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# On the global real analytic coordinates for Teichmüller spaces

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## 1. Introduction.

Let $G$ be a Fuchsian group acting on the unit disk $\boldsymbol{D}$. The group $G$ is called type $(g, m)$, if the quotient space $\boldsymbol{D}/G$ is conformally equivalent to a compact Riemann surface of genus $g$ with $m$ disjoint disks removed. Then the Euler-Poincaré characteristic $\chi(\boldsymbol{D}/G)$ is $2-2g-m$. From now on we only consider those types $(g, m)$ satisfying $\chi(\boldsymbol{D}/G)<0$ or $2g+m\geqq 3$. A Fuchsian group is marked by choosing a system of generators. Let $G$ be a marked Fuchsian group of type $(g, m)$. Then all other marked Fuchsian groups of this type are considered as deformations of $G$ and they form the Teichmüller space $\boldsymbol{T}(g, m)$. The Teichmüller space $\boldsymbol{T}(g, m)$ has the structure of a real analytic manifold of dimension $6g-6+3m$.

Keen [**5**] found that $9g-9+4m$ absolute values of traces of hyperbolic elements in a marked Fuchsian group give global real analytic coordinates for $\boldsymbol{T}(g, m)$. These absolute values have a geometric interpretation on $\boldsymbol{D}/G$ as lengths of certain closed geodesics. But this number of parameters is not minimal. Seppälä and Sorvali [8] showed that $6g-4$ multipliers (corresponding to absolute values of traces) of hyperbolic elements in a marked Fuchsian group give global real analytic coordinates for $\boldsymbol{T}(g, 0)$. Recently S. Wolpert proved the result, which is equivalent to the following: *any* $6g-6$ *absolute values of traces of elements in a marked Fuchsian group can not give global* (*even locally*) *real analytic coordinates for* $\boldsymbol{T}(g, 0)$. Hence either $6g-4$ or $6g-5$ is the minimal number of such parameters for $\boldsymbol{T}(g, 0)$.

Sorvali [**9**] showed that $6g-6+3m$ multipliers of hyperbolic elements in a marked Fuchsian group give global real analytic coordinates for $\boldsymbol{T}(g, m)$ with $gm\neq 0$. In this case this number of these parameters is minimal.

In this paper, first we show that $3m-6$ absolute values of traces of hyperbolic elements in a marked Fuchsian group give global real analytic coordinates for $\boldsymbol{T}(0, m)$ (Theorem 4.1). Next for $\boldsymbol{T}(g, 0)$, we find $6g-4$ absolute values of traces giving global real analytic coordinates by the same method used in the case of $\boldsymbol{T}(0, m)$ (Theorem 5.1). Finally we show that $6g-6+3m$ absolute values of traces give global real analytic coordinates for $\boldsymbol{T}(g, m)$ with $m\neq 0$ (Theorem 6.3). Hence in the case of $m\neq 0$, we see that the minimal number of these parameters is equal to the dimension of $\boldsymbol{T}(g, m)$. The method of proofs of our theorems is due to an idea of Keen [4] and [5]. This is different from the methods of Seppälä and Sorvali [8] and Sorvali [9].

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## 2. Definitions.

Every biholomorphic mapping $g$ of the unit disk $\boldsymbol{D}$ is a linear transformation of the form $g(z)=(az+b)/(\bar{b}z+\bar{a})$, where $a, b\in\boldsymbol{C}$ and $|a|^2-|b|^2=1$. Each of the matrices

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -a & -b \\ -\bar{b} & -\bar{a} \end{pmatrix}$$

of $SL(2, \boldsymbol{C})$ is called the matrix representation of $g$. We call $|a+\bar{a}|$ the absolute value of trace of $g$ and denote it by $|\mathrm{tr}(g)|$. If $b\neq 0$, then $\{z\in\boldsymbol{D} \mid |\bar{b}z+\bar{a}|=1\}$ is called the isometric circle of $g$ and denoted by $I(g)$. These transformations form a group, which is denoted by $\boldsymbol{M}$. An element $g$ of $\boldsymbol{M}$ is called hyperbolic, parabolic and elliptic if $|\mathrm{tr}(g)|>2$, $=2$ and $<2$, respectively. A discrete subgroup $G$ of $\boldsymbol{M}$ is called a Fuchsian group. A hyperbolic element $g$ of $\boldsymbol{M}$ has two disjoint fixed points $p(g)$ and $q(g)$ on the unit circle $\boldsymbol{S}$. $q(g)=\lim_{n\to\infty}g^n(z)$ for all $z$ in $\boldsymbol{D}$ and $q(g)$ is called the attracting fixed point of $g$. $p(g)=\lim_{n\to\infty}g^{-n}(z)$ for all $z$ in $\boldsymbol{D}$ and $p(g)$ is called the repelling fixed point of $g$. The circle in $\boldsymbol{D}$ orthogonal to $\boldsymbol{S}$ from $p(g)$ to $q(g)$ is called the axis of $g$ and denoted by $\mathrm{ax}(g)$.

Let $G$ be a Fuchsian group of type $(g, m)$. Then $G$ has the following representation (see [2]):

$$G=\langle A_1, B_1, \cdots, A_g, B_g, E_1, \cdots, E_m \mid E_m\circ\cdots\circ E_1B_g^{-1}A_g^{-1}B_gA_g\circ\cdots\circ B_1^{-1}A_1^{-1}B_1A_1=I\rangle.$$

A Fuchsian group $G$ together with a system of generators $\mathcal{S}=(A_1, B_1, \cdots, A_g, B_g, E_1, \cdots, E_m)$ is called a marked Fuchsian group $(G, \mathcal{S})$. Two marked Fuchsian groups $(G_1, \mathcal{S}_1)$ with $\mathcal{S}_1=(A_{11}, \cdots, E_{1m})$ and $(G_2, \mathcal{S}_2)$ with $\mathcal{S}_2=(A_{21}, \cdots, E_{2m})$ are called conformally equivalent if there is an element $h$ of $\boldsymbol{M}$ such that $hA_{1i}h^{-1}=A_{2i}$, $hB_{1i}h^{-1}=B_{2i}$ and $hE_{1j}h^{-1}=E_{2j}$, $(i=1, \cdots, g; j=1, \cdots, m)$. The equivalence class of $(G, \mathcal{S})$ is denoted by $[G, \mathcal{S}]$. The set of the equivalence classes $[G, \mathcal{S}]$ of marked Fuchsian groups of type $(g, m)$ is called the Teichmüller space of type $(g, m)$ and denoted by $\boldsymbol{T}(g, m)$. We can introduce a topology on $\boldsymbol{T}(g, m)$ such that $\boldsymbol{T}(g, m)$ becomes a real analytic manifold of dimension $6g-6+3m$ (see [**1**]).

Keen [**4**] showed that every Fuchsian groups of type $(g, m)$ with $g, m\geqq 0$ and $2g+m\geqq 3$ has the generators such that the fixed points of generators are arranged clockwise on $\boldsymbol{S}$ in the following order:

$$p(C_1),\ p(A_1),\ q(B_1),\ q(A_1),\ p(B_1),\ q(C_1),\ \cdots,\ p(C_g),\ p(A_g),$$
$$q(B_g),\ q(A_g),\ p(B_g),\ q(C_g),\ p(E_1),\ q(E_1),\ \cdots,\ p(E_m),\ q(E_m),$$

where $C_i=B_i^{-1}A_i^{-1}B_iA_i$ for $i=1, \cdots, g$ (see Figure 1). We call this property

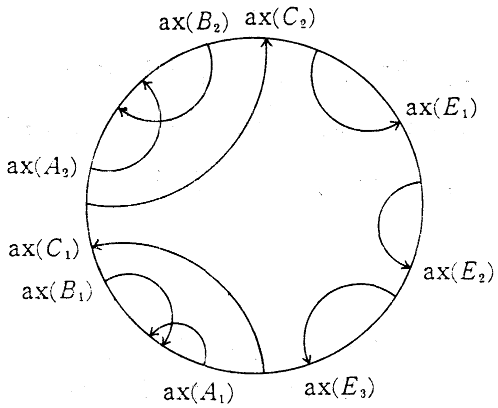


Figure 1. $G$ has type (2, 3).

of fixed points the property $(*)$. We may assume without loss of generality that we only consider marked Fuchsian groups with the property $(*)$.

For each $i$ $(i=2, \cdots, g)$, we see that the axes of $A_1$, $A_i$ and $A_iA_1$ are disjoint and that $p(A_1)$, $q(A_1)$, $p(A_i)$, $q(A_i)$, $q(A_iA_1)$ and $p(A_iA_1)$ are arranged clockwise on $\boldsymbol{S}$ in this order. This is shown as follows: We normalize $q(A_1)=-1$, $p(A_1)=1$ and $\mathrm{Im}(q(A_i))=\mathrm{Im}(p(A_i))$. Let $r_1$ be the end point of $I(A_1)$ lying in the right side of $\mathrm{ax}(A_1)$, and $s_1$ be the open arc on $\boldsymbol{S}$ connecting $p(A_1)$ and $r_1$ which does not contain $q(A_1)$. Let $r_i$ be the end point of $I(A_i^{-1})$ lying in the right side of $\mathrm{ax}(A_i)$, and $s_i$ be the open arc on $\boldsymbol{S}$ connecting $q(A_i)$ and $r_i$ which does not contain $p(A_i)$ (see Figure 2). Then $A_iA_1(s_i)\subsetneqq s_i$ and $(A_iA_1)^{-1}(s_1)=A_1^{-1}A_i^{-1}(s_1)\subsetneqq s_1$. Thus $q(A_iA_1)$ lies in $s_i$ and $p(A_iA_1)$ lies in $s_1$. Hence $p(A_1)$, $q(A_1)$, $p(A_i)$, $q(A_i)$, $q(A_iA_1)$ and $p(A_iA_1)$ are arranged clockwise on $\boldsymbol{S}$ in this order. This order is preserved by every element of $\boldsymbol{M}$.

Thus the system $(A_1, A_i, (A_iA_1)^{-1})$ generates a marked Fuchsian group of type (0, 3) with the property $(*)$. Similarly we see that the following systems of generators generate marked Fuchsian groups of type (0, 3) with the property $(*)$:

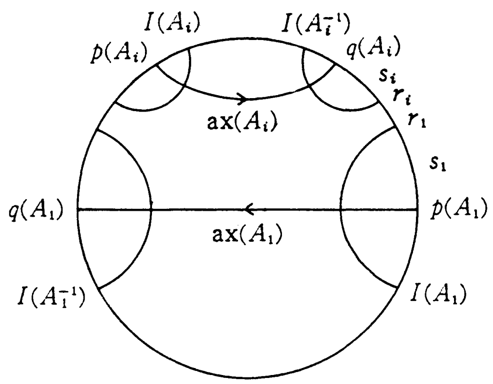


Figure 2.

$(B_1^{-1}, A_i, B_1A_i^{-1})$, $(B_1^{-1}, B_i^{-1}, B_1B_i)$, $(A_1, B_i^{-1}, A_1^{-1}B_i)$, $(A_1, E_j, (E_jA_1)^{-1})$,

$(B_1^{-1}, E_j, B_1E_j^{-1})$, $(E_1, E_k, (E_kE_1)^{-1})$, $(E_2, E_l, (E_lE_2)^{-1})$,

for $i=2, \cdots, g$; $j=1, \cdots, m$; $k=2, \cdots, m$; $l=3, \cdots, m$.

## 3. Keen's results.

In this section, we consider marked Fuchsian groups of types (0, 3) and (1, 1). The general marked Fuchsian group of type $(g, m)$ is obtained from these basic groups by means of the amalgamated product. Thus these groups are called building blocks. Keen showed the following two lemmas which play an important role in this paper.

LEMMA 3.1 (Keen [**4**]). *Let $G$ be a marked Fuchsian group of type* (0, 3) *and have a system of generators $(E_1, E_2, E_3)$ (see Figure* 3). *Then the absolute values of traces of $E_1$, $E_2$ and $E_3$ determine $G$ up to conjugation by a Möbius transformation. If $G$ is normalized by conditions $q(E_1)=-1$, $p(E_1)=1$ and $q(E_2)=i$, then $E_1$ and $E_2$ have the following matrix representations:*

$$E_1=\begin{pmatrix} t_1 & \sqrt{t_1^2-1} \\ \sqrt{t_1^2-1} & t_1 \end{pmatrix}$$

*and*

$$E_2=\begin{pmatrix} t_2+i\dfrac{t_3-t_1t_2}{\sqrt{t_1^2-1}} & \dfrac{t_3-t_1t_2}{\sqrt{t_1^2-1}}-i\sqrt{t_2^2-1} \\ \dfrac{t_3-t_1t_2}{\sqrt{t_1^2-1}}+i\sqrt{t_2^2-1} & t_2-i\dfrac{t_3-t_1t_2}{\sqrt{t_1^2-1}} \end{pmatrix},$$

*where $2t_i=-|\mathrm{tr}(E_i)|$ for $i=1, 2, 3$.*

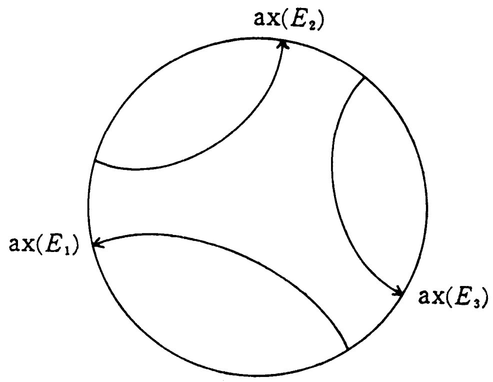


Figure 3.

REMARK 3.2 (Keen [4]). We can not arbitrarily take the matrix representations of $E_1$, $E_2$ and $E_3$. We have to take their matrices such that $\mathrm{tr}(E_1)\mathrm{tr}(E_2)\mathrm{tr}(E_3)<0$.

In Keen [4], $E_1$, $E_2$ and $E_3$ are represented by matrices such that $\mathrm{tr}(E_1)>0$, $\mathrm{tr}(E_2)>0$ and $\mathrm{tr}(E_3)<0$. However in this paper, we take their matrices whose traces are all negative. Let

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$$

be the matrix representation of $E_2$ in Lemma 3.1. A direct calculation gives

$$\mathrm{Re}(b)=\mathrm{Im}(a)<\mathrm{Re}(a)<\mathrm{Im}(b)<0\,,$$

which will be useful in later computations.

LEMMA 3.3 (Keen [4]). *Let $G$ be a marked Fuchsian group of type* (1, 1) *and have a system of generators* $(A_1, B_1, E_1)$ *(see Figure 4). Then the absolute values of traces of $A_1$, $B_1$ and $B_1A_1$ determine $G$ up to conjugation by a Möbius transformation. If $G$ is normalized by conditions $q(A_1)=-1$, $p(A_1)=1$ and* $\mathrm{ax}(A_1)\cap\mathrm{ax}(B_1)=\{0\}$, *then $A_1$ and $B_1$ have the following matrix representations:*

$$A_1=\begin{pmatrix} t & \sqrt{t^2-1} \\ \sqrt{t^2-1} & t \end{pmatrix}$$

*and*

$$B_1=\begin{pmatrix} s & \exp(i\varphi)\sqrt{s^2-1} \\ \exp(-i\varphi)\sqrt{s^2-1} & s \end{pmatrix},$$

*where* $2t=-|\mathrm{tr}(A_1)|$, $2s=-|\mathrm{tr}(B_1)|$ *and* $\varphi\in(0, \pi)$ *is the intersection angle between* $\mathrm{ax}(A_1)$ *and* $\mathrm{ax}(B_1)$. *This angle $\varphi$ is uniquely determined by*

$$\sqrt{(s^2-1)(t^2-1)}\cos\varphi=r-st\,,$$

*where* $2r=|\mathrm{tr}(B_1A_1)|$.

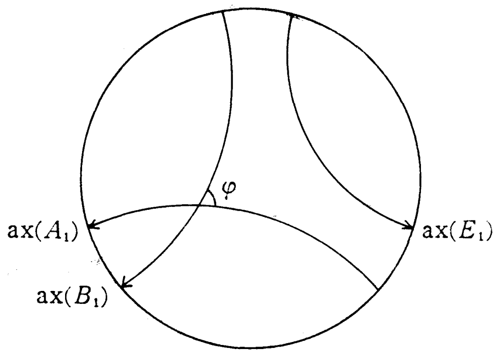


Figure 4.

In Keen [4], $A_1$ and $B_1$ are represented by matrices such that $\mathrm{tr}(A_1)$, $\mathrm{tr}(B_1)$ and $\mathrm{tr}(B_1A_1)$ are all positive.

## 4. A parametrization of type (0, $m$).

In this section, we observe the type (0, $m$). The case of $m=3$ is considered in Lemma 3.1. Thus we consider the case of $m\geqq 4$. Let $G$ be a marked Fuchsian group of type (0, $m$) with $m\geqq 4$. We normalize $G$ as $q(E_1)=-1$, $p(E_1)=1$ and $q(E_2)=i$. Under this situation, we find absolute values of traces of hyperbolic elements which uniquely determine $G$.

The system $(E_1, E_2, (E_2E_1)^{-1})$ generates a group of type (0, 3) with the property (*). Then by Lemma 3.1, $E_1$ and $E_2$ are uniquely determined by $\mathrm{tr}(E_1)$ $(<-2)$, $\mathrm{tr}(E_2)$ $(<-2)$ and $\mathrm{tr}(E_2E_1)$ $(<-2)$. Next we will uniquely determine $E_3$. The fixed point $q(E_3)$ is $\exp(i\theta)$ for some $\theta\in(0, \pi/2)$. Let $Q$ be the linear transformation which maps $-1$, 1 and $\exp(i\theta)$ to $-1$, 1 and $i$, respectively. The system $(QE_1Q^{-1}, QE_3Q^{-1}, Q(E_3E_1)^{-1}Q^{-1})$ generates a group of type (0, 3) with the property (*) such that $q(QE_1Q^{-1})=-1$, $p(QE_1Q^{-1})=1$ and $q(QE_3Q^{-1})=i$. Thus by Lemma 3.1, $QE_3Q^{-1}$ is determined by $\mathrm{tr}(E_1)(<-2)$, $\mathrm{tr}(E_3)(<-2)$ and $\mathrm{tr}(E_3E_1)(<-2)$. The matrix representations of $Q$ are

$$\frac{\pm 1}{\sqrt{2\sin\theta(1+\sin\theta)}}\begin{pmatrix}1+\sin\theta & -\cos\theta \\ -\cos\theta & 1+\sin\theta\end{pmatrix}.$$

Thus if $\theta$ is obtained, then $E_3$ is uniquely determined. Now we will show that $\theta$ is obtained from above traces and $\mathrm{tr}(E_3E_2)$. Since both $\mathrm{tr}(E_2)$ and $\mathrm{tr}(E_3)$ are negative, $\mathrm{tr}(E_3E_2)$ is taken to be negative by Remark 3.2. Set $2k=\mathrm{tr}(E_3E_2)$,

$$E_2=\begin{pmatrix}a & b \\ \bar{b} & \bar{a}\end{pmatrix} \quad \text{and} \quad QE_3Q^{-1}=\begin{pmatrix}p & q \\ \bar{q} & \bar{p}\end{pmatrix}.$$

Then we have

$$E_3 = \begin{pmatrix} \mathrm{Re}(p)+i\dfrac{\mathrm{Im}(p)-\mathrm{Im}(q)\cos p}{\sin\theta} & \mathrm{Re}(q)+i\dfrac{\mathrm{Im}(q)-\mathrm{Im}(p)\cos\theta}{\sin\theta} \\ \mathrm{Re}(q)-i\dfrac{\mathrm{Im}(q)-\mathrm{Im}(p)\cos\theta}{\sin\theta} & \mathrm{Re}(p)-i\dfrac{\mathrm{Im}(p)-\mathrm{Im}(q)\cos\theta}{\sin\theta} \end{pmatrix}$$

and thus

$$\alpha\sin\theta+\beta\cos\theta=\gamma,$$

where

$$\alpha=\mathrm{Re}(a)\mathrm{Re}(p)+\mathrm{Re}(b)\mathrm{Re}(q)-k,$$
$$\beta=\mathrm{Im}(a)\mathrm{Im}(q)-\mathrm{Im}(b)\mathrm{Im}(p),$$
$$\gamma=\mathrm{Im}(a)\mathrm{Im}(p)-\mathrm{Im}(b)\mathrm{Im}(q).$$

Since $\mathrm{Re}(b)=\mathrm{Im}(a)<\mathrm{Re}(a)<\mathrm{Im}(b)<0$, $\mathrm{Re}(q)=\mathrm{Im}(p)<\mathrm{Re}(p)<\mathrm{Im}(q)<0$ and $k<0$, we obtain that

(1) $\alpha>\gamma>0$,
(2) $\gamma>\beta$.

Thus

$$\sin(\theta+\theta_0)=\frac{\gamma}{\sqrt{\alpha^2+\beta^2}}$$

where $\theta_0\in[0, 2\pi)$ satisfies

$$\cos\theta_0=\frac{\alpha}{\sqrt{\alpha^2+\beta^2}} \quad\text{and}\quad \sin\theta_0=\frac{\beta}{\sqrt{\alpha^2+\beta^2}}.$$

Since $0<\theta+\theta_0<5\pi/2$ and $\gamma>0$, $\theta+\theta_0$ is equal to $\theta_1$, $\theta_2$ or $\theta_3$, where $0<\theta_1<\pi/2<\theta_2<2\pi<\theta_3<5\pi/2$ and $\sin\theta_i=\gamma/\sqrt{\alpha^2+\beta^2}$; $i=1, 2, 3$. Since $\cos\theta_0>0$, $\theta_0$ is either in $[0, \pi/2)$ or $(3\pi/2, 2\pi)$.

In the case of $\theta_0$ in $[0, \pi/2)$, (2) implies that $\sin\theta_0<\sin(\theta+\theta_0)$. Then we can take $\theta$ as $\theta_1-\theta_0$. On the other hand, (1) implies that $\sin(\theta_0+\pi/2)>\sin\theta_2$ and thus $\theta_2-\theta_0\notin(0, \pi/2)$. It is trivial that $\theta_3-\theta_0\notin(0, \pi/2)$. Hence $\theta$ is uniquely determined.

In the case of $\theta_0\in(3\pi/2, 2\pi)$, we uniquely determine $\theta$ as $\theta_3-\theta_0$.

If $m=4$, then $E_4$ is determined by the relation $E_4\circ\cdots\circ E_1=I$. If $m\geqq 5$, then we similarly determine $E_4, \cdots, E_{m-1}$. And $E_m$ is determined by the relation $E_m\circ\cdots\circ E_1=I$.

In our construction, these traces are real analytically corresponding to the entries of $E_1, E_2, \cdots, E_m$ up to conjugation by a Möbius transformation. Now we have the following theorem.

THEOREM 4.1. *Let $G$ be a marked Fuchsian group of type* $(0, m)$. *Then absolute values of traces of the following* $3m-6$ *hyperbolic elements determine $G$ up to conjugation by a Möbius transformation,*

(i) $E_i$  $i=1, \cdots, m-1$,

(ii) $E_iE_1$, $i=2, \cdots, m-1$,
(iii) $E_iE_2$, $i=3, \cdots, m-1$,

*where in the case of* $m=3$, (iii) *is omitted. These values give global real analytic coordinates for* $\boldsymbol{T}(0, m)$.

## 5. A parametrization of type $(g, 0)$.

Here we observe the type $(g, 0)$. Let $G$ be a marked Fuchsian group of this type. We normalize $G$ as $q(A_1)=-1$, $p(A_1)=1$ and $\mathrm{ax}(A_1)\cap \mathrm{ax}(B_1)=\{0\}$. We find absolute values of traces of hyperbolic elements which uniquely determine $G$ by the following three steps.

*The first step*: Here we will determine $A_1, A_2, \cdots, A_g$ and $B_1$. The system $(A_1, B_1, C_1^{-1})$, $C_1=B_1^{-1}A_1^{-1}B_1A_1$ generates a group of type $(1, 1)$ with the property $(*)$. By Lemma 3.3, $A_1$ and $B_1$ are determined by $\mathrm{tr}(A_1)$ $(<-2)$, $\mathrm{tr}(B_1)$ $(<-2)$ and $\mathrm{tr}(B_1A_1)$ $(>2)$. Let $\varphi$ be the intersection angle between $\mathrm{ax}(A_1)$ and $\mathrm{ax}(B_1)$. Next we will see that $A_2$ is determined by $\mathrm{tr}(A_1)$, $\mathrm{tr}(A_2)$, $\mathrm{tr}(A_2A_1)$ and $\mathrm{tr}(A_2B_1^{-1})$. The fixed point $q(A_2)$ is $\exp(i\theta)$ for some $\theta$ in $(0, \pi)$. We take $Q$ as in the same manner stated in Section 4. Then by Lemma 3.1, $QA_2Q^{-1}$ is determined by $\mathrm{tr}(A_1)$ $(<-2)$, $\mathrm{tr}(A_2)$ $(<-2)$ and $\mathrm{tr}(A_2A_1)$ $(<-2)$. Since both $\mathrm{tr}(A_2)$ and $\mathrm{tr}(B_1)$ are negative, $\mathrm{tr}(A_2B_1^{-1})$ is negative by Remark 3.2. Set $2k=\mathrm{tr}(A_2B_1^{-1})$,

$$B_1=\begin{pmatrix} s & \exp(i\varphi)\sqrt{s^2-1} \\ \exp(-i\varphi)\sqrt{s^2-1} & s \end{pmatrix} \quad \text{and} \quad QA_2Q^{-1}=\begin{pmatrix} p & q \\ \bar{q} & \bar{p} \end{pmatrix}.$$

Then we have

$$\alpha\sin\theta+\beta\cos\theta=\gamma\,,$$

where

$$\alpha=2k-\mathrm{Re}(p)s-\mathrm{Re}(q)s\cos\varphi\,,$$
$$\beta=\mathrm{Im}(p)s\sin\varphi\,,$$
$$\gamma=\mathrm{Im}(q)s\sin\varphi\,.$$

Thus

$$\sin(\theta+\theta_0)=\frac{\gamma}{\sqrt{\alpha^2+\beta^2}}$$

where $\theta_0\in[0, 2\pi)$ satisfies

$$\cos\theta_0=\frac{\alpha}{\sqrt{\alpha^2+\beta^2}} \quad \text{and} \quad \sin\theta_0=\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\,.$$

Since $0<\sin(\theta+\theta_0)<\sin\theta_0$, we obtain that $\theta_0\in(0, \pi)$ and $\theta+\theta_0\in(\pi/2, \pi)$. Thus we uniquely obtain $\theta$. Hence $A_2$ is uniquely determined.

If $g\geqq 3$, then $A_3, \cdots, A_g$ are similarly determined.

*The second step*: In the case of $g=2$, we skip this step. We will determine $B_2, B_3, \cdots, B_{g-1}$. The system $(A_1, B_2^{-1}, A_1^{-1}B_2)$ generates a group of type (0, 3) with the property (*). The fixed point $q(B_2^{-1})=\exp(i\theta)$ for some $\theta$ in $(0, \pi)$. We take $Q$ as in the same manner in Section 4. Then by Lemma 3.1, $QB_2^{-1}Q^{-1}$ is determined by $\mathrm{tr}(A_1)\,(<-2)$, $\mathrm{tr}(B_2)\,(<-2)$ and $\mathrm{tr}(B_2A_1^{-1})\,(<-2)$. Moreover $\theta$ is obtained from above traces and $\mathrm{tr}(B_2B_1)\,(<-2)$. Therefore $B_2$ is uniquely determined.

If $g\geqq 4$, then we similarly determine $B_3, \cdots, B_{g-1}$.

*The third step*: We will determine $B_g$. After the above two steps $C_g=B_g^{-1}A_g^{-1}B_gA_g$ is determined by the relation. We take $\mathrm{tr}(B_g)\,(<-2)$ and $\mathrm{tr}(B_gA_g)\,(>2)$. We construct the marked group of type (1, 1) with the property (*), $\langle \tilde{A}_g, \tilde{B}_g, \tilde{C}_g^{-1} \mid \tilde{C}_g^{-1}\tilde{B}_g^{-1}\tilde{A}_g^{-1}\tilde{B}_g\tilde{A}_g=I\rangle$ which satisfies the following conditions:

$$q(\tilde{A}_g)=-1, \qquad p(\tilde{A}_g)=1, \qquad \mathrm{ax}(\tilde{A}_g)\cap\mathrm{ax}(\tilde{B}_g)=\{0\},$$
$$\mathrm{tr}(\tilde{A}_g)=\mathrm{tr}(A_g), \qquad \mathrm{tr}(\tilde{B}_g)=\mathrm{tr}(B_g), \qquad \mathrm{tr}(\tilde{B}_g\tilde{A}_g)=\mathrm{tr}(B_gA_g).$$

The transformation $T$ which maps $q(C_g)$, $p(C_g)$ and $q(A_g)$ to $q(\tilde{C}_g)$, $p(\tilde{C}_g)$ and $q(\tilde{A}_g)$, respectively, is uniquely determined. These six fixed points are determined by $|\mathrm{tr}(B_g)|$, $|\mathrm{tr}(B_gA_g)|$ and absolute values of traces used in the first and second steps. Thus $T$ is determined by these values. Hence we can determine $B_g=T^{-1}\tilde{B}_gT$ by these values.

From these three steps we obtain the following theorem.

THEOREM 5.1. *Let $G$ be a marked Fuchsian group of type* $(g, 0)$. *Then the absolute values of traces of the following* $6g-4$ *hyperbolic elements determine $G$ up to conjugation by a Möbius transformation,*

(i) $A_i, B_i$, $\quad i=1, \cdots, g$,
(ii) $A_iA_1, A_iB_1^{-1}$, $\quad i=2, \cdots, g$,
(iii) $B_iB_1, B_iA_1^{-1}$, $\quad i=2, \cdots, g-1$,
(iv) $B_1A_1, B_gA_g$,

*where in the case of* $g=2$, (iii) *is omitted. These values give global real analytic coordinates for* $\boldsymbol{T}(g, 0)$.

Seppälä and Sorvali [8] conjectured $6g-4$ is the minimal number of such parameters.

## 6. A parametrization of type $(g, m)$ with $m\neq 0$.

Let $G$ be a marked Fuchsian group of type $(g, m)$ with $gm\neq 0$. We normalize $G$ as $q(A_1)=-1$, $p(A_1)=1$ and $\mathrm{ax}(A_1)\cap\mathrm{ax}(B_1)=\{0\}$. Then using the first and second steps in Section 5, we determine $A_1, B_1, \cdots, A_g, B_g$ by $3+$

$3(2g-2)=6g-3$ traces. If $m=1$, then $E_1$ is determined by the relation. If $m\geqq 2$, then we determine $E_j$ by $\mathrm{tr}(A_1)(<-2)$, $\mathrm{tr}(E_j)(<-2)$, $\mathrm{tr}(E_jA_1)(<-2)$ and $\mathrm{tr}(E_jB_1^{-1})(<-2)$ for $j=1, \cdots, m-1$. Hence we have the following theorem.

THEOREM 6.1. *Let G be a marked Fuchsian group of type* $(g, m)$ *with* $gm \neq 0$. *Then the absolute values of traces of the following* $6g-6+3m$ *hyperbolic elements determine G up to conjugation by a Möbius transformation,*

(i) $A_i, B_i, E_j,\quad i=1, \cdots, g;\ j=1, \cdots, m-1,$

(ii) $A_iA_1, A_iB_1^{-1}, B_iB_1, B_iA_1^{-1},\quad i=2, \cdots, g,$

(iii) $E_jA_1, E_jB_1^{-1},\quad j=1, \cdots, m-1,$

(iv) $B_1A_1,$

*where in the case of* $g=1$, (ii) *is omitted and in the case of* $m=1$, *both* (iii) *and* $E_j$ *in* (i) *are omitted. These values give global real analytic coordinates for* $\boldsymbol{T}(g, m)$.

REMARK 6.2. In Theorems 4.1, 5.1 and 6.1, if all traces of (i) are taken to be negative, then each one of (ii) and (iii) is negative and each one of (iv) is positive.

Finally from Theorems 4.1 and 6.1, we conclude the following main theorem.

THEOREM 6.3. *The Teichmüller space* $\boldsymbol{T}(g, m)$ *with* $m\neq 0$ *and* $2g+m\geqq 3$ *has global real analytic coordinates which consist of* $6g-6+3m$ *absolute values of traces of hyperbolic elements in marked Fuchsian groups.*

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