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The q-bracket product and quantum enveloping algebras of classical types

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1. Introduction.

Let A be an algebra over a commutative ring R and let $x_1, \cdots, x_n \in A$. We say A is a *polynomial algebra in* $x_1, \cdots, x_n$ if the set of all monomials $x_1{}^{a_1} \cdots x_n{}^{a_n}$, $a_i \in \boldsymbol{N}$, forms a free base for the R-module A. Note that the concept depends on the total ordering on generators.

If L is a finite dimensional Lie algebra over a field k with a base $z_1, \cdots, z_N$, the famous Poincaré-Birkhoff-Witt theorem tells that the universal enveloping algebra $U(L)$ is a polynomial algebra in $z_1, \cdots, z_N$.

Let $(a_{ij})_{1 \leq i, j \leq n}$ be a symmetrizable generalized Cartan matrix. The corresponding quantum enveloping algebra $\hat{U}$ was introduced by Drinfeld [**2**, **3**] and Jimbo [**4**]. We follow Lusztig's formulation [**6**].

Take integers $d_i \neq 0$ such that $d_i a_{ij} = d_j a_{ji}$. Let k be a field with $q \in k^\times$ such that $q^{4d_i} \neq 1$ $(1 \leqq i \leqq n)$. $\hat{U}$ is the k-algebra (associative with 1) with generators e_i, f_i, k_i, k_i^{-1} $(1 \leqq i \leqq n)$ and relations

$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \qquad k_i k_j = k_j k_i\,, \tag{1.1}$$

$$k_i e_j k_i^{-1} = q^{d_i a_{ij}} e_j, \qquad k_i f_j k_i^{-1} = q^{-d_i a_{ij}} f_j\,, \tag{1.2}$$

$$e_i f_j - f_j e_i = \delta_{ij} \frac{k_i^2 - k_i^{-2}}{q^{2d_i} - q^{-2d_i}}\,, \tag{1.3}$$

$$\sum_{\nu=0}^{1-a_{ij}} \begin{bmatrix} 1-a_{ij} \\ \nu \end{bmatrix}_{q^{2d_i}} e_i^{1-a_{ij}-\nu} e_j (-e_i)^\nu = 0 \qquad (i \neq j)\,, \tag{1.4}$$

$$\sum_{\nu=0}^{1-a_{ij}} \begin{bmatrix} 1-a_{ij} \\ \nu \end{bmatrix}_{q^{2d_i}} f_i^{1-a_{ij}-\nu} f_j (-f_i)^\nu = 0 \qquad (i \neq j)\,. \tag{1.5}$$

Here, we use the notations

$$\begin{bmatrix} m \\ n \end{bmatrix}_t = \frac{[m]_t}{[n]_t [m-n]_t} \in \boldsymbol{Z}[t, t^{-1}]\,,$$

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$$[m]_t = (t-t^{-1})(t^2-t^{-2})\cdots(t^m-t^{-m})$$

for integers $m \geqq n \geqq 0$.

It is known that we have

$$\hat{U} = \hat{U}^+ \otimes \hat{U}^0 \otimes \hat{U}^-$$

where $\hat{U}^+$ (resp. $\hat{U}^-$) is the k-algebra with generators e_i (resp. f_i) ($1 \leqq i \leqq n$) and relation (1.4) (resp. (1.5)), and $\hat{U}^0 = k[k_1, \cdots, k_n, k_1^{-1}, \cdots, k_n^{-1}]$ (cf. Yamane [**9**]). It is natural to ask whether $\hat{U}^+$ and $\hat{U}^-$ are polynomial algebras over k in our sense. Recently, Lusztig [**10, 11**] answers this question affirmatively in case k is of characteristic zero and q is trancsendental over the prime field $\boldsymbol{Q}$ (see also Corollary 3.7a). We consider the problem over a general base field (or a commutative ring). The case of type (A_n)

$$(a_{ij}) = \begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}$$

is studied by some authors [**7**], [**9**]. Our main result (Sections 4, 7) tells that $\hat{U}^+$ and $\hat{U}^-$ are polynomial algebras in cases of type (B_n), (C_n), and (D_n), too:

$$\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -2 & 2 \end{pmatrix} (B_n), \qquad \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & \ddots & & & \\ & -1 & \ddots & -1 & & \\ & & \ddots & 2 & -1 & -1 \\ & & & -1 & 2 & 0 \\ & & & -1 & 0 & 2 \end{pmatrix} (D_n).$$

(The matrix of type (C_n) is the transpose of the matrix of type (B_n).)

Let us be more precise. We concentrate on $\hat{U}^+$ (not on the whole $\hat{U}$). The defining relation (1.4) tells that $\hat{U}^+$ is defined over $\boldsymbol{Z}[q^M, q^{-M}]$, where $M \geqq 2$, an integer such that

$$\begin{bmatrix} 1-a_{ij} \\ \nu \end{bmatrix}_{q^{2d_i}} \in \boldsymbol{Z}[q^M, q^{-M}] \qquad (i \neq j,\ 0 \leqq \nu \leqq 1-a_{ij}).$$

(Forget the condition $q^{4d_i} \neq 1$.) To avoid this redundancy, we make a substitution $q \to q^{1/M}$, or $d_i \to d_i/M$. In each case of our concern, we take

$$\begin{aligned} &d_1 = \cdots = d_n = 1/2 && (A_n), (D_n), \\ &d_1 = \cdots = d_{n-1} = 1/2, \quad d_n = 1/4 && (B_n), \\ &d_1 = \cdots = d_{n-1} = 1/2, \quad d_n = 1 && (C_n). \end{aligned}$$

Let R be a commutative ring with $q \in R^{\times}$. With these data, let $\mathcal{A}_n$, $\mathcal{B}_n$, $\mathcal{C}_n$, and $\mathcal{D}_n$ be the R-algebra with generators e_i ($1 \leqq i \leqq n$) and relation (1.4) corresponding to (A_n), (B_n), (C_n), and (D_n), respectively. We need the *q-bracket product* to describe the polynomial generators. When x, y are elements of an R-algebra, we put

$$[x, y] = xy - qyx, \qquad [x, y]_2 = xy - q^2yx.$$

Basic properties of the q-bracket product are developed in §2. For elements $x_1, \cdots, x_m$ in an R-algebra, we define inductively

$$[x_1, \cdots, x_m] = [[x_1, \cdots, x_{m-1}], x_m].$$

Similarly, we define $[x_1, \cdots, x_m]_2$.

THEOREM 1.6. a) *If* $1+q^2 \in R^\times$, $\mathcal{A}_n$ *is a polynomial algebra in*

$$[e_i, e_{i+1}, \cdots, e_j] \qquad (1 \leqq i \leqq j \leqq n).$$

b) *If* $1+q^2$, $1+q+q^2 \in R^\times$, $\mathcal{B}_n$ *is a polynomial algebra in*

$$[e_i, e_{i+1}, \cdots, e_j] \qquad (1 \leqq i \leqq j \leqq n),$$
$$[[e_j, \cdots, e_n], [e_i, \cdots, e_n]] \qquad (1 \leqq i \leqq j \leqq n).$$

c) *If* $1+q^2$, $1+q^4$, $1+q^2+q^4 \in R^\times$, $\mathcal{C}_n$ *is a polynomial algebra in*

$$e_n, \quad [e_i, e_{i+1}, \cdots, e_j] \quad (1 \leqq i \leqq j \leqq n-1),$$
$$[[e_i, \cdots, e_{n-1}], e_n]_2 \quad (1 \leqq i \leqq n-1),$$
$$[[e_i, \cdots, e_{n-1}], e_n, [e_i, \cdots, e_{n-1}]]_2 \quad (1 \leqq i \leqq n-1),$$
$$[[[e_i, \cdots, e_{n-1}], e_n]_2, [e_j, \cdots, e_{n-1}]] \quad (1 \leqq i \leqq j \leqq n-1).$$

d) *If* $1+q^2 \in R^\times$, $\mathcal{D}_n$ *is a polynomial algebra in*

$$[e_i, e_{i+1}, \cdots, e_j] \quad (1 \leqq i \leqq j \leqq n), \quad (i, j) \neq (n-1, n),$$
$$[e_i, \cdots, e_{n-2}, e_n] \quad (1 \leqq i \leqq n-2),$$
$$[[e_i, \cdots, e_{n-1}], [e_j, \cdots, e_{n-2}, e_n]] \quad (1 \leqq i < j \leqq n-2).$$

More precisely, we should specify some total ordering on the generators. (We cannot use arbitrary orderings (Proposition 5.6).) The description of the relevant orderings, as well as the proof, will be given in Section 4 (Theorems 4.5, 4.15, 4.23) for a), b), d), and in Section 7 (Theorem 7.10) for c).

CONVENTIONS. We work over a commutative ring R. An algebra means an associative R-algebra with 1, and $\otimes$ means $\otimes_R$. We fix an element q in $R^\times$, the group of units in R. When we refer to lexicographical orderings (mostly for indices (ij)), we always read from left to right.

2. The q-bracket product.

DEFINITION 2.1. We put

$$[x, y] = xy - qyx, \quad (x, y) = xy - yx, \quad (x, y)_a = xy - ayx$$

for $a \in R$, and x, y in some algebra.

LEMMA 2.2 (generalized Jacobi identity). *we have*

$$(x, y)_a z + ay(x, z)_b + ab(y, z)_c x = x(y, z)_c + c(x, z)_b y + bcz(x, y)_a$$

for $a, b, c \in R$, and x, y, z in some algebra.

This is easily verified.

COROLLARY 2.3. *Let $a, b, c \in R$ and let x, y, z be elements in some algebra.*

a) *If $(x, z)_b = 0$, then $((x, y)_a, z)_{bc} = (x, (y, z)_c)_{ab}$.*

b) *If $(x, y)_a = 0$, then $(x, (y, z)_{ac})_{ab} = a(y, (x, z)_b)_c$.*

c) *If $(y, z)_c = 0$, then $((x, y)_{ac}, z)_{bc} = c((x, z)_b, y)_a$.*

This corollary will frequently be used throughout the paper.

DEFINITION 2.4. For x, y, z in some algebra, we put

$$[x, y, z] = [[x, y], z] .$$

It equals $[x, [y, z]]$ if $(x, z) = 0$. When $[x, y, x] = 0$, we write

$$x \to y \quad \text{or} \quad y \leftarrow x .$$

Throughout the rest of § 2, we assume that

$$1 + q^2 \in R^{\times} .$$

PROPOSITION 2.5. *If $y \to x$, $y \to z$, and $(x, z) = 0$ in some algebra, then we have*

a) $(y, [x, y, z]) = 0$,

b) $([y, z], [x, y]) = (q - q^{-1}) y [x, y, z]$.

PROOF. Let $U = [x, y]$, $V = [y, z]$, $W = [x, y, z]$. Since $[y, U] = 0$, we have by (2.3 b)

$$(y, [U, z])_{q^2} = q(U, [y, z]), \quad \text{or} \quad (y, W)_{q^2} = q(U, V) .$$

Similarly, by (2.3 c), we have

$$(W, y)_{q^2} = q(U, V) ,$$

since $[V, y] = 0$. Hence $0 = (y, W)_{q^2} - (W, y)_{q^2} = (1 + q^2)(y, W)$. Since $1 + q^2 \in R^{\times}$, the assertion will follow from the above identities. Q. E. D.

PROPOSITION 2.6. *Let $c \in R$. If $x \to y$, $x \to z$, and $(y, z)_c = 0$ in some algebra,* *then we have*

$$([x, y], [x, z])_c = 0 = ([y, x], [z, x])_c .$$

PROOF. Let $U=[x, y]$ and $V=[x, z]$. Since $(y, z)_c=0$, we have by (2.3a)

$$((y, x)_{q^{-1}}, z)_{qc} = (y, [x, z])_{q^{-1}c}, \qquad \text{or}$$

$$(U, z)_{qc}+q(y, V)_{q^{-1}c} = 0 . \tag{2.6.1}$$

Since $[U, x]=0$, we have

$$(U, V)_{q^2c} = (U, [x, z])_{q^2c} = q(x, (U, z)_{qc}) \tag{2.6.2}$$

by (2.3b). Similarly,

$$(U, V)_{q^{-2}c} = (x, (y, V)_{q^{-1}c}) \tag{2.6.3}$$

since $(x, V)_{q^{-1}}=0$. It follows from (2.6.1)-(2.6.3) that

$$0 = (U, V)_{q^2c}+q^2(U, V)_{q^{-2}c} = (1+q^2)(U, V)_c .$$

Hence $(U, V)_c=0$ by assumption. To see $([y, x], [z, x])_c=0$, use replacement $q \to q^{-1}$. Q.E.D.

PROPOSITION 2.7. *Let x, y, z be elements in some algebra. Assume $(x, z)=0$.*

a) *If $x \to y \to z$, then $[x, y] \to z$ and $x \to [y, z]$.*
b) *If $x \leftarrow y \leftarrow z$, then $[x, y] \leftarrow z$ and $x \leftarrow [y, z]$.*

PROOF. a) Since $(x, z)=0$, we have $x \to [y, z]$ obviously. Since $[[y, z], y]=0$, it follows from Proposition 2.6 that

$$0 = [[x, [y, z]], [x, y]]=[[x, y], z, [x, y]] ,$$

i.e., $[x, y] \to z$. b) is similar. Q.E.D.

DEFINITION 2.8. For x, y in some algebra, we write

$$x \Rightarrow y \qquad \text{or} \qquad y \Leftarrow x$$

if $[x, y, x]$ commutes with x, or equivalently if $x \to (x, y)$.

PROPOSITION 2.9. *Let x, y, z be elements in some algebra. Assume $x \leftarrow y \Leftarrow z$ and $(x, z)=0$.*

a) *We have $[x, y] \Leftarrow z$.*
b) *If $y \to z$ in addition, we have $x \Leftarrow [y, z]$.*

PROOF. We put

$$w = [z, [x, y], z] = [x, [z, y, z]] .$$

This commutes with z, since so do x and $[z, y, z]$. This yields a). If $y \to z$ in addition, $[z, y, z]$ commutes with y by Proposition 2.5, hence it commutes with $[y, z]$, too. It follows that

$$[w, [y, z]] = [[x, y, z], [z, y, z]] .$$

This commutes with y, by Proposition 2.5 again. Thus

$$0=(y, [w, [y, z]])=([y, w], [y, z]).$$

The second equality follows from (2.3 a), since $(y, [y, z])_{q^{-1}}=0$. We have

$$[y, w]=[y, [z, [x, y, z]]]=[[y, z], x, [y, z]]$$

since $[x, y, z]$ commutes with y. The fact that $[y, w]$ commutes with $[y, z]$ will mean $[y, z]\Rightarrow x$, i.e. b). Q.E.D.

3. Algebraic independence of Lie elements.

We introduce the notion of quantized Lie elements in some loose sense. The algebraic independence theorem (3.7) will play a key role.

Let $\mathfrak{g}$ be the Kac-Moody algebra over $\boldsymbol{C}$ corresponding to the generalized Cartan matrix $(a_{ij})_{1\leqq i,j\leqq n}$. (We are interested in case when $\mathfrak{g}$ is finite dimensional simple.) It is the complex Lie algebra with generators $\bar{e}_i$, $\bar{f}_i$, h_i $(1\leqq i\leqq n)$ and relations

$$(3.1)\qquad (h_i, h_j)=0,$$

$$(3.2)\qquad (h_i, \bar{e}_j)=a_{ij}\bar{e}_j, \qquad (h_i, \bar{f}_j)=-a_{ij}\bar{f}_j,$$

$$(3.3)\qquad (\bar{e}_i, \bar{f}_j)=\delta_{ij}h_i,$$

$$(3.4)\qquad (\mathrm{ad}\,\bar{e}_i)^{1-a_{ij}}(\bar{e}_j)=0 \qquad (i\neq j),$$

$$(3.5)\qquad (\mathrm{ad}\,\bar{f}_i)^{1-a_{ij}}(\bar{f}_j)=0 \qquad (i\neq j).$$

It is known [**5**] we have

$$\mathfrak{g}=\mathfrak{n}^+\oplus\mathfrak{h}\oplus\mathfrak{n}^-$$

where $\mathfrak{h}=\boldsymbol{C}h_1\oplus\cdots\oplus\boldsymbol{C}h_n$ and $\mathfrak{n}^{\pm}$ is the Lie algebra with generators $\bar{e}_i$ (resp. $\bar{f}_i$) and relation (3.4) (resp. (3.5)).

Recall that we take $d_i\in(1/M)\boldsymbol{Z}$ $(1\leqq i\leqq n)$ in such a way that $d_ia_{ij}=d_ja_{ji}$ and

$$\begin{bmatrix}1-a_{ij}\\ \nu\end{bmatrix}_{q^{2d_i}}\in\boldsymbol{Z}[q, q^{-1}] \qquad (i\neq j,\ 0\leqq\nu\leqq 1-a_{ij}).$$

Let $\hat{U}_0^+$ be the $\boldsymbol{Z}[q, q^{-1}]$-algebra with generators e_i $(1\leqq i\leqq n)$ and relation (1.4). We put

$$\hat{U}^+=R\otimes_{\boldsymbol{Z}[q,q^{-1}]}\hat{U}_0^+.$$

It is easy to see

$$U_{\boldsymbol{C}}(\mathfrak{n}^+)=\boldsymbol{C}\otimes_{\boldsymbol{Z}[q,q^{-1}]}\hat{U}_0^+$$

if we view $\boldsymbol{C}$ as a $\boldsymbol{Z}[q, q^{-1}]$-algebra via $q\mapsto 1$.

DEFINITION 3.6. Define $\mathcal{L}\subset\hat{U}_0^+$ to be the smallest $\boldsymbol{Z}[q, q^{-1}]$-submodule such that

i) $e_i \in \mathcal{L}$ $(1 \leqq i \leqq n)$,

ii) if $x, y \in \mathcal{L}$ and $f(q) \in \boldsymbol{Z}[q, q^{-1}]$ with $f(1)=1$, then $(x, y)_{f(q)} \in \mathcal{L}$.

The map $e_i \mapsto \bar{e}_i$ will induce a natural map

$$\mathcal{L} \longrightarrow \mathfrak{n}^+, \qquad x \longmapsto \bar{x}$$

which induces a surjective $\boldsymbol{C}$-linear map $\boldsymbol{C} \otimes_{\boldsymbol{Z}[q, q^{-1}]} \mathcal{L} \to \mathfrak{n}^+$.

We use the representation theory of Lusztig [**6**] to deduce the following theorem. Let $\boldsymbol{Q}(q)$ be the transcendental extension of $\boldsymbol{Q}$ in one variable.

THEOREM 3.7 (cf. [**10**], Proposition 1.10). *Assume $\mathfrak{g}$ is finite dimensional semisimple. Let $z_1, \cdots, z_N \in \mathcal{L}$. If $\bar{z}_1, \cdots, \bar{z}_N \in \mathfrak{n}^+$ are linearly independent over $\boldsymbol{C}$, then the monomials*

$$z_1{}^{a_1} \cdots z_N{}^{a_N}, \qquad a_i \in \boldsymbol{N}$$

are linearly independent over $\boldsymbol{Q}(q)$ in $\boldsymbol{Q}(q) \otimes_{\boldsymbol{Z}[q, q^{-1}]} \hat{U}_0^+$.

PROOF. Construct the algebra $\hat{U}$ over $\boldsymbol{Q}(q)$. Take a simple highest weight $\hat{U}$-module M of highest weight $(q^{d_1 m_1}, \cdots, q^{d_n m_n})$, $m_i \in \boldsymbol{N}$. (To be precise, we should construct $\hat{U}$ over $\boldsymbol{Q}(q^{1/M})$, where $d_i \in (1/M)\boldsymbol{Z}$.) There is a lattice $M_{\mathcal{A}}$ of M, where $\mathcal{A} = \boldsymbol{Q}[q, q^{-1}]$, which is stable under the action of e_i, f_i, $k_i^{\pm}$ [**6**, §4]. In particular, $M_{\mathcal{A}}$ is a $\hat{U}_0^+$-module. Assume there is a non-trivial linear relation

$$\sum_{a_i \in N} g_{a_1, \cdots, a_N}(q) z_1{}^{a_1} \cdots z_N{}^{a_N} = 0$$

over $\boldsymbol{Q}(q)$ in $\hat{U}$. We can assume that $g_{a_1, \cdots, a_N}(q)$ are elements in $\mathcal{A}$ having no non-trival common divisor. Thus $g_{a_1, \cdots, a_N}(1) \neq 0$ for some N-tuple $(a_1, \cdots, a_N)$. This means

$$t = \sum_{a_i \in N} g_{a_1, \cdots, a_N}(1) \bar{z}_1{}^{a_1} \cdots \bar{z}_N{}^{a_N} \neq 0$$

in $U_C(\mathfrak{n}^+)$, by the P-B-W theorem. But t acts as zero on $\boldsymbol{C} \otimes_{\mathcal{A}} M_{\mathcal{A}}$. The main theorem [**6**, Theorem 4.12] tells that $\boldsymbol{C} \otimes_{\mathcal{A}} M_{\mathcal{A}}$ is a simple $\mathfrak{g}$-module with highest weight $(m_1, \cdots, m_n)$. Since we can take $(m_1, \cdots, m_n)$ arbitrarily, it follows that t acts as zero on every finite dimensional $\mathfrak{g}$-module. This will yield $t=0$, a contradiction. Q.E.D.

COROLLARY 3.7a (cf. [**10**], Proposition 1.13 and [**11**], Proposition 4.2). *Assume $\mathfrak{g}$ is finite dimensional semisimple. $\boldsymbol{Q}(q) \otimes_{\boldsymbol{Z}[q, q^{-1}]} \hat{U}_0^+$ is a polynomial algebra over $\boldsymbol{Q}(q)$. More precisely, if we take elements $z_1, \cdots, z_N$ in $\mathcal{L}$ such that $\bar{z}_1, \cdots, \bar{z}_N$ form a base for $\mathfrak{n}^+$, then $\boldsymbol{Q}(q) \otimes_{\boldsymbol{Z}[q, q^{-1}]} \hat{U}_0^+$ is a polynomial algebra over $\boldsymbol{Q}(q)$ in $z_1, \cdots, z_N$. (In particular, the total ordering on generators can be chosen arbitrarily.)*

PROOF. The dimension argument of [**10**], Proposition 1.13 works as it stands. (See also [8], §3.2.) Q.E.D.

COROLLARY 3.8. *Assume* $\mathfrak{g}$ *is finite dimensional semisimple. Let* $R_0 \subset \boldsymbol{Q}(q)$ *be a subring containing* q, q^{-1}. *If there are elements* $z_1, \cdots, z_N \in \mathcal{L}$ *such that*

i) $\bar{z}_1, \cdots, \bar{z}_N \in \mathfrak{n}^+$ *are linearly independent over* $\boldsymbol{C}$,

ii) $R_0 \otimes_{\boldsymbol{Z}[q,q^{-1}]} \hat{U}_0^+$ *is spanned over* R_0 *by all monomials* $z_1^{a_1} \cdots z_N^{a_N}$, $a_i \in \boldsymbol{N}$, *then* $R_0 \otimes_{\boldsymbol{Z}[q,q^{-1}]} \hat{U}_0^+$ *is a polynomial algebra over* R_0 *in* $z_1, \cdots, z_N$.

PROOF. The monomials are linearly independent over R_0, since so are over $\boldsymbol{Q}(q)$. Q. E. D.

REMARK 3.9. With the assumption of 3.8, $\hat{U}^+$ is a polynomial algebra in $z_1, \cdots, z_N$ if R is a commutative R_0-algebra. When $R_0 = \boldsymbol{Z}[q, q^{-1}, f(q)^{-1}]$ for some $f(q) \neq 0$, this simply means $f(q) \in R^\times$.

REMARK 3.10. One checks easily condition i) of 3.8 is fulfilled for the elements described in Theorem 1.6. In fact, they correspond bijectively with the positive roots in each case.

4. The structure of algebras $\mathcal{A}_n$, $\mathcal{B}_n$, and $\mathcal{D}_n$.

Let $\mathcal{A}_n$, $\mathcal{B}_n$, and $\mathcal{D}_n$ be the algebra with generators e_i $(1 \leq i \leq n)$ and relation (1.4) corresponding to (A_n), (B_n), and (D_n) respectively. Recall that we take

$$d_1 = \cdots = d_n = 1/2 \qquad (A_n), (D_n)$$

$$d_1 = \cdots = d_{n-1} = 1/2, \quad d_n = 1/4 \quad (B_k)$$

in (1.4). We analyze the structure of these algebras to deduce Theorem 1.6.

LEMMA 4.1. *For* x, y, z *in an algebra, we have*

a) $[x, y, x] = -q \sum_{\nu=0}^{2} \begin{bmatrix} 2 \\ \nu \end{bmatrix}_q x^{2-\nu} y (-x)^\nu$,

b) $[x, (x, y), x] = -q \sum_{\nu=0}^{3} \begin{bmatrix} 3 \\ \nu \end{bmatrix}_{q^{1/2}} x^{3-\nu} y (-x)^\nu$.

This is easily verified. It follows that the relation (1.4) can be represented in the following diagrams:

(4.2 a) $$e_1 \rightleftarrows e_2 \rightleftarrows \cdots\cdots \rightleftarrows e_n \qquad (A_n),$$

(4.2 b) $$e_1 \rightleftarrows \cdots\cdots \rightleftarrows e_{n-1} \overset{\longrightarrow}{\underset{\Longleftarrow}{}} e_n \qquad (B_n),$$

(4.2 c) $$\begin{tikzcd}[column sep=small, row sep=small] & & & e_{n-1} \arrow[dl, shift left] \\ e_1 \arrow[r, shift left] & \cdots\cdots \arrow[l, shift left] \arrow[r, shift left] & e_{n-2} \arrow[l, shift left] \arrow[ur, shift left] \arrow[dr, shift left] & \\ & & & e_n \arrow[ul, shift left] \end{tikzcd} \qquad (D_n).$$

In the above diagrams, we understand e_i and e_j commute with each other unless there is any arrow between them.

Thus $\mathcal{A}_n$, $\mathcal{B}_n$, and $\mathcal{D}_n$ are the algebras with generators e_i ($1 \leqq i \leqq n$) and relation (4.2 a), b), and c) respectively.

Recall that the iterated q-bracket product is defined inductively

$$[x_1, \cdots, x_m] = [[x_1, \cdots, x_{m-1}], x_m]$$

for x_i in some algebra. It is easy to see we have

$$[[x_1, \cdots, x_i], [x_{i+1}, \cdots, x_m]] = [x_1, \cdots, x_m] \quad (1 \leqq i < m)$$

if x_j commutes with x_k for $j-k>1$.

DEFINITION 4.3. We put in the algebras $\mathcal{A}_n$, $\mathcal{B}_n$, and $\mathcal{D}_n$

$$e_{ij} = [e_i, e_{i+1}, \cdots, e_{j-1}] \quad (1 \leqq i < j \leqq n+1).$$

It follows that we have $[e_{ij}, e_{jk}] = e_{ik}$ ($i<j<k$) in $\mathcal{A}_n$.

PROPOSITION 4.4. *If* $1+q^2 \in R^\times$, *the following identities hold in* $\mathcal{A}_n$.

a) $[e_{ij}, e_{jk}] = e_{ik}$,

$[e_{ik}, e_{ij}] = [e_{jk}, e_{ik}] = 0 \quad (1 \leqq i < j < k \leqq n+1)$.

b) $(e_{ij}, e_{kl}) = (e_{il}, e_{jk}) = 0$,

$(e_{jl}, e_{ik}) = (q-q^{-1}) e_{il} e_{jk} \quad (1 \leqq i < j < k < l \leqq n+1)$.

PROOF. We have $e_{ij} \rightleftarrows e_{jk}$ ($i<j<k$) by iterated application of Proposition 2.7. This proves a). We have $e_{ij} \rightleftarrows e_{jk} \rightleftarrows e_{kl}$ and $(e_{ij}, e_{kl}) = 0$ ($i<j<k<l$) in addition. Since $e_{il} = [e_{ij}, e_{jk}, e_{kl}]$, b) follows from Proposition 2.5. Q.E.D.

When X is a totally ordered set, an *ordered monomial in* X will mean a product, or a word, $x_1 x_2 \cdots x_r$, where $x_i \in X$ and $x_1 \leqq x_2 \leqq \cdots \leqq x_r$, $r \geqq 0$.

THEOREM 4.5. *Let* $\bar{\mathcal{A}}_n$ *be the algebra with generators* e_{ij} ($1 \leqq i < j \leqq n+1$) *and relations* a), b) *of Proposition* 4.4.

a) $\bar{\mathcal{A}}_n$ *is a polynomial algebra in* e_{ij}, *arranged in the lexicographical ordering.*

b) *There is a surjective algebra map* $\mathcal{A}_n \to \bar{\mathcal{A}}_n$, $e_i \mapsto e_{i,i+1}$.

c) *This is an isomorphism if* $1+q^2 \in R^\times$.

PROOF. b) is easy and left to the reader. c) follows from Proposition 4.4. a) Let $(ij) < (kl)$ with $i<j$, $k<l$. By means of the defining relations a), b) of 4.4, we can write $e_{kl} e_{ij}$ as a linear combination of (at most) two ordered monomials. For instance,

$$e_{jk} e_{ij} = q^{-1} e_{ij} e_{jk} - q^{-1} e_{ik} \quad (i<j<k),$$

$$e_{jl} e_{ik} = e_{ik} e_{jl} + (q-q^{-1}) e_{il} e_{jk} \quad (i<j<k<l).$$

It follows easily that $\bar{\mathcal{A}}_n$ is spanned by all ordered monomials in e_{ij}. By c) and Corollary 3.8, we see $\bar{\mathcal{A}}_n$ is a polynomial algebra in e_{ij} under the assumption $1+q^2\in R^\times$, especially when $R=\boldsymbol{Z}[q, q^{-1}, (1+q^2)^{-1}]$. However, the linear indepedence over $\boldsymbol{Z}[q, q^{-1}, (1+q^2)^{-1}]$ will imply the linear independence over $\boldsymbol{Z}[q, q^{-1}]$. Hence $\bar{\mathcal{A}}_n$ is always a polynomial algebra. Q.E.D.

REMARK 4.6. One can also use the diamond lemma of Bergman [**1**] instead of Corollary 3.8. In general $\bar{\mathcal{A}}_n$ is a strict quotient of $\mathcal{A}_n$ (cf. Yamane [**9**]).

Next we turn to $\mathcal{B}_n$.

PROPOSITION 4.7. *In the algebra $\mathcal{B}_n$, we put*

$$y_i=[e_i, \cdots, e_k] \;(=e_{i,n+1}) \quad (1\leqq i\leqq n).$$

If $1+q^2\in R^\times$, the following identities hold in $\mathcal{B}_n$.

a) $[y_i, e_i]=0$, $[e_i, y_{i+1}]=y_i$ $(1\leqq i<n)$.

b) $(y_i, e_j)=0$ *if* $j<i-1$ *or* $i<j<n$.

c) $[y_j, y_i]$ *commutes with* y_i *and* y_j $(1\leqq i<j\leqq n)$.

d) $[y_i, [y_k, y_j]]=[[y_k, y_i], y_j]$, $[y_j, [y_k, y_i]]=[[y_j, y_i], y_k]$

$$(1\leqq i<j<k\leqq n).$$

PROOF. a) We have $y_i=[e_i, y_{i+1}]$ by definition, and $e_i\rightarrow y_{i+1}$ by iterated application of Proposition 2.7. Hence $[y_i, e_i]=[e_i, y_{i+1}, e_i]=0$. b) This is obvious if $j<i-1$. If $i<j<n$, we have $y_i=[e_{ij}, e_j, y_{j+1}]$ with $e_{ij}\leftarrow e_j\rightarrow y_{j+1}$ and $(e_{ij}, y_{j+1})=0$. Hence $(y_i, e_j)=0$ by Proposition 2.5. c) We have

$$e_{ij} \underset{\Longleftarrow}{\overset{\longrightarrow}{}} y_j$$

by iterated application of Proposition 2.9. Hence $[y_j, y_i]=[y_j, e_{ij}, y_j]$ commutes with y_j, e_{ij}, and in particular with $y_i=[e_{ij}, y_j]$. d) We have

$$\begin{aligned}
&[y_i, [y_k, y_j]]=[[e_{ij}, y_j], [y_k, y_j]]\\
&=[[e_{ij}, [y_k, y_j]], y_j] \quad (\text{since } y_j \text{ commutes with } [y_k, y_j])\\
&=[[y_k, [e_{ij}, y_j]], y_j] \quad (\text{since } e_{ij} \text{ commutes with } y_k)\\
&=[[y_k, y_i], y_j].
\end{aligned}$$

Similarly we have

$$[y_j, [y_k, y_i]]=[[y_j, y_i], y_k]$$

by noting $y_j=[e_{jk}, y_k]$ and $(e_{jk}, y_i)=0$. Q.E.D.

DEFINITION 4.8. Let $\mathcal{B}_n^\circ$ be the algebra with generators $y_1, \cdots, y_n$ and relations c), d) of Proposition 4.7.

There is an algebra map $\mathcal{A}_{n-1}\to\mathcal{B}_n$, $e_i\mapsto e_i$ $(1\leqq i\leqq n-1)$. If $1+q^2\in R^\times$, there is also an algebra map $\mathcal{B}_n^\circ\to\mathcal{B}_n$, $y_i\mapsto e_{i,n+1}$ $(1\leqq i\leqq n)$. The following lemma is easily verified by using a), b) of Proposition 4.7.

LEMMA 4.9. *If $1+q^2\in R^\times$, the following R-linear maps are surjective:*

$$\mathcal{A}_{n-1}\otimes\mathcal{B}_n^\circ\longrightarrow\mathcal{B}_n,\quad x\otimes y\mapsto\bar{x}\bar{y},$$

$$\mathcal{B}_n^\circ\otimes\mathcal{A}_{n-1}\longrightarrow\mathcal{B}_n,\quad y\otimes x\mapsto\bar{y}\bar{x}$$

where $\bar{x}$ and $\bar{y}$ denote the images in $\mathcal{B}_n$ under the canonical homomorphisms.

Since we know the structure of $\mathcal{A}_{n-1}$, we are rednced to the study of $\mathcal{B}_n^\circ$. We use the induction on n. There is a canonical algebra map $\mathcal{B}_{n-1}^\circ\to\mathcal{B}_n^\circ$, $y_i\to y_{i+1}$ $(1\leqq i<n)$. Put

$$z_1=y_1,\quad z_i=[y_i,\,y_1]\quad(1<i\leqq n).$$

We claim the monomials $z_1{}^{a_1}\cdots z_n{}^{a_n}$, $a_i\in\boldsymbol{N}$, span the right $\mathcal{B}_{n-1}^\circ$-module $\mathcal{B}_n^\circ$. The following identities follow from definition.

(4.10 a) $\quad y_iz_i=z_iy_i,\quad z_i=[y_i,\,z_1]\quad(1<i\leqq n).$

(4.10 b) $\quad [y_j,\,z_k]=[z_j,\,y_k],\quad [z_1,\,[y_k,\,y_j]]=[z_k,\,y_j]\quad(1<j<k).$

(4.10 c) $\quad z_1z_i=z_iz_1\quad(1<i\leqq n).$

We also have

(4.10 d) $\quad y_kz_j+qz_ky_j+q^2z_1[y_k,\,y_j]=[y_k,\,y_j]z_1+qy_jz_k+q^2z_jy_k\quad(1<j<k)$

by the Jacobi identity (2.2) among y_k, y_j, z_1 $(a=b=c=q)$.

LEMMA 4.11. *If $1+q+q^2\in R^\times$, the following identities hold in $\mathcal{B}_n^\circ$ for $1<j<k\leqq n$.*

a) $y_jz_k=z_ky_j+(1-q)(z_jy_k-z_1[y_k,\,y_j])$,

b) $y_kz_j=z_jy_k+(1-q^{-1})(z_ky_j-z_1[y_k,\,y_j])$,

c) $[y_k,\,y_j]z_1=(q^{-1}-1+q)z_1[y_k,\,y_j]+(1-q)z_jy_k+(1-q^{-1})z_ky_j$,

d) $z_kz_j=qz_jz_k+(1-q)^2z_1(q^{-1}z_1[y_k,\,y_j]+z_jy_k-q^{-1}z_ky_j)$.

PROOF. The three identities (4.10 b), d) can be thought of as a set of linear equations among 6 variables

$$y_jz_k,\quad y_kz_j,\quad [y_k,\,y_j]z_1,\quad z_ky_j,\quad z_jy_k,\quad z_1[y_k,\,y_j].$$

By "solving" these equations, we have

$$(1+q+q^2)(y_jz_k-z_ky_j)=(1-q^3)(z_jy_k-z_1[y_k,\,y_j]).$$

Division by $1+q+q^2$ yields a). b) and c) follow from a) and (4.10 b). To get d), apply $[-,\,z_1]$ to a). We have

$$z_j z_k = z_k z_j + (1-q)(z_j z_k - z_1[[y_k, y_j], z_1])$$

since z_1 commutes with z_j and z_k. We will get d) from this by using c).

Q. E. D.

LEMMA 4.12. *If $1+q+q^2 \in R^\times$, the monomials $z_1^{a_1} \cdots z_n^{a_n}$, $a_i \in \boldsymbol{N}$, span the right $\mathcal{B}^\circ_{n-1}$-module $\mathcal{B}^\circ_n$.*

PROOF. For $r \geqq 0$, put

$$P_r = \sum_{a_1+\cdots+a_n=r} z_1^{a_1} \cdots z_n^{a_n} \mathcal{B}^\circ_{n-1}.$$

We claim

$$y_i P_r \subset P_r \quad (1<i\leqq n), \qquad z_i P_r \subset P_{r+1} \quad (1\leqq i\leqq n).$$

In fact, this is obvious if $r=0$, and follows from (4.10 a), c) and Lemma 4.11 a), b), d) if $r=1$. The claim follows easily by induction on r (and i). It follows that the sum $\sum_r P_r$ is a left ideal of $\mathcal{B}^\circ_n$ containing 1, whence the assertion.

Q. E. D.

DEFINITION 4.13. In $\mathcal{B}^\circ_n$, we put

$$z_{ii} = y_i \quad (1\leqq i\leqq n),$$

$$z_{ij} = [y_j, y_i] \quad (1\leqq i<j\leqq n).$$

Lemma 4.12 means the (ordered) monomials in $z_{11}, z_{12}, \cdots, z_{1n}$ span the right $\mathcal{B}^\circ_{n-1}$-module $\mathcal{B}^\circ_n$ (if $1+q+q^2 \in R^\times$). Hence the following proposition follows by induction.

PROPOSITION 4.14. *If $1+q+q^2 \in R^\times$, the ordered monomials in z_{ij} $(1\leqq i\leqq j\leqq n)$, relative to the lexicographical ordering, span the R-module $\mathcal{B}^\circ_n$.*

THEOREM 4.15. a) *If $1+q+q^2 \in R^\times$, $\mathcal{B}^\circ_n$ is a polynomial algebra in z_{ij} $(1\leqq i\leqq j\leqq n)$, relative to the lexicographical ordering.*

b) *If $1+q^2$, $1+q+q^2 \in R^\times$, the R-linear maps of Lemma 4.9 are bijective, yielding*

$$\mathcal{A}_{n-1} \otimes \mathcal{B}^\circ_n \cong \mathcal{B}_n \cong \mathcal{B}^\circ_n \otimes \mathcal{A}_{n-1},$$

and $\mathcal{B}_n$ is a polynomial algebra in e_{ij} $(1\leqq i<j\leqq n)$ and z_{kl} $(1\leqq k\leqq l\leqq n)$ if we arrange them as follows: Let $e_{ij}<z_{kl}$ for all (ij), (kl) (or reversely $z_{kl}<e_{ij}$). Give the lexicographical orderings on both subsets $\{e_{ij}\}$ and $\{z_{kl}\}$.

PROOF. b) The latter part follows from Corollary 3.8 and Proposition 4.14, yielding the first part. a) This follows from b), similarly as Theorem 4.5 a).

Q. E. D.

Finally, we analyze the structure of $\mathcal{D}_n$. This goes parallel to the previous case.

DEFINITION 4.16. In $\mathcal{D}_n$, we put

$$y_i^{(1)} = [e_i, \cdots, e_{n-1}] \quad (=e_{in}) \qquad (1 \leqq i \leqq n-1),$$
$$y_i^{(2)} = [e_i, \cdots, e_{n-2}, e_n] \qquad (1 \leqq i \leqq n-2),$$
$$y_{n-1}^{(2)} = e_n .$$

PROPOSITION 4.17. *If* $1+q^2 \in R^\times$, *we have*

a) $[y_j^{(*)}, e_j] = 0, \quad y_j^{(*)} = [e_j, y_{j+1}^{(*)}] \qquad (1 \leqq j \leqq n-2)$,

b) $(e_i, y_j^{(*)}) = 0$ *unless* $j=i, i+1 \qquad (1 \leqq i \leqq n-2,\ 1 \leqq j \leqq n-1)$,

c) $[y_j^{(*)}, y_i^{(*)}] = 0 \qquad (1 \leqq i < j \leqq n-1), \qquad *=1, 2$,

d) $(y_i^{(1)}, y_i^{(2)}) = 0 \qquad (1 \leqq i \leqq n-1)$,

e) $[y_i^{(1)}, y_j^{(2)}] = [y_i^{(2)}, y_j^{(1)}] \qquad (1 \leqq i < j \leqq n-1)$.

PROOF. Note that the diagram (4.2 c) contains two copies of diagram (4.2 a) of length $n-1$. a)-c) follow from Proposition 4.4. d) This is obvious if $i=n-1$. If $i \leqq n-2$, we have $e_{i,n-1} \to e_{n-1}$ and $e_{i,n-1} \to e_n$. Since $(e_{n-1}, e_n)=0$, $y_i^{(1)}=[e_{i,n-1}, e_{n-1}]$, and $y_i^{(2)}=[e_{i,n-1}, e_n]$, the claim will follow from Proposition 2.6 ($c=1$). e) This follows from d), since $y_i^{(*)}=[e_{ij}, y_j^{(*)}]$. Q.E.D.

DEFINITION 4.18. Let $\mathcal{D}_n^\circ$ be the algebra with generators $y_i^{(*)}$ ($*=1, 2$, $1 \leqq i \leqq n-1$) and relations c), d), e) of 4.17.

It follows that there is a natural algebra map $\mathcal{D}_n^\circ \to \mathcal{D}_n$ if $1+q^2 \in R^\times$. On the other hand, we have an algebra map $\mathcal{A}_{n-2} \to \mathcal{D}_n$, $e_i \mapsto e_i$ ($1 \leqq i \leqq n-2$). Just as Lemma 4.9, we have:

LEMMA 4.19. *If* $1+q^2 \in R^\times$, *these algebra maps induce surjective R-linear maps*

$$\mathcal{A}_{n-2} \otimes \mathcal{D}_n^\circ \longrightarrow \mathcal{D}_n , \qquad \mathcal{D}_n^\circ \otimes \mathcal{A}_{n-2} \longrightarrow \mathcal{D}_n .$$

DEFINITION 4.20. In the algebra $\mathcal{D}_n^\circ$, we put

$$w_{i,i-1} = y_i^{(1)} , \qquad w_{ii} = y_i^{(2)} \qquad (1 \leqq i \leqq n-1),$$
$$w_{ij} = [y_i^{(1)}, y_j^{(2)}] = [y_i^{(2)}, y_j^{(1)}] \qquad (1 \leqq i < j \leqq n-1).$$

PROPOSITION 4.21. *Assume* $1+q^2 \in R^\times$.

a) *The ordered monomials in* w_{ij} $(i-1 \leqq j \leqq n-1)$, *relative to an arbitrary ordering, span a subalgebra* $\mathcal{S}_i$ *of* $\mathcal{D}_n^\circ$ $(1 \leqq i \leqq n-1)$.

b) *We have*

$$\mathcal{S}_1 \mathcal{S}_2 \cdots \mathcal{S}_{n-1} = \mathcal{S}_{n-1} \mathcal{S}_{n-2} \cdots \mathcal{S}_1 = \mathcal{D}_n^\circ .$$

PROOF. There is a natural algebra map $\mathcal{D}_{n-1}^\circ \to \mathcal{D}_n^\circ$, $y_i^{(*)} \mapsto y_{i+1}^{(*)}$ $(1 \leqq i \leqq n-2)$. We claim a) for $i=1$, and that

$$\mathcal{D}_n^\circ = \mathcal{S}_1\mathcal{D}_{n-1}^\circ = \mathcal{D}_{n-1}^\circ\mathcal{S}_1$$

where we identify $\mathcal{D}_{n-1}^\circ$ with its image in $\mathcal{D}_n^\circ$. This claim follows from the next lemma (the latter part from a)-c), and the first part from d), e)). The assertion follows by induction. (In particular, some other (but not all) orderings are admissible in b).)

LEMMA 4.22. *Assume $1+q^2\in R^\times$, and let $w_i=w_{1i}$ $(0\leqq i\leqq n-1)$. The following identities hold in $\mathcal{D}_n^\circ$.*

a) $[y_i^{(1)}, w_0]=[y_i^{(2)}, w_1]=0$, $[w_0, y_i^{(2)}]=[w_1, y_i^{(1)}]=w_i$ $(1\leqq i\leqq n-1)$,

b) $[y_i^{(*)}, w_i]=0$ $(1<i\leqq n-1,\ *=1, 2)$,

c) $(y_i^{(*)}, w_j)=0$, $(y_j^{(*)}, w_i)=(q-q^{-1})y_i^{(*)}w_j$ $(1<i<j\leqq n-1)$,

d) $(w_0, w_1)=0$,

e) $[w_j, w_i]=0$ *unless* $(ij)=(01)$ $(0\leqq i<j\leqq n-1)$.

PROOF. a) and d) follow from definition. b) $[y_i^{(1)}, w_i]=[y_i^{(1)}, w_0, y_i^{(2)}]=0$ by a). Similar for $*=2$. c) Let $*=1$. We have

$$\begin{aligned}
&(y_i^{(1)}, w_j)_{q^2}=(y_i^{(1)}, [w_0, y_j^{(2)}])_{q^2}\\
&=q(w_0, [y_i^{(1)}, y_j^{(2)}]) \qquad (\text{by (2.3 b), since } [y_i^{(1)}, w_0]=0)\\
&=q(w_0, [y_i^{(2)}, y_j^{(1)}])\\
&=q([w_0, y_i^{(2)}], y_j^{(1)}) \qquad (\text{by (2.3 a), since } [y_j^{(1)}, w_0]=0)\\
&=q(w_i, y_j^{(1)}).\\
&(w_j, y_i^{(1)})_{q^2}=([w_1, y_j^{(1)}], y_i^{(1)})_{q^2}\\
&=q([w_1, y_i^{(1)}], y_j^{(1)}) \qquad (\text{by (2.3 c), since } [y_j^{(1)}, y_i^{(1)}]=0)\\
&=q(w_i, y_j^{(1)}).
\end{aligned}$$

Hence

$$0=(y_i^{(1)}, w_j)_{q^2}-(w_j, y_i^{(1)})_{q^2}=(1+q^2)(y_i^{(1)}, w_j).$$

The claim will follow from these identities, since $1+q^2\in R^\times$. The case $*=2$ is similar. e) If $1<j$, then $[w_j, w_0]=[w_1, y_j^{(1)}, w_0]=0$ by a) and d). Similarly, $[w_j, w_1]=0$. If $1<i<j$, then $[w_j, w_i]=[w_j, w_0, y_i^{(2)}]=0$, since $[w_j, w_0]=0$ and $(w_j, y_i^{(2)})=0$. Q.E.D.

As a consequence of Proposition 4.21, we have the following result similarly as Theorem 4.15.

THEOREM 4.23. *Assume $1+q^2\in R^\times$.*

a) *The subalgebra $\mathcal{S}_i\subset\mathcal{D}_n^\circ$ is a polynomial algebra in w_{ij} $(i-1\leqq j\leqq n-1)$, relative to an arbitrary total ordering $(1\leqq i\leqq n-1)$.*

b) *The multiplication of* $\mathcal{D}_n^\circ$ *induces an* R*-linear isomorphism*

$$\mathcal{S}_1 \otimes \mathcal{S}_2 \otimes \cdots \otimes \mathcal{S}_{n-1} \xrightarrow{\cong} \mathcal{D}_n^\circ .$$

(Some other orderings are admissible.)

c) *The canonical algebra maps* $\mathcal{A}_{n-2} \to \mathcal{D}_n$ *and* $\mathcal{D}_n^\circ \to \mathcal{D}_n$ *(see* 4.19) *induce* R*-linear isomorphisms*

$$\mathcal{A}_{n-2} \otimes \mathcal{D}_n^\circ \cong \mathcal{D}_n \cong \mathcal{D}_n^\circ \otimes \mathcal{A}_{n-2} .$$

d) $\mathcal{D}_n^\circ$ *is a polynomial algebra in* w_{ij} $(1 \leqq i \leqq n-1,\ i-1 \leqq j \leqq n-1)$ *if we arrange them so that* $i<k$ *implies* $w_{ij} < w_{kl}$ *for any* j, l *(or reversely).*

e) $\mathcal{D}_n$ *is a polynomial algebra in* e_{ij} $(1 \leqq i < j \leqq n-1)$ *and* w_{kl} $(1 \leqq k \leqq n-1,$ $k-1 \leqq l \leqq n-1)$ *if we arange them as follows. Let* $e_{ij} < w_{kl}$ *for all* (ij), (kl) *(or reversely). Give the lexicographical orderings on both subsets* $\{e_{ij}\}$ *and* $\{w_{kl}\}$. *(Some other orderings are admissible.)*

5. Some modification of $\mathcal{B}_n^\circ$.

In this section, we give some comments on the algebra $\mathcal{B}_n^\circ$. (The algebra $\mathcal{D}_n^\circ$ admits a similar observation, too.) Recall (4.8) that $\mathcal{B}_n^\circ$ is the algebra with generators $y_1, \cdots, y_n$ and relations

(5.1) $\qquad [y_j, y_i]$ commutes with y_i and y_j $(i<j)$,

$$[y_i, [y_k, y_j]] = [[y_k, y_i], y_j] \quad (i<j<k), \tag{5.2}$$

$$[y_j, [y_k, y_i]] = [[y_j, y_i], y_k] \quad (i<j<k). \tag{5.3}$$

It would be natural to add to one more relation

$$[y_k, [y_j, y_i]] = [[y_k, y_j], y_i] \quad (i<j<k). \tag{5.4}$$

PROPOSITION 5.5 (cf. Theorem 4.5). *Let* $\bar{\mathcal{B}}_n^\circ$ *be the algebra with generators* y_i $(1 \leqq i \leqq n)$ *and relations* (5.1)-(5.4).

a) $\bar{\mathcal{B}}_n^\circ$ *is a polynomial algebra in* z_{ij} $(1 \leqq i \leqq j \leqq n)$, *arranged in the lexicographical ordering (see Definition* 4.13 *for* z_{ij}).

b) *If* $1+q+q^2 \in R^\times$, *the projection* $\mathcal{B}_n^\circ \to \bar{\mathcal{B}}_n^\circ$ *is an isomorphism.*

PROOF. b) We claim (5.4) follows from (5.2) and (5.3) if $1+q+q^2 \in R^\times$. We have

$$([y_k, y_j], y_i)_{q^2} - (y_k, [y_j, y_i])_{q^2} = q([y_k, y_i], y_j)$$

by the Jacobi identity among y_k, y_j, y_i. Hence

$$\begin{aligned}&(1+q)\{([y_k, y_j], y_i)_{q^2} - (y_k, [y_j, y_i])_{q^2}\}\\ &= q[[y_k, y_i], y_j] - q[y_j, [y_k, y_i]] = q[y_i, [y_k, y_j]] - q[[y_j, y_i], y_k]\end{aligned}$$

by (5.2), (5.3). The claim will follow since

$$(1+q)(u, v)_{q^2}-q[v, u]=(1+q+q^2)[u, v]$$

for u, v in an algebra.

a) We claim the identities a)-d) of Lemma 4.11 hold in $\mathscr{B}_n^\circ$. The assertion will follow from this (cf. the proofs of Theorems 4.5 and 4.15). We have

$$(1+q+q^2)(y_j z_k - z_k y_j)=(1-q^3)(z_j y_k - z_1[y_k, y_j])$$

(proof of 4.11). Similarly we have

$$(1+q)(y_j z_k - z_k y_j)=(1-q^2)(z_j y_k - z_1[y_k, y_j])$$

from (5.2)-(5.4). This yields a) of 4.11. b)-d) will follow as before. Q.E.D.

In particular, $\mathscr{B}_3^\circ$ is a polynomial algebra in

$$y_1,\quad [y_2, y_1],\quad [y_3, y_1],\quad y_2,\quad [y_3, y_2],\quad y_3 .$$

This ordering looks unnatural. One sees a "natural" ordering causes some "unnatural" restrictions on q.

PROPOSITION 5.6. *$\mathscr{B}_3^\circ$ is a polynomial algebra in*

$$y_1,\quad y_2,\quad y_3,\quad [y_2, y_1],\quad [y_3, y_1],\quad [y_3, y_2]$$

if and only if $1-q+q^2,\ 1-q^2+q^3 \in R^\times$.

PROOF. "If" part will follow from the following explicit expressions. Let $z_{ij}=[y_j, y_i]$ $(i<j)$.

i) $y_j y_i = q y_i y_j + z_{ij} \qquad (i<j)$,

ii) $$z_{13}y_2=\frac{1}{1-q+q^2}(q y_2 z_{13}+(1-q)y_1 z_{23}-q(1-q)y_3 z_{12}),$$

$$z_{23}y_1=\frac{1}{1-q+q^2}(q y_1 z_{23}+(1-q)y_3 z_{12}-q(1-q)y_2 z_{13}),$$

$$z_{12}y_3=\frac{1}{1-q+q^2}(q y_3 z_{12}+(1-q)y_2 z_{13}-q(1-q)y_1 z_{23}),$$

iii) $$z_{13}z_{12}=q z_{12}z_{13}+\frac{(1-q)^2}{1-q+q^2}(y_1 y_3 z_{12}-q y_1 y_2 z_{13}+q^2 y_1{}^2 z_{23}),$$

$$z_{23}z_{12}=q\frac{1-q^2+q^3}{1-q+q^2}z_{12}z_{23}+\frac{(1-q)^2}{1-q+q^2}(y_2 y_3 z_{12}-q y_2{}^2 z_{13}+q^3 y_1 y_2 z_{23}),$$

$$z_{23}z_{13}=q z_{13}z_{23}+\frac{(1-q)^2}{1-q^2+q^3}(y_3{}^2 z_{12}-q^2 y_2 y_3 z_{13}+q^3 y_1 y_3 z_{23}).$$

These identities hold "really" in $\mathscr{B}_3^\circ$ if $1-q+q^2,\ 1-q^2+q^3 \in R^\times$. In general, by multiplying by the denominators, we obtain identites which hold in $\mathscr{B}_3^\circ$. If it is a polynomial algebra in $y_1, y_2, y_3, z_{12}, z_{13}, z_{23}$, it follows that $1-q+q^2$, $1-q^2+q^3$ should be units, since they divide q. Q.E.D.

We end this section with the following proposition.

PROPOSITION 5.7. *Let* $K=[y_3, [y_2, y_1]]=[[y_3, y_2], y_1]$. *Then* K *is central in* $\mathscr{B}_3^\circ$, *and we have*

a) $[[y_3, y_2], [y_3, y_1]] = (1-q)y_3K$,

b) $[[y_3, y_2], [y_2, y_1]] = (1-q)y_2K$,

c) $[[y_3, y_1], [y_2, y_1]] = (1-q)y_1K$.

PROOF. First we claim

a)' $[[y_3, y_2], [y_3, y_1]] = [y_3, K] = [K, y_3]$,

b)' $[[y_3, y_2], [y_2, y_1]] = [y_2, K] = [K, y_2]$,

c)' $[[y_3, y_1], [y_2, y_1]] = [y_1, K] = [K, y_1]$.

Take b)' for instance. We have

$$[[y_3, y_2], [y_2, y_1]] = [y_2, [[y_3, y_2], y_1]] = [y_2, K]$$

since $[y_3, y_2]$ commutes with y_2. Similarly, it equals

$$[[y_3, [y_2, y_1]], y_2] = [K, y_2].$$

This yields b)'. Similarly, a)' and c)' are proved by using (5.2) and (5.3). It follows we have $(1+q)(y_i, K)=0$ $(i=1, 2, 3)$. It is enough to show $(y_i, K)=0$. To do so, use the identity

$$([y_3, y_2], y_1)_{q^2}-(y_3, [y_2, y_1])_{q^2} = q([y_3, y_1], y_2)$$

(proof of 5.5 b)). Application of $[y_3, -]$ yields

$$(y_3, K)_{q^2} = (1+q)[[y_3, y_2], [y_3, y_1]] = (1-q^2)y_3K$$

where the second equality follows from a)'. It follows that $(y_3, K)=0$. Similarly, application of $[-, y_1]$ yields that $(y_1, K)=0$. Finally apply $[y_2, -]$ to get

$$\begin{aligned}(1+q)((y_3, y_2), [y_2, y_1])_{q^2} &= q([y_2, [y_3, y_1]], y_2)\\ &= q([[y_2, y_1], y_3], y_2) = q([y_2, y_1], (y_3, y_2)).\end{aligned}$$

Hence

$$0 = (1+q+q^2)[(y_3, y_2), [y_2, y_1]] = (1+q+q^2)(K, y_2)$$

since y_2 commutes with $[y_2, y_1]$. Since $(1+q)(y_2, K)=0$, it follows that $(y_2, K)=0$. Q.E.D.

6. Some calculations of the q-bracket product.

To deal with C_n, the algebra $\hat{U}^+$ of type (C_n), we require some more technical calculations of the q-bracket product. (This is the reason why we separate this case.) This section collects the necessary results to be used in the next section.

For an integer i, and elements x, y in an algebra, we put

$$[x, y]_i = xy - q^i yx .$$

Thus $[\ ,\ ]_1 = [\ ,\ ]$ and $[\ ,\ ]_0 = (\ ,\ )$. We put

$$[x, y, z]_i = [[x, y]_i, z]_i$$

as in Definition 2.4.

DEFINITION 6.1. $x \xrightarrow{i} y$ (or $y \xleftarrow{i} x$, $x \underset{i}{\rightarrow} y$) means $[x, y, x]_i = 0$. $x \overset{i}{\Rightarrow} y$ means that $[x, y, x]_i$ commutes with x, or $x \xrightarrow{i} (x, y)$.

PROPOSITION 6.2. *Assume* $1+q^{2i} \in R^{\times}$. *If* $x \leftarrow y \xleftarrow{i} z$ *and* $(x, z)=0$ *in an algebra, then we have* $x \leftarrow [y, z]_i$.

PROOF. We have $z \xrightarrow{i} y$ and $z \xrightarrow{i} [x, y]$. Since $[y, [x, y]] = 0$, it follows from Proposition 2.6 that

$$0 = [[y, z]_i, [[x, y], z]_i] = [[y, z]_i, x, [y, z]_i] .$$

Q. E. D.

In the rest of this section, assume that

$$1+q^2,\ 1+q^4,\ 1+q^2+q^4 \in R^{\times} ,$$

and we are given elements e_1, e_2, e_3 in an algebra such that

$$e_1 \rightleftarrows e_2 \underset{2}{\overset{2}{\rightleftarrows}} e_3, \qquad (e_1, e_3) = 0 .$$

(This is the defining relation for C_3 (7.1).) We have $e_1 \rightleftarrows [e_2, e_3]_2$ by Proposition 6.2. One of the main results, Corollary 6.7, tells that

$$[e_1, e_2] \underset{2}{\overset{2}{\rightleftarrows}} e_3 .$$

The proof is not so easy as Proposition 2.9. We put

$$y_1 = [e_1, e_2], \qquad y_2 = e_2 ,$$
$$z_i = [y_i, e_3]_2, \qquad x_i = [z_i, y_i]_2 \qquad (i=1, 2).$$

PROPOSITION 6.3.

a) $[y_1, e_1] = [z_1, e_1] = 0$,

b) $[y_2, y_1] = [z_2, z_1] = 0$,

c) x_2 *commutes with* y_2 *and* z_2.

PROOF. a) and b) follow, since we have

$$e_1 \rightleftarrows y_2, \qquad e_1 \rightleftarrows z_2\,,$$

with $y_1=[e_1, y_2]$, $z_1=[e_1, z_2]$. c) This is a special case of Proposition 4.7 c).
Q.E.D.

We list up consequences of Corollary 2.3 and Proposition 6.3:

Since $[y_2, y_1]=0$, *we have*

$$[y_2, [y_1, e_3]_2]_3 = q[y_1, [y_2, e_3]_2], \qquad [[y_1, z_2], y_2] = (y_1, [z_2, y_2]_2),$$

$$[[z_1, y_2], y_1]_3 = q([z_1, y_1]_2, y_2), \qquad [[z_2, y_2]_2, y_1]_2 = q[[z_2, y_1], y_2]\,.$$

Since $[z_2, z_1]=0$, *we have*

$$[z_2, [z_1, y_2]]_3 = q(z_1, [z_2, y_2]_2), \qquad [z_2, [z_1, y_2]_3]_3 = q[z_1, [z_2, y_2]_2]_2\,,$$

$$[z_1, [y_1, z_2]] = ([z_1, y_1]_2, z_2)\,.$$

Since $[z_1, e_1]=0$, *we have*

$$[z_1, [e_1, y_2]]_2 = q(e_1, [z_1, y_2])\,.$$

Let us rewrite these identities by setting

$$u = [z_1, y_2]\,.$$

LEMMA 6.4.

a) $[y_2, z_1]_3 = q[y_1, z_2]$,

b) $[y_2, u]_3 = q(y_1, x_2)$,

c) $[u, y_1]_3 = q(x_1, y_2)$,

d) $[x_2, y_1]_2 = q[[z_2, y_1], y_2]$,

e) $[z_2, u]_3 = q(z_1, x_2)$,

f) $[z_2, [z_1, y_2]_3]_3 = q[z_1, x_2]_2$,

g) $[u, z_1]_3 = q(x_1, z_2)$,

h) $x_1 = q(e_1, u)$.

In b), g) we are using a).

PROPOSITION 6.5.

a) $[y_2, u] = 0$,

b) $[z_2, u] = 0$,

c) $[u, y_1] = 0$,

d) $[u, z_1] = 0$.

PROOF. a) Start with the Jacobi identity

$$[e_1, [z_2, y_2]_2]_2 = [[e_1, z_2], y_2]_3 + q[z_2, [e_1, y_2]], \qquad \text{i.e.,}$$

$$[e_1, x_2]_2 = [z_1, y_2]_3 + q[z_2, y_1]. \tag{6.5.1}$$

Apply $[-, y_2]$ to (6.5.1) by noting

$$[[e_1, x_2]_2, y_2] = [[e_1, y_2], x_2]_2 = [y_1, x_2]_2 \qquad (\text{since } (x_2, y_2)=0),$$

$$[[z_1, y_2]_3, y_2] = [[z_1, y_2], y_2]_3 = [u, y_2]_3,$$

$$[q[z_2, y_1], y_2] = [x_2, y_1]_2 \qquad (6.4 \text{ d}).$$

We get

$$[y_1, x_2]_2 = [u, y_2]_3 + [x_2, y_1]_2, \qquad \text{or}$$

$$[u, y_2]_3 = (1+q^2)(y_1, x_2). \tag{6.5.2}$$

It follows from Lemma 6.4 b) and (6.5.2) that

$$q[u, y_2]_3 = (1+q^2)[y_2, u]_3, \qquad \text{or} \qquad (1+q^2+q^4)[y_2, u] = 0$$

yielding the assertion.

b) Apply $[-, z_2]$ to (6.5.1) by noting

$$[[e_1, x_2]_2, z_2] = [[e_1, z_2], x_2]_2 \qquad (\text{since } x_2 \text{ commutes with } z_2,$$

$$= [z_1, x_2]_2 = q^{-1}[z_2, [z_1, y_2]_3]_3 \qquad (6.4 \text{ f}),$$

$$[q[z_2, y_1], z_2] = [z_2, [y_2, z_1]_3] \qquad (6.4 \text{ a}).$$

We get

$$q^{-1}[z_2, [z_1, y_2]_3]_3 = [[z_1, y_2]_3, z_2] + [z_2, [y_2, z_1]_3], \qquad \text{or}$$

$$(1+q^2)[z_2, [z_1, y_2]_3] = q[z_2, [y_2, z_1]_3].$$

This means

$$0 = [z_2, (1+q^2)[z_1, y_2]_3 - q[y_2, z_1]_3] = (1+q^2+q^4)[z_2, u]$$

yielding the assertion.

c) Apply $(-, y_2)$ to (6.4 h) by noting

$$(x_1, y_2) = q^{-1}[u, y_1]_3 \qquad (6.4 \text{ c}),$$

$$(q(e_1, u), y_2) = [[e_1, y_2], u] = [y_1, u] \qquad (\text{by (2.3 c), since } [y_2, u]=0).$$

It follows that

$$[u, y_1]_3 = q[y_1, u], \qquad \text{i.e.,} \quad (1+q^2)[u, y_1] = 0.$$

d) This is similar as c). Apply $(-, z_2)$ to (6.4 h), and use (6.4 g). Q.E.D.

COROLLARY 6.6. *x_1 commutes with y_1 and z_1.*

PROOF. This follows, since $x_1 = q(e_1, u)$ (6.4 h), $[u, y_1] = [u, z_1] = 0$, and $[y_1, e_1] = [z_1, e_1] = 0$ (6.3 a). Q.E.D.

Since $x_1 = [y_1, e_3, y_1]_2$, this means $y_1 = [e_1, e_2] \overset{2}{\Rightarrow} e_3$. By looking at the above arguments carefully, one sees we have proved:

COROLLARY 6.7. *Assume* $1+q^2$, $1+q^2+q^4 \in R^\times$. *If* $e_1 \rightleftarrows e_2 \overset{2}{\Rightarrow} e_3$ *and* $(e_1, e_3)=0$ *in an algebra, then we have* $[e_1, e_2] \overset{2}{\Rightarrow} e_3$.

7. The structure of the algebra $\mathcal{C}_n$.

We analyze the structure of $\mathcal{C}_n$ to deduce Theorem 1.6 c). Let (a_{ij}) be the Cartan matrix of type (C_n). Since we take

$$d_1 = \cdots = d_{n-1} = 1/2, \qquad d_n = 1,$$

it follows from Lemma 4.1 that the defining realtion (1.4) corresponding to (a_{ij}) is expressed in the diagram

$$e_1 \rightleftarrows \cdots \rightleftarrows e_{n-1} \underset{2}{\overset{2}{\rightleftarrows}} e_n . \tag{7.1}$$

Here, as in (4.2), we understand e_i and e_j commute with each other unless there is any arrow between them.

DEFINITION 7.2. Let $\mathcal{C}_n$ be the algebra with generators e_i $(1 \leqq i \leqq n)$ and relation (7.1).

NOTATION 7.3. In the algebra $\mathcal{C}_n$, we put

$$y_i = [e_i, \cdots, e_{n-1}] \quad (= e_{in}),$$
$$z_i = [y_i, e_n]_2, \quad x_i = [z_i, y_i]_2 \qquad (1 \leqq i \leqq n-1),$$
$$u_{ij} = [z_i, y_j] \qquad (1 \leqq i < j \leqq n-1).$$

(The convention 4.3 is valid in $\mathcal{C}_n$, too.)

PROPOSITION 7.4. *Assume* $1+q^2$, $1+q^4$, $1+q^2+q^4 \in R^\times$. *The following identities hold in* $\mathcal{C}_n$.

a) $[y_i, e_i] = [z_i, e_i] = 0$, $y_i = [e_i, y_{i+1}]$, $z_i = [e_i, z_{i+1}]$ $(1 \leqq i \leqq n-2)$.

b) $[y_i, e_n]_2 = z_i$, $[e_n, z_i]_2 = 0$ $(1 \leqq i \leqq n-1)$.

c) $(e_i, y_j) = (e_i, z_j) = 0$ *unless* $j=i$, $i+1$ $(1 \leqq i \leqq n-2,\ 1 \leqq j \leqq n-1)$.

d) x_i *commutes with* y_i *and* z_i $(1 \leqq i \leqq n-1)$.

e) $[y_j, y_i] = [z_j, z_i] = 0$ $(1 \leqq i < j \leqq n-1)$.

f) $[y_j, z_i]_3 = q[y_i, z_j]$ $(1 \leqq i < j \leqq n-1)$.

g) $[y_j, u_{ij}] = [z_j, u_{ij}] = 0$ $(1 \leqq i < j \leqq n-1)$.

h) $[u_{ij}, y_i] = [u_{ij}, z_i] = 0$ $(1 \leqq i < j \leqq n-1)$.

i) $(u_{ik}, y_j) = (u_{ik}, z_j) = 0$ $(1 \leqq i < j < k \leqq n-1)$.

PROOF. a), c), e) follow from Proposition 4.4, since we have two diagrams of type (A_{n-1}):

$$e_1 \rightleftarrows \cdots \rightleftarrows e_{n-2} \rightleftarrows y_{n-1},$$

$$e_1 \rightleftarrows \cdots \rightleftarrows e_{n-2} \rightleftarrows z_{n-1} \qquad \text{(by Proposition 6.2).}$$

We have $y_i \overset{2}{\underset{2}{\rightleftarrows}} e_n$ by iterated application of Corollary 6.7, yielding b), d). If $1 \leqq i < j \leqq n-1$, we have

$$e_{ij} \rightleftarrows y_j \overset{2}{\underset{2}{\rightleftarrows}} e_n, \qquad (e_{ij}, e_n) = 0.$$

By noting $y_i = [e_{ij}, y_j]$, one sees f), g), h) follow from Lemma 6.4 a) and Proposition 6.5. Finally, let $1 \leqq i < j < k \leqq n-1$. We have

$$[e_{jk}, u_{ik}] = u_{ij},$$

since $[e_{jk}, [z_i, y_k]] = [z_i, [e_{jk}, y_k]]$ (because e_{jk} commutes with z_i). We claim

a)′ $[y_j, u_{ik}]_2 = q(u_{ij}, y_k)$,

b)′ $[u_{ik}, y_j]_2 = q(u_{ij}, y_k)$,

c)′ $[z_j, u_{ik}]_2 = q(u_{ij}, z_k)$,

d)′ $[u_{ik}, z_j]_2 = q(u_{ij}, z_k)$.

It is easy to see i) is a consequence of a)′-d)′ (cf. the proof of Lemma 4.22 c)). Now, a)′-d)′ follow from the following identities which are consequences of Corollary 2.3:

a)″ $[[e_{jk}, y_k], u_{ik}]_2 = q([e_{jk}, u_{ik}], y_k)$ (since $[y_k, u_{ik}]=0$),

b)″ $[[z_i, y_k], y_j]_2 = q([z_i, y_j], y_k)$ (since $[y_k, y_j]=0$),

c)″ $[[e_{jk}, z_k], u_{ik}]_2 = q([e_{jk}, u_{ik}], z_k)$ (since $[z_k, u_{ik}]=0$),

d)$''_1$ $[[z_i, y_k], z_j]_2 = (z_i, [y_k, z_j]_3)$ (since $[z_i, z_j]_{-1}=0$),

d)$''_2$ $([z_i, y_j], z_k) = (z_i, [y_j, z_k])$ (since $[z_i, z_k]_{-1}=0$).

To deduce d)′, use d)$''_1$, d)$''_2$, and f). Q.E.D.

The above proof shows e) and i) imply

(7.4 j) $$(y_k, u_{ij}) = (q-q^{-1})u_{ik}y_j, \qquad (z_k, u_{ij}) = (q-q^{-1})u_{ik}z_j$$
$$(1 \leqq i < j < k \leqq n-1).$$

DEFINITION 7.5. *Let C_n° be the algebra with generators y_i, z_i $(1 \leqq i \leqq n-1)$ and relations* d)-i) *of Proposition* 7.4.

We interpret $x_i = [z_i, y_i]_2$ and $u_{ij} = [z_i, y_j]$ $(i<j)$ in the defining relations. Proposition 7.4 means there is a natural algebra map $C_n^\circ \to C_n$ if $1+q^2$, $1+q^4$,

$1+q^2+q^4 \in R^\times$. On the other hand, there are canonical algebra maps

$$\mathcal{A}_{n-2} \longrightarrow \mathcal{C}_n, \qquad e_i \longmapsto e_i \quad (1 \leqq i \leqq n-2),$$
$$\mathcal{A}_1 \longrightarrow \mathcal{C}_n, \qquad e_1 \longmapsto e_n .$$

The following corollary follows easily from a)-c) of Proposition 7.4.

COROLLARY 7.6. *With the assumption of* 7.4, *the canonical algebra maps induce a surjective R-linear map*

$$\mathcal{A}_{n-2} \otimes \mathcal{A}_1 \otimes \mathcal{C}_n^\circ \longrightarrow \mathcal{C}_n .$$

(*One can permute the three factors freely in the left side.*)

LEMMA 7.7. *In the algebra* $\mathcal{C}_n^\circ$, *we have*

$$[u_{ik}, u_{ij}] = [u_{jk}, u_{ik}] = 0 \qquad (1 \leqq i < j < k \leqq n-1).$$

PROOF. $[u_{ik}, z_i, y_j]=0$ by h), i), and $[z_j, y_k, u_{ik}]=0$ by g), j) of Proposition 7.4. Q.E.D.

PROPOSITION 7.8. *The ordered monomials in*

$$y_i, \quad z_i \quad x_i, \quad u_{ij} \quad (i < j \leqq n-1),$$

relative to an arbitrary total ordering, span a subalgebra $\mathcal{S}_i$ *of* $\mathcal{C}_n^\circ$ $(1 \leqq i \leqq n-1)$.

This follows easily from Proposition 7.4 d), h), and Lemma 7.7.

PROPOSITION 7.9. *We have*

$$\mathcal{C}_n^\circ = \mathcal{S}_1 \mathcal{S}_2 \cdots \mathcal{S}_{n-1} = \mathcal{S}_{n-1} \cdots \mathcal{S}_2 \mathcal{S}_1 .$$

(*Some other orderings are admissible, too.*)

PROOF. Similarly as Proposition 4.21, consider the algebra map

$$\mathcal{C}_{n-1}^\circ \longrightarrow \mathcal{C}_n^\circ, \qquad y_i, z_i \longmapsto y_{i+1}, z_{i+1} \quad (1 \leqq i \leqq n-2).$$

It is enough to show

$$\mathcal{C}_n^\circ = \mathcal{S}_1 \mathcal{C}_{n-1}^\circ = \mathcal{C}_{n-1}^\circ \mathcal{S}_1 .$$

Here, note that the subalgebra $\mathcal{S}_1$ is generated by y_1, z_1, u_{1j} $(1 < j \leqq n-1)$, and $\mathcal{C}_{n-1}^\circ$ by y_k, z_k $(1 < k \leqq n-1)$. The first equality $\mathcal{C}_n^\circ = \mathcal{S}_1 \mathcal{C}_{n-1}^\circ$ follows from the following identities:

$$y_k y_1 = q y_1 y_k, \qquad z_k z_1 = q z_1 z_k \qquad (7.4\ \text{e}),$$
$$y_k z_1 = q^{-1} z_1 y_k - q^{-1} u_{1k} ,$$
$$z_k y_1 = q^{-1} y_1 z_k + (q - q^{-3}) z_1 y_k + q^{-3} u_{1k} \qquad (7.4\ \text{f}),$$
$$y_k u_{1j} = \begin{cases} u_{1j} y_k & (k < j), \\ q u_{1j} y_k & (k = j), \\ u_{1j} y_k + (q - q^{-1}) u_{1k} y_j & (k > j) \end{cases} \qquad (7.4\ \text{g}), \text{i}), \text{j}),$$

$z_k u_{1j}=$ similar as above.

The second equality $\mathcal{C}_n^\circ=\mathcal{C}_{n-1}^\circ\mathcal{S}_1$ follows similarly. Q.E.D.

Just as in §4, the above arguments give rise to the following structure thoerem for $\mathcal{C}_n^\circ$ and $\mathcal{C}_n$, yielding Theorem 1.6 c). Put

$$u_{i,i-2}=y_i,\quad u_{i,i-1}=z_i,\quad u_{ii}=x_i \qquad (1\leqq i\leqq n-1).$$

THEOREM 7.10. a) *The subalgebra* $\mathcal{S}_i\subset\mathcal{C}_n^\circ$ *is a polynomial algebra in* u_{ij} $(i-2\leqq j\leqq n-1)$, *relative to an arbitrary total ordering* $(1\leqq i\leqq n-1)$.

b) *The multiplication of* $\mathcal{C}_n^\circ$ *induces an R-linear isomorphism*

$$\mathcal{S}_1\otimes\cdots\otimes\mathcal{S}_{n-1}\xrightarrow{\cong}\mathcal{C}_n^\circ .$$

(*Some other orderings are admissible, too.*)

c) $\mathcal{C}_n^\circ$ *is a polynomial algebra in* u_{ij} $(1\leqq i\leqq n-1,\ i-2\leqq j\leqq n-1)$ *if we arrange them so that* $i<k$ *implies* $u_{ij}<u_{kl}$ *for any* j, l (*or reversely*).

d) *Assume* $1+q^2, 1+q^4, 1+q^2+q^4\in R^\times$. *The canonical algebra maps* $\mathcal{A}_{n-2}\to\mathcal{C}_n$, $\mathcal{A}_1\to\mathcal{C}_n$, *and* $\mathcal{C}_n^\circ\to\mathcal{C}_n$ (*see* 7.6) *induce an R-linear isomorphism*

$$\mathcal{A}_{n-2}\otimes\mathcal{A}_1\otimes\mathcal{C}_n^\circ\xrightarrow{\cong}\mathcal{C}_n .$$

(*The three factors in the left side can be permuted arbitrarily.*)

e) *With the assumption of* d), $\mathcal{C}_n$ *is a polynomial algebra in* e_{ij} $(1\leqq i<j\leqq n-1)$, e_n, u_{kl} $(1\leqq k\leqq n-1,\ k-2\leqq l\leqq n-1)$ *if we arrange them as follows: Let* $e_{ij}<e_n<u_{kl}$ *for any* (ij), (kl). *Give the lexicographical orderings on both subsets* $\{e_{ij}\}$ *and* $\{u_{kl}\}$.

Comparison of this theorem with Theorem 4.15 plus Proposition 5.5 suggests that the algebra $\mathcal{C}_n^\circ$ corresponds with $\bar{\mathcal{B}}_n^\circ$ in fact. The algebra corresponding to $\mathcal{B}_n^\circ$ is obtained by weakening condition (7.4 i) as follows:

$$(y_i, u_{jk})=(u_{ij}, y_k),\quad (z_i, u_{jk})=(u_{ij}, z_k) \qquad (1\leqq i<j<k\leqq n-1).$$

These identities hold in $\mathcal{C}_n^\circ$, and imply (7.4 i) if $1+q^2\in R^\times$, under the conditions (7.4 e), f).

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