J. Math. Soc. Japan
Vol. 42, No. 2, 1990

Joint spectra of strongly hyponormal operators on Banach spaces

Dedicated to Professor Emeritus Eiitiro Homma with respect

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(Received Dec. 12, 1988)

1. Introduction.

The joint spectrum for a commuting n-tuple in functional analysis has its origin in functional calculus which appeared in J. L. Taylor's paper [**23**] in 1970. In the case of operators on Hilbert spaces, in [**25**] F.-H. Vasilescu characterized the joint spectrum for a commuting pair and in [**11**] R. Curto did it for a commuting n-tuple.

For those on a Banach space, in [**18**] and [**19**] A. McIntosh, A. Pryde and W. Ricker characterized the joint spectrum for a strongly commuting n-tuple of operators. In [**5**] M. Chō proved that the joint spectrum for such an n-tuple is the joint approximate point spectrum of it.

The aim of this paper is to give a characterization of the joint spectrum for a doubly commuting n-tuple of strongly hyponormal operators on a uniformly convex and uniformly smooth space.

Let E^n be the complex exterior algebra on n-generators $e_1, \cdots, e_n$ with product $\wedge$. Then E^n is graded: $E^n=\oplus_{k=-\infty}^{\infty} E_k^n$, where $E_k^n \wedge E_1^n \subset E_{k+1}^n$ and $\{e_{j_1}\wedge\cdots\wedge e_{j_k} : 1\leqq j_1<\cdots<j_k\leqq n\}$ is a basis for $E_k^n (k\geqq 1)$, while $E_0^n \cong \boldsymbol{C}$ and $E_k^n=(0)$ for $k<0$ and $k>n$. Let X be a complex Banach space and $\boldsymbol{T}=(T_1, \cdots, T_n)$ be a commuting n-tuple of bounded linear operators on X. Let $E_k^n(X)=E_k^n \otimes X$ and define $D_k^{(n)}: E_k^n(X)\rightarrow E_{k-1}^n(X)$ by $D_k^{(n)}(x\otimes e_{j_1}\wedge\cdots\wedge e_{j_k})=\sum_{i=1}^k (-1)^{i+1} T_{j_i}x\otimes e_{j_1}\wedge\cdots\wedge \check{e}_{j_i}\wedge\cdots\wedge e_{j_k}$ when $k>0$ (here $\check{}$ means deletion), and $D_k^{(n)}=0$ when $k\leqq 0$ and $k>n$. A straightforward computation shows that $D_k^{(n)}\circ D_{k+1}^{(n)}=0$ for all k, so that $\{E_k^n(X), D_k^{(n)}\}_{k\in \boldsymbol{Z}}$ is a chain complex, called the Koszul complex for $\boldsymbol{T}=(T_1, \cdots, T_n)$ and denoted by $E(X, \boldsymbol{T})$. Of course, the mapping $D_k^{(n)}$ depends on $\boldsymbol{T}=(T_1, \cdots, T_n)$. We denote it by $D_k^{(n)}(\boldsymbol{T})$, if necessary.

We define $\boldsymbol{T}=(T_1, \cdots, T_n)$ to be invertible in case its associated Koszul complex is exact (that is, $\mathrm{Ker}(D_k^{(n)})=R(D_{k+1}^{(n)})$ for all k). The Taylor spectrum $\sigma(\boldsymbol{T})$ for $\boldsymbol{T}=(T_1, \cdots, T_n)$ is the set of $z\in \boldsymbol{C}^n$ such that $\boldsymbol{T}-z=(T_1-z_1, \cdots, T_n-z_n)$ is not invertible.

A point $z \in \boldsymbol{C}^n$ is in the joint approximate point spectrum $\sigma_\pi(\boldsymbol{T})$ of $\boldsymbol{T}$ if there exists a sequence $\{x_k\}$ of unit vectors in X such that

$$\|(T_i - z_i)x_k\| \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty \text{ for } i=1, 2, \cdots, n.$$

For an operator $T \in B(X)$, the spectrum and the approximate point spectrum of T are denoted by $\sigma(T)$ and $\sigma_\pi(T)$, respectively.

We denote by X^* the dual space of X. Let us set

$$\pi = \{(x, f) \in X \times X^* : \|f\| = f(x) = \|x\| = 1\}.$$

The spatial numerical range $V(T)$ and the numerical range $V(B(X), T)$ of T are defined by

$$V(T) = \{f(Tx) : (x, f) \in \pi\}$$

and

$$V(B(X), T) = \{\mathcal{F}(T) : \mathcal{F} \in B(X)^* \text{ and } \|\mathcal{F}\| = \mathcal{F}(I) = 1\},$$

respectively. The following results are well-known for $T \in B(X)$:

(1) $\operatorname{co} \sigma(T) \subset \overline{V(T)}$ and $\overline{\operatorname{co}}\, V(T) = V(B(X), T)$,

where $\operatorname{co} E$, $\bar{E}$ and $\overline{\operatorname{co}}\, E$ are the convex hull, the closure and the closed convex hull of E, respectively. Also

(2) $V(T) \subset V(T^*) \subset \overline{V(T)}$.

If $V(H) \subset \boldsymbol{R}$, then H is called hermitian. Hence, H is hermitian iff H^* is hermitian. An operator $T \in B(X)$ is called hyponormal if there are hermitian operators H and K such that $T = H + iK$ and the commutator $C = i(HK - KH) \geqq 0$, meaning that $V(C) \subset \boldsymbol{R}^+ = \{a \in \boldsymbol{R} : a \geqq 0\}$. A hyponormal operator $T = H + iK$ is called strongly hyponormal if H^2 and K^2 are hermitian. It holds that if T is strongly hyponormal, then $T - \lambda$ is also for every $\lambda \in \boldsymbol{C}$. For an operator $T = H + iK$, we denote the operator $H - iK$ by $\bar{T}$.

Remark 1. There is an hermitian operator H such that H^2 is not hermitian. However, if H is a hermitian, then

$$V(H^2) \subset \{z \in \boldsymbol{C} : \operatorname{Re} z \geqq 0\}.$$

Hence, if T is a strongly hyponormal operator, then

$$V(\bar{T}T) \subset \boldsymbol{R}^+.$$

For commuting operators T_1 and T_2 such that $T_j = H_j + iK_j$ $(j=1, 2)$, T_1 and T_2 are called doubly commuting if $\bar{T}_1 T_2 = T_2 \bar{T}_1$. It is easy to see that if T_1 and T_2 are doubly commuting then H_1 and K_1 commute with H_2 and K_2.

For a commuting n-tuple $\boldsymbol{T} = (T_1, \cdots, T_n)$ such that $T_j = H_j + iK_j$ $(j=1, \cdots, n)$, a point $z = (z_1, \cdots, z_n) \in \boldsymbol{C}^n$ is in the complete star spectrum $\sigma_{cs}(\boldsymbol{T})$ of $\boldsymbol{T}$ if there is some partition $\{j_1, \cdots, j_k\} \cup \{l_1, \cdots, l_m\} = \{1, \cdots, n\}$ such that

$$\sum_{\mu=1}^{k}\overline{(T_{j_\mu}-z_{j_\mu})}(T_{j_\mu}-z_{j_\mu})+\sum_{\nu=1}^{m}(T_{l_\nu}-z_{l_\nu})\overline{(T_{l_\nu}-z_{l_\nu})}$$

is not invertible. In particular, the set

$$\left\{(z_1, \cdots, z_n)\in \boldsymbol{C}^n : \sum_{j=1}^{n}(T_j-z_j)\overline{(T_j-z_j)} \text{ is not invertible}\right\}$$

is called the right spectrum of $\boldsymbol{T}=(T_1, \cdots, T_n)$ and denoted by $\sigma_r(\boldsymbol{T})$. It is clear that $\sigma_\pi(\boldsymbol{T})\subset\sigma(\boldsymbol{T})\cap\sigma_{cs}(\boldsymbol{T})$ for a commuting n-tuple $\boldsymbol{T}=(T_1, \cdots, T_n)$.

A Banach space X is called uniformly convex if to each $\varepsilon>0$, there corresponds a $\delta>0$ such that the conditions $\|x\|=\|y\|=1$ and $\|x-y\|\geqq\varepsilon$ imply that $(1/2)\|x+y\|\leqq 1-\delta$.

We set, for $t>0$:

$$\rho(t)=\sup\{(1/2)(\|x+y\|+\|x-y\|)-1 : \|x\|=1, \|y\|\leqq t\}.$$

A Banach space X is called uniformly smooth if

$$\frac{\rho(t)}{t}\longrightarrow 0 \quad \text{as} \quad t\longrightarrow 0.$$

REMARK 2. A Banach space X is uniformly smooth iff X^* is uniformly convex. See Beauzamy [**3**] for details.

We give an example of a doubly commuting n-tuple of strongly hyponormal operators on a uniformly convex and uniformly smooth space.

Let $\mathcal{H}$ be a complex Hilbert space. Let $\mathcal{C}_p$ be the Schatten p-class for $1<p<\infty$. Then it is well-known that the space $\mathcal{C}_p$ is uniformly convex and uniformly smooth, and is a 2-sided ideal of $B(\mathcal{H})$. When A and B^* are hyponormal operators on $\mathcal{H}$, the derivation $\delta_{A,B}=\delta_{H,H'}+i\delta_{K,K'}$ is a hyponormal operator on $\mathcal{C}_p$, where $A=H+iK$ and $B=H'+iK'$. Moreover,

$$V(B(\mathcal{C}_p), \delta_{A,B})=\overline{W(A)}-\overline{W(B)},$$

where $W(T)$ is a usual numerical range of an operator T on a Hilbert space $\mathcal{H}$. See Shaw [**21**].

Let $\mathcal{L}_A$ denote the left multiplication induced by $A\in B(\mathcal{H})$. Then if $A=H+iK$ is a hyponormal operator, then $\mathcal{L}_A=\mathcal{L}_H+i\mathcal{L}_K$ is a strongly hyponormal operator. Let $\boldsymbol{A}=(A_1, \cdots, A_n)$ be a doubly commuting n-tuple of hyponormal operators on $\mathcal{H}$. Then $\boldsymbol{T}=(\mathcal{L}_{A_1}, \cdots, \mathcal{L}_{A_n})$ is a doubly commuting n-tuple of strongly hyponormal operators on a uniformly convex and uniformly smooth space $\mathcal{C}_p$ $(1<p<\infty)$.

We use the following results.

THEOREM A ([**17**], Theorem 2.5). *Let X be uniformly convex and let H be a hermitian, non-negative operator on X. If there are sequences $\{x_n\}\subset X$ and $\{f_n\}\subset X^*$ such that $\|x_n\|=\|f_n\|=1$ for each n with $f_n(x_n)\to 1$ and $f_n(Hx_n)\to 0$,* *then* $Hx_n \to 0$.

THEOREM B ([17], Theorem 2.7). *Let* X *be uniformly convex and let* $T=H+iK$ *be a hyponormal operator on* X. *If* $\{x_n\}$ *is a bounded sequence in* X *such that* $Tx_n \to 0$, *then* $Hx_n \to 0$ *and* $Kx_n \to 0$.

2. Joint spectra of doubly commuting n-tuples.

LEMMA 1. *Let* $T=H+iK$ *be a strongly hyponormal operator. Then,* $\sigma(\bar{T}T) \cup \sigma(T\bar{T}) \subset \boldsymbol{R}^+$.

PROOF. Since T is strongly hyponormal, the proof follows from $\sigma(\bar{T}T)-\{0\}=\sigma(T\bar{T})-\{0\}$ and $\sigma(\bar{T}T) \subset \overline{V(\bar{T}T)} \subset \boldsymbol{R}^+$.

LEMMA 2. *Let* X *be uniformly convex. Let* $\boldsymbol{T}=(T_1, \cdots, T_n)$ *be a doubly commuting n-tuple of strongly hyponormal operators on* X. *If* $\sum_{j=1}^k \bar{T}_j T_j + \sum_{j=k+1}^n T_j \bar{T}_j$ *is not invertible* $(1 \leqq k \leqq n)$, *then* $\sum_{j=1}^n T_j \bar{T}_j$ *is not invertible.*

PROOF. Put $\boldsymbol{S}=(\bar{T}_1 T_1, \cdots, \bar{T}_k T_k, T_{k+1}\bar{T}_{k+1}, \cdots, T_n \bar{T}_n)$. Then $\boldsymbol{S}$ is a commuting n-tuple. It is clear that 0 is in the boundary of the spectrum $\sigma(\sum_{j=1}^k \bar{T}_j T_j + \sum_{j=k+1}^n T_j \bar{T}_j)$. Hence, 0 is in the approximate point spectrum of $\sum_{j=1}^k \bar{T}_j T_j + \sum_{j=k+1}^n T_j \bar{T}_j$. So by the spectral mapping theorem for the joint approximate point spectrum, there exists $\alpha=(\alpha_1, \cdots, \alpha_n) \in \sigma_\pi(\boldsymbol{S})$ such that $\sum_{j=1}^n \alpha_j = 0$. Since $(\bigcup_{j=1}^k \sigma(\bar{T}_j T_j)) \cup (\bigcup_{j=k+1}^n \sigma(T_j \bar{T}_j))$ is contained in $\boldsymbol{R}^+$, it follows that $\alpha_j=0$ for every $j=1, \cdots, n$. Therefore, there exists a sequence $\{x_m\}$ of unit vectors in X such that

$$\bar{T}_j T_j x_m \longrightarrow 0 \quad \text{and} \quad T_l \bar{T}_l x_m \longrightarrow 0 \quad \text{for } j=1, \cdots, k \text{ and } l=k+1, \cdots, n.$$

If $T_j=H_j+iK_j$, then $C_j=i(H_jK_j-K_jH_j) \geqq 0$ for $j=1, \cdots, k$. Choose a linear functional $f_m \in X^*$ such that $\|f_m\|=f_m(x_m)=1$ for each m. Since then $f_m((H_j^2+K_j^2)x_m) \geqq 0$, $f_m(C_j x_m) \geqq 0$ and

$$f_m(\bar{T}_j T_j x_m) = f_m((H_j^2+K_j^2+C_j)x_m) \longrightarrow 0 \qquad \text{for} \quad j=1, \cdots, k,$$

it follows that

$$f_m(C_j x_m) \longrightarrow 0 \qquad \text{for} \quad j=1, \cdots, k.$$

Hence, by Theorem A, it follows that $C_j x_m \to 0$ and

$$(H_j^2+K_j^2)x_m \longrightarrow 0 \qquad \text{for} \quad j=1, \cdots, k.$$

Therefore, it follows that $T_j \bar{T}_j x_m = (H_j^2+K_j^2-C_j)x_m \to 0$ for $j=1, \cdots, n$.

THEOREM 3. *Let* X *be uniformly convex. Let* $\boldsymbol{T}=(T_1, \cdots, T_n)$ *be a doubly commuting n-tuple of strongly hyponormal operators on* X. *Then*

$$\sigma_{cs}(\boldsymbol{T})=\sigma_r(\boldsymbol{T})=\{(z_1,\cdots,z_n)\in\boldsymbol{C}^n:(\bar{z}_1,\cdots,\bar{z}_n)\in\sigma_\pi(\boldsymbol{S})\},$$

where $\boldsymbol{S}=(\bar{T}_1,\cdots,\bar{T}_n)$.

PROOF. It is clear that

$$\{(z_1,\cdots,z_n)\in\boldsymbol{C}^n:(\bar{z}_1,\cdots,\bar{z}_n)\in\sigma_\pi(\boldsymbol{S})\}\subset\sigma_r(\boldsymbol{T})\subset\sigma_{cs}(\boldsymbol{T}).$$

Since $\boldsymbol{T}-z=(T_1-z_1,\cdots,T_n-z_n)$ is a doubly commuting n-tuple of strongly hyponormal operators for every $z=(z_1,\cdots,z_n)\in\boldsymbol{C}^n$, it suffices to prove that if $0\in\sigma_{cs}(\boldsymbol{T})$ then $0\in\sigma_\pi(\boldsymbol{S})$. By the definition of the complete star spectrum and Lemma 2 it follows that $\sum_{j=1}^n T_j\bar{T}_j$ is not invertible and there exists a sequence $\{x_m\}$ of unit vectors in X such that

$$T_j\bar{T}_jx_m\longrightarrow 0\qquad\text{for}\quad j=1,\cdots,n.$$

Since T_j is hyponormal on a uniformly convex space X, by Theorem B it follows that $\bar{T}_j^2x_m\to 0$ for $j=1,\cdots,n$. Also by the spectral mapping theorem for the joint approximate point spectrum, there exists a sequence $\{y_m\}$ of unit vectors in X such that $\bar{T}_jy_m\to 0$ for $j=1,\cdots,n$. Therefore, we have that $0\in\sigma_\pi(\boldsymbol{S})$.

We now explain a recursive method of obtaining the $D_k^{(n)}$'s. We split the basis of E_k^n into

$$B_1=\{e_{j_1}\wedge\cdots\wedge e_{j_k}:1\leqq j_1<\cdots<j_k\leqq n-1\}$$

and

$$B_2=\{e_{j_1}\wedge\cdots\wedge e_{j_{k-1}}\wedge e_n:1\leqq j_1<\cdots<j_{k-1}\leqq n-1\}$$

for $k\geqq 1$, $n>1$.

Then E_k^{n-1} is precisely the subspace of E_k^n generated by B_1 and a natural isomorphism can be established between E_{k-1}^{n-1} and the subspace of E_k^n generated by B_2. E_k^n can then be identified in a natural way with $E_k^{n-1}\oplus E_{k-1}^{n-1}$ ($k\geqq 1$, $n>1$). It is not hard to see that $D_k^{(n)}$ takes the matrix form:

$$D_k^{(n)}=\begin{pmatrix}D_k^{(n-1)} & (-1)^{k+1}\mathrm{diag}(T_n)\\ 0 & D_{k-1}^{(n-1)}\end{pmatrix}\qquad(n>1,\ k\geqq 1),$$

where $\mathrm{diag}(T_n)$ is meant to be a diagonal matrix with constant diagonal entry T_n.

For a doubly commuting n-tuple $\boldsymbol{T}=(T_1,\cdots,T_n)$ of hyponormal operators, define $\bar{D}_k^{(n)}(\boldsymbol{T}):E_{k-1}^n(X)\to E_k^n(X)$ by

$$\bar{D}_k^{(n)}(\boldsymbol{T})={}^t(D_k^{(n)}(\boldsymbol{S})),\qquad\text{where }\boldsymbol{S}=(\bar{T}_1,\cdots,\bar{T}_n).$$

Let $D_k=D_k^{(n)}(\boldsymbol{T})$ and $\bar{D}_k=\bar{D}_k^{(n)}(\boldsymbol{T})$ for every k. Then it is easy to see that

$$(\bar{D}_kD_k+D_{k+1}\bar{D}_{k+1})D_{k+1}\bar{D}_{k+1}=D_{k+1}\bar{D}_{k+1}(\bar{D}_kD_k+D_{k+1}\bar{D}_{k+1})=(D_{k+1}\bar{D}_{k+1})^2.$$

LEMMA 4. *Let* $\boldsymbol{T}=(T_1,\cdots,T_n)$ *be a doubly commuting n-tuple of hyponormal operators. If* $\bar{D}_kD_k+D_{k+1}\bar{D}_{k+1}$ *is invertible for every k, then* $E(X,\boldsymbol{T})$ *is exact.*

PROOF. It suffices to prove that $\mathrm{Ker}(D_k)\subset R(D_{k+1})$. Let x be in $\mathrm{Ker}(D_k)$. Put $y=\bar{D}_{k+1}(\bar{D}_k D_k+D_{k+1}\bar{D}_{k+1})^{-1}x$. Then $y\in E^n_{k+1}(X)$ and

$$\begin{aligned} D_{k+1}y &= D_{k+1}\bar{D}_{k+1}(\bar{D}_k D_k+D_{k+1}\bar{D}_{k+1})^{-1}x \\ &= (\bar{D}_k D_k+D_{k+1}\bar{D}_{k+1})^{-1}D_{k+1}\bar{D}_{k+1}x \\ &= (\bar{D}_k D_k+D_{k+1}\bar{D}_{k+1})^{-1}(\bar{D}_k D_k+D_{k+1}\bar{D}_{k+1})x = x\,. \end{aligned}$$

It follows that $x\in R(D_{k+1})$. Hence, $R(D_{k+1})=\mathrm{Ker}(D_k)$ for every k.

THEOREM 5. *Let X be uniformly convex. Let $\boldsymbol{T}=(T_1, \cdots, T_n)$ be a doubly commuting n-tuple of hyponormal operators on X. Then $\sigma(\boldsymbol{T})\subset\sigma_{\mathrm{cs}}(\boldsymbol{T})$.*

PROOF. It suffices to prove that if $0\notin\sigma_{\mathrm{cs}}(\boldsymbol{T})$, then $0\notin\sigma(\boldsymbol{T})$. An easy computation shows that

$$\bar{D}_k D_k+D_{k+1}\bar{D}_{k+1}=\begin{pmatrix}\bar{D}_k^{(n-1)}D_k^{(n-1)}+D_{k+1}^{(n-1)}\bar{D}_{k+1}^{(n-1)}+\mathrm{diag}(T_n\bar{T}_n) & 0 \\ 0 & \bar{D}_{k-1}^{(n-1)}D_{k-1}^{(n-1)}+D_k^{(n-1)}\bar{D}_k^{(n-1)}+\mathrm{diag}(\bar{T}_n T_n)\end{pmatrix}.$$

Hence, this formula shows that if $0\notin\sigma_{\mathrm{cs}}(\boldsymbol{T})$, then $\bar{D}_k D_k+D_{k+1}\bar{D}_{k+1}$ is invertible for every k. So, by Lemma 4, it follows that $E(X, \boldsymbol{T})$ is exact.

LEMMA 6 ([**23**], Theorem 3.6). *Let $\boldsymbol{T}=(T_1, \cdots, T_n)$ be a commuting n-tuple of operators on a Banach space X. Then $\sigma(\boldsymbol{T})=\sigma(\boldsymbol{T}^*)$, where $\boldsymbol{T}^*=(T_1^*, \cdots, T_n^*)$.*

THEOREM 7. *Let X be uniformly convex and uniformly smooth. Let $\boldsymbol{T}=(T_1, \cdots, T_n)$ be a doubly commuting n-tuple of strongly hyponormal operators on X. Then*

$$\sigma(\boldsymbol{T})=\sigma_{\mathrm{cs}}(\boldsymbol{T})=\{(z_1, \cdots, z_n)\in\boldsymbol{C}^n : (\bar{z}_1, \cdots, \bar{z}_n)\in\sigma_\pi(\boldsymbol{S})\}\,,$$

where $\boldsymbol{S}=(\bar{T}_1, \cdots, \bar{T}_n)$.

PROOF. By Theorems 3 and 5, it suffices to prove that if $0\in\sigma_\pi(\boldsymbol{S})$, then $0\in\sigma(\boldsymbol{T})$. Since 0 belongs to $\sigma_\pi(\boldsymbol{S})$, there exists a sequence $\{x_k\}$ of unit vectors in X such that

$$\bar{T}_j x_k \longrightarrow 0 \qquad \text{for} \quad j=1, \cdots, n\,.$$

Since 0 belongs to $\sigma(\sum_{j=1}^n T_j\bar{T}_j)$, it also belongs to $\sigma((\sum_{j=1}^n T_j\bar{T}_j)^*)=\sigma(\sum_{j=1}^n \bar{T}_j^* T_j^*)$. Also $(\bar{T}_1^*, \cdots, \bar{T}_n^*)$ is a doubly commuting n-tuple of strongly hyponormal operators on a uniformly convex space X^*. From the proof of Lemma 2 there exists a sequence $\{f_k\}$ of unit vectors in X^* such that

$$\bar{T}_j^* T_j^* f_k \longrightarrow 0 \qquad \text{for} \quad j=1, \cdots, n\,.$$

Since $\bar{T}_j^*$ is a hyponormal operator on a uniformly convex space X^*. By Theorem B, it follows that

$$T_j^{*2} f_k \longrightarrow 0 \quad \text{for} \quad j=1, \cdots, n.$$

Hence, by the spectral mapping theorem for the joint approximate point spectrum, it follows that $0 \in \sigma_\pi(\boldsymbol{T}^*)$, where $\boldsymbol{T}^*=(T_1^*, \cdots, T_n^*)$. Therefore, from Lemma 6 it follows that $0 \in \sigma(\boldsymbol{T})$.

ACKNOWLEDGEMENT. The author would like to express his cordial thanks to the referee, Professors T. Huruya and K-S. Saito for their kind advices.

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