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On the stability of Riemannian manifolds

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§ 0. Introduction.

A map $f:(M, g)\rightarrow(N, h)$ from a compact Riemannian manifold (M, g) into a complete Riemannian manifold (N, h) is *harmonic* if it is a critical point for the energy integral $E(f)=\int_{M}|df|^{2}dv_{g}$.

The identity map of a compact Riemannian manifold is always a harmonic map. Any harmonic map has its Jacobi operator determined by the second variational formula of the energy integral of the harmonic map. The Jacobi operator of the identity map of a compact manifold is a linear elliptic self-adjoint operator of second order on the vector fields of the manifold. So we consider the first eigenvalue of the Jacobi operator of the identity map. We call a Riemannian manifold *stable* if the first eigenvalue of the Jacobi operator of the identity map is non-negative and *unstable* otherwise.

The stability of Riemannian manifolds has been studied by many people. Mostly they studied which Riemannian manifolds are stable or unstable. We consider the stability problem from the different point of view. We are interested in the problem how stability of a compact Riemannian manifold depends on its Riemannian metric. For example the three-dimensional sphere is unstable with its standard Riemannian metric but there also exists a Riemannian metric which makes the three-sphere stable. We formulate the problem as follows:

We consider all the possible Riemannian metrics on a given compact manifold. Then what are the possible signs of the first eigenvalues of Jacobi operators of the identity map of the manifold?

In this paper we give the complete answer to the problem if the dimension of the manifold is less than or equal to three and a partial result if the dimension of the manifold is greater than three.

The essential part of the proof is that we can construct an unstable Riemannian metric on the Euclidean ball of dimension greater than or equal to three.

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map theory.

REMARK. It is an easy consequence of the result by G. Z. Gao and S. T. Yau [2] that there exists a Riemannian metric on the three-sphere which makes it stable.

§ 1. Definitions and statement of results.

From here on we let all the manifolds which appear in this paper be compact, connected and without boundary unless otherwise specified. In this section we give definitions necessary in this paper. Refer to [**1**] for precise definitions. We give the statement of our Main Theorem at the end of this section.

DEFINITION (Jacobi operator). For a harmonic map $f:(M, g)\to(N, h)$ we define the *Jacobi operator* J_f of f as the following differential operator on $\Gamma(f^{-1}TN)$.

$$J_f V := \Delta^f V - \operatorname{trace} R^N(V, df)df$$

for $V\in\Gamma(f^{-1}TN)$. Here $\Delta^f : \Gamma(f^{-1}TN)\to\Gamma(f^{-1}TN)$ is the differential operator which can be written in the following way using the pull-back connection ∇^f of the Levi-Civita connection on (N, h) by f.

$$\Delta^f V := -\sum_{i=1}^{\dim M}\{\nabla^f_{e_i}\nabla^f_{e_i}V-\nabla^f_{\nabla^f_{e_i}e_i}V\}$$

for $V\in\Gamma(f^{-1}TN)$, where $\{e_i\}_{i=1}^{\dim M}$ is the orthonormal frame of (M, g) and R^N is the curvature tensor of (N, h).

The Jacobi operator appears in the second variational formula of the energy functional. See the proof of [**1**, Proposition (4.3)] for the derivation of the formula.

PROPOSITION (Second variational formula). *Let $f:(M, g)\to(N, h)$ be a harmonic map and $f_{s,t}$ be a smooth two-parameter variation of f such that $f_{0,0}=f$. Let $V, W\in\Gamma(f^{-1}TN)$ be the variational vector fields of $f_{s,t}$ with respect to s and t, i.e.,*

$$V=\left.\frac{\partial f_{s,t}}{\partial s}\right|_{(s,t)=(0,0)}, \qquad W=\left.\frac{\partial f_{s,t}}{\partial t}\right|_{(s,t)=(0,0)}.$$

Then the second variational formula of the energy functional is given by

$$\left.\frac{\partial^2 E(f_{s,t})}{\partial s\partial t}\right|_{(s,t)=(0,0)}=\int_M h(J_f V, W)dv_g.$$

DEFINITION (Stability of harmonic maps). For a harmonic map $f:(M, g)\to(N, h)$ let J_f be its Jacobi operator.

(1) We call the map f *stable* if for any $V\in\Gamma(f^{-1}TN)$

$$\int_M h(J_f V,\ V)dv_g \geqq 0.$$

(2) We call the map f *strongly stable* if for any non-zero $V\in\Gamma(f^{-1}TN)$

$$\int_M h(J_f V,\ V)dv_g > 0.$$

(3) We call the map f *unstable* otherwise.

The identity map id: $(M, g)\to(M, g)$ of a Riemannian manifold (M, g) is always a harmonic map. So we can define the stability of a Riemannian manifold in terms of its identity map.

DEFINITION (Stability of Riemannian manifolds). Let (M, g) be a Riemannian manifold and id its identity map. We call (M, g) *stable* (respectively *strongly stable, unstable*) if id is stable (respectively strongly stable, unstable).

Since the Jacobi operator is an elliptic self-adjoint differential operator of second order, there exists the first eigenvalue of the Jacobi operator. For a harmonic map $f:(M, g)\to(N, h)$ we write λ_1^f for the first eigenvalue of J_f and especially when f is the identity map id of a compact Riemannian manifold (M, g) we write $\lambda_1^{(M,g)}$ for the first eigenvalue. In the above case of the identity map, $J_{(M,g)}=J_{\mathrm{id}}$ can be written in the following way using the Ricci tensor $\mathrm{Ric}=\mathrm{Ric}_{(M,g)}$ of (M, g).

$$J_{(M,g)}V = J_{\mathrm{id}}V = \Delta^{\mathrm{id}} - \mathrm{Ric}(V)$$

for $V\in\Gamma(TM)$. The next definition is important to our Main Theorem.

DEFINITION (The quantity $\Lambda(M)$). For a compact manifold M we write $\Lambda(M)$ for the set of possible signs of first eigenvalues of Jacobi operators, i.e.,

$$\Lambda(M) := \{\mathrm{sign}(\lambda_1^{(M,g)}) \mid g\in\mathrm{Riem}(M)\} \subset \{1, 0, -1\}$$

where $\mathrm{Riem}(M)$ is the set of all Riemannian metrics on M.

The relations between $\Lambda(M)$ and the stability are:

(1) $\Lambda(M)\ni 1$ if and only if there exists a Riemannian metric g on M such that (M, g) is strongly stable.

(2) $\Lambda(M)\subset\{0, 1\}$ if and only if any Riemannian metric g on M makes (M, g) stable.

(3) $\Lambda(M)\ni -1$ if and only if there exists a Riemannian metric g on M which makes (M, g) unstable.

Now we can state our Main Theorem.

MAIN THEOREM. *Let M be a compact manifold without boundary and let $\Lambda(M)$ as above. Then the following statements hold.*

(1) *If* $\dim M=1$ *then* $\Lambda(M)=\{0\}$.

(2) *If* $\dim M=2$ *then* $\Lambda(M)$ *is determined by the Euler number* $\chi(M)$ *of* M. *If* $\dim M=2$ *and* $\chi(M)\geqq 0$ *then* $\Lambda(M)=\{0\}$. *If* $\dim M=2$ *and* $\chi(M)<0$ *then* $\Lambda(M)=\{1\}$.

(3) *If* $\dim M=3$ *then* $\Lambda(M)=\{1, 0, -1\}$.

(4) *If* $\dim M\geqq 4$ *then* $\Lambda(M)\ni -1$.

§2. Proof of Main Theorem.

In this section we prove Main Theorem. First we show that we can construct a Riemannian metric on the Euclidean ball of dimension greater than or equal to three which is unstable.

LEMMA 2.1. *For an integer $n\geqq 3$ there exist a positive real number $L=L(n)$, a smooth function ρ with a compact support in the n-dimensional Euclidean ball $(B^n(L), g_0)$ of the radius of L and a smooth vector field Z' with a compact support in $(B^n(L), g_0)$ which satisfy the following condition.*

Let J' be the Jacobi operator of the identity map of $(B^n(L), g')$ and dv' the volume element of $(B^n(L), g')$ where $g'=e^{2\rho}g_0$. Then

$$\int_{B^n(L)} g'(J'Z', Z')dv' < 0. \tag{2-1}$$

PROOF. In this proof we denote by $\{C_i(n)\}_{i=1,2,\dots}$ the constants which depend only on n and we put $B^n(L)=\{x=(x_1, \cdots, x_n)\in \boldsymbol{R}^n \mid |x|<L\}$. We take a constant l such that $l\geqq 4$ and put $L=2l$. And we put $Z'=\operatorname{grad}\rho$ for a function ρ on $B^n(L)$. Later we shall show that we can choose l and a smooth function ρ so that the lemma holds. We rewrite the right-hand side of (2-1) using only ρ and g_0 as

$$\begin{aligned}\text{(2-2)}\qquad & \int_{B^n(L)}\{\operatorname{trace}'\langle\nabla' Z', \nabla' Z'\rangle' - \operatorname{Ric}'(Z', Z')\}dv' \\ &= \int_{B^n(L)}\{\operatorname{trace}\langle\nabla \operatorname{grad}\rho, \nabla \operatorname{grad}\rho\rangle - 2n|\operatorname{grad}\rho|^2 \\ &\qquad + (n-8)(\nabla d\rho)(\operatorname{grad}\rho, \operatorname{grad}\rho)\}e^{n\rho}dv\end{aligned}$$

where trace′, Ric′ and $\langle\ ,\ \rangle'$ are the trace, the Ricci tensor and the Riemannian metric itself with respect to g' respectively. Those symbols without "′" are the symbols related to g_0. We take a Lipschitz function ρ_1 on $B^n(L)$ such that

$$\rho_1 = \max\left\{\frac{1}{n}\left(1-\frac{|x|}{l}\right), 0\right\}$$

for $x\in \boldsymbol{R}^n$. First we estimate the integral on the right-hand side of (2-2) with $\rho=\rho_1$. Later we take a smooth function ρ near ρ_1 and estimate the difference. Here are the estimates of the terms on the right-hand side of (2-2).

$$\int_{B^n(L)} \operatorname{trace}\langle \nabla \operatorname{grad} \rho_1, \nabla \operatorname{grad} \rho_1 \rangle e^{n\rho_1} dv \leqq C_1(n) l^{n-4}.$$

$$\int_{B^n(L)} 2n |\operatorname{grad} \rho_1|^2 e^{n\rho_1} dv \geqq C_2(n) l^{n-2}.$$

$$\int_{B^n(L)} (n-8)(\nabla d\rho_1)(\operatorname{grad} \rho_1, \operatorname{grad} \rho_1) e^{n\rho_1} dv = 0.$$

Therefore we can take $l_0>0$ and $C_3(n)>0$ such that

$$\begin{aligned}(2\text{-}3) \qquad \int_{B^n(L)} \{&\operatorname{trace}\langle \nabla \operatorname{grad} \rho_1, \nabla \operatorname{grad} \rho_1 \rangle - 2n |\operatorname{grad} \rho_1|^2 \\ &+ (n-8)(\nabla d\rho_1)(\operatorname{grad} \rho_1, \operatorname{grad} \rho_1)\} e^{n\rho_1} dv \leqq -C_3(n) l^{n-2}\end{aligned}$$

for $l \geqq l_0$.

Now we take a smooth ρ and estimate the difference. We take $\rho\in C^\infty(B^n(2l))$ satisfying the following conditions:

(1) The function ρ depends only on $|x|$.

$$(2) \qquad 0 \leqq \rho(x) \leqq \frac{1}{n}$$

for any $x\in B^n(2l)$.

$$(3) \qquad \rho(x) = \rho_1(x)$$

for any x such that $1 \leqq |x| \leqq l-1$.

$$(4) \qquad \rho(x) = 0$$

for any x such that $l+1 \leqq |x| \leqq 2l$.

$$(5) \qquad |\operatorname{grad} \rho| \leqq \frac{C_4(n)}{l}$$

for any $x\in B^n(2l)$.

$$(6) \qquad \left| \frac{\partial}{\partial x_i} \operatorname{grad} \rho \right| \leqq \frac{C_5(n)}{l}$$

for any $x\in B^n(2l)$ and $i=1, \cdots, n$.

$\rho(x)$ differs from $\rho_1(x)$ when $|x|<1$ or when $l-1<|x|<l+1$. The integral on $B^n(1)$ is negligible when l is large enough. So all we have to estimate is the difference of the integral on $B^n(l+1)\setminus B^n(l-1)$. Under the above conditions of ρ it is easy to show that there exist constants $l_1>0$ and $C_6(n)>0$ such that

$$(2\text{-}4) \quad \left| \int_{B^n(l+1)\setminus B^n(l-1)} g_1(J_1 Z_1, Z_1) dv_1 - \int_{B^n(l+1)\setminus B^n(l-1)} g'(J'Z', Z') dv' \right| < C_6(n) l^{n-3}$$

for $l>l_1$ where those symbols with subscript "1" are the symbols related to ρ_1 and those symbols with "$'$" are the symbols related to ρ. From the formulas (2-3) and (2-4) we can conclude that if we take l large enough $L=2l$ satisfies the statement of the lemma with ρ constructed as above and $Z'=\operatorname{grad}\rho$. ■

Now we get down to the main part of the proof.

Case (1). $\dim M=1$.

This is an easy case because any (M, g) is isometric to a certain flat one-dimensional sphere. So we omit the proof of this case.

Case (2). $\dim M=2$.

The following theorem by H. Urakawa is essential for the proof of this case (cf. [**5**, Proposition 2.1]).

THEOREM 2.2. *Let $f:(M, g)\to(N, h)$ be a holomorphic map between Kähler manifolds (M, g), (N, h). Then*

$$\int_M h(J_f V, V)dv_g = \frac{1}{2}\int_M \operatorname{trace} h(DV, DV)dv_g,$$

where $DV(X):=\tilde{\nabla}_{f_*JX}V-J\tilde{\nabla}_{f_*X}V$, $V\in\Gamma(f^{-1}TN)$, $X\in\Gamma(TM)$, $\tilde{\nabla}$ is the pull-back connection of the Levi-Civita connection on (N, h) and J stands for complex structures of M and of N. In particular,

(1) *such map $f:(M, g)\to(N, h)$ is stable.*

(2) $\operatorname{Ker}(J_f)=\{V\in\Gamma(f^{-1}TN)\,|\,\tilde{\nabla}_{f_*JX}V=J\tilde{\nabla}_{f_*X}V \quad \forall X\in\Gamma(TM)\}$.

Let (M, g) and (N, h) be as in Theorem 2.2 and $\mathrm{id}:(M, g)\to(M, g)$ be the identity map of (M, g). The following statements are easy consequences of Theorem 2.2.

(1) $\lambda_1^f\geqq 0$.

(2) Let $H^0(M, f^{-1}T^{1,0}N)$ be the $\boldsymbol{C}$-linear vector space consisting of the holomorphic sections of the pull-back bundle of the holomorphic vector bundle of N. Then there exists an isomorphism between $\operatorname{Ker}(J_f)$ and $H^0(M, f^{-1}T^{1,0}N)$ as $\boldsymbol{R}$-linear vector spaces. Especially in case of id $H^0(M, f^{-1}T^{1,0}N)=H^0(M, T^{1,0}M)$ and the above isomorphism $i:\operatorname{Ker}(J_{(M,g)})\to H^0(M, T^{1,0}M)$ is given by

$$i(V)=\frac{1}{2}(V-\sqrt{-1}JV)$$

for $V\in\operatorname{Ker}(J_{(M,g)})$.

(3) $\lambda_1^{(M,g)}>0$ if and only if $H^0(M, T^{1,0}M)=\{0\}$.

Now we apply the above results to two-dimensional M's. If M is orientable M is a Kähler manifold.

If $\chi(M)<0$ and M is orientable then it is well known that $H^0(M, T^{1,0}M)=\{0\}$. So $\lambda_1^{(M,g)}>0$ for any Riemannian metric g on M and $\Lambda(M)=1$.

If $\chi(M)<0$ and M is nonorientable then $\chi(\tilde{M})<0$ for the orientable covering manifold $\tilde{M}$ of M. $\lambda_1^{(\tilde{M},\tilde{g})}>0$ for any Riemannian metric g on M where $(\tilde{M}, \tilde{g})$ is the orientable Riemannian covering manifold of (M, g). So $\lambda_1^{(M,g)}\geqq\lambda_1^{(\tilde{M},\tilde{g})}>0$ for any Riemannian metric g on M and $\Lambda(M)=1$.

If $\chi(M)\geqq 0$ and M is orientable then M is diffeomorphic to a sphere S^2 or a torus T^2. From the well-known facts that $\dim_C H^0(S^2, T^{1,0}S^2)=3$ and that $\dim_C H^0(T^2, T^{1,0}T^2)=2$ we can conclude that $\lambda_1^{(M,g)}=0$ for any Riemannian metric g on M and that $\Lambda(M)=0$.

If $\chi(M)\geqq 0$ and M is nonorientable then $\chi(\tilde{M})\geqq 0$ for its orientable covering manifold $\tilde{M}$. Therefore for any Riemannian metric g on M $\lambda_1^{(M,g)}\geqq\lambda_1^{(\tilde{M},\tilde{g})}=0$ where $(\tilde{M}, \tilde{g})$ is the orientable Riemannian covering manifold of (M, g). In order to determine whether $\lambda_1^{(M,g)}>0$ or $\lambda_1^{(M,g)}=0$ we have to investigate whether vector fields in $\mathrm{Ker}(J_{(\tilde{M},\tilde{g})})$ are compatible with the covering map. We use the fact that (M, g) is conformally equivalent to the manifold (M, g_0) with the constant curvature (see [4]). Their orientable covering Riemannian manifolds $(\tilde{M}, \tilde{g})$ and $(\tilde{M}, \tilde{g}_0)$ are also conformally equivalent. The conformalities determine the complex structures on the manifolds and the complex structures determine the kernels of $J_{(\tilde{M},\tilde{g})}$ and $J_{(\tilde{M},\tilde{g}_0)}$. Therefore they have the same kernel of Jacobi operators. So in order to investigate whether the vector fields in $\mathrm{Ker}(J_{(\tilde{M},\tilde{g})})$ are compatible with the covering map we only have to consider the case of the manifolds with constant curvature under the condition $\chi(M)\geqq 0$. It is easy to show that they are always compatible. This shows that $\lambda_1^{(M,g)}=0$ and we can conclude that $\Lambda(M)=0$.

Case (3). $\dim M=3$.

We use next two theorems along with Lemma 2.1. The first one is by L. Z. Gao and S. T. Yau (cf. [**2**, Main Theorem]).

THEOREM 2.3. *Every compact three-dimensional manifold without boundary admits a metric with negative Ricci curvature.*

The second theorem is by K. Kodaira and D. C. Spencer (cf. [**3**, Theorem 2]). We only show how it applies to our case.

THEOREM 2.4. *Let $\{g_t\}_{t\in R}$ be a smooth family of Riemannian metrics on M. Then $\lambda_1^{(M,g_t)}$ is a continuous function of t.*

That $\Lambda(M)\ni 1$ is an easy consequence of Theorem 2.3 and the definition of Jacobi operator.

That $\Lambda(M)\ni -1$ is derived from Lemma 2.1. By Lemma 2.1 we have a three-dimensional ball $(B^3(L), g')$ and a vector field Z' on the ball which satisfy

(2-1). We can imbed this ball into M. We extend the Riemannian metric g' on the ball to a Riemannian metric g on M appropriately and the vector field Z' on the ball to the vector field Z on M being zero outside the ball. M with the Riemannian metric g constructed as above is unstable because Z satisfies

$$\int_M \langle J_{(M,g)}Z, Z\rangle dv_g < 0 .$$

This implies $\Lambda(M) \ni -1$.

Lastly we show that $\Lambda(M) \ni 0$. For a compact three-dimensional manifold M we know from the above argument that there exist Riemannian metrics g_0 and g_1 such that $\lambda_1^{(M,g_0)}>0$ and $\lambda_1^{(M,g_1)}<0$. We consider a smooth family $\{g_t\}_{t\in[0,1]}$ of Riemannian metrics on M such that

$$g_t = (1-t)g_0 + tg_1$$

for $t\in[0, 1]$. Then Theorem 2.4 guarantees that there exists $s\in(0, 1)$ such that

$$\lambda_1^{(M,g_s)} = 0 .$$

This concludes Case (3).

Case (4). $\dim M \geqq 4$.

That $\Lambda(M) \ni -1$ is an easy consequence of Lemma 2.1 as in Case (3).

This completes the whole proof of the Main Theorem.

§ 3. Conjectures.

Unfortunately we can not determine $\Lambda(M)$ completely in case of M such that $\dim M \geqq 4$. But the author believes that the following conjecture holds.

CONJECTURE. *For every compact manifold M without boundary such that* $\dim M \geqq 4$ *we have*

$$\Lambda(M) = \{1, 0, -1\} .$$

The author also believes that the solution of the next conjecture will be the essential step toward the solution of the above conjecture.

CONJECTURE. *For every integer $n \geqq 4$ there exists a Riemannian metric on n-dimensional sphere which makes the sphere with the Riemannian metric strongly stable.*

References

[1] J. Eells and L. Lemaire, Selected Topics in Harmonic Maps, CBMS Regional Conf. Ser. in Math., **50**, Amer. Math. Soc., 1983.
[2] L. Z. Gao and S. T. Yau, The existence of negatively Ricci curved metrics on three

manifolds, Invent. Math., **85** (1986), 637-652.

[3] K. Kodaira and D. C. Spencer, On deformations of complex analytic structures III, Ann. of Math., **71** (1960), 43-76.

[4] J. L. Kazdan and F. W. Warner, Curvature functions for compact 2-manifolds, Ann. of Math., **99** (1974), 14-47.

[5] H. Urakawa, Stability of harmonic maps and eigenvalues of Laplacian, in "Curvature and Topology of Riemannian manifolds, Proceedings, Katata", 1985, pp. 285-307.

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