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Metric deformation of non-positively curved manifolds

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§ 1. Introduction and statement of results.

There are strong relations between the topology and the curvature of a Riemannian manifold. For example, let M be a compact Riemannian manifold of negative curvature. Then every abelian subgroup of $\pi_1(M)$ must be cyclic, which is not necessarily true for a manifold of non-positive curvature.

A natural question is under what conditions a metric of non-positive curvature can be deformed to a metric of negative curvature. For this question, we have the following results.

THEOREM 1. *Let (M, g) be a complete Riemannian manifold with $K_g \leqq 0$, where K_g denotes the sectional curvature of (M, g), and p a point in M. Then there is a positive number R which is determined by the metric g and its derivatives around p, such that the following holds; suppose $K_g < 0$ on $M \setminus B_R(p)$, then there is a metric $\bar{g}$ such that $K_{\bar{g}} < 0$ and $g = \bar{g}$ on $M \setminus B_R(p)$, where we put $B_R(p) = \{q \in M;\ d(p, q) < R\}$.*

In general, the number R in Theorem 1 is much smaller than $i(p)$, the injectivity radius at p, but for two dimensional manifolds, we have a better result.

THEOREM 2. *Let (M, g) be a complete Riemannian manifold of two dimension with $K_g \leqq 0$. Suppose there is a point p in M such that $K_g < 0$ on $M \setminus B_{i(p)}(p)$. Then there is a complete metric $\bar{g}$ such that $K_{\bar{g}} < 0$ and $g = \bar{g}$ on $M \setminus B_{i(p)}(p)$.*

As a corollary to Theorem 2, we have the following result for $\boldsymbol{R}^2$.

COROLLARY OF THEOREM 2. *Let $(\boldsymbol{R}^2, g)$ be a complete metric on $\boldsymbol{R}^2$ with $K_g \leqq 0$. Suppose there is a compact set $A \subset \boldsymbol{R}^2$ with $K_g < 0$ on $\boldsymbol{R}^2 \setminus A$. Then there is a complete metric $\bar{g}$ on $\boldsymbol{R}^2$ with $K_{\bar{g}} < 0$ and $g = \bar{g}$ on $\boldsymbol{R}^2 \setminus B$ for some compact set $B \subset \boldsymbol{R}^2$.*

Generally, it is not possible to change a metric of non-positive curvature to a metric of negative curvature, because there is a topological obstruction between them as is stated before. But if the set of points at which K_g takes the zero is contained in a topologically trivial ball, then it is likely that we can deform the metric to a metric of negative curvature. This consideration leads us to the following conjecture.

CONJECTURE. *Theorem 2 is true for the case of dimension* $n \geqq 3$.

We consider an application of Theorem 1. We recall the well known corollary to Margulis-Heinze theorem; let M be a complete Riemannian manifold of dimension n with $-1 \leqq K < 0$, then $\mathrm{vol}(M) \geqq V(n)$, where $V(n)$ is a positive number depending only on the dimension n ([**1**]). Using Theorem 1, we can generalize the curvature condition as follows.

APPLICATION OF THEOREM 1. *Let M be a complete Riemannian manifold of dimension n with $-1 \leqq K < 0$ except for discrete points in M, then* $\mathrm{vol}(M) \geqq V(n)$, *where $V(n)$ is the same number as in the above fact.*

As for the non-negative manifolds, we have the same results as in the non-positive cases.

THEOREM 3. *Let (M, g) be a complete Riemannian manifold with $K_g \geqq 0$ and p a point in M. Then there is a positive number R which is determined by the metric g and its derivatives around p, such that the following holds; suppose $K_g > 0$ on $M \setminus B_R(p)$, then there is a metric $\bar{g}$ such that $K_{\bar{g}} > 0$ and $g = \bar{g}$ on $M \setminus B_R(p)$.*

THEOREM 4. *Let (M, g) be a complete Riemannian manifold of two dimension with $K_g \geqq 0$. Suppose there is a point p in M such that $K_g > 0$ on $M \setminus B_{i(p)}(p)$. Then there is a complete metric $\bar{g}$ such that $K_{\bar{g}} > 0$ and $g = \bar{g}$ on $M \setminus B_{i(p)}(p)$.*

About the deformation of a metric with non-negative curvature, there is a following conjecture. Let M be a compact manifold with non-negative curvature and suppose there is a point with positive curvature. Then M will admit a metric of positive curvature. Ehrlich has pointed out the difficulty of this problem in [**3**].

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§ 2. Proof of Theorem 1.

We need some definitions; We put $B_\eta(p) = \{q \in M;\ d(p, q) < \eta\}$ and $\phi(q) = (1/2) d_p^2(q)$ where $d_p(q) = d(p, q)$.

We change the metric

$$\bar{g}(q) = g(q) r(\phi(q)) \qquad \text{for } q \in B_\eta(p), \tag{2.1}$$

where r is a certain function with $2 \geqq r \geqq 1$ and $|r'| \leqq 1$.

We denote as follows

$$(2.2) \qquad \begin{cases} r(q) = r(\phi(q)) \\ r'(q) = r'(\phi(q)) \\ r''(q) = r''(\phi(q)). \end{cases}$$

For the curvature $\bar{K}$ of $\bar{g}$, we show

LEMMA 1. *Let $q \in B_\eta(p)$ and σ_q a 2-plane in T_qM. Then*

$$(2.3) \qquad |K(\sigma)-\bar{K}(\sigma)-r'(q)| \leqq C\{(1+|r'|)|r-1|+(|r'|+|r'|^2+|r''|)d_p^2(q)\}$$

where C is a certain positive constant which does not depend on q.

PROOF. It is well known how the curvature tensor changes by the conformal change of the metric [2]. Let ξ, η be a orthonormal base of σ_q. Then we have

$$(2.4) \qquad \begin{aligned} K(\sigma)-\bar{K}(\sigma)-r' = {} & K(\sigma)\Big(1-\frac{1}{r}\Big)+r'\Big(\frac{1-r^2}{2r^2}\Big)(\mathrm{Hess}\,\phi(\xi, \xi)+\mathrm{Hess}\,\phi(\eta, \eta)) \\ & +\frac{r'}{2}(\mathrm{Hess}\,\phi(\xi, \xi)+\mathrm{Hess}\,\phi(\eta, \eta)-2)+\frac{r'^2}{4r^3}d_p^2 \\ & +\Big(\frac{r''}{2r^2}-\frac{3r'^2}{4r^3}\Big)(\langle\xi, \mathrm{grad}\,\phi\rangle^2+\langle\eta, \mathrm{grad}\,\phi\rangle^2). \end{aligned}$$

Since $\mathrm{grad}\,\phi = d_p\,\mathrm{grad}\,d_p$ and $|\mathrm{grad}\,d_p|=1$, we have

$$(2.5) \qquad \langle\xi, \mathrm{grad}\,\phi\rangle^2+\langle\eta, \mathrm{grad}\,\phi\rangle^2 \leqq d_p^2.$$

We need the estimation of the Hessian [**5**];

If $-\delta^2 \leqq K \leqq \Delta^2$ and $d_p(q) \leqq \pi/2\Delta$, then

$$\Delta d_p(q)\cot(\Delta d_p(q)) \leqq \mathrm{Hess}\,\phi(q)(\xi, \xi) \leqq \delta d_p(q)\coth(\delta d_p(q)).$$

Hence

$$(2.6) \qquad |\mathrm{Hess}\,\phi(\xi, \xi)-1| \leqq C' d_p^2(q),$$

$$(2.7) \qquad |\mathrm{Hess}\,\phi(\xi, \xi)| \leqq C',$$

where C' is a constant depending only on $g(B_p(\pi/2\Delta))$. From (2.4)-(2.7) and $r \geqq 1$, $|r'| \leqq 1$, we have

$$|K(\sigma)-\bar{K}(\sigma)-r'| \leqq C\{(1+|r'|)|r-1|+(|r'|+|r'|^2+|r''|)d_p^2(q)\}.$$

Lemma 1 is shown.

Now we construct a suitable function $r(t)$ to have $\bar{K}<0$. Let $f(t)$ be a function with following properties, (2.8) $$f(0)=0$$

(2.9) $$f(t)=0 \qquad (1<t)$$

(2.10) $$0 \leqq f(t) \leqq 1$$

(2.11) $$f'(t)>0 \qquad \left(0 \leqq t \leqq \frac{1}{2}\right)$$

(2.12) $$|f'(t)| \leqq 1$$

(2.13) $$|f''(t)| \leqq 1 .$$

Putting $r^{(\alpha\beta)}(t)=\beta f(t/\alpha)+1$ for $0<\alpha<1$, $0<\beta<\alpha$, we define a metric $g^{(\alpha\beta)}$

(2.14) $$g^{(\alpha\beta)}(q)=g(q)r^{(\alpha\beta)}(q).$$

We now want to explain the method to make the curvature $K^{(\alpha\beta)}$ of $g^{(\alpha\beta)}$ negative. As the first step, we make $K^{(\alpha\beta)}<0$ near p; If we take α smaller, $K^{(\alpha\beta)}$ decreases at p. Therefore, taking α sufficiently small we have $K^{(\alpha\beta)}<0$ in a small neighborhood N of p for any β. But at the same time, $K^{(\alpha\beta)}$ increases and may become positive in $M\setminus N$. Thus, as the second step we need to reduce the increase of $K^{(\alpha\beta)}$ in $M\setminus N$. We can make $|K-K^{(\alpha\beta)}|$ arbitrary small if we take β small. Therefore, we obtain $K^{(\alpha\beta)}<0$ on M by taking β small, if we assume $K<0$ in $M\setminus N$ (see the assumption in the Lemma 2 below). In fact, we have

LEMMA 2. *If we assume on the curvature K_g the condition* (2.16) *below, then there are α_1 and β_1 such that $K^{(\alpha_1\beta_1)}<0$.*

PROOF. We denote $C\{(1+|r'|)|r-1|+(|r'|+|r'|^2+|r''|)d_p^2(q)\}$ as $C(r(q))$ for simplicity. We devide the cases for q as follows;

⟨Case 1⟩: $0\leqq\phi(q)\leqq\dfrac{\alpha^2}{2}$

⟨Case 2⟩: $\dfrac{\alpha^2}{2}\leqq\phi(q)\leqq\alpha$

⟨Case 3⟩: $\alpha\leqq\phi(q)$.

⟨*Case* 1⟩: In this case, we have

PROPOSITION 1. *For a point q of the Case* 1,

(2.15) $$\frac{|C(r^{(\alpha\beta)}(q))|}{|r^{(\alpha\beta)\prime}(q)|}\longrightarrow 0 \qquad (\alpha\longrightarrow 0).$$

PROOF OF PROPOSITION 1. From $f'(t)>0$ $(0\leqq t\leqq 1/2)$, there is a constant C' such that

$$\left|\frac{f(\phi(q)/\alpha)}{f'(\phi(q)/\alpha)}\right| \leqq C', \qquad \left|\frac{f''(\phi(q)/\alpha)}{f'(\phi(q)/\alpha)}\right| \leqq C', \qquad \left|\frac{r^{(\alpha\beta)}-1}{r^{(\alpha\beta)\prime}}\right| \leqq C'\alpha$$

and

$$\left|\frac{r^{(\alpha\beta)\prime\prime}}{r^{(\alpha\beta)\prime}}\right| \leqq \frac{C'}{\alpha}.$$

Using $|r^{(\alpha\beta)\prime}|<\beta/\alpha<1$ and $d_p(q)\leqq\alpha$ we have this proposition.

The estimates (2.3) and (2.15) imply there is a positive constant α_0 such that $K^{(\alpha\beta)}(\sigma_q)<0$ for α with $0<\alpha\leqq\alpha_0$ and q with $\phi(q)\leqq\alpha^2/2$. Therefore the case 1 is shown for α with $0<\alpha\leqq\alpha_0$.

Now we assume the following condition on the metric g;

(2.16) *There is a positive constant* α_1 *with* $0<\alpha_1\leqq\alpha_0$ *such that* $K(\sigma_q)<0$ *for every 2-plane in* T_qM *with* $\alpha_1\leqq d_p(q)$.

⟨*Case* 2⟩: At first we need the following proposition.

PROPOSITION 2. *If* $d_p(q)$ *is bounded, then for any fixed* α,

$$|C(r^{(\alpha\beta)}(q))| \longrightarrow 0 \qquad (\beta \longrightarrow 0). \tag{2.17}$$

The proof is easy and we omit it.

From (2.3), (2.16), (2.17), we have $K^{(\alpha_1\beta_1)}(\sigma_q)<0$ for q with $\alpha_1^2/2\leqq\phi(q)\leqq\alpha_1$ for sufficiently small β_1. Therefore the case 2 is shown.

⟨*Case* 3⟩: For a point q of this case, $g^{(\alpha_1\beta_1)}(q)=g(q)$. Therefore $K^{(\alpha_1\beta_1)}(\sigma_q)<0$ follows directly from (2.16). Thus the case 3 is shown for α_1 and β_1.

From the above argument for these three cases, there are positive numbers α_1, β_1 such that $K^{(\alpha_1\beta_1)}<0$, if we assume the condition (2.16) on the metric g. Hence Lemma 2 is shown.

PROOF OF THEOREM 1. We put $R=\alpha_1$. Then $K_g(\sigma_q)<0$ on $M\setminus B_R(p)$ imply the condition (2.16). Therefore, using the Lemma 2, we have the Theorem 1.

§ 3. The proof of Theorem 2.

We identify $B_{i(p)}(p)$ and $B_{i(p)}(0)\subset T_pM$ by $\exp_p$ and introduce the polar coordinate (r, θ) on $B_{i(p)}(p)$. Then the metric g can be written as follows on $B_{i(p)}(p)$.

$$ds^2 = dr^2+h^2(r, \theta)d\theta^2 . \tag{3.1}$$

In order to obtain a metric with $K<0$, we change the given metric to a new metric $\bar{g}$ as follows,

$$d\bar{s}^2 = f^2(r)dr^2+h^2(r, \theta)d\theta^2 , \tag{3.2}$$

where $h(r, \theta)$ is the same as in (3.1), $f(r)$ is some positive function with $f(0)=1$ and $f^{(n)}(0)=0$. We shall choose $f(r)$ such that $d\bar{s}^2$ has negative sectional curvature. We denote $f'=df/dr$, $h'=\partial h/\partial r$, $h''=\partial^2 h/\partial r^2$. Then we have $\bar{R}^r_{\theta r\theta}=h(f'h'-fh'')/f^3$,

$$K_{\bar{g}} = \frac{\bar{R}^r_{\theta r\theta}}{h^2} = \frac{f'h'-fh''}{f^3 h} \quad \text{and} \tag{3.3}$$

$$K_g = -\frac{h''}{h}. \tag{3.4}$$

Thus, the assumption $K_g \leqq 0$ implies

$$h''(r, \theta) \geqq 0 \qquad \text{for} \quad 0 \leqq r \leqq i(p). \tag{3.5}$$

Further, we have

$$h'(r, \theta) > 0 \qquad \text{for} \quad 0 < r \leqq i(p). \tag{3.6}$$

Since the set $\{q \in M;\ K_g(q)=0\}$ is closed, there is a positive number ε such that $K_g(q)<0$ for $M \setminus B_{i(p)-\varepsilon}$. Then we have the following lemma.

LEMMA 3. *There are functions $\{f_i(r)\}$, $(i=1, 2, \cdots)$ with the following properties;*

$$f_i(r) = 1 \qquad \text{for} \quad 0 \leqq r \leqq \frac{1}{i}, \quad i(p)-\varepsilon \leqq r \leqq i(p). \tag{3.7}$$

$$\frac{f_i'(r)}{f_i(r)} < \frac{h''(r, \theta)}{h'(r, \theta)} \qquad \text{for} \quad \frac{1}{i} < r \leqq i(p), \quad 0 \leqq \theta \leqq 2\pi. \tag{3.8}$$

$$|f_i(r)-1|,\ |f_i'(r)-0|,\ |f_i''(r)-0|,\ |f_i'''(r)-0| \leqq \frac{1}{i}, \quad \text{for } 0 \leqq r \leqq i(p). \tag{3.9}$$

PROOF OF LEMMA 3. From (3.4) and (3.6), we have

$$\frac{h''(r, \theta)}{h'(r, \theta)} \geqq 0 \qquad \text{for} \quad 0 \leqq r \leqq i(p), \tag{3.10}$$

$$\frac{h''(r, \theta)}{h'(r, \theta)} > 0 \qquad \text{for} \quad i(p)-\varepsilon \leqq r \leqq i(p). \tag{3.11}$$

Then from (3.10), (3.11), it is easy to show that there are functions $f_i(r)$ with the properties (3.7), (3.8) and (3.9).

Using functions $f_i(r)$ in Lemma 3, we define metrics g_i as follows,

$$ds_i^2 = f_i^2(r)dr^2 + h^2(r, \theta)d\theta^2 \qquad \text{on} \quad B_{i(p)}(p),$$

and

$$g_i = g \quad \text{on} \quad M \setminus B_{i(p)}(p).$$

Then (3.7) and (3.8) imply

$$K_{g_i} < 0 \qquad \text{for} \quad M \setminus B_{1/i}(p). \tag{3.12}$$

Now we denote the constant R in the Theorem 1 for these metrics $\{g_i\}$ by $\{R_i\}$. Looking over the proof of Theorem 1 carefully, it is known from (3.9) that there is a positive number δ such that $R_i>\delta$ for all i. Then taking i large, the condition (3.12) implies the assumption in Theorem 1 and hence we obtain a metric $\bar{g}$ with $K_{\bar{g}}<0$ on M.

PROOF OF COROLLARY. It is easy to show from Theorem 2.

PROOF OF APPLICATION. In the proof of Theorem 1, it follows from the definition of $r^{(\alpha\beta)}(t)$ that $r^{(\alpha\beta)}(t)\to 1$ $(\beta\to 0)$, so that $g^{(\alpha\beta)}(q)\to g(q)$ $(\beta\to 0)$, thus $\mathrm{vol}_{g^{(\alpha\beta)}}(M)\to\mathrm{vol}_g(M)$ $(\beta\to 0)$. From Theorem 1, we may assume $K_{g^{(\alpha\beta)}}<0$ and then it follows that $\mathrm{vol}_{g^{(\alpha\beta)}}(M)\geqq V(n)$ where $V(n)$ is some constant depending on the dimention n of M. Hence, we have $\mathrm{vol}_g(M)\geqq V(n)$.

PROOF OF THEOREMS 3 AND 4. We can show them in the same ways as to prove Theorems 1 and 2.

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