J. Math. Soc. Japan
Vol. 42, No. 2, 1990

Consistency of Menas' conjecture

By Yo MATSUBARA

(Received March 7, 1989)

In this paper we will prove the consistency of the following conjecture of Menas [8] with ZFC. Menas' conjecture: *For every regular uncountable cardinal κ and λ a cardinal $>\kappa$, if X is a stationary subset of $\mathcal{P}_\kappa\lambda$ then X splits into $\lambda^{<\kappa}$ many disjoint stationary subsets.* We will prove the consistency of the conjecture by showing that it holds in L, the class of constructible sets.

Baumgartner and Taylor [1] have shown the consistency of the failure of Menas' conjecture with ZFC. Thus we can conclude that Menas' conjecture is independent of ZFC. Throughout this paper we let κ denote a regular uncountable cardinal and λ a cardinal $>\kappa$.

Baumgartner and DiPrisco proved that if $0^\#$ does not exist then every stationary subset of $\mathcal{P}_\kappa\lambda$ splits into λ many disjoint stationary subsets. In [6], we have proved the following, strengthening their result slightly using generic ultrapowers.

THEOREM 1. *If there is a stationary subset of $\mathcal{P}_\kappa\lambda$ which does not split into λ many disjoint stationary subsets, then $b^\#$ exists for every bounded subset b of λ.*

The proof of Theorem 1 was based on the following two results.

THEOREM 2 (Foreman [2]). *If I is a countably complete λ^+-saturated ideal on $\mathcal{P}_\kappa\lambda$ then I is precipitous.*

THEOREM 3 ([6]). *If there is a precipitous ideal on $\mathcal{P}_\kappa\lambda$ then $b^\#$ exists for every bounded subset b of λ.*

Let $\mathrm{NS}(\kappa, \lambda)$ denote the nonstationary ideal on $\mathcal{P}_\kappa\lambda$. Thus $\mathrm{NS}(\kappa, \lambda)$ is a κ-complete normal idea. If X is a stationary subset of $\mathcal{P}_\kappa\lambda$ which does not split into λ many disjoint stationary subsets then $\mathrm{NS}(\kappa, \lambda)|X$ is a λ-saturated κ-complete normal ideal on $\mathcal{P}_\kappa\lambda$ where

$$\mathrm{NS}(\kappa, \lambda)|X=\{Y\subseteq\mathcal{P}_\kappa\lambda:\ Y\cap X\in\mathrm{NS}(\kappa, \lambda)\}.$$

Thus by Theorem 2, the existence of a stationary subset of $\mathcal{P}_\kappa\lambda$ which does not split into λ many disjoint stationary subsets implies the existence of a precipitous ideal on $\mathcal{P}_\kappa\lambda$.

Unfortunately the above results do not provide us with a method to split stationary subsets of $\mathcal{P}_\kappa\lambda$ into $\lambda^{<\kappa}$ many disjoint stationary subsets when $\lambda^{<\kappa}>\lambda$. The next result is the tool to overcome this difficulty.

LEMMA 4. *If $\lambda^{<\kappa}=2^\lambda$ and $\kappa^{<\kappa}<2^\lambda$ then every stationary subset of $\mathcal{P}_\kappa\lambda$ splits into $\lambda^{<\kappa}$ many disjoint stationary subsets.*

PROOF. We will use the following well-known result of Kueker [4].

Kueker's Theorem: *For every $W\subseteq\mathcal{P}_\kappa\lambda$, W contains a cub (closed and unbounded) subset of $\mathcal{P}_\kappa\lambda$ iff there exists a function $f: [\lambda]^{<\omega}\to\mathcal{P}_\kappa\lambda$ such that $\{s\in\mathcal{P}_\kappa\lambda: f''[s]^{<\omega}\subseteq\mathcal{P}(s)\}\subseteq W$. For each $f: [\lambda]^{<\omega}\to\mathcal{P}_\kappa\lambda$, we let $A(f)=\{s\in\mathcal{P}_\kappa\lambda: f''[s]^{<\omega}\subseteq\mathcal{P}(s)\}$.*

Now assume $\lambda^{<\kappa}=2^\lambda$ and $\kappa^{<\kappa}<\lambda^{<\kappa}$. Let X be a stationary subset of $\mathcal{P}_\kappa\lambda$. Let $\langle f_\alpha: \alpha<2^\lambda\rangle$ enumerate the functions from $[\lambda]^{<\omega}$ into $\mathcal{P}_\kappa\lambda$.

CLAIM 1. *Given any function $f: [\lambda]^{<\omega}\to\mathcal{P}_\kappa\lambda$ we have*

$$|\{\alpha<2^\lambda: A(f_\alpha)\subseteq A(f)\}| = 2^\lambda.$$

PROOF OF CLAIM 1. For each $B\subseteq[\lambda]^{<\omega}$, choose $f_B: [\lambda]^{<\omega}\to\mathcal{P}_\kappa\lambda$ to be a function such that $f_B(a)=f(a)$ for each $a\in B$ and $f_B(a)\supsetneqq f(a)$ for each $a\in[\lambda]^{<\omega}\setminus B$. It is clear that for each $B\subseteq[\lambda]^{<\omega}$, $A(f_B)\subseteq A(f)$ and $f_B\neq f_{B'}$ provided $B\neq B'$.

CLAIM 2. *If Y is an unbounded subset of $\mathcal{P}_\kappa\lambda$ then $|Y|=2^\lambda$.*

PROOF OF CLAIM 2. Since $\mathcal{P}_\kappa\lambda\subseteq\bigcup_{s\in Y}\mathcal{P}(s)$, we must have $\lambda^{<\kappa}\leqq\kappa^{<\kappa}|Y|$. But since $\lambda^{<\kappa}=2^\lambda$ and $\kappa^{<\kappa}<\lambda^{<\kappa}$, we must have $|Y|=2^\lambda$.

Suppose X is a stationary subset of $\mathcal{P}_\kappa\lambda$. Now for each $\alpha<2^\lambda$, we will pick a sequence $\langle s^\alpha_\beta: \beta<\alpha\rangle$ of distinct elements from X by induction on α. We will describe the α-th stage of induction. Since $X\cap A(f_\alpha)$ is stationary, by Claim 2 we have $|X\cap A(f_\alpha)|=2^\lambda$. Hence $|X\cap A(f_\alpha)\setminus\{s^\eta_\beta: \eta<\alpha \text{ and } \beta<\eta\}|=2^\lambda$. Pick α many distinct elements from $X\cap A(f_\alpha)\setminus\{s^\eta_\beta: \eta<\alpha \text{ and } \beta<\eta\}$. Let these elements form $\langle s^\alpha_\beta: \beta<\alpha\rangle$. For each $\gamma<2^\lambda$, define $X_\gamma=\{s^\alpha_\gamma: \gamma<\alpha<2^\lambda\}$. It is clear that $\langle X_\gamma: \gamma<2^\lambda\rangle$ are pairwise disjoint subsets of X. We will show that X_γ is a stationary subset of $\mathcal{P}_\kappa\lambda$ for each $\gamma<2^\lambda$.

Fix $\gamma<2^\lambda$. Let C be a cub subset of $\mathcal{P}_\kappa\lambda$. By Kuecker's theorem there is a function $f: [\lambda]^{<\omega}\to\mathcal{P}_\kappa\lambda$ such that $A(f)\subseteq C$. By Claim 1 there is some $\alpha<2^\lambda$ such that $A(f_\alpha)\subseteq A(f)$ and $\alpha>\gamma$. Thus $s^\alpha_\gamma\in X_\gamma\cap A(f_\alpha)\subseteq X_\gamma\cap C$. Hence X_γ is a stationary subset of $\mathcal{P}_\kappa\lambda$. □

The next result is a direct consequence of Theorem 1 and Lemma 4.

THEOREM 5. *If* GCH *(the generalized continuum hypothesis) holds and there is a set b of ordinals such that $b^{\#}$ does not exist then every stationary subset of $\mathcal{P}_\kappa\lambda$ splits into $\lambda^{<\kappa}$ many disjoint stationary subsets, provided* $\sup b<\lambda$.

COROLLARY 6. (i) $L\models$*Menas' conjecture; Hence Menas' conjecture is consistent with* ZFC.

(ii) $L[U]\models$*If U is a normal measure on a measurable cardinal δ, then for every regular uncountable cardinal κ and λ a cardinal* $>\max(\kappa, \delta^+)$, *every stationary subset of $\mathcal{P}_\kappa\lambda$ splits into $\lambda^{<\kappa}$ many disjoint stationary subsets.*

REMARK. We will improve (ii) in Corollary 12.

PROOF OF COROLLARY 6. (i) One instance of Theorem 5 says that if GCH holds and $0^{\#}$ does not exist then Menas' conjecture holds. This is also a direct consequnce of the above mentioned result of Baumgartner and DiPrisco and Lemma 4. Since GCH and $\neg\exists 0^{\#}$ hold in L, Menas' conjecture holds in L.

(ii) Since U can be coded by a subset of δ^+ in $L[U]$, it is easy to see that $L[U]\models$ there is a subset b of δ^+ such that $b^{\#}$ does not exist. By Silver's work [9], we know that $L[U]\models$GCH. Thus Theorem 5 implies that the conclusion holds. □

In [7] we have discussed the following question: *Can $\mathcal{P}_\kappa\lambda$ be split into $\lambda^{<\kappa}$ many disjoint stationary subsets?* We gave an affirmative answer when κ is an inaccessible cardinal. The following corollary of Lemma 4 answers this question under GCH.

COROLLARY 7. *If* GCH *holds then for any regular uncountable cardinal κ and $\lambda>\kappa$, $\mathcal{P}_\kappa\lambda$ splits into $\lambda^{<\kappa}$ many disjoint stationary subsets.*

PROOF. By the work of Jech and DiPrisco we know that $\mathcal{P}_\kappa\lambda$ splits into λ many disjoint stationary subsets (see Theorem 1 of [7]). Thus we may assume that $\lambda^{<\kappa}>\lambda$. Using GCH we see that $\lambda^{<\kappa}=2^\lambda$ and $\kappa^{<\kappa}<\lambda^{<\kappa}$. By Lemma 4 every stationary subset of $\mathcal{P}_\kappa\lambda$ splits into $\lambda^{<\kappa}$ many disjoint stationary subsets. □

Recently Gitik [3] showed the consistency of a strong failure of Menas' conjecture by proving the following theorem.

Gitik's Theorem: *If the existence of a supercompact cardinal is consistent with* ZFC, *then it is consistent that for a regular cardinal κ and some $\lambda>\kappa$ there is a stationary subset X of $\mathcal{P}_\kappa\lambda$ such that* $\mathrm{NS}(\kappa, \lambda)|X$ *is κ^+-saturated i.e. X does not split into κ^+ many disjoint stationary subsets.*

Gitik asks if the existence of such a nonsplitting set is consistent with GCH. We give a partial answer to this question.

THEOREM 8. *If* GCH *holds below a regular uncountable cardinal* κ *and* $\lambda^{<\kappa}>\lambda$, *then for every stationary subset* X *of* $\mathcal{P}_\kappa\lambda$, $\mathrm{NS}(\kappa, \lambda)|X$ *cannot be* λ^+*-saturated.*

PROOF. Assume there is a stationary subset X of $\mathcal{P}_\kappa\lambda$ such that $\mathrm{NS}(\kappa, \lambda)|X$ is λ^+-saturated. By Lemma 4 and the hypothesis of this theorem, it suffices to prove that $2^\lambda=\lambda^+$ in order to derive a contradiction. The following lemma completes our proof.

LEMMA 9. *If* GCH *holds below* κ *and* $\mathcal{P}_\kappa\lambda$ *carries a* κ*-complete* λ^+*-saturated normal ideal, then* $2^\lambda=\lambda^+$.

This is Theorem 19 of [6]. Since the proof of this result is short we will reproduce it here for completeness.

PROOF OF LEMMA 9. Let I be a κ-complete λ^+-saturated normal ideal on $\mathcal{P}_\kappa\lambda$. Let G be a generic filter on $\mathcal{P}(\mathcal{P}_\kappa\lambda)/I$. Let $j: V\to M\cong Ult(V, G)$ be the canonical elementary embedding into the transitive collapse of $Ult(V, G)$. Since $\{s\in\mathcal{P}_\kappa\lambda: 2^{|s|}=|s|^+\}\in G$, we must have $M\models 2^{|\lambda|}=|\lambda|^+$. By the λ^+-saturatedness of I, $(|\lambda|^+)^M=(\lambda^+)^V$. Hence $V[G]\models|\mathcal{P}(\lambda)\cap M|\leqq|(\lambda^+)|^V$.

For each $x\in\mathcal{P}(\lambda)\cap V$, define $f_x: \mathcal{P}_\kappa\lambda\to\mathcal{P}_\kappa\lambda$ by $f_x(s)=s\cap x$. Thus, $[f_x]\in\mathcal{P}([\mathrm{id}])\cap M$ where id denotes the identity function on $\mathcal{P}_\kappa\lambda$. Furthermore, for each distinct $x, y\in\mathcal{P}(\lambda)\cap V$, $[f_x]\neq[f_y]$. Using the fact $[\mathrm{id}]=j''\lambda$, we have $V[G]\models|\mathcal{P}(\lambda)\cap V|\leqq|\mathcal{P}(\lambda)\cap M|$. Thus we conclude $V[G]\models|\mathcal{P}(\lambda)\cap V|\leqq|(\lambda^+)^V|$. This implies $V\models 2^\lambda=\lambda^+$. □

Recently we have learned the following result from Magidor which indicates that consistency strength of the existence of precipitous ideal on $\mathcal{P}_\kappa\lambda$ is quite high.

THEOREM 10 (Magidor [5]). *If* $\mathcal{P}_\kappa\lambda$ *carries a precipitous ideal then there is an inner model of a measurable cardinal, say* δ *such that* $o(\delta)=\delta^{++}$ *where* $o(\delta)$ *is the Mitchell order of* δ.

Theorem 10 together with Lemma 4 gives the following.

COROLLARY 11. *If* GCH *holds and there is no inner model of a measurable cardinal* δ *such that* $o(\delta)=\delta^{++}$ *then Menas' conjecture holds.*

We can now improve (ii) of Corollary 6 using the fact that $L[U]$ satisfies the hypothesis of Corollary 11.

COROLLARY 12. $L[U]\models$ *If* U *is a normal measure on some measurable cardinal then Menas' conjecture holds.*

References

[1] J. E. Baumgartner and A. D. Taylor, Saturation properties of ideals in generic extensions. I, Trans. Amer. Math. Soc., **270** (1982), 557-573.

[2] M. Foreman, Potent axioms, Trans. Amer. Math. Soc., **86** (1986), 1-26.

[3] M. Gitik, Nonsplitting subset of $\mathcal{P}_\kappa(\kappa^+)$, J. Symbolic Logic, **50** (1985), 881-894.

[4] D. Kueker, Countable approximations and Lowenheim-Skolem theorems, Ann. Math. Logic, **11** (1977), 57-103.

[5] M. Magidor, private communication.

[6] Y. Matsubara, Menas' conjecture and generic ultrapowers, Ann. Pure Appl. Logic, **36** (1987), 225-234.

[7] Y. Matsubara, Splitting $\mathcal{P}_\kappa\lambda$ into stationary subsets, J. Symbolic Logic, **53** (1988), 385-389.

[8] T. K. Menas, On strong compactness and supercompactness, Ann. Math. Logic, **7** (1975), 327-359.

[9] J. H. Silver, The consistency of the GCH with the existence of a measurable cardinal, in "Axiomatic Set Theory", (D. Scott ed.), Proc. Sympos. Pure Math., **13** (1971), 383-390.

Yo MATSUBARA
Department of Mathematics
Occidental College
1600 Campus Road
Los Angeles, California 90041
U. S. A.