$ we need certain information concerning the automorphism groups of the graphs of Hermitian forms over finite fields. Let Γ be such a graph. It is known [**BCN**], [**IS1**] that the group $\mathrm{Aut}(\Gamma)$ contains a subgroup $G(d, q)$ isomorphic to the semidirect product $N\lambda H$, where N is the elementary abelian group of order q^{d^2} and H is the factorgroup of $\Gamma L_d(q^2)$ by the subgroup consisting of scalar matrices whose orders divide $q+1$. Moreover, H is the stabilizer in $G(d, q)$ of some vertex x of Γ. In its action on the set of cliques passing through x the group H induces $P\Gamma L_d(q^2)$. The kernel of the action has order $q-1$ and acts regularly on the set $\Sigma-\{x\}$ for each clique Σ passing through x.

LEMMA 4.1. $\mathrm{Aut}(\Gamma)=G(d, q)$.

PROOF. At first let $d=2$. Then the graph $\tilde{\Gamma}$ constructed from Γ as in Section 2 is the graph of the dual polar space of type ${}^2A_3(q)$. It is clear that each automorphism of Γ can be extended to an automorphism of $\tilde{\Gamma}$ in a unique way. On the other hand there is a unique way to construct $\tilde{\Gamma}$ from Γ. So $\mathrm{Aut}(\Gamma)$ is the stabilizer of a vertex in the group $\mathrm{Aut}(\tilde{\Gamma})$. It is known [**Cam**] that $\mathrm{Aut}(\tilde{\Gamma})\cong P\Gamma U_4(q)$, hence $\mathrm{Aut}(\Gamma)\cong G(2, q)$.

Now suppose that $d\geqq 3$. Let $x\in\Gamma$ and F be the stabilizer of the vertex x in the group $\mathrm{Aut}(\Gamma)$. Then F preserves the structure $\pi(x)$ of the projective space $PG(d-1, q^2)$ consisting of the subgraphs $\Delta(x, y)$ for $1\leqq d(x, y)\leqq d-1$. So the group induced by the action of F on the set of cliques containing x is a subgroup of $P\Gamma L_d(q^2)$. Let K be the kernel of this action. Since any two cliques containing x are contained in a subgraph $\Delta(u, v)$ for $d(u, v)=2$, which is isomorphic to the graph related to $\mathrm{Her}(2, q)$, the group K acts faithfully and semiregularly on each set $\Sigma-\{x\}$ where Σ is a clique containing x. □

By Lemma 4.1 and the properties of the group $G(d, q)$ we have the following.

COROLLARY 4.2. *Let Γ be the graph related to the scheme* $\mathrm{Her}(d, q)$, $d\geqq 3$, *x, u be vertices of Γ and τ be a collineation of $\pi(x)$ onto $\pi(u)$. Let $y\in\Gamma_1(x)$*

and z be an arbitrary vertex from $(\Delta(x, y))^{\tau}-\{u\}$. *Then the group* Aut$(\Gamma)$ *contains a unique automorphism which maps x to u, y to z and induces the collineation* τ. □

If $d=2$ then the structure $\pi(x)$ is trivial and we have the following.

COROLLARY 4.3. *Let* Γ *be the graph related to the scheme* Her$(2, q)$, x, u *be vertices of* Γ *and* τ *be a bijection from the set of cliques passing through* x *onto the set of cliques passing through* u. *Let* $y\in\Gamma_1(x)$ *and* z *be an arbitrary vertex from* $(\Delta(x, y))^{\tau}-\{u\}$. *Then the group* Aut$(\Gamma)$ *contains at most one automorphism which maps* x *to* u, y *to* z *and induces the mapping* τ. □

Now we can prove the main result of the section which implies Theorem A (ii) in the case $d\geqq 4$.

PROPOSITION 4.4. *Let* Γ *be a distance-regular graph whose parameters coincide with those of the graph* Π *related to the scheme* Her(d, q), $d\geqq 4$. *Then* $\Gamma\cong\Pi$.

PROOF. We will use induction on d. Let x, u be vertices of Γ and Π respectively and τ be a collineation of $\pi(x)$ onto $\pi(u)$. Notice that since $d\geqq 4$ $\pi(x)$ and $\pi(u)$ are isomorphic. Let $y\in\Gamma_1(x)$ and v be an arbitrary vertex from $(\Delta(x, y))^{\tau}-\{u\}$.

As before let V be the space generated by the vectors x^* for $x\in\Gamma$. Let U be the analogous space for the graph Π. Let us define a linear mapping α from V onto U as follows. By Lemma 1.1 the set $\Gamma_1(x)$ is a basis of V, hence it is sufficient to define α on $\Gamma_1(x)$. Let $\Xi=\Delta(x, t)$ for some $t\in\Gamma_{d-1}(x)$ such that $y\in\Xi$. By induction we may suppose that Ξ is isomorphic to the graph related to Her$(d-1, q)$. By Corollary 4.2 there is a unique isomorphism α_{Ξ} of Ξ onto $\tau(\Xi)$ such that $\alpha_{\Xi}(x)=u$, $\alpha_{\Xi}(y)=v$ and on the set of cliques from Σ α_{Ξ} induces the restriction of τ on this set. Notice that α_{Ξ} can be considered as a linear mapping between the subspaces of V and U generated by the corresponding sets of vectors.

Now we can define the mapping α. Namely, for each vertex $a\in\Gamma_1(x)$ put $\alpha(a)=\alpha_{\Xi}(a)$ where Ξ is any subgraph of type $\Delta(x, t)$, $t\in\Gamma_{d-1}(x)$ passing through y and a. Since any two hyperplanes in a projective space intersect in a subspace of codimension 2, Corollary 4.2 in the case $d\geqq 5$ and Corollary 4.3 in the case $d=4$ imply that $\alpha(a)$ does not depend on the choice of Ξ.

Since α is a linear mapping, it is defined on the set of all vertices of Γ. By definition α maps cliques from $\Gamma_1(x)$ onto cliques from $\Pi_1(u)$. Hence α is orthogonal. So to prove that α induces an isomorphism from Γ onto Π it is sufficient to prove that $\alpha(a)\in\Pi$ for each $a\in\Gamma$.

Now let $\{s, t\}$ be an edge of Γ such that for any hyperplane $\Phi \in \pi(s) \cap \pi(t)$ and for any $r \in \Phi$ we have $\alpha(r) \in \Pi$. For any line in $\pi(s)$ there exists a hyperplane in $\pi(s) \cap \pi(t)$ which contains this line. So α defines a collineation τ_s from $\pi(s)$ onto $\pi(\alpha(s))$. Let Ξ be any hyperplane from $\pi(s)$ and $r \in \Xi_1(s)$. By Corollary 4.2 there is a unique isomorphism φ from Ξ onto $\tau_s(\Xi)$ which maps s onto $\alpha(s)$, r onto $\alpha(r)$ and on the set of cliques from Ξ passing through s induces the restriction of τ_s. By Corollaries 4.2 and 4.3 the linear mapping φ coincides with the restriction of α on each hyperplane of Ξ passing through r. Since $d \geqq 4$, these hyperplanes cover $\Xi_1(s)$. Thus for each vertex $a \in \Xi$ we have $\alpha(a) \in \Pi$.

The preceding arguments mean that if α is "good" on all hyperplanes from $\pi(s) \cap \pi(t)$ then it is "good" on all hyperplanes from $\pi(s)$. In this way we can pass by connectivity from x to any other vertex of Γ. $\square$

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