J. Math. Soc. Japan
Vol. 41, No. 4, 1989

Estimates on the stability of minimal surfaces and harmonic maps

By Makoto SAKAKI

(Received April 22, 1988)
(Revised Aug. 10, 1988)

0. Introduction.

Let $f: M \rightarrow N$ be a minimal immersion of a manifold M into a Riemannian manifold N. We say that M is stable if the second variation of the volume functional of M is positive for every variation of f which leaves the boundary values fixed. Let $N^n(a)$ denote the n-dimensional simply-connected space form of constant curvature a. Barbosa and do Carmo [**1**], [**2**] discussed the stability of simply-connected compact minimal surfaces with piecewise C^1 boundary in $N^n(a)$, whose result was improved in our previous paper [**9**] as follows.

THEOREM 0.1 ([**9**]). *Let $f: M \rightarrow N^n(a)$ be a minimal immersion of a 2-dimensional simply-connected compact manifold M with piecewise C^1 boundary into $N^n(a)$. If the second fundamental form A of f satisfies $\int_M (|a|+(1/2)|A|^2)dM < (4/3)\pi$, then M is stable.*

When $a \geqq 0$, Theorem 0.1 is proved in a little different way (cf. [**1**], Hoffman and Osserman [**6**]). In [**2**] it is asked if the argument of Theorem 0.1 can be generalized or not for a general ambient space. The first aim of this paper is to give a positive answer to this question. Let G_2N denote the Grassmann bundle over a Riemannian manifold N of 2-dimensional tangent subspaces to N. The Riemannian structure of G_2N is defined in Section 1.

THEOREM 0.2. *Let $f: M \rightarrow N$ be a minimal immersion of a 2-dimensional simply-connected compact manifold M with piecewise C^1 boundary ∂M into a Riemannian manifold N. Suppose that the sectional curvature of N is bounded and the sectional curvature of G_2N is bounded from above. Then there is a positive constant c_1 depending only on N such that if the second fundamental form A of f satisfies $\int_M (1+(1/2)|A|^2)dM < c_1$, then M is stable.*

If we omit the hypothesis that M is simply-connected and assume the positivity of the injectivity radius of G_2N, we obtain the following estimate (cf.

Hoffman [5], Tanno [12]).

THEOREM 0.3. *Let $f: M\to N$ be a minimal immersion of a 2-dimensional compact manifold M with piecewise C^1 boundary ∂M into a Riemannian manifold N. Suppose that the sectional curvature of N is bounded, the sectional curvature of G_2N is bounded from above, and the injectivity radius of G_2N is positive. Then there is a positive constant c_2 depending only on N such that if the second fundamental form A of f satisfies $\int_M(1+(1/2)|A|^2)dM<c_2$, then M is stable.*

The definitions of c_1 and c_2 are given in Section 3, and we can see that c_1 is greater than c_2 for each ambient space.

As in the case of a minimal immersion, a harmonic map is called stable if the second variation of the energy functional is positive for every variation of the map which leaves the boundary values fixed. In Section 4 we use the method in [1], [2] and prove the following estimate on the stability of some harmonic maps from 2-dimensional simply-connected compact Riemannian manifolds with piecewise C^1 boundary.

THEOREM 0.4. *Let $f:(M, ds^2)\to N$ be a harmonic map from a 2-dimensional simply-connected compact Riemannian manifold (M, ds^2) with piecewise C^1 boundary ∂M to a Riemannian manifold N such that $|df|$ vanishes only at isolated points, and assume that the sectional curvature of N is not greater than $a>0$. If $(1/2)\int_M|df|^2dM<(2/a)\pi$, then f is stable.*

The author would like to express his hearty thanks to Professor S. Tanno for his constant encouragement and advice, and to the referee for useful comments.

1. The Gauss map.

In this section we follow [7] and recall the definition and some properties of the Gauss map. We use for positive integers m, n $(m<n)$, the following ranges of indices:

$$(1.1)\qquad 1\leqq i, j, k, \cdots\leqq m, \qquad m+1\leqq \alpha, \beta, \cdots\leqq n, \qquad 1\leqq B, C, \cdots\leqq n\,.$$

Let N be an n-dimensional Riemannian manifold. Denote its $O(n)$-bundle of orthonormal frames by $O(N)\to N$ on which the canonical form $\theta=(\theta^B)$ and the Levi-Civita connection form $\omega=(\omega_C^B)$ are chosen. Let $G_mN\to N$ be the Grassmann bundle over N of m-dimensional tangent subspaces to N. Let $\{e_B\}$ be the local orthonormal frame of N which is dual to $\{\theta^B\}$. Set $\varphi^B=\theta^B$ and $\varphi^{\alpha i}=\omega_i^\alpha$ at $(x, [e_1, \cdots, e_m])\in G_mN$. Then $\{\varphi^B, \varphi^{\alpha i}\}$ is a local orthonormal coframe of

G_mN. We denote by $\{E_B, E_{\alpha i}\}$ the local orthonormal frame of G_mN which is dual to $\{\varphi^B, \varphi^{\alpha i}\}$.

Let $f: M \to N$ be an isometric immersion of an m-dimensional Riemannian manifold M into N, and let T_pM denote the tangent space of M at p. Then the Gauss map $\gamma_f: M \to G_mN$ of f is defined by $\gamma_f(p) = f_* T_pM$ for $p \in M$. We choose $\{\theta^B\}$ such that $f^*\theta^\alpha = 0$ for all α on M. Let h^α_{ij} be the components of the second fundamental form of f. Let ds^2 be the metric of M. We denote by R the curvature tensor of N. Then the metric $d\tilde{s}^2$ on M induced by γ_f from G_mN is given as follows:

$$d\tilde{s}^2 = ds^2 + \sum_{i,j,k,\alpha} h^\alpha_{ij} h^\alpha_{ik} f^*\theta^j \otimes f^*\theta^k . \tag{1.2}$$

The tension field $\tau(\gamma_f)$ of γ_f is given as follows:

$$\tau(\gamma_f) = \sum_{i,j,\alpha,B} R^\alpha_{ijB} h^\alpha_{ij} E_B + \sum_{i,\alpha} h^\alpha_{ii} E_\alpha + \sum_{i,j,\alpha} (h^\alpha_{jji} - R^\alpha_{jij}) E_{\alpha i} , \tag{1.3}$$

where h^α_{ijk} are the components of the covariant derivative of the second fundamental form of f.

2. A priori estimates.

In this section we show two lemmas for proving Theorems 0.2 and 0.3. In the following we shall use the range (1.1) of indices for $m=2$, unless otherwise stated.

LEMMA 2.1. *Let $f: M \to N$ be a minimal immersion of a 2-dimensional manifold M into an n-dimensional Riemannian manifold N, and let A denote the second fundamental form of f. Assume that the sectional curvature of N is bounded from above by a and below by b. Then*

$$|\tau(\gamma_f)|^2 \leqq \left(\frac{32n-69}{18}\right)(a-b)^2 |A|^2 + \frac{1}{2}(n-2)(a-b)^2 .$$

PROOF. We may choose an orthonormal basis $\{e_B\}$ for $T_{f(p)}N$ for each $p \in M$ such that e_1, e_2 are tangent to $f(M)$ and the components h^α_{ij} of A satisfy

$$(h^3_{ij}) = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \quad (h^4_{ij}) = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}, \quad (h^5_{ij}) = \cdots = (h^n_{ij}) = 0 \tag{2.1}$$

for some λ and μ (see [**9**]). Using this fact with (1.3) we have at p

$$\begin{aligned} |\tau(\gamma_f)|^2 &= \sum_B \Big(\sum_{i,j,\alpha} R^\alpha_{ijB} h^\alpha_{ij} \Big)^2 + \sum_{i,\alpha} \Big(\sum_j R^\alpha_{jij} \Big)^2 \\ &= (-R^3_{221}\lambda + R^4_{121}\mu)^2 + (R^3_{112}\lambda + R^4_{212}\mu)^2 + \sum_\alpha \{ (R^3_{11\alpha} - R^3_{22\alpha})\lambda \\ &\quad + (R^4_{12\alpha} + R^4_{21\alpha})\mu \}^2 + \sum_\alpha \{ (R^\alpha_{212})^2 + (R^\alpha_{121})^2 \}, \end{aligned} \tag{2.2}$$

where R denotes the curvature tensor of N. From the hypothesis for the sectional curvature of N,

$$|R^B_{CBD}| \leqq \frac{1}{2}(a-b), \qquad |R^B_{CDE}| \leqq \frac{2}{3}(a-b) \tag{2.3}$$

if B, C, D and E are different from one another (see [3]). By (2.2) and (2.3) we have

$$|\tau(\gamma_f)|^2 \leqq \left\{\frac{3}{2}(a-b)^2+\frac{16}{9}(n-3)(a-b)^2\right\}(|\lambda|+|\mu|)^2+\frac{1}{2}(n-2)(a-b)^2$$

$$\leqq \left(\frac{32n-69}{18}\right)(a-b)^2(2\lambda^2+2\mu^2)+\frac{1}{2}(n-2)(a-b)^2,$$

which implies our assertion. Q. E. D.

REMARK. If we do not use (2.1), we have a worse estimate (cf. [2], [9], [10]).

Under the same notation and condition as in Lemma 2.1, let ds^2 be the metric on M induced by f from N, and let $d\tilde{s}^2$ be the metric on M induced by γ_f from G_2N. We denote by $\tilde{K}$ and $\tilde{H}$ the Gaussian curvature of $(M, d\tilde{s}^2)$ and the mean curvature vector of the isometric immersion $\gamma_f: (M, d\tilde{s}^2)\to G_2N$, respectively. Assume that the sectional curvature of G_2N is not greater than $d>0$.

LEMMA 2.2. *Under the same notation and condition as above,*

(i) $d\tilde{s}^2=\left(1+\frac{1}{2}|A|^2\right)ds^2$,

(ii) $|\tilde{H}| \leqq c = \dfrac{(a-b)(32n-69)}{6(110n-240)^{1/2}}$,

(iii) $\tilde{K} \leqq c^2+d$.

PROOF. The first statement is easily shown from (1.2) and the minimality of f.

Let $\{\varepsilon_i\}$ be a local orthonormal frame of (M, ds^2), and set $\varepsilon_i'=\varepsilon_i/(1+(1/2)|A|^2)^{1/2}$. Then we can see from (i) that $\{\varepsilon_i'\}$ is a local orthonormal frame of $(M, d\tilde{s}^2)$. Let ${}^G\nabla$ and ${}^M\nabla$ be the Riemannian connection of G_2N and (M, ds^2), respectively. We denote by $(\ )^\perp$ the projection to the normal space of $(M, d\tilde{s}^2)$ in G_2N. Then we calculate

$$2\tilde{H}=\sum_i({}^G\nabla_{\gamma_{f*}\varepsilon_i'}\gamma_{f*}\varepsilon_i')^\perp=\frac{1}{1+(1/2)|A|^2}\sum_i({}^G\nabla_{\gamma_{f*}\varepsilon_i}\gamma_{f*}\varepsilon_i)^\perp \tag{2.4}$$

$$=\frac{1}{1+(1/2)|A|^2}\sum_i\{{}^G\nabla_{\gamma_{f*}\varepsilon_i}\gamma_{f*}\varepsilon_i-\gamma_{f*}({}^M\nabla_{\varepsilon_i}\varepsilon_i)\}^\perp=\frac{\tau(\gamma_f)^\perp}{1+(1/2)|A|^2}.$$

Therefore from (2.4) and Lemma 2.1 we have

$$|\tilde{H}| \leqq \frac{|\tau(\gamma_f)|}{2+|A|^2} \leqq \frac{(a-b)\{(64n-138)|A|^2+18(n-2)\}^{1/2}}{6(2+|A|^2)}$$
$$\leqq \max_{t\geqq 0} \frac{(a-b)\{(64n-138)t+18(n-2)\}^{1/2}}{6(2+t)},$$

which implies (ii).

We choose a local orthonormal frame $\{F_i, F_\alpha\}$ of G_2N such that $F_i=\gamma_{f*}\varepsilon'_i$ on $\gamma_f(M)$, where $3\leqq\alpha\leqq 3n-4$. Let $\tilde{h}^\alpha_{ij}$ be the components of the second fundamental form of $\gamma_f:(M, d\tilde{s}^2)\to G_2N$. Then by the Gauss equation and (ii) we have

$$\tilde{K} \leqq d+\sum_\alpha\{\tilde{h}^\alpha_{11}\tilde{h}^\alpha_{22}-(\tilde{h}^\alpha_{12})^2\} \leqq d+\sum_\alpha \frac{1}{4}(\tilde{h}^\alpha_{11}+\tilde{h}^\alpha_{22})^2 \leqq c^2+d\,.$$

Thus the proof is complete. Q.E.D.

REMARK. It seems hard to obtain an estimate like (iii) by the same method as in [1], [2] and [9].

3. Stability of minimal surfaces.

PROOF OF THEOREM 0.2. Let ϕ be a piecewise smooth function on M such that $\phi=0$ on ∂M, and let ν be a unit normal vector field to $f(M)$. We shall consider the second variation $I(\phi\nu, \phi\nu)$ of the area functional of M for the variational vector field $\phi\nu$. Let ds^2 and $d\tilde{s}^2$ be defined as in Section 2. We denote by ${}^M\nabla$ and ${}^\perp\nabla$ the Riemannian connection of (M, ds^2) and the normal connection of the normal bundle of (M, ds^2) induced by f, respectively. Let $\{\varepsilon_i\}$ be a local orthonormal frame of (M, ds^2) and let R be the curvature tensor of N. From the hypothesis we may assume that the sectional curvature of N is bounded from above by a and below by b. Then by the second variational formula for minimal submanifolds, we have

$$\text{(3.1)}\qquad I(\phi\nu, \phi\nu) = \int_M (|{}^\perp\nabla(\phi\nu)|^2-\sum_i\langle R(\phi\nu, \varepsilon_i)\varepsilon_i, \phi\nu\rangle-|A^{\phi\nu}|^2)dM$$
$$\geqq \int_M\{|{}^M\nabla\phi|^2-(2a+|A|^2)\phi^2\}dM\,.$$

Let ${}^M\tilde{\nabla}$ and $d\tilde{M}$ denote the Riemannian connection of $(M, d\tilde{s}^2)$ and the area element of $(M, d\tilde{s}^2)$, respectively. Then by Lemma 2.2 (i)

$$\text{(3.2)}\qquad |{}^M\tilde{\nabla}\phi|_1^2=\frac{1}{1+(1/2)|A|^2}|{}^M\nabla\phi|^2, \qquad d\tilde{M}=\left(1+\frac{1}{2}|A|^2\right)dM\,.$$

Here and in what follows, $|\ |_1$ is the norm with respect to $d\tilde{s}^2$. From (3.1) and (3.2) we have

$$(3.3)\qquad I(\phi\nu,\phi\nu)\geqq\int_M\Big(|{}^M\tilde{\nabla}\phi|_1^2-\frac{2a+|A|^2}{1+(1/2)|A|^2}\phi^2\Big)d\tilde{M}\geqq\int_M(|{}^M\tilde{\nabla}\phi|_1^2-\eta\phi^2)d\tilde{M}\,,$$

where

$$\eta=\eta(a)=\begin{cases}2a & a\geqq 1\\ 2 & a<1.\end{cases}$$

Let $\tilde{\lambda}_1(M)$ denote the first eigenvalue of the Laplacian of $(M, d\tilde{s}^2)$. The inequality (3.3) says that if $\tilde{\lambda}_1(M)>\eta$, then M is stable.

We assume that N is n-dimensional. Let c and d be defined as in Lemma 2.2. Set

$$c_1=\min\Big\{\frac{8\pi}{2c^2+2d+\eta},\ \frac{4\pi}{\eta}\Big\},$$

which is positive and depends only on the geometry of N. Using Proposition 3.3 and 3.10 of [1] with Lemma 2.2(iii) we can see that $\tilde{\lambda}_1(M)>\eta$ if $\tilde{a}(M)<c_1$, where $\tilde{a}(M)$ denotes the area of $(M, d\tilde{s}^2)$. It is obvious that $\tilde{a}(M)=\int_M(1+(1/2)|A|^2)dM$. Therefore, Theorem 0.2 is proved. Q.E.D.

For the proof of Theorem 0.3, we recall a theorem for the estimate of the first eigenvalue of the Laplacian.

Let $f: M\to L$ be an isometric immersion of a 2-dimensional compact Riemannian manifold M with piecewise C^1 boundary into a Riemannian manifold L. Let H be the mean curvature vector of f and let $R(L)$ be the injectivity radius of L. We denote by $a(M)$ and $\lambda_1(M)$ the area of M and the first eigenvalue of the Laplacian of M, respectively. Set $F(t)=\min\{t, \pi/2\}$ for $t\in R$.

THEOREM 3.1 ([12]). *Under the same notation as above, assume that the sectional curvature of L is not greater than ξ^2 where $\xi>0$. Suppose the following:*

$$2|H|\leqq\kappa,\qquad a(M)\leqq\frac{\pi(1-t)}{\xi^2}\Big\{\sin\Big(F\Big(\frac{\xi R(L)}{2}\Big)\Big)\Big\}^2,$$

$$\kappa\cdot\arcsin\Big(\xi\Big(\frac{a(M)}{\pi(1-t)}\Big)^{1/2}\Big)\leqq\frac{\xi t}{2(3-t)}$$

for some t such that $0<t\leqq t_0=(9-\sqrt{57})/2$. Then

$$\lambda_1(M)\geqq\frac{1}{4}\Big[\frac{\xi t}{2(3-t)}\Big\{\arcsin\Big(\xi\Big(\frac{a(M)}{\pi(1-t)}\Big)^{1/2}\Big)\Big\}^{-1}-\kappa\Big]^2.$$

PROOF OF THEOREM 0.3. We use the same notation as in the proof of Theorem 0.2. Let t_0 be as in Theorem 3.1. Set

$$c_3=\frac{\pi(1-t)}{d}\Big\{\sin\Big(F\Big(\frac{\sqrt{d}t}{4(\sqrt{\eta}+c)(3-t)}\Big)\Big)\Big\}^2,\quad c_4=\frac{\pi(1-t)}{d}\Big\{\sin\Big(F\Big(\frac{\sqrt{d}R(G_2N)}{2}\Big)\Big)\Big\}^2,$$

and

$$c_2 = \max_{0<t\leq t_0} \min\{c_3, c_4\}.$$

Apparently c_2 is positive under the hypothesis and depends only on the geometry of N. We may apply Theorem 3.1 to the isometric immersion $\gamma_f: (M, d\tilde{s}^2) \to G_2N$ because we have Lemma 2.2(ii). Then we find that if $\tilde{a}(M) < c_2$, then $\tilde{\lambda}_1(M) > \eta$, which implies the stability of M by (3.3). Thus the proof is complete. Q. E. D.

Next we consider the value of c_2 in Theorem 0.3 when the ambient space is the unit sphere by use of another Gauss map. In this case the value of c_1 in Theorem 0.2 is $4\pi/3$ (see Theorem 0.1).

PROPOSITION 3.2. *Let $f: M \to S^n$ be a minimal immersion of a 2-dimensional compact oriented manifold M with piecewise C^1 boundary ∂M into the n-dimensional unit sphere S^n. There is a positive number c_5 such that if the second fundamental form A of f satisfies $\int_M (1+(1/2)|A|^2)dM < c_5$, then M is stable.*

PROOF. We use the same notation as in the proof of Theorem 0.2, except for the replacement of N by S^n and the definition of $d\tilde{s}^2$. Let $G_2(n+1)$ denote the Grassmann manifold of 2-dimensional oriented planes through the origin in $\boldsymbol{R}^{n+1}$, and let $\phi: S^n \to \boldsymbol{R}^{n+1}$ be the inclusion. Then the Gauss map $g: M \to G_2(n+1)$ of $\phi \circ f: M \to \boldsymbol{R}^{n+1}$ is defined naturally. We identify $G_2(n+1)$ with the complex hyperquadric $Q_{n-1} = \{[Z] \in \boldsymbol{CP}^n;\ Z \in \boldsymbol{C}^{n+1},\ Z^2 = 0\}$ in the n-dimensional complex projective space $\boldsymbol{CP}^n$ of constant holomorphic sectional curvature 2. We denote by $d\tilde{s}^2$ the metric on M induced by g from $G_2(n+1)$. The map g is a conformal harmonic map satisfying $d\tilde{s}^2 = (1+(1/2)|A|^2)ds^2$ (see [**4**, p. 18], [**6**, p. 446]). Therefore we find as in Lemma 2.2(ii) that $g: (M, d\tilde{s}^2) \to G_2(n+1)$ is a minimal immersion (cf. [**4**, p. 16]). Computing as in (3.1), (3.2) and (3.3) we have

$$I(\phi\nu, \phi\nu) \geqq \int_M \{|{}^M\nabla\phi|^2 - (2+|A|^2)\phi^2\} dM \geqq \int_M (|{}^M\tilde{\nabla}\phi|_1^2 - 2\phi^2) d\tilde{M}.$$

So the condition $\tilde{\lambda}_1(M) > 2$ implies the stability of M.

As $G_2(n+1)$ is a simply-connected symmetric space (see [**13**, p. 284]) and its maximum of the sectional curvature is 2 (see [**8**, p. 82]), the injectivity radius of $G_2(n+1)$ is equal to $\pi/\sqrt{2}$. We use Theorem 3.1 for the isometric immersion $g: (M, d\tilde{s}^2) \to G_2(n+1)$. Then we find that the inequality

$$\int_M \left(1+\frac{1}{2}|A|^2\right) dM < c_5 = \max_{0<t\leq t_0} \frac{\pi(1-t)}{2} \left(\sin\frac{t}{4(3-t)}\right)^2$$

yields $\tilde{\lambda}_1(M) > 2$. Thus the proof is complete. Q. E. D.

REMARK. (i) In [**5**] and [**12**], under the same notation as in Theorem 0.3, the integral $\int_M (a+(1/2)|A|^2)^2 dM$ is used for the estimate instead of

$\int_M (1+(1/2)|A|^2)dM$, where a is chosen to be nonnegative.

(ii) By the computation we can see that c_6 in Proposition 3.2 is a little greater than 0.0027. So Proposition 3.2 is better than Corollary of [12] as a uniform estimate.

(iii) The same method as above is not available to the stability of surfaces with constant mean curvature, because their Gauss map may not be conformal and we cannot obtain a uniform estimate like Lemma 2.2.

4. Stability of harmonic maps.

For the proof of Theorem 0.4, we need the following lemma.

LEMMA 4.1. *Let* $f:(M, ds^2)\to N$ *be a harmonic map from a 2-dimensional Riemannian manifold* (M, ds^2) *to a Riemannian manifold* N *such that* $|df|$ *vanishes only at isolated points, and assume that the sectional curvature of* N *is not greater than* $a>0$. *Let* $\bar{K}$ *be the Gaussian curvature of* $(M', d\bar{s}^2)$, *where* $d\bar{s}^2=(1/2)|df|^2 ds^2$ *and* M' *is the subset of* M *where* $|df|$ *is non-zero. Then* $\bar{K}\leqq a$ *on* M'.

PROOF. Let K be the Gaussian curvature of (M, ds^2) and let Δ be the Laplacian of (M, ds^2). Let ${}^M\nabla$ and ${}^N\nabla$ be the Riemannian connection of (M, ds^2) and N, respectively. We use ∇ to denote the connection on $T^*M\otimes f^{-1}TN$ induced from ${}^M\nabla$ and ${}^N\nabla$. Then $\bar{K}$ is given by

$$\bar{K}=\frac{2}{|df|^2}K+\frac{2}{|df|^4}\left(-\frac{1}{2}\Delta|df|^2+\frac{2}{|df|^2}\left|\frac{1}{2}{}^M\nabla|df|^2\right|^2\right). \tag{4.1}$$

By the Weitzenböck formula (see [4])

$$-\frac{1}{2}\Delta|df|^2=-|\nabla df|^2+\sum_{i,j}\langle R(f_*\varepsilon_i, f_*\varepsilon_j)f_*\varepsilon_j, f_*\varepsilon_i\rangle-K|df|^2, \tag{4.2}$$

where R is the curvature tensor of N and $\{\varepsilon_i\}$ is an orthonormal basis for T_pM with respect to ds^2 at a point p under consideration on M'. We assume that N is n-dimensional. We choose an orthonormal basis $\{e_B\}$ for $T_{f(p)}N$, and set $f_i^B=(df)_i^B$ and $f_{ij}^B=(\nabla df)_{ij}^B$. From (4.1), (4.2) and the hypothesis

$$\bar{K}\leqq a+\frac{4}{|df|^6}\left\{-\frac{1}{2}|df|^2\sum_{i,j,B}(f_{ij}^B)^2+\sum_j\left(\sum_{i,B}f_i^B f_{ij}^B\right)^2\right\}. \tag{4.3}$$

Set $T^{BC}=\sum_i f_i^B f_i^C$. Then the $(n\times n)$-matrix (T^{BC}) is symmetric and can be assumed to be diagonal for a suitable choice of $\{e_B\}$. Then $\sum_i f_i^B f_i^C=\delta^{BC}\lambda_B$ for some λ_B. This equation implies that the vectors $(f_i^B)=(f_1^B, f_2^B)$ are orthogonal to one another in $\boldsymbol{R}^2$. So we may choose $\{\varepsilon_i\}$ and $\{e_B\}$ such that

$$(f_i^1)=(\lambda, 0), \qquad (f_i^2)=(0, \mu), \qquad (f_i^3)=\cdots=(f_i^n)=0 \tag{4.4}$$

for some λ and μ. Using that $f_{11}^B+f_{22}^B=0$, $f_{12}^B=f_{21}^B$ and (4.4) we can see that

$$(4.5)\qquad -\frac{1}{2}|df|^2\sum_{i,j,B}(f^B_{ij})^2+\sum_j(\sum_{i,B}f^B_i f^B_{ij})^2$$
$$\leqq -(\lambda^2+\mu^2)\{(f^1_{11})^2+(f^1_{12})^2+(f^2_{11})^2+(f^2_{12})^2\}+(\lambda f^1_{11}+\mu f^2_{12})^2+(\lambda f^1_{12}-\mu f^2_{11})^2$$
$$=-(\lambda f^2_{11}+\mu f^1_{12})^2-(\lambda f^2_{12}-\mu f^1_{11})^2\leqq 0\,.$$

Hence by (4.3) and (4.5), $\bar{K}\leqq a$ on M'. Q.E.D.

REMARK. The author could not show the inequality (4.5) only from that $f^B_{11}+f^B_{22}=0$ and $f^B_{12}=f^B_{21}$ (cf. [**9**]).

PROOF OF THEOREM 0.4. Let ϕ be a piecewise smooth function on M such that $\phi=0$ on ∂M, and let ν be a unit section of $f^{-1}TN$. We shall consider the second variation $I(\phi\nu, \phi\nu)$ of the energy functional for the variational vector field $\phi\nu$. Let ${}^M\nabla$, ${}^N\nabla$, R and $\{\varepsilon_i\}$ be defined as in the proof of Lemma 4.1. We denote by ${}^f\nabla$ the connection on $f^{-1}TN$ induced from ${}^N\nabla$. Then by the second variational formula for harmonic maps, we have

$$I(\phi\nu, \phi\nu)=\int_M(|{}^f\nabla(\phi\nu)|^2-\sum_i\langle R(\phi\nu, f_*\varepsilon_i)f_*\varepsilon_i, \phi\nu\rangle)dM$$
$$\geqq\int_M(|{}^M\nabla\phi|^2-a\phi^2|df|^2)dM.$$

Let ${}^M\bar{\nabla}$ and $d\bar{M}$ be the Riemannian connection of $(M, d\bar{s}^2)$ and the area element of $(M, d\bar{s}^2)$, respectively, where $d\bar{s}^2$ is defined as in Lemma 4.1. Then as in the proof of Theorem 0.2, we have

$$(4.6)\qquad I(\phi\nu, \phi\nu)\geqq\int_M(|{}^M\bar{\nabla}\phi|_2^2-2a\phi^2)d\bar{M}\,,$$

where $|\ |_2$ is the norm with respect to $d\bar{s}^2$. We denote by $\bar{a}(M)$ and $\bar{\lambda}_1(M)$ the area of $(M, d\bar{s}^2)$ and the first eigenvalue of the Laplacian of $(M, d\bar{s}^2)$, respectively. Using Proposition 3.3 and 3.10 of [1] with Lemma 4.1 we find that if $\bar{a}(M)<2\pi/a$, then $\bar{\lambda}_1(M)>2a$, which implies the stability of f by (4.6). The fact that $\bar{a}(M)=(1/2)\int_M|df|^2dM$ completes the proof. Q.E.D.

EXAMPLE. Let $f: S^2\to S^3$ be the inclusion of S^2 as an equator of S^3. Then f is harmonic. For any $\varepsilon>0$, let D_ε be a geodesic disk on S^2 with area $2\pi+\varepsilon$. Then it is easy to see that f is unstable on D_ε for any $\varepsilon>0$. By choosing ε arbitrarily close to 0, we find that Theorem 0.4 is strict in this case.

Let $f:(M, ds^2)\to N$ be a conformal harmonic map from a 2-dimensional Riemannian manifold (M, ds^2) to a Riemannian manifold N (for example, the Gauss map of a minimal surface in a space form). Then the map $f:(M, d\bar{s}^2)\to N$ becomes a minimal immersion, where $d\bar{s}^2=(1/2)|df|^2ds^2$. Applying Theorem 3.1

to this isometric immersion, we find easily the following fact.

PROPOSITION 4.2. *Let* $f: M \to N$ *be a conformal harmonic map from a 2-dimensional compact Riemannian manifold* M *with piecewise* C^1 *boundary to a Riemannian manifold* N, *and assume that the sectional curvature of* N *is bounded from above and the injectivity radius of* N *is positive. Then there is a positive constant* c_6 *depending only on* N *such that if* $(1/2)\int_M |df|^2 dM < c_6$, *then* f *is stable.*

References

[1] J.L. Barbosa and M. do Carmo, Stability of minimal surfaces and eigenvalues of the Laplacian, Math. Z., **173** (1980), 13-28.
[2] J.L. Barbosa and M. do Carmo, Stability of minimal surfaces in spaces of constant curvature, Bol. Soc. Brasil. Math., **11** (1980), 1-10.
[3] M. Berger, Sur quelques varietes Riemanniennes suffisamment pincees, Bull. Soc. Math. France, **88** (1960), 57-71.
[4] J. Eells and L. Lemaire, Selected Topics in Harmonic Maps, CBMC Regional Conf. Ser., **50**, Amer. Math. Soc., 1983.
[5] D. Hoffman, Lower bounds on the first eigenvalue of the Laplacian of Riemannian submanifolds, Minimal submanifolds and geodesics, Kaigai Publ., Tokyo, 1978, pp. 61-73.
[6] D. Hoffman and R. Osserman, The area of the generalized Gaussian image and the stability of minimal surfaces in S^n and R^n, Math. Ann., **260** (1982), 437-452.
[7] G.R. Jensen and M. Rigoli, Harmonic Gauss maps, Pacific J. Math., **136** (1989), 261-282.
[8] K. Ogiue, Differential geometry of Kaehler submanifolds, Adv. in Math., **13** (1974), 73-114.
[9] M. Sakaki, Remarks on the rigidity and stability of minimal submanifolds, to appear in Proc. Amer. Math. Soc.
[10] J. Simons, Minimal varietes in riemannian manifolds, Ann. of Math., **88** (1968), 62-105.
[11] J. Spruck, Remarks on the stability of minimal submanifolds of R^n, Math. Z., **144** (1975), 169-174.
[12] S. Tanno, Remarks on Sobolev inequalities and stability of minimal submanifolds, J. Math. Soc. Japan, **35** (1983), 323-329.
[13] J.A. Wolf, Spaces of constant curvature, McGraw-Hill, New-York, 1967.

Makoto SAKAKI
Department of Mathematics
Tokyo Institute of Technology
Oh-okayama, Meguro-ku, Tokyo 152
Japan