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# On the cylinder homomorphisms of Fano complete intersections

By Ichiro SHIMADA

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## § 0. Introduction.

Let $X$ be a smooth complete intersection of dimension $n \geqq 3$. In this paper, a complete interseciton always means a complete intersection of hypersurfaces in a projective space over the complex number field $\boldsymbol{C}$. Let $\mathcal{C}(X)$ (resp. $\mathcal{L}(X)$) denote the variety of all conics (resp. lines) on $X$. Then we have the cylinder homomorphisms

$$\begin{aligned} \Psi_C : H_{n-2}(\mathcal{C}(X), \boldsymbol{Q}) &\longrightarrow H_n(X, \boldsymbol{Q}) \\ [\gamma] &\longmapsto [\bigcup_{t \in \gamma} C_t] \\ \Psi_{\mathcal{L}} : H_{n-2}(\mathcal{L}(X), \boldsymbol{Q}) &\longrightarrow H_n(X, \boldsymbol{Q}) \\ [\gamma] &\longmapsto [\bigcup_{s \in \gamma} L_s] \end{aligned}$$

where $C_t$ is the conic corresponding to $t \in \mathcal{C}(X)$ and $L_s$ is the line corresponding to $s \in \mathcal{L}(X)$.

A complete intersection $X$ is called a Fano complete intersection if its anti-canonical line bundle $-K_X$ is very ample. The purpose of this paper is to prove the following:

THEOREM. *If $X$ is a general Fano complete intersection of dimension $\geqq 3$, then $\Psi_C$ and $\Psi_{\mathcal{L}}$ are surjective.*

COROLLARY. *If $n=2m-1$, the Abel-Jacobi maps $J^{m-1}(\mathcal{C}(X)) \to J^m(X)$ and $J^{m-1}(\mathcal{L}(X)) \to J^m(X)$ are surjective for a general Fano complete intersection $X$.*

We say that a property holds for a general complete intersection if it holds for all complete intersections belonging to some Zariski open dense subset of the Hilbert scheme of complete intersections of given dimension and multi-degree.

It is known that every Fano complete intersection is covered by conics on it (cf. [**9**]), and every Fano complete intersection of index $\geqq 2$ is covered by lines on it (cf. [**21**]). (See § 1 for the definition of the index.) These facts are the basis of our whole argument.

The method of our proof is as follows. The Hodge level of $H^n(X)$ is defined to be $\max\{|p-q| \mid p+q=n, h^{p,q}(X)>0\}$. In § 2, we prove our Theorem in the case where $H^n(X)$ has the Hodge level $n-2$ by using the method of infinitesimal Abel-Jacobi mapping (cf. [5], [7], [12], [22]). If $X$ is a hypersurface, an essentially same result as ours in this section has been obtained in [5].

In § 3, we prove a general result about a cylinder homomorphism (Proposition in § 3). Let $\mathcal{V}$ be a smooth projective variety. Let $\widetilde{W}$ be a general hyperplane section of $\mathcal{V}$ and $W$ be a general hyperplane section of $\widetilde{W}$. We consider the cylinder homomorphism $\Psi(\widetilde{W})$ (resp. $\Psi(W)$) associated to a family of subschemes in $\mathcal{V}$ contained in $\widetilde{W}$ (resp. $W$). Our Proposition says that, under certain conditions, if the vanishing cycles of $W$ in $\widetilde{W}$ are contained in the image of $\Psi(W)$, then the vanishing cycles of $\widetilde{W}$ in $\mathcal{V}$ are contained in the image of $\Psi(\widetilde{W})$. This result enables us to prove Theorem by induction with respect to the dimension of $X$. The proof of Proposition is quite topological and different from the method of infinitesimal Abel-Jacobi mapping.

For a Fano hypersurface of index 1, the cylinder homomorphism $\Psi_{\mathcal{L}}$ has been studied and its kernel is determined in [13], [14], [15]. Also there are many cases where cylinder homomorphisms are known to be isomorphisms (cf. [2], [3], [4], [6], [7], [12], [20], [22]. See also the forthcoming paper [17]).

In [18], Shioda studied a cylinder homomorphism associated to the family of lines on a hypersurface with 'inductive' structure. On the other hand, it is known that, if $X$ is a smooth cubic hypersurface, then $\mathcal{L}(X)$ is smooth (cf. [1]). Combined these, we get the surjectivity of $\Psi_{\mathcal{L}}$ for a smooth cubic hypersurface via the monodromy argument. Unfortunately, if $X$ is a hypersurface with inductive structure and of degree $\geqq 4$, then $\mathcal{L}(X)$ is singular and this method cannot be applied.

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## § 1. Preliminaries.

Let $X$ be a smooth complete intersection in $\boldsymbol{P}^N$ of dimension $n \geqq 3$ and multi-degree $(a_1, a_2, \cdots, a_d)$, with $2 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_d$. A complete intersection $X$ is a Fano complete intersection if and only if $f := N+1-\sum_{i=1}^{d} a_i > 0$. This number $f$ is called the index of a Fano complete intersection $X$.

LEMMA 0. *The Hodge level of* $H^n(X)$ *is* $n-2$ *if and only if* $1 \leqq f \leqq a_d$.

PROOF. See [23], SGA 7II, Corollaire 2.8. in Exposé XI.

LEMMA 1. *If $X$ is general, the variety $\mathcal{C}(X)$ of all conics on $X$ is smooth and complete.*

PROOF. Let $V$ be the variety of all conics in $\boldsymbol{P}^N$, defined as in [**19**]. This variety $V$ is smooth and complete. Since $\mathcal{C}(X)$ is a closed subscheme of $V$, $\mathcal{C}(X)$ is complete. Let $Q$ be the variety of all complete intersections in $\boldsymbol{P}^N$ of multi-degree $(a_1, a_2, \cdots, a_d)$. Suppose that

$$a_1 = \cdots = a_{d_1} < a_{d_1+1} = \cdots = a_{d_1+d_2} < \cdots$$
$$\cdots < a_{d_1+d_2+\cdots+d_{\nu-1}+1} = \cdots = a_{d_1+d_2+\cdots+d_\nu},$$

where $d_1+d_2+\cdots+d_\nu=d$. We put $a_{(i)}:=a_{d_1+\cdots+d_i}$. Then we have a sequence of morphisms

$$Q = Q_{(\nu)} \xrightarrow{\gamma_{(\nu)}} Q_{(\nu-1)} \xrightarrow{\gamma_{(\nu-1)}} \cdots \xrightarrow{\gamma_{(2)}} Q_{(1)} \xrightarrow{\gamma_{(1)}} Q_{(0)} = \operatorname{Spec} \boldsymbol{C}$$

where $Q_{(i)}$ is the variety of all complete intersections in $\boldsymbol{P}^N$ of multi-degree $(a_1, a_2, \cdots, a_{d_1+\cdots+d_i})$. Let

$$\begin{array}{ccc} \mathfrak{X}_{(i)} & \hookrightarrow & \boldsymbol{P}^N \times Q_{(i)} \\ \downarrow{\scriptstyle \pi_{(i)}} & & \\ Q_{(i)} & & \end{array}$$

be the universal family. Then the coherent sheaf $\pi_{(i)*}\mathcal{O}(a_{(i+1)})$ is locally free because the function $\dim_{k(y)} H^0(X_y, \mathcal{O}(a_{(i+1)}))$ of $y \in Q_{(i)}$ is constant where $X_y$ is the fibre of $\pi_{(i)}$ over $y \in Q_{(i)}$. Hence we have a smooth morphism

$$\operatorname{Grass}(d_{i+1}, \pi_{(i)*}\mathcal{O}(a_{(i+1)})) \longrightarrow Q_{(i)}.$$

It is obvious that $Q_{(i+1)}$ is a Zariski open dense subset of $\operatorname{Grass}(d_{i+1}, \pi_{(i)*}\mathcal{O}(a_{(i+1)}))$. Hence the morphism $\gamma_{(i+1)}$ is smooth. In particular, $Q$ is smooth. Let $Z_{(i)} \subset V \times Q_{(i)}$ be the incidence correspondence

$$Z_{(i)} := \{(C, X) \in V \times Q_{(i)} \mid C \subset X\}$$

with the natural projection $\beta_{(i)}: Z_{(i)} \to Q_{(i)}$. Then we have a commutative

$$\begin{array}{ccc} Z_{(i+1)} & \xrightarrow{\beta_{(i+1)}} & Q_{(i+1)} \\ \downarrow{\scriptstyle \alpha_{(i+1)}} & & \downarrow{\scriptstyle \gamma_{(i+1)}} \\ Z_{(i)} & \xrightarrow[\beta_{(i)}]{} & Q_{(i)} \end{array}$$

where $Z_{(0)}=V$. We shall show that the natural morphism $\alpha_{(i+1)}$ is smooth. Let

$$\begin{array}{ccc} \mathfrak{C} & \hookrightarrow & \boldsymbol{P}^N \times V \\ \downarrow & & \\ V & & \end{array}$$

be the universal family of conics. We put

$$\mathfrak{C}_{(i)} := \mathfrak{C}\times_V Z_{(i)}$$
$$\tilde{\mathcal{X}}_{(i)} := \mathcal{X}_{(i)}\times_{Q_{(i)}} Z_{(i)}\,.$$

Then we have the following commutative diagram:

$$\begin{array}{ccccc}
\mathfrak{C}_{(i)} & \hookrightarrow & \tilde{\mathcal{X}}_{(i)} & \hookrightarrow & \boldsymbol{P}^N\times Z_{(i)} \\
\downarrow & \swarrow{\scriptstyle \tilde{\pi}_{(i)}} & & & \\
Z_{(i)} & & & &
\end{array}$$

where $\tilde{\pi}_{(i)}$ is the natural projection. Consider the coherent sheaf

$$\mathcal{F}_{(i)} := \mathcal{O}(a_{(i+1)})\otimes\mathcal{I}_{\mathfrak{C}_{(i)}}$$

on $\tilde{\mathcal{X}}_{(i)}$, where $\mathcal{I}_{\mathfrak{C}_{(i)}}$ is the ideal sheaf of $\mathfrak{C}_{(i)}$ in $\tilde{\mathcal{X}}_{(i)}$. This sheaf is flat over $Z_{(i)}$ by construction. Take an arbitrary point $z\in Z_{(i)}$. Let $(C_z, X_z)$ be the corresponding pair of a conic and a complete intersection which are defined over $k(z)$. Then we see that the kernel

$$K_1 := \ker(H^0(\boldsymbol{P}^N_{k(z)}, \mathcal{O}(a_{(i+1)})) \longrightarrow H^0(X_z, \mathcal{O}(a_{(i+1)}))$$

of the restriction to $X_z$ is contained in the kernel

$$K_2 := \ker(H^0(\boldsymbol{P}^N_{k(z)}, \mathcal{O}(a_{(i+1)})) \longrightarrow H^0(C_z, \mathcal{O}(a_{(i+1)}))$$

of the restriction to $C_z$, and the dimensions of both spaces over $k(z)$ are independent of $z$. We have

$$\dim_{k(z)}H^0(X_z, \mathcal{F}_{(i),z}) = \dim_{k(z)}K_2 - \dim_{k(z)}K_1\,.$$

Thus $\tilde{\pi}_{(i)*}\mathcal{F}_{(i)}$ is locally free. It is easy to see that $Z_{(i+1)}$ is a Zariski open dense subset of $\mathrm{Grass}(d_{i+1}, \tilde{\pi}_{(i)*}\mathcal{F}_{(i)})$. Hence $\alpha_{(i+1)}$ is smooth. Since $Z_{(0)}=V$ is smooth, so is $Z:=Z_{(\nu)}$. We put $\beta:=\beta_{(\nu)}$, $\alpha=\alpha_{(\nu)}\circ\alpha_{(\nu-1)}\circ\cdots\circ\alpha_{(1)}$, and consider the following diagram:

$$\begin{array}{ccc}
Z & \xrightarrow{\ \beta\ } & Q \\
{\scriptstyle\alpha}\downarrow & & \\
V & &
\end{array} \tag{1.0}$$

By the theorem of generic smoothness, $\beta^{-1}(X)=\mathcal{C}(X)$ is smooth for a general $X\in Q$. □

Next, we investigate the normal bundle $N_{C/X}$ of a smooth conic $C$ on $X$. Let $\mathcal{O}_C(k)$ denote the unique line bundle of degree $k$ on $C\cong\boldsymbol{P}^1$.

LEMMA 2. *Let $Z$ be the incidence correspondence defined as in the proof of Lemma* 1. *For a general pair $(C, X)\in Z$, the normal bundle $N_{C/X}$ of $C$ in $X$ is* *given as follows.*

$$N_{C/X} \cong \begin{cases} \mathcal{O}_C(1)^{\oplus 2(f-1)} \oplus \mathcal{O}_C^{\oplus n-2f+1} & \textit{if } 1 \leqq f \leqq (n+1)/2 \\ \mathcal{O}_C(2)^{\oplus 2f-n-1} \oplus \mathcal{O}_C(1)^{\oplus 2(n-f)} & \textit{if } (n+1)/2 \leqq f \leqq n \\ \mathcal{O}_C(2)^{\oplus n-2} \oplus \mathcal{O}_C(4) & \textit{if } f=n+1 \ (\textit{i.e. } X=\boldsymbol{P}^n). \end{cases}$$

PROOF. This Lemma is obvious if $X=\boldsymbol{P}^n$. We assume $f \leqq n$. We may assume that $X$ and $C$ are smooth. As is well known, every vector bundle on $\boldsymbol{P}^1$ can be written as a direct sum of line bundles. Thus, we can set $N_{C/X} \cong \bigoplus_{i=1}^{n-1} \mathcal{O}_C(b_i)$, where $b_1 \leqq b_2 \leqq \cdots \leqq b_{n-1}$. We have the exact sequence

$$0 \longrightarrow N_{C/X} \longrightarrow N_{C/\boldsymbol{P}^N} \xrightarrow{\lambda} N_{X/\boldsymbol{P}^N}|_C \longrightarrow 0$$

and isomorphisms

$$N_{C/\boldsymbol{P}^N} \cong \mathcal{O}_C(2)^{\oplus N-2} \oplus \mathcal{O}_C(4), \qquad N_{X/\boldsymbol{P}^N}|_C \cong \bigoplus_{i=1}^{d} \mathcal{O}_C(2a_i).$$

Hence the integers $\{b_i\}$ satisfy the following conditions:

$$\sum_{i=1}^{n-1} b_i = 2\Big(N - \sum_{i=1}^{d} a_i\Big) = 2(f-1), \qquad b_{n-2} \leqq 2, \ b_{n-1} \leqq 4.$$

Thus, in order to prove Lemma 2, it is enough to show that, for a general pair $(C, X) \in Z$, following equalities hold:

$$h^0(C, N_{C/X} \otimes \mathcal{O}_C(-3)) = 0, \tag{1.1}$$

$$h^0(C, N_{C/X} \otimes \mathcal{O}_C(-2)) = \max(0, 2f-n-1), \tag{1.2}$$

$$h^0(C, N_{C/X} \otimes \mathcal{O}_C(-1)) = 2(f-1). \tag{1.3}$$

Choose homogeneous coordinates $(x_0, x_1, \cdots, x_N)$ in $\boldsymbol{P}^N$ such that $C$ is defined by

$$x_0^2 - x_1 x_2 = x_3 = \cdots = x_N = 0.$$

If $X$ is defined by $f_1 = f_2 = \cdots f_d = 0$, where $f_i$ is homogeneous of degree $a_i$, then each $f_i$ is of the form

$$f_i = (x_0^2 - x_1 x_2)\tilde{h}_i(x_0, x_1, x_2) + x_3 \tilde{g}_{3i}(x_0, x_1, x_2) + \cdots + x_N \tilde{g}_{Ni}(x_0, x_1, x_2) + \sum A_{i_0 i_1 \cdots i_N}(x_0^{i_0} x_1^{i_1} \cdots x_N^{i_N}),$$

where $A_{i_0 i_1 \cdots i_N} \in \boldsymbol{C}$, $i_0 + i_1 + \cdots + i_N = a_i$, and $i_3 + i_4 + \cdots + i_N \geqq 2$. Then the morphism

$$N_{C/\boldsymbol{P}^N} (\cong \mathcal{O}_C(4) \oplus \mathcal{O}_C(2)^{\oplus(N-2)}) \xrightarrow{\lambda} N_{X/\boldsymbol{P}^N}|_C (\cong \bigoplus_{i=1}^{d} \mathcal{O}_C(2a_i))$$

is given by the matrix $(h_i, g_{\nu i})$, $1 \leqq i \leqq d$, $3 \leqq \nu \leqq N$, where

$$h_i := \tilde{h}_i|_C \in H^0(C, \mathcal{O}_C(2a_i - 4))$$
$$g_{\nu i} := \tilde{g}_{\nu i}|_C \in H^0(C, \mathcal{O}_C(2a_i - 2)).$$

For $s=1, 2, 3$, $H^0(C, N_{C/X}\otimes\mathcal{O}_C(-s))$ is the kernel of the morphism

$$\lambda^0(-s): H^0(C, N_{C/\boldsymbol{P}^N}\otimes\mathcal{O}_C(-s)) \longrightarrow H^0(C, N_{X/\boldsymbol{P}^N}|_C\otimes\mathcal{O}_C(-s))$$

given by $(h_i, g_{\nu i})$. It is easy to see that $H^0(C, N_{C/X}\otimes\mathcal{O}_C(-3))=0$ unless $h_1=\cdots=h_d=0$. Thus the equality (1.1) holds for a general $X$ containing $C$. The equality (1.2) (resp. (1.3)) holds if and only if the linear map $\lambda^0(-2)$ (resp. $\lambda^0(-1)$) has the maximal rank. Because

$$h^0(C, N_{C/\boldsymbol{P}^N}\otimes\mathcal{O}_C(-s))-h^0(C, N_{X/\boldsymbol{P}^N}|_C\otimes\mathcal{O}_C(-s))$$
$$=\begin{cases} (N+1)-\left(2\sum_{i=1}^{d} a_i-d\right)=2f-n-1 & \text{if } s=2 \\ 2N-2\sum_{i=1}^{d} a_i=2(f-1) & \text{if } s=1. \end{cases}$$

The subset

$$M_s=\{(h_i, g_{\nu i}) \mid \lambda^0(-s) \textit{ has the maximal rank,}\}$$

of $\bigoplus_{i=1}^{d}\{H^0(C, \mathcal{O}_C(2a_i-4))\oplus H^0(C, \mathcal{O}_C(2a_i-2))^{\oplus(N-2)}\}$ is Zariski open. Therefore, if $M_1$ and $M_2$ are non-empty, the equalities (1.2) and (1.3) hold for a general $X$ containing $C$. It is not difficult to find examples of elements in $M_1$ and $M_2$. □

REMARK 1. For a general Fano complete intersection $X$ of dimension $n$ and index $f$, and a general line $L$ on $X$, the normal bundle $N_{L/X}$ is given as follows:

$$N_{L/X}\cong\begin{cases} \mathcal{O}_L(-1)\oplus\mathcal{O}_L^{\oplus(n-2)} & \text{if } 1=f \\ \mathcal{O}_L^{\oplus(n-f+1)}\oplus\mathcal{O}_L(1)^{\oplus(f-2)} & \text{if } 2\leqq f\leqq n+1 \end{cases}$$

## §2. Proof of the theorem.

Let $X$, $\mathcal{C}(X)$, and $\Psi_C$ be as in the introduction. We shall prove that the dual cylinder homomorphism

$$\Psi_C^*: H^n(X, \boldsymbol{Q}) \longrightarrow H^{n-2}(\mathcal{C}(X), \boldsymbol{Q})$$

is injective. We have the incidence correspondence

$$\begin{array}{ccc} \Omega & \xrightarrow{\quad q \quad} & X \\ {\scriptstyle p}\downarrow & & \\ \mathcal{C}(X), & & \end{array}$$

where $\Omega$ is $\{(C, x)\in\mathcal{C}(X)\times X \mid x\in C\}$ and $p$, $q$ are the natural projections. The dual cylinder homomorphism $\Psi_C^*$ is by definition the composition of maps

$$H^n(X, \boldsymbol{Q}) \xrightarrow{q^*} H^n(\Omega, \boldsymbol{Q}) \xrightarrow{p_*} H^{n-2}(\mathcal{C}(X), \boldsymbol{Q}).$$

**2.a. Monodoromy argument.** Let $(a_1, a_2, \cdots, a_d)$ be the multi-degree of

$X$ in $\boldsymbol{P}^N$, with $2\leqq a_1\leqq a_2\leqq\cdots\leqq a_d$. If $X$ is Fano and general, we have a smooth Fano complete intersection $Y$ in $\boldsymbol{P}^N$ of multi-degree $(a_1, a_2, \cdots, a_{d-1})$, and a Lefschetz pencil $\{X_t\}_{t\in\boldsymbol{P}^1}$ on $Y$ cut out by hypersurfaces of degree $a_d$ which contains $X=X_o$ as the member corresponding to $o\in\boldsymbol{P}^1$. Recall that $V$ is the variety of all conics in $\boldsymbol{P}^N$. We set

$$\tilde{\mathcal{C}}:=\{(C, t)\in V\times\boldsymbol{P}^1 \mid C\subset X_t\}$$

and let $\pi:\tilde{\mathcal{C}}\to\boldsymbol{P}^1$ be the natural projection. By Lemma 1, we may assume that there is a non-empty Zariski open subset $U$ on $\boldsymbol{P}^1$ such that

0) $o\in U$,
1) $X_t$ is smooth for all $t$ on $U$, and
2) $\pi$ is smooth over $U$.

Then the fundamental group $\pi_1(U, o)$ acts both on $H^n(X, Q)$ and $H^{n-2}(\mathcal{C}(X), \boldsymbol{Q})$, and the dual cylinder homomorphism $\Psi_{\mathcal{C}}^*$ is $\pi_1(U, o)$-equivariant. Consider the Lefschetz decomposition of $H^n(X, \boldsymbol{Q})$. If $n$ is odd, the action of $\pi_1(U, o)$ on the vanishing cocycles $H^n(X, \boldsymbol{Q})=H^n_{\text{prim}}(X, \boldsymbol{Q})$ is irreducible (cf. [**11**]). If $n$ is even, the primitive decomposition $H^n(X, \boldsymbol{Q})=H^n_{\text{prim}}(X, \boldsymbol{Q})\oplus\boldsymbol{Q}[\omega]^{n/2}$ coincides with the decomposition into the vanishing cocycles and the invariant cocycles, where $[\omega]\in H^2(X, \boldsymbol{Q})$ is the cohomology class of a hyperplane section. In this case, the action of $\pi_1(U, o)$ is irreducible on $H^n_{\text{prim}}(X, \boldsymbol{Q})$ and trivial on $\boldsymbol{Q}[\omega]^{n/2}$. Hence, in order to prove the injectivity of $\Psi_{\mathcal{C}}^*$, it is enough to show the following two claims:

CLAIM 1. *If $n$ is even, $\Psi_{\mathcal{C}}^*([\omega]^{n/2})$ is not zero.*

CLAIM 2. *The composition*

$$\Psi^*_{\mathcal{C},\text{prim}}: H^n_{\text{prim}}(X, \boldsymbol{Q}) \longrightarrow H^n(X, \boldsymbol{Q}) \longrightarrow H^{n-2}(\mathcal{C}(X), \boldsymbol{Q})$$

*of the inclusion and $\Psi_{\mathcal{C}}^*$ is not-trivial.*

**2.b. Proof of Claim 1.** We set $n=2m$. We fix a $(N-m)$-plane $\boldsymbol{P}^{N-m}$ in $\boldsymbol{P}^N$, and set $W=X\cap\boldsymbol{P}^{N-m}$. The Poincaré dual of the homology class $[W]\in H_n(X, \boldsymbol{Q})$ is $[\omega]^m\in H^n(X, \boldsymbol{Q})$. It is enough to show that there exists an algebraic cycle $\Xi\subset\mathcal{C}(X)$ of dimension $m-1$ such that the intersection number $\Psi_{\mathcal{C}}([\Xi])\cdot[W]$ is not zero. Let $\Gamma$, $\Gamma_1$ be the closed subvarieties of $V$ defined as follows:

$$\Gamma:=\{C\in V \mid C\cap\boldsymbol{P}^{N-m}\neq\emptyset\},$$
$$\Gamma_1:=\{C\in V \mid \dim(C\cap\boldsymbol{P}^{N-m})\geqq 1\}.$$

The codimension of $\Gamma$ in $V$ is $m-1$. Recall that in the diagram (1.0), the natural projection $\alpha$ is smooth. Then we see that $\alpha^{-1}(\Gamma)$ is a subvariety of $Z$

of codimension $m-1$. The morphism $\beta$ maps $\alpha^{-1}(\Gamma)$ onto $Q$ surjectively, because every Fano complete intersection is covered by conics (cf. [**9**]). Thus, for a general $X\in Q$, the intersection $\alpha^{-1}(\Gamma)\cap\beta^{-1}(X)$ is a closed subvariety of codimension $m-1$ in $\beta^{-1}(X)=\mathcal{C}(X)$. On the other hand, the codimension of $\Gamma_1$ in $V$ is more than $m-1$. For a general $X\in Q$, the codimension of $\alpha^{-1}(\Gamma_1)\cap\beta^{-1}(X)$ in $\beta^{-1}(X)=\mathcal{C}(X)$ is more than $m-1$. Therefore, for a general $X$, we have a closed $(m-1)$-dimensional subvariety $\Xi$ of $\mathcal{C}(X)$ which intersects with $\alpha^{-1}(\Gamma)\cap\beta^{-1}(X)$ at points and does not intersect with $\alpha^{-1}(\Gamma_1)\cap\beta^{-1}(X)$. The subvariety $q(p^{-1}(\Xi))$ of $X$ intersects with $W$ at points. The homology class of $q(p^{-1}(\Xi))$ is just $\Psi_{\mathcal{C}}([\Xi])$. This completes the proof of Claim 1. □

REMARK 2. It is known that every Fano complete intersection of index $f\geqq 2$ is covered by lines (cf. [**21**], the proof of Lemma 1 in Lecture 4). For a Fano complete intersection $X$ of index $f=1$, we can easily see that a subvariety of $X$ of codimension 1 is covered by lines. Hence the above argument can be applied to the family of lines.

**2.c. Proof of Claim 2 for the case where $H^{n-1,1}(X)\neq 0$.** In this subsection, we assume that the Hodge level of $H^n(X)$ is $n-2$. The map $\Psi^*_{\mathcal{C},\mathrm{prim}}$: $H^n_{\mathrm{prim}}(X, \boldsymbol{Q})\to H^{n-2}(\mathcal{C}(X), \boldsymbol{Q})$ is a morphism of Hodge structure of type $(-1, -1)$. We denote the $(n-1, 1)$-part of $\Psi^*_{\mathcal{C},\mathrm{prim}}$ by $\varphi$: $H^{n-1,1}(X)\to H^{n-2,0}(\mathcal{C}(X))$. We shall prove that $\varphi$ is a non-zero map. Take a general point $o\in\mathcal{C}(X)$, and let $T^*_{o,\mathcal{C}(X)}$ be the cotangent space of $\mathcal{C}(X)$ at $o$. We define the infinitesimal dual cylinder map

$$\tau: H^1(X, \Omega^{n-1}_X)\longrightarrow \bigwedge^{n-2} T^*_{o,\mathcal{C}(X)}$$

at $o$ to be the composition of the following maps:

$$H^1(X, \Omega^{n-1}_X)\xrightarrow{\sim} H^{n-1,1}(X), \qquad (\textit{Dolbeault isomorphism})$$

$$H^{n-1,1}(X)\underset{\varphi}{\longrightarrow} H^{n-2,0}(\mathcal{C}(X)),$$

$$H^{n-2,0}(\mathcal{C}(X))\xrightarrow{\sim} H^0(\mathcal{C}(X), \Omega^{n-2}_{\mathcal{C}(X)}) \qquad (\textit{Dolbeault isomorphism}) \text{ and}$$

$$H^0(\mathcal{C}(X), \Omega^{n-2}_{\mathcal{C}(X)})\longrightarrow \bigwedge^{n-2} T^*_{o,\mathcal{C}(X)} \qquad (\textit{restriction at } o).$$

We show that $\tau$ is non-trivial. Let $C_o$ be the conic on $X$ corresponding to $o\in\mathcal{C}(X)$. The map $\tau$ can be described as the composition of five maps (cf. [**22**] p. 21, [**10**] p. 826):

$$\tau_1: H^1(X, \Omega^{n-1}_X)\longrightarrow H^1(C_o, \Omega^{n-1}_X|_{C_o}),$$

$$\tau_2: H^1(C_o, \Omega^{n-1}_X|_{C_o})\longrightarrow H^1(C_o, \Omega^1_{C_o}\otimes \bigwedge^{n-2} N^*_{C_o/X}),$$

$$\tau_3: H^1(C_o, \Omega^1_{C_o}\otimes \bigwedge^{n-2} N^*_{C_o/X})\xrightarrow{\sim} H^0(C_o, \bigwedge^{n-2} N_{C_o/X})^*,$$

$$\tau_4 \colon H^0(C_o, \overset{n-2}{\wedge} N_{C_o/X})^* \longrightarrow (\overset{n-2}{\wedge} H^0(C_o, N_{C_o/X}))^*\,, \qquad \text{and}$$

$$\tau_5 \colon (\overset{n-2}{\wedge} H^0(C_o, N_{C_o/X}))^* \xrightarrow{\sim} \overset{n-2}{\wedge} T^*_{o,C(X)}\,.$$

Here $\tau_1$ is the restriction map, $\tau_2$ is derived from the exact sequence:

$$0 \longrightarrow \overset{n-1}{\wedge} N^*_{C_o/X} \longrightarrow \Omega^{n-1}_X|_{C_o} \longrightarrow \Omega^1_{C_o} \otimes \overset{n-2}{\wedge} N^*_{C_o/X} \longrightarrow 0\,,$$

$\tau_3$ is the Kodaira-Serre duality, $\tau_4$ is the dual of the natural map: $\tau_4^* \colon \overset{n-2}{\wedge} H^0(C_o, N_{C_o/X}) \to H^0(C_o, \overset{n-2}{\wedge} N_{C_o/X})$, and $\tau_5$ is derived from the canonical isomorphism $H^0(C_o, N_{C_o/X}) \cong T_{o,C(X)}$. In order to show that $\tau$ is non-trivial, it is enough to prove the following inequality:

$$\operatorname{codim}\{\operatorname{im}(\tau_2 \circ \tau_1) \subset H^1(C_o, \Omega^1_{C_o} \otimes \overset{n-2}{\wedge} N^*_{C_o/X})\} < \dim(\operatorname{im} \tau_4)\,. \tag{2.1}$$

Note that $\dim(\operatorname{im}\tau_4) = \dim(\operatorname{im}\tau_4^*)$. By Lemma 2, it is easy to see that $\tau_4^*$ is surjective, and

$$h^0(C_o, \overset{n-2}{\wedge} N_{C_o/X}) = 2nf - 4f - n + 3\,. \tag{2.2}$$

Next, we compute the left-hand side of (2.1). Recall that $Y$ is a smooth complete intersection in $\boldsymbol{P}^N$ of multi-degree $(a_1, \cdots, a_{d-1})$ which contains $X$ as a hyperplane section of degree $a_d$. We have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & N^*_{X/Y}|_C & \xrightarrow{\sim} & N^*_{X/Y}|_C & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N^*_{C/Y} & \longrightarrow & \Omega^1_Y|_C & \longrightarrow & \Omega^1_C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow\wr & & \\
0 & \longrightarrow & N^*_{C/X} & \longrightarrow & \Omega^1_X|_C & \longrightarrow & \Omega^1_C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

(Here, we omit $o$ in $C_o$.) By taking $n$-th exterior product of the middle row and tensoring $N_{X/Y}|_C$, we get the commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigwedge^{n-1} N^*_{C/X} & \longrightarrow & \Omega^{n-1}_X|_C & \longrightarrow & \bigwedge^{n-2} N^*_{C/X}\otimes\Omega^1_C & \longrightarrow & 0 \\
 & & \downarrow\wr & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigwedge^{n} N^*_{C/Y}\otimes N_{X/Y} & \longrightarrow & \Omega^{n}_Y\otimes N_{X/Y}|_C & \longrightarrow & \bigwedge^{n-1} N^*_{C/Y}\otimes\Omega^1_C\otimes N_{X/Y} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \Omega^{n}_X\otimes N_{X/Y}|_C & \xrightarrow{\sim} & \bigwedge^{n-1} N^*_{C/X}\otimes\Omega^1_C\otimes N_{X/Y} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

with exact lines and rows. Then, we have the commutative diagram of co-homology groups:

$$
\begin{array}{ccccc}
H^0(X, \Omega^n_X\otimes N_{X/Y}) & \xrightarrow{\sigma_1} & H^0(C, \Omega^n_X\otimes N_{X/Y}|_C) & \xrightarrow{\sigma_2} & H^0(C, \bigwedge^{n-1} N^*_{C/X}\otimes\Omega^1_C\otimes N_{X/Y}) \\
\downarrow & & \downarrow & & \downarrow \\
H^1(X, \Omega^{n-1}_X) & \xrightarrow{\tau_1} & H^1(C, \Omega^{n-1}_X|_C) & \xrightarrow{\tau_2} & H^1(C, \bigwedge^{n-2} N^*_{C/X}\otimes\Omega^1_C) \\
\downarrow & & \downarrow & & \downarrow \\
H^1(X, \Omega^n_Y\otimes N_{X/Y}) & \longrightarrow & H^1(C, \Omega^n_Y\otimes N_{X/Y}|_C) & \longrightarrow & H^1(C, \bigwedge^{n-1} N^*_{C/Y}\otimes\Omega^1_C\otimes N_{X/Y}),
\end{array}
$$

where the vertical sequences are exact. It is clear that the map $\sigma_2$ in this diagram is an isomorphism. The line bundle $\Omega^n_X\otimes N_{X/Y}$ is isomorphic to $\mathcal{O}_{P^N}(-f+a_d)|_X$. Because the restriction map $H^0(\boldsymbol{P}^N, \mathcal{O}_P(-f+a_d))\to H^0(C, \mathcal{O}_P(-f+a_d)|_C)$ is surjective, the map $\sigma_1$ in this diagram is also surjective. Therefore we have

$$
\begin{aligned}
\operatorname{codim}\{\operatorname{im}(\tau_2\circ\tau_1)\subset H^1(C, \Omega^1_C\otimes \textstyle\bigwedge^{n-2} N^*_{C/X})\} &\leqq h^1(C, \textstyle\bigwedge^{n-1} N^*_{C/Y}\otimes\Omega^1_C\otimes N_{X/Y}) \\
&= h^0(C, \textstyle\bigwedge^{n-1} N_{C/Y}\otimes N^*_{X/Y}) \qquad (\textit{Kodaira-Serre duality})
\end{aligned}
$$

Since $Y$ is a Fano complete intersection of index $f+a_d$, we can use Lemma 2 to compute $N_{C/Y}$. We set $N_{C/Y}\cong\bigoplus_{i=1}^n\mathcal{O}_C(c_i)$. Then we have

$$
\begin{aligned}
h^0(C, \textstyle\bigwedge^{n-1} N_{C/Y}\otimes N^*_{X/Y}) &= \sum_{i=1}^n h^0(C, \mathcal{O}_C(2(f+a_d-1)-c_i-2a_d)) \\
&= \sum_{i=1}^n \max(0, 2f-1-c_i) \\
&= \begin{cases} 0 & \textit{if } f=1,\ (n+2)/2\leqq f+a_d, \\ n-1 & \textit{if } f=2,\ d=1,\ (\textit{i.e. } Y=\boldsymbol{P}^{n+1}), \\ 2nf-n-2f-2a_d+2 & \textit{otherwise.} \end{cases}
\end{aligned}
$$

By virtue of Lemma 0, we may assume $f\leqq a_d$. Comparing the above results with (2.2), we see that (2.1) holds under this assumption.

The case that the Hodge level of $H^n(X)$ is less than $n-2$ is dealt with in the next section.

REMARK 3. Let $X$, $Y$ be as above. For a general line $L$ on $X$, we have the following results:

$$\dim(\operatorname{im}\tau_4)=h^0(L, \overset{n-2}{\wedge}N_{L/X})=nf-n-2f+3\,,$$
$$\operatorname{codim}\{\operatorname{im}(\tau_2\circ\tau_1)\subset H^1(L, \Omega_L^1\otimes\overset{n-2}{\wedge}N_{L/X}^*)\}\leqq h^0(L, \overset{n-1}{\wedge}N_{L/Y}\otimes N_{X/Y}^*)$$
$$=\begin{cases}0 & \text{if } 1=f\\ nf-n-f-a_d+2 & \text{if } 2\leqq f\,.\end{cases}$$

## § 3. Geometry of vanishing cycles.

Let $\mathcal{V}$ be a smooth complex projective variety of dimension $n+1$. Let $\mathcal{F}$ be a variety parametrizing a flat family $\{P_u\}_{u\in\mathcal{F}}$ of $k$-dimensional subschemes of $\mathcal{V}$. For any subvariety $\mathcal{S}$ of $\mathcal{V}$, we put

$$\mathcal{F}(\mathcal{S}):=\{u\in\mathcal{F}\mid P_u\subset\mathcal{S}\}.$$

Let $L$ and $M$ be very ample line bundles on $\mathcal{V}$. For a smooth member $\widetilde{W}$ of $|L|$, we denote by $V_n(\widetilde{W}/\mathcal{V}, \boldsymbol{Q})$ the subspace of $H_n(\widetilde{W}, \boldsymbol{Q})$ generated by vanishing cycles of $\widetilde{W}$ in $\mathcal{V}$. For a smooth member $W$ of $|M|_{\widetilde{W}}|$, we define the subspace $V_{n-1}(W/\widetilde{W}, \boldsymbol{Q})$ of $H_{n-1}(W, \boldsymbol{Q})$ in the same way. We have the cylinder homomorphisms

$$\Psi(\widetilde{W})\colon H_{n-2k}(\mathcal{F}(\widetilde{W}), \boldsymbol{Q})\longrightarrow H_n(\widetilde{W}, \boldsymbol{Q}) \qquad \text{and}$$
$$\Psi(W)\colon H_{n-1-2k}(\mathcal{F}(W), \boldsymbol{Q})\longrightarrow H_{n-1}(W, \boldsymbol{Q})\,.$$

Let $\iota_*\colon H_n(W, \boldsymbol{Q})\to H_n(\widetilde{W}, \boldsymbol{Q})$ be the natural map induced from the inclusion $\iota\colon W\hookrightarrow\widetilde{W}$. The main result of this section is the following:

PROPOSITION. *Suppose that, if we take a general member* $\widetilde{W}\in|L|$ *and a general member* $W\in|M|_{\widetilde{W}}|$, *then*

(a) $\mathcal{F}(W)$ *is smooth and complete, and*

(b) *the image of* $\Psi(W)$ *contains* $V_{n-1}(W/\widetilde{W}, \boldsymbol{Q})$.

*Then we have*

$$\operatorname{im}\Psi(\widetilde{W})+\operatorname{im}\iota_*\supset V_n(\widetilde{W}/\mathcal{V}, \boldsymbol{Q})\,.$$

*for a general* $\widetilde{W}\in|L|$.

Before proving Proposition, we shall show how to deduce Theorem from Proposition.

PROOF OF THEOREM. We fix the multi-degree $(a_1, \cdots, a_d)$ of a Fano complete intersection $X$ and prove Theorem by induction with respect to the dimension $n$ of $X$. In §1, we have proved Theorem for a general Fano complete intersection $X$ with $f \leqq a_d$, i.e. $n \leqq a_d - d - 1 + \sum_{i=1}^{d} a_d$. Now assume that Theorem is true for a general Fano complete intersection of multi-degree $(a_1, \cdots, a_d)$ and dimension $n-1$. We apply Proposition to the following case: $\mathcal{V}$ is a general complete intersection of multi-degree $(a_1, \cdots, a_{d-1})$ and dimension $n+1$, $L=\mathcal{O}(a_d)$, $M=\mathcal{O}(1)$, and $\mathcal{F}$ is $\mathcal{C}(\mathcal{V})$ or $\mathcal{L}(\mathcal{V})$. Then the assumption (a) of Proposition is satisfied by Lemma 1, and the assumption (b) is satisfied by the induction hypothesis. If $n$ is odd, the image of $\iota_*: H_n(W, \boldsymbol{Q}) \to H_n(\widetilde{W}, \boldsymbol{Q})$ is zero. If $n$ is even, the image is the space generated by the homology class of an intersection of $\widetilde{W}$ and a linear subspace of $\boldsymbol{P}^N$ of codimension $n/2$. Thus we have $H_n(\widetilde{W}, \boldsymbol{Q}) = V_n(\widetilde{W}/\mathcal{V}, \boldsymbol{Q}) \oplus \operatorname{im} \iota_*$. Since Claim 1 in §2 holds for a general Fano complete intersection of any dimension, we see that $\operatorname{im} \Psi(\widetilde{W}) \supset \operatorname{im} \iota_*$. Now by Proposition, we see that Theorem is true for a general $\widetilde{W} \in |L|$, i.e. for a general Fano complete intersection of multi-degree $(a_1, \cdots, a_d)$ and dimension $n$. □

PROOF OF PROPOSITION. First, we shall show some general lemmas. Let $\varphi$ and $h$ be holomorphic functions defined in a small neighborhood of the origin $o$ of $\boldsymbol{C}^{n+1}$ such that $\varphi(o)=h(o)=0$. We denote by $T_o$ the holomorphic tangent space of $\boldsymbol{C}^{n+1}$ at $o$. Suppose that $o$ is a non-degenerate critical point of $\varphi$. Then we have the Hessian

$$\varphi_{**}: T_o \times T_o \longrightarrow \boldsymbol{C}$$

of $\varphi$ at $o$, which is a non-degenerate symmetric bilinear form of $T_o$. Suppose also that $o$ is not a critical point of $h$. Let $(dh)_o^{\perp}$ be the kernel of $(dh)_o \in T_o^*$.

LEMMA 3. *Assume that $\varphi$ and $h$ satisfy the following condition:*

(#) *the restriction of $\varphi_{**}$ from $T_o$ to $(dh)_o^{\perp}$ remains non-degenerate.*

*Then there is a local coordinate system $(w_1, \cdots, w_{n+1})$ of $\boldsymbol{C}^{n+1}$ around $o$ such that*

$$\varphi = w_1^2 + \cdots + w_{n+1}^2, \tag{3.1}$$

$$\frac{\partial h}{\partial w_i} \equiv 0 \quad \text{for } i=1, \cdots, n, \tag{3.2}$$

$$\frac{\partial h}{\partial w_{n+1}}(0) \neq 0. \tag{3.3}$$

PROOF. We have a local coordinate system $(z_1, \cdots, z_{n+1})$ of $\boldsymbol{C}^{n+1}$ with the center $o$ such that $h=z_{n+1}$. For $s \in \boldsymbol{C}$ which is small enough, we denote by $\varphi_s$ the restriction of $\varphi$ to the hypersurface $h^{-1}(s)$. The critical points of $\varphi_s$ are given by

$$\left\{(z_1, \cdots, z_{n+1}) \mid z_{n+1}=s, \frac{\partial\varphi}{\partial z_1}(z)=\cdots=\frac{\partial\varphi}{\partial z_n}(z)=0\right\}.$$

We put $y_i=\partial\varphi/\partial z_i$ $(i=1, \cdots, n)$. By the condition (#), the $n\times n$ matrix

$$\left(\frac{\partial^2\varphi}{\partial z_i\partial z_j}\right)_{i,j=1,\cdots,n}=\left(\frac{\partial y_j}{\partial z_i}\right)_{i,j=1,\cdots,n}$$

is non-degenerate at $o$. Hence $(y_1, \cdots, y_n, z_{n+1})$ is a local coordinate system of $\boldsymbol{C}^{n+1}$ with the center $o$. (Note that $y_1=\cdots=y_n=0$ at $o$ because $o$ is a critical point of $\varphi$.) Since $\varphi_s(y_1, \cdots, y_n)=\varphi(y_1, \cdots, y_n, s)$ has a critical point at $y_1=\cdots y_n=0$, the Taylor expansion of $\varphi$ is of the form

$$\varphi(y_1, \cdots, y_n, z_{n+1})=z_{n+1}^2\cdot\varphi_1(z_{n+1})+\sum_{i,j=1}^{n} y_i y_j\cdot H_{ij}(y_1, \cdots, y_n, z_{n+1})$$

where $H_{ij}=H_{ji}$. We see that $\varphi_1(0)\neq 0$ and the matrix $(H_{ij}(0, \cdots, 0))_{i,j=1,\cdots,n}$ is non-degenerate, because the critical point $o$ of $\varphi$ is non-degenerate. In the same way as the proof of lemma of Morse (cf. [**16**] p. 6), we can get a local coordinate system $(w_1, \cdots, w_{n+1})$ such that

$$\varphi=w_1^2+\cdots+w_{n+1}^2, \qquad w_{n+1}=z_{n+1}\cdot\sqrt{\varphi_1(z_{n+1})}.$$

The function $z_{n+1}\mapsto w_{n+1}=z_{n+1}\cdot\sqrt{\varphi_1(z_{n+1})}$ has its inverse in a small neighborhood of $w_{n+1}=0$. Since $h=z_{n+1}$, we get (3.2) and (3.3). □

We take a sufficiently small polydisk $\Delta^{n+1}$ in $\boldsymbol{C}^{n+1}$ with the center $o$. We put

$$\tilde{V}_\varepsilon=\varphi^{-1}(\varepsilon)\cap\Delta^{n+1}, \qquad V_{\varepsilon,s}=\varphi^{-1}(\varepsilon)\cap h^{-1}(s)\cap\Delta^{n+1}.$$

We fix $\varepsilon\neq 0$ which is small enough. From Lemma 3, we have

LEMMA 4. *Under the condition* (#) *in Lemma* 3, *we have a small disk* $\Delta$ *in* $\boldsymbol{C}$ *with the center* 0, *and two values* $s_{+1}, s_{-1}\in\Delta$ *such that*

(i) $V_{\varepsilon,s}$ *is smooth for all* $s\in\Delta\setminus\{s_{+1}, s_{-1}\}$, *and*

(ii) $V_{\varepsilon,s_i}$ $(i=\pm 1)$ *has one and only one singular point* $p_i$, *which is an ordinary double point.*

Note that, if $(w_1, \cdots, w_{n+1})$ is the local coordinate system in Lemma 3, then $h$ is a function $h(w_{n+1})$ of one variable $w_{n+1}$, and we have $s_{+1}=h(\sqrt{\varepsilon})$, $s_{-1}=h(-\sqrt{\varepsilon})$.

Now we shall consider the vanishing cycles of $\tilde{V}_\varepsilon$ and $V_{\varepsilon,s}$. For simplicity, we assume $\varepsilon$ to be a small positive real number. Then the vanishing cycle $[\tilde{S}_\varepsilon^+]\in H_n(\tilde{V}_\varepsilon, \boldsymbol{Z})$ corresponding to the ordinary double point $o\in\tilde{V}_0$ is represented by the $n$-dimensional sphere

$$\tilde{S}_\varepsilon=\{(w_1, \cdots, w_{n+1}) \mid w_1^2+\cdots+w_{n+1}^2=\varepsilon,\ w_i\in\boldsymbol{R}\ (i=1, \cdots, n+1)\}\subset\tilde{V}_\varepsilon$$

with an orientation + (cf. [**11**]). We define a path $\gamma_0: [-1, 1] \to \Delta$ connecting $s_{-1}$ and $s_{+1}$ by $\gamma_0(v) = h(v\cdot\sqrt{\varepsilon}) \in \Delta$. We put

$$S_{\varepsilon,\gamma_0(v)} = \{(w_1, \cdots, w_{n+1}) \mid w_1^2 + \cdots w_n^2 = (1-v^2)\varepsilon, \\ w_{n+1} = v\cdot\sqrt{\varepsilon},\ w_i \in \boldsymbol{R}\ (i=1, \cdots, n+1)\} \\ \subset \tilde{S}_\varepsilon \cap V_{\varepsilon,\gamma_0(v)}\,.$$

We choose an orientation + of the $(n-1)$-dimensional sphere $S_{\varepsilon,\gamma_0(v)}$. It is easy to see that

$$H_{n-1}(V_{\varepsilon,\gamma_0(v)}, \boldsymbol{Z}) = \boldsymbol{Z}[S^+_{\varepsilon,\gamma_0(v)}] \quad \text{for } v \in (-1, 1), \qquad \text{and}$$
$$S_{\varepsilon,\gamma_0(-1)} = \{p_{-1}\}\,, \quad S_{\varepsilon,\gamma_0(+1)} = \{p_{+1}\}\,.$$

We also have

$$\tilde{S}_\varepsilon = \bigcup_{v \in [-1,1]} S_{\varepsilon,\gamma_0(v)}\,.$$

By deforming the above construction continuously, we get the following:

LEMMA 5. *Let $\gamma: [-1, 1] \to \Delta$ be a path satisfying the following three conditions:*

(C1) $\gamma(-1) = s_{-1}$, $\gamma(1) = s_{+1}$.

(C2) *$\gamma$ is of $C^\infty$, and if $v \neq v'$, then $\gamma(v) \neq \gamma(v')$.*

(C3) *$\gamma$ can be deformed to $\gamma_0$ preserving the properties* (C1), (C2).

*Then we have an $(n-1)$-dimensional sphere $S_{\varepsilon,\gamma(v)}$ in $V_{\varepsilon,\gamma(v)}$ for each $v \in (-1, 1)$ such that,*

(i) *with an orientation +, $S^+_{\varepsilon,\gamma(v)}$ represents the vanishing cycle corresponding to both of the two ordinary double points $p_{-1} \in V_{\varepsilon,\gamma(-1)}$ and $p_{+1} \in V_{\varepsilon,\gamma(+1)}$,*

(ii) *$\{p_{-1}\} \cup \bigcup_{v \in (-1,1)} S_{\varepsilon,\gamma(v)} \cup \{p_{+1}\}$ is an n-dimensional sphere in $\tilde{V}_\varepsilon$, and that*

(iii) *with an appropriate orientation, this n-dimensional sphere represents the vanishing cycle $[\tilde{S}^+_\varepsilon]$ in $H_n(\tilde{V}_\varepsilon, \boldsymbol{Z})$.*

Let $\{\tilde{W}_t\}_{t \in \boldsymbol{P}^1}$ be a general Lefschetz pencil of the members of $|L|$. Suppose that $\tilde{W}_0$ has an ordinary double point $o \in \tilde{W}_0$. Let $\{H_s\}_{s \in \boldsymbol{P}^1}$ be a general Lefschetz pencil of the members of $|M|$ such that $H_0$ is smooth and $o \in H_0$. We may assume that $o \in \mathcal{V}$ is not contained in the base loci of these pencils. Let $\varphi(w) = t$ (resp. $h(w) = s$) be the local defining equation of $\tilde{W}_t$ (resp. $H_s$) in a small neighborhood of $o \in \mathcal{V}$. By the assumption of generality, we may assume that

(3.4) *$\varphi$ and $h$ satisfy the condition* ($\sharp$) *in Lemma* 3.

We fix a small positive real number $\varepsilon$. Let $\{W_{\varepsilon,s}\}_{s \in \boldsymbol{P}^1}$ be the pencil cut out on $\tilde{W}_\varepsilon$ by $\{H_s\}_{s \in \boldsymbol{P}^1}$. We may also assume that

(3.5) *$\{W_{\varepsilon,s}\}_{s \in \boldsymbol{P}^1}$ is a Lefschetz pencil.*

By the assumption (3.4), we can apply Lemmas 4, 5 to the local geometry of $\widetilde{W}_\varepsilon$ and $W_{\varepsilon,s}$ around $o\in\mathcal{V}$. We continue to use the notation $[\tilde{S}_\varepsilon^+]\in H_n(\widetilde{W}_\varepsilon, \boldsymbol{Z})$, the vanishing cycle corresponding to the ordinary double point $o\in\widetilde{W}_0$. It is enough to show that

$$\operatorname{im}\Psi(\widetilde{W}_\varepsilon)+\operatorname{im}\iota_* \ni [\tilde{S}_\varepsilon^+].$$

Note that because of (3.5), the image of the natural map

$$H_n(W_{\varepsilon,s}, \boldsymbol{Q}) \to H_n(\widetilde{W}_\varepsilon, \boldsymbol{Q})$$

is independent of $s\in\boldsymbol{P}^1$, and this image is just $\operatorname{im}\iota_*$. By Lemma 4 and the assumption (3.5), we have a small neighborhood $\Delta\subset\boldsymbol{P}^1$ of $s=0$ and two points $s_{+1}$, $s_{-1}\in\Delta$ such that (i) $W_{\varepsilon,s}$ is smooth for $s\in\Delta\setminus\{s_{-1}, s_{+1}\}$, and that (ii) $W_{\varepsilon,s_i}$ $(i=\pm1)$ has one and only one singular point $p_i$, which is an ordinary double point. Let $\gamma$: $[-1, 1]\to\Delta$ be a path which satisfies the three conditions in Lemma 5. By Lemma 5, we have an $(n-1)$-dimensional sphere $S_{\varepsilon,\gamma(v)}\subset W_{\varepsilon,\gamma(v)}$ for each $v\in(-1, 1)$ which has the three properties in Lemma 5. In particular, we see that

$$[S_{\varepsilon,\gamma(v)}^+] \in V_{n-1}(W_{\varepsilon,\gamma(v)}/\widetilde{W}_\varepsilon, \boldsymbol{Q}).$$

Let $\tilde{\mathcal{F}}_\varepsilon\subset\boldsymbol{P}^1\times\mathcal{F}$ be the incidence correspondence

$$\tilde{\mathcal{F}}_\varepsilon = \{(s, u)\in\boldsymbol{P}^1\times\mathcal{F} \mid P_u\subset W_{\varepsilon,s}\}$$

with the natural projection $\Pi$: $\tilde{\mathcal{F}}_\varepsilon\to\boldsymbol{P}^1$. By the assumption (a) of Proposition, there is a Zariski open dense subset $U\subset\boldsymbol{P}^1$ over which $\Pi$ is proper and smooth. Then, for $s\in U$, $H_{n-1-2k}(\mathcal{F}(W_{\varepsilon,s}), \boldsymbol{Q})$ has a $\boldsymbol{Q}$-Hodge structure of weight $n-1-2k$, and if $W_{\varepsilon,s}$ is also smooth, then the cylinder homomorphism

$$\Psi_s := \Psi(W_{\varepsilon,s}) : H_{n-1-2k}(\mathcal{F}(W_{\varepsilon,s}), \boldsymbol{Q}) \longrightarrow H_{n-1}(W_{\varepsilon,s}, \boldsymbol{Q})$$

is a morphism of Hodge structure of type $(k, k)$. Let $\omega\in H^2(\mathcal{F}(W_{\varepsilon,s}), \boldsymbol{Q})$ be the restriction of a polarization class of $\tilde{\mathcal{F}}_\varepsilon$ to $\mathcal{F}(W_{\varepsilon,s})$. By construction, $\omega$ is invariant under the monodromy action of $\pi_1(U, s)$. Let

$$L_* : H_{\cdot+2}(\mathcal{F}(W_{\varepsilon,s}), \boldsymbol{Q}) \longrightarrow H_\cdot(\mathcal{F}(W_{\varepsilon,s}), \boldsymbol{Q})$$

be the cap product with $\omega$. We have the Lefschetz decomposition

$$H_{n-1-2k}(\mathcal{F}(W_{\varepsilon,s}), \boldsymbol{Q}) = \bigoplus L_*^\nu P_{n-1-2k+2\nu}(\mathcal{F}(W_{\varepsilon,s}), \boldsymbol{Q})$$

where $P_{n-1-2k+2\nu}(\mathcal{F}(W_{\varepsilon,s}), \boldsymbol{Q})$ is the primitive part of $H_{n-1-2k+2\nu}(\mathcal{F}(W_{\varepsilon,s}), \boldsymbol{Q})$. Note that this decomposition is compatible with the monodromy action of $\pi_1(U, s)$. On the other hand, we have the decomposition

$$H_{n-1}(W_{\varepsilon,s}, \boldsymbol{Q}) = V_{n-1}(W_{\varepsilon,s}/\widetilde{W}_\varepsilon, \boldsymbol{Q})\oplus I_{n-1}(W_{\varepsilon,s}/\widetilde{W}_\varepsilon, \boldsymbol{Q})$$

where $I_{n-1}(W_{\varepsilon,s}/\widetilde{W}_\varepsilon, \boldsymbol{Q})$ is the space of invariant cycles (cf. [**11**]). This decomposition is also compatible with the monodromy action of $\pi_1(U, s)$, and $\pi_1(U, s)$ acts irreduciblly on $V_{n-1}(W_{\varepsilon,s}/\widetilde{W}_\varepsilon, \boldsymbol{Q})$ and trivially on $I_{n-1}(W_{\varepsilon,s}/\widetilde{W}_\varepsilon, \boldsymbol{Q})$. Hence any $\pi_1(U, s)$-invariant subspace of $H_{n-1}(W_{\varepsilon,s}, \boldsymbol{Q})$ which does not contain $V_{n-1}(W_{\varepsilon,s}/\widetilde{W}_\varepsilon, \boldsymbol{Q})$ must be contained in $I_{n-1}(W_{\varepsilon,s}/\widetilde{W}_\varepsilon, \boldsymbol{Q})$. By assumption (b) of Proposition, there is at least one $\mu$ such that

$$\Psi_s(L_*^\mu P_{n-1-2k+2\mu}(\mathcal{F}(W_{\varepsilon,s}), \boldsymbol{Q})) \supset V_{n-1}(W_{\varepsilon,s}/\widetilde{W}_\varepsilon, \boldsymbol{Q}). \tag{3.6}$$

We denote by $\Psi_s^\mu$ the restriction of $\Psi_s$ to $L_*^\mu P_{n-1-2k+2\mu}(\mathcal{F}(W_{\varepsilon,s}), \boldsymbol{Q})$. Because $L_*^\mu P_{n-1-2k+2\mu}(\mathcal{F}(W_{\varepsilon,s}), \boldsymbol{Q})$ has a polarized $\boldsymbol{Q}$-Hodge structure and $\Psi_s^\mu$ is a morphism of Hodge structure, we have the orthogonal decomposition

$$L_*^\mu P_{n-1-2k+2\mu}(\mathcal{F}(W_{\varepsilon,s}), \boldsymbol{Q}) = \ker(\Psi_s^\mu) \oplus \ker(\Psi_s^\mu)^\perp.$$

Note that

(3.7) *this decomposition is compatible with the monodromy action of* $\pi_1(U, s)$, and that

(3.8) *the natural isomorphism* $(\ker \Psi_s^\mu)^\perp \simeq \operatorname{im} \Psi_s^\mu$ *is* $\pi_1(U, s)$*-equivariant.*

Now we can take the path $\gamma: [-1, 1] \to \Delta$ which satisfies an additional condition:

(C4) $\gamma(v) \in \Delta \cap U$ for $v \in (-1, 1)$.

We devide the situation into two cases

*Case* 1. $V_{n-1}(W/\widetilde{W}, \boldsymbol{Q}) \neq 0$, *for general* $\widetilde{W}$ *and* $W$.
*Case* 2. $V_{n-1}(W/\widetilde{W}, \boldsymbol{Q}) = 0$, *for general* $\widetilde{W}$ *and* $W$.

In Case 1, we see that $[S_{\varepsilon,\gamma(v)}^+] \neq 0$ in $H_{n-1}(W_{\varepsilon,\gamma(v)}, \boldsymbol{Q})$, because every vanishing cycle of $W_{\varepsilon,\gamma(v)}$ in $\widetilde{W}_\varepsilon$ is conjugate to each other by the action of the global monodromy (cf. [**11**]). By (3.6), there is a unique cycle $[\tau_{\gamma(v)}] \in (\ker \Psi_{\gamma(v)}^\mu)^\perp$ with coefficients in $\boldsymbol{Q}$ for each $v \in (-1, 1)$ such that

$$\Psi_{\gamma(v)}([\tau_{\gamma(v)}]) = [S_{\varepsilon,\gamma(v)}^+]. \tag{3.9}$$

Let $\eta$ be a positive real number $<1$ which is sufficiently close to 1. Let $\omega_1$: $[0, 2\pi] \to U$ (resp. $\omega_{-1}$: $[0, 2\pi] \to U$) be a small circle with the center $s_{+1} = \gamma(1)$ (resp. $s_{-1} = \gamma(-1)$) and $\omega_{+1}(0) = \omega_{+1}(2\pi) = \gamma(\eta)$ (resp. $\omega_{-1}(0) = \omega_{-1}(2\pi) = \gamma(-\eta)$) whose interior does not contain any point of $\boldsymbol{P}^1 \setminus U$ except $s_{+1}$ (resp. $s_{-1}$).

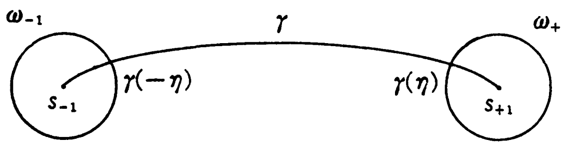


Figure 1.

Let

$$w_{+1,*}: H_{n-1-2k}(\mathcal{F}(W_{\varepsilon,\gamma(\eta)}), \boldsymbol{Z}) \longrightarrow H_{n-1-2k}(\mathcal{F}(W_{\varepsilon,\gamma(\eta)}), \boldsymbol{Z})$$
$$w'_{+1,*}: H_{n-1}(W_{\varepsilon,\gamma(\eta)}, \boldsymbol{Z}) \longrightarrow H_{n-1}(W_{\varepsilon,\gamma(\eta)}, \boldsymbol{Z})$$

be the local monodromies along $\omega_{+1}$. We shall show that there is a cycle $[\beta_0]\in H_{n-1-2k}(\mathfrak{F}(W_{\varepsilon,\gamma(\eta)}), \boldsymbol{Q})$ such that

$$[\tau_{\gamma(\eta)}]=q\cdot([\beta_0]-w_{+1,*}([\beta_0])) \quad \text{in } H_{n-1-2k}(\mathfrak{F}(W_{\varepsilon,\gamma(\eta)}), \boldsymbol{Q})$$

where $q\in \boldsymbol{Q}^{\times}$. Since the intersection form of $H_{n-1}(W_{\varepsilon,\gamma(\eta)}, \boldsymbol{Q})$ restricted to $V_{n-1}(W_{\varepsilon,\gamma(\eta)}/\widetilde{W}_{\varepsilon}, \boldsymbol{Q})$ is non-degenerate, there is a cycle $[\alpha]\in V_{n-1}(W_{\varepsilon,\gamma(\eta)}/\widetilde{W}_{\varepsilon}, \boldsymbol{Q})$ such that $[\alpha]\cdot[S^{+}_{\varepsilon,\gamma(\eta)}]\neq 0$. Let $[\beta_0]$ be the unique element of $(\ker \Psi^{\mu}_{\gamma(\eta)})^{\perp}$ such that $\Psi_{\gamma(\eta)}([\beta_0])=[\alpha]$. By the Picard-Lefschetz formula, we have

$$w'_{+1,*}([\alpha])=[\alpha]\pm([\alpha]\cdot[S^{+}_{\varepsilon,\gamma(\eta)}])[S^{+}_{\varepsilon,\gamma(\eta)}].$$

By (3.7) and (3.8), we see that

$$[\beta_0]-w_{+1,*}([\beta_0])\in(\ker \Psi^{\mu}_{\gamma(\eta)})^{\perp} \quad \text{and}$$

$$\Psi_{\gamma(\eta)}([\beta_0]-w_{+1,*}([\beta_0]))=\pm([\alpha]\cdot[S^{+}_{\varepsilon,\gamma(\eta)}])[S^{+}_{\varepsilon,\gamma(\eta)}].$$

Hence we have

$$[\tau_{\gamma(\eta)}]=\pm 1/([\alpha]\cdot[S^{+}_{\varepsilon,\gamma(\eta)}])\cdot([\beta_0]-w_{+1,*}([\beta_0])).$$

Let $i$: $H_{n-1-2k}(\mathfrak{F}(W_{\varepsilon,\gamma(\eta)}), \boldsymbol{Z})\to H_{n-1-2k}(\mathfrak{F}(W_{\varepsilon,\gamma(\eta)}), \boldsymbol{Q})$ be the natural map. Note that $\ker i$ is the torsion part of $H_{n-1-2k}(\mathfrak{F}(W_{\varepsilon,\gamma(\eta)}), \boldsymbol{Z})$. There is an integer $N$ such that $N\cdot(\ker i)=0$. Now we have topological cycles $[T_{\gamma(\eta)}]$, $[B_0]\in H_{n-1-2k}(\mathfrak{F}(W_{\varepsilon,\gamma(\eta)}), \boldsymbol{Z})$ such that

$$i([T_{\gamma(\eta)}])=M_1\cdot([\tau_{\gamma(\eta)}]),\quad i([B_0])=M_2\cdot([\beta_0]) \quad \text{and}$$

$$[T_{\gamma(\eta)}]=[B_0]-w_{+1,*}([B_0])$$

where $M_1$, $M_2$ are non-zero integers. For each $\theta\in[0, 2\pi]$, we can construct a topological $(n-1-2k)$-cycle $B_\theta$ in $\mathfrak{F}(W_{\varepsilon,\omega_{+1}(\theta)})$ such that $B_0$ represents $[B_0]$ in $H_{n-1-2k}(\mathfrak{F}(W_{\varepsilon,\omega_{+1}(0)}, \boldsymbol{Z})$ and $B_\theta$ deforms continuously as $\theta$ moves. Then $B_{2\pi}$ represents $w_{+1,*}([B_0])$, and we get a topological $(n-2k)$-chain $\widetilde{B}_{+1}:=\cup_{\theta\in[0,2\pi]}B_\theta$ in $\widetilde{\mathfrak{F}}_{\varepsilon}$, contained in $\cup_{\theta\in[0,2\pi]}\mathfrak{F}(W_{\varepsilon,\omega_{+1}(\theta)})=\Pi^{-1}(\omega_{+1}([0, 2\pi]))$, with the orientation satisfying $\partial\widetilde{B}_{+1}=B_0-B_{2\pi}$. For each $v\in[-\eta, \eta]$, we also have a topological $(n-1-2k)$-cycle $T_{\gamma(v)}$ in $\mathfrak{F}(W_{\varepsilon,\gamma(v)})$ which deforms continuously in $v$ and satisfies

$$\text{(3.10)} \qquad i([T_{\gamma(v)}])=M_1\cdot[\tau_{\gamma(v)}] \quad \textit{for each } v\in[-\eta, \eta].$$

We get a topological $(n-2k)$-chain $\widetilde{T}:=\cup_{v\in[-\eta,\eta]}T_{\gamma(v)}$ in $\widetilde{\mathfrak{F}}_{\varepsilon}$ with the orientation satisfying $\partial\widetilde{T}=T_{\gamma(\eta)}-T_{\gamma(-\eta)}$. Since $\partial\widetilde{B}_{+1}=B_0-B_{2\pi}$ and $T_{\gamma(\eta)}$ represent the same homology class in $H_{n-1-2k}(\mathfrak{F}(W_{\varepsilon,\gamma(\eta)}), \boldsymbol{Z})$, we have a topological $(n-2k)$-chain $J_{\gamma(\eta)}\subset\mathfrak{F}(W_{\varepsilon,\gamma(\eta)})$ such that $\partial J_{\gamma(\eta)}=B_0-B_{2\pi}-T_{\gamma(\eta)}$. Then $\partial((-\widetilde{B}_{+1})+J_{\gamma(\eta)}+\widetilde{T})=-T_{\gamma(-\eta)}$. Now, around $\gamma(-1)=s_{-1}$, we can construct in the same way a topological $(n-2k)$-chain $\widetilde{B}_{-1}$ contained in $\Pi^{-1}(\omega_{-1}([0, 2\pi]))$ such that $\partial\widetilde{B}_{-1}$ is a topological $(n-1-2k)$-cycle contained in $\mathfrak{F}(W_{\varepsilon,\gamma(-\eta)})$ which represents the same homology class as $[-T_{\gamma(-\eta)}]$ in $H_{n-1-2k}(\mathfrak{F}(W_{\varepsilon,\gamma(-\eta)}), \boldsymbol{Z})$. Let $J_{\gamma(-\eta)}\subset\mathfrak{F}(W_{\varepsilon,\gamma(-\eta)})$ be a topological $(n-2k)$-chain such that $\partial J_{\gamma(-\eta)}=\partial\widetilde{B}_{-1}+T_{\gamma(-\eta)}$. Then

$$\partial(-\tilde{B}_{+1}+J_{\gamma(\eta)}+\hat{T}+J_{\gamma(-\eta)}-\tilde{B}_{-1})=0\,.$$

Thus we get a topological $(n-2k)$-*cycle* in $\tilde{\mathcal{F}}_{\varepsilon}$, which we will denote by $\Gamma$.

There is a natural morphism $j:\tilde{\mathcal{F}}_{\varepsilon}\to\mathcal{F}(\widetilde{W}_{\varepsilon})$ induced by the inclusion $\mathcal{F}(W_{\varepsilon,s})\hookrightarrow\mathcal{F}(\widetilde{W}_{\varepsilon})$. Because of (3.9) and (3.10), we have

$$\Psi_{\gamma(v)}(i([T_{\gamma(v)}])=M_1\cdot([S^{+}_{\varepsilon,\gamma(v)}]) \quad in\ H_{n-1}(W_{\varepsilon,s},\boldsymbol{Q}) \tag{3.11}$$

for $v\in[-\eta,\eta]$. Recall that the $n$-dimensional sphere $\{p_{-1}\}\cup\bigcup_{v\in(-1,1)}S_{\varepsilon,\gamma(v)}\cup\{p_{+1}\}$ with an appropriate choice of the orientation represents $[\tilde{S}^{+}_{\varepsilon}]$ in $H_n(\widetilde{W}_{\varepsilon},\boldsymbol{Z})$. We see that

$$\Psi(\widetilde{W}_{\varepsilon})(j_*[\Gamma])-M_1\cdot([\tilde{S}^{+}_{\varepsilon}]) \quad\text{or}\quad \Psi(\widetilde{W}_{\varepsilon})(j_*[\Gamma])+M_1\cdot([\tilde{S}^{+}_{\varepsilon}])$$

can be represented by a topological $n$-cycle $C_{+1}+C_{-1}$ with coefficients in $\boldsymbol{Q}$ such that the support of $C_{+1}$ (resp. $C_{-1}$) is contained in $\bigcup_{s\in\delta_{+1}}W_{\varepsilon,s}\subset\widetilde{W}_{\varepsilon}$ (resp. $\bigcup_{s\in\delta_{-1}}W_{\varepsilon,s}\subset\widetilde{W}_{\varepsilon}$) where $\delta_{+1}=\omega_{+1}([0,2\pi])\cup\gamma([\eta,1])$ (resp. $\delta_{-1}=\omega_{-1}([0,2\pi])\cup\gamma([-1,-\eta])$).

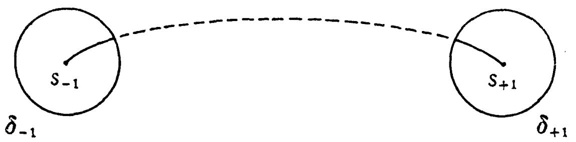


Figure 2.

In fact, the topological cycle $C_{+1}+C_{-1}$ is constructed as follows. For a topological chain $A$ in $\tilde{\mathcal{F}}_{\varepsilon}$, let $\phi(A)$ denote the topological chain $\bigcup_{u\in j(A)}P_u\subset\widetilde{W}_{\varepsilon}$. By (3.11), the cycle $M_1\cdot[S^{+}_{\varepsilon,\gamma(v)}]-[\phi(T_{\gamma(v)})]\in H_{n-1}(W_{\varepsilon,\gamma(v)},\boldsymbol{Z})$ is contained in the torsion part of $H_{n-1}(W_{\varepsilon,\gamma(v)},\boldsymbol{Z})$. Let $N'$ be a non-zero integer such that $N'\cdot$(torsion part of $H_{n-1}(W_{\varepsilon,\gamma(v)},\boldsymbol{Z})$)$=0$, which is independent of $v\in[-\eta,\eta]$. Then, because of $N'\cdot M_1\cdot[S^{+}_{\varepsilon,\gamma(v)}]=N'\cdot[\phi(T_{\gamma(v)})]$ in $H_{n-1}(W_{\varepsilon,\gamma(v)},\boldsymbol{Z})$, we have a topological $n$-chain $I_{\gamma(v)}\subset W_{\varepsilon,\gamma(v)}$ such that

$$\partial I_{\gamma(v)}=N'\cdot\phi(T_{\gamma(v)})-N'\cdot M_1\cdot S^{+}_{\varepsilon,\gamma(v)}\,.$$

Let $b:\widetilde{W}'_{\varepsilon}\to\widetilde{W}_{\varepsilon}$ be the blowing up of $\widetilde{W}_{\varepsilon}$ along the base locus of the pencil $\{W_{\varepsilon,s}\}_{s\in\boldsymbol{P}^1}$. We have the natural morphism $\widetilde{W}'_{\varepsilon}\to\boldsymbol{P}^1$, which is topologically trivial over $\gamma([-\eta,\eta])\subset\boldsymbol{P}^1$. Thus we may assume that $I_{\gamma(v)}$ deforms continuously as $v$ moves. Let $\tilde{M}_{\varepsilon}$ be the $n$-chain $\bigcup_{v\in[-\eta,\eta]}S_{\varepsilon,\gamma(v)}$ in $\widetilde{W}_{\varepsilon}$ with the orientation satisfying $\partial\tilde{M}_{\varepsilon}=S^{+}_{\varepsilon,\gamma(\eta)}-S^{+}_{\varepsilon,\gamma(-\eta)}$. We have an $(n+1)$-chain $\tilde{I}_{\varepsilon}=b(\bigcup_{v\in[-\eta,\eta]}I_{\gamma(v)})$ in $\widetilde{W}_{\varepsilon}$ with the orientation satisfying

$$\partial\tilde{I}_{\varepsilon}=I_{\gamma(\eta)}-I_{\gamma(-\eta)}+N'\cdot M_1\cdot\tilde{M}_{\varepsilon}-N'\cdot\phi(\tilde{T})\,.$$

Let $\tilde{D}_{\varepsilon,+1}$ (resp. $\tilde{D}_{\varepsilon,-1}$) be the $n$-chain

$$\{p_{+1}\}\cup\bigcup_{v\in[\eta,1)}S_{\varepsilon,\gamma(v)}\ (\text{resp. } \{p_{-1}\}\cup\bigcup_{v\in(-1,-\eta]}S_{\varepsilon,\gamma(v)})$$

with the orientation satisfying $\partial\tilde{D}_{\varepsilon,+1}=-S^{+}_{\varepsilon,\gamma(\eta)}$ (resp. $\partial\tilde{D}_{\varepsilon,-1}=S^{+}_{\varepsilon,\gamma(-\eta)}$). Then we see that

$$\tilde{D}_{\varepsilon,+1}+\tilde{M}_{\varepsilon}+\tilde{D}_{\varepsilon,-1}=+\tilde{S}^{+}_{\varepsilon} \quad \text{or} \quad -\tilde{S}^{+}_{\varepsilon}, \quad \text{and}$$

$$\begin{aligned} &N'\cdot\phi(j(\Gamma))-N'\cdot M_1\cdot(\tilde{D}_{\varepsilon,+1}+\tilde{M}_{\varepsilon}+\tilde{D}_{\varepsilon,-1})\\ &=-\partial\tilde{I}_{\varepsilon}+N'\cdot\phi(j(-\tilde{B}_{+1}+J_{\gamma(\eta)}))+I_{\gamma(\eta)}-N'\cdot M_1\cdot\tilde{D}_{\varepsilon,+1}\\ &\quad+N'\cdot\phi(j(-\tilde{B}_{-1}+J_{\gamma(-\eta)}))-I_{\gamma(-\eta)}-N'\cdot M_1\cdot\tilde{D}_{\varepsilon,-1}. \end{aligned}$$

Then we have

$$C_{+1}=\frac{1}{N'}(N'\cdot\phi(j(-\tilde{B}_{+1}+J_{\gamma(\eta)}))+I_{\gamma(\eta)}-N'\cdot M_1\cdot\tilde{D}_{\varepsilon,+1}), \quad \text{and}$$

$$C_{-1}=\frac{1}{N'}(N'\cdot\phi(j(-\tilde{B}_{-1}+J_{\gamma(-\eta)}))-I_{\gamma(-\eta)}-N'\cdot M_1\cdot\tilde{D}_{\varepsilon,-1}).$$

It is easy to see that the cycle $C_i$ can be deformed to the cycle contained in $W_{\varepsilon,\gamma(i)}$ $(i=\pm 1)$. Hence $[C_{+1}]$, $[C_{-1}]\in\operatorname{im}\iota_*$. Thus we get

$$M_1\cdot[\tilde{S}^{+}_{\varepsilon}]\in\operatorname{im}\Psi(\tilde{W}_{\varepsilon})+\operatorname{im}\iota_*.$$

In Case 2, i. e., $V_{n-1}(W/\tilde{W}, \boldsymbol{Q})=0$ for general $\tilde{W}$ and $W$, we shall prove that

$$V_n(\tilde{W}/\mathcal{V}, \boldsymbol{Q})\subset\operatorname{im}\iota_*$$

for a general $\tilde{W}$. It is enough to show that $[\tilde{S}^{+}_{\varepsilon}]\in\operatorname{im}\iota_*$. By the assumption of this case, we may assume that the homology class $[S^{+}_{\varepsilon,\gamma(v)}]\in H_{n-1}(W_{\varepsilon,\gamma(v)}, \boldsymbol{Z})$ is a torsion for $v\in[-\eta, \eta]$. Hence there is an topological $n$-chain $F_{\varepsilon,\gamma(v)}\subset W_{\varepsilon,\gamma(v)}$ such that $\partial F_{\varepsilon,\gamma(v)}=N''\cdot S^{+}_{\varepsilon,\gamma(v)}$ where $N''$ is a non-zero integer which does not depend on $v\in[-\eta, \eta]$. We may assume that the $n$-chain $F_{\varepsilon,\gamma(v)}$ deforms continuously as $v$ moves, and we get an $(n+1)$-chain $\tilde{F}_{\varepsilon}=\bigcup_{v\in[-\eta,\eta]}F_{\varepsilon,\gamma(v)}$. Now, by the similar argument as in Case 1, we have a topological $n$-cycle $E_{+1}+E_{-1}$ in $\tilde{W}'_{\varepsilon}$ such that the support of $E_{+1}$ (resp. $E_{-1}$) is contained in $\bigcup_{v\in[\eta,1]}W_{\varepsilon,\gamma(v)}\subset\tilde{W}'_{\varepsilon}$ (resp. $\bigcup_{v\in[-1,-\eta]}W_{\varepsilon,\gamma(v)}\subset\tilde{W}'_{\varepsilon}$), and

$$N''\cdot\tilde{S}^{+}_{\varepsilon}-(E_{+1}+E_{-1}) \quad \text{or} \quad -N''\cdot\tilde{S}^{+}_{\varepsilon}-(E_{+1}+E_{-1})$$

is the boundary $\partial\tilde{F}_{\varepsilon}$. Because $[E_{+1}]$, $[E_{-1}]\in\operatorname{im}\iota_*$, we see that $N''\cdot[\tilde{S}^{+}_{\varepsilon}]\in\operatorname{im}\iota_*$. □

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Ichiro SHIMADA
Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Tokyo 113
Japan