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On Q_{ab}-rationality of Eisenstein series of weight 3/2

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§ 0. Introduction.

Let F be a totally real algebraic number field and K a subfield of $\boldsymbol{C}$. A holomorphic Hilbert modular form f of integral weight or half integral weight over F is called K-rational if all Fourier coefficients of f at ∞ belong to K. In [7], Shimura proved in a general frame work that the orthogonal complements of the spaces of cusp forms (the spaces of Eisenstein series) are generated by $\boldsymbol{Q}_{ab}$-rational ones except the following two cases: (1) $F=\boldsymbol{Q}$ and the weight is 2; (2) $F=\boldsymbol{Q}$ and the weight is 3/2. Here $\boldsymbol{Q}_{ab}$ is the maximal abelian extention of $\boldsymbol{Q}$. In the first exceptional case, the Fourier coefficients of Eisenstein series are classically well known and the assertion is true (Hecke [1]). As for the second exceptional case, Pei [3] has given generators of the orthogonal complements of cusp forms in the space of holomorphic modular forms of weight 3/2. Therefore we can verify that the assertion is also correct in this case by his results and [6] Proposition 1.5. Nevertheless his construction is quite complicate and rather technical. The purpose of this paper is to give a more conceptual and shorter proof to the above fact for the weight 3/2 using the recent results of Shimura [9]. To be more precise, let N be a positive integer divisible by 4, and $\Gamma(N)$ the principal congruence elliptic modular group of level N. Let $\mathcal{N}\mathcal{H}(3/2, \Gamma(N))$ be the orthogonal complement of the cusp forms in the space of holomorphic modular forms of weight 3/2. Then

THEOREM. *The space* $\mathcal{N}\mathcal{H}(3/2, \Gamma(N))$ *is generated by* $\boldsymbol{Q}_{ab}$-*rational modular forms.*

NOTATION AND PRELIMINARY REMARKS.

(1) As usual, we denote by $\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{Q}$ and $\boldsymbol{Z}$, the real number field, the complex number field, the rational number field and the ring of rational integers. We also denote by $\boldsymbol{R}_+$ the set of positive real numbers and by $\boldsymbol{Q}_{ab}$ the maximal abelian extension of $\boldsymbol{Q}$ in $\boldsymbol{C}$.

(2) We put $i=\sqrt{-1}$. For two complex numbers $z(\neq 0)$ and α, we put

$$z^{\alpha}=\exp(\alpha(\log|z|+i\arg(z))),$$

by taking $\arg(z)$ so that $-\pi<\arg(z)\leqq\pi$. We also write

$$e(z)=\exp(2\pi iz), \qquad z\in\boldsymbol{C}.$$

For a complex number z, we sometimes use the expression

$$z=x+iy, \qquad x, y\in\boldsymbol{R},$$

without mentioning it.

(3) We denote by $\boldsymbol{H}$ the upper half complex plane:

$$\boldsymbol{H}=\{z\in\boldsymbol{C} \mid \operatorname{Im}(z)>0\}.$$

We put

$$GL_2^+(\boldsymbol{R})=\{\alpha\in GL_2(\boldsymbol{R}) \mid \det(\alpha)>0\}.$$

Then $GL_2^+(\boldsymbol{R})$ acts on $\boldsymbol{H}$ by

$$\alpha z=\frac{az+b}{cz+d}, \qquad \alpha=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in GL_2^+(\boldsymbol{R}), \quad z\in\boldsymbol{H}.$$

We also put

$$P=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in GL_2^+(\boldsymbol{R}) \mid c=0\right\}.$$

(4) For an integer a and an odd integer b, we denote by $\left(\frac{a}{b}\right)$ the quadratic residue symbol, which coincides with the ordinary one if b is an odd prime. For a non-zero integer a, the map "$b\mapsto\left(\frac{a}{b}\right)$" is a Dirichlet character defined modulo $|a|$ or modulo $4|a|$. We denote this Dirichlet character by $\left(\frac{a}{*}\right)$. For a positive integer b, the map "$a\mapsto\left(\frac{a}{b}\right)$" is also a Dirichlet character defined modulo b. We denote it by $\left(\frac{*}{b}\right)$. For further properties of the quadratic residue symbol, see [4]. We also denote by φ the Euler function.

(5) The proofs of statements of §1 can be found in [7], unless other references are given.

§1. Automorphic eigenforms of half-integral weight.

Let N be a positive integer. We define congruence subgroups $\Gamma_0(N)$, $\Gamma^0(N)$ and $\Gamma(N)$ of $SL_2(\boldsymbol{Z})$ by

$$\Gamma_0(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(\boldsymbol{Z}) \mid c\equiv 0 \pmod{N}\right\},$$

$$\Gamma^0(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(\boldsymbol{Z}) \mid b\equiv 0 \pmod{N}\right\},$$

$$\Gamma(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(\boldsymbol{Z}) \mid a\equiv d\equiv 1 \pmod{N},\ b\equiv c\equiv 0 \pmod{N}\right\}.$$

We call a subgroup Γ of $SL_2(\boldsymbol{Z})$ a congruence modular group if $\Gamma \supset \Gamma(N)$ for some N. Then we see easily that

$$\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0(N) \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \Gamma^0(N).$$

The theta function is defined by

$$\theta(z) = \sum_{n=-\infty}^{\infty} \mathrm{e}(n^2 z), \qquad z \in \boldsymbol{H}.$$

For $\gamma \in \Gamma_0(4)$, we put

$$j(\gamma, z) = \frac{\theta(\gamma z)}{\theta(z)}, \qquad z \in \boldsymbol{H}.$$

Then by definition, it holds that

$$j(\gamma\gamma', z) = j(\gamma, \gamma' z) \cdot j(\gamma', z), \qquad \gamma, \gamma' \in \Gamma_0(4).$$

The following lemma is well known ([**4**]).

LEMMA 1.1. *For* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$, *we have*

$$j(\gamma, z) = \varepsilon_d^{-1} \left(\frac{c}{d}\right) (cz+d)^{1/2},$$

where ε_d *is given by*

$$\varepsilon_d = \begin{cases} 1 & (d \equiv 1 \pmod 4), \\ i & (d \equiv 3 \pmod 4). \end{cases}$$

For $\gamma \in \Gamma^0(4)$, we put

$$j'(\gamma, z) = \frac{\theta(\delta\gamma z)}{\theta(\delta z)}, \qquad z \in \boldsymbol{H},$$

where $\delta = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}^{-1}$. Then we see easily that

$$j'(\gamma\gamma', z) = j'(\gamma, \gamma' z) \cdot j'(\gamma', z), \qquad \gamma, \gamma' \in \Gamma^0(4),$$

and

$$j'(\gamma, z) = \varepsilon_a^{-1} \left(\frac{c}{d}\right) (cz+d)^{1/2}, \qquad \text{if } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^0(4).$$

This implies

$$j'(\gamma, z) = j(\gamma, z), \qquad \text{for } \gamma \in \Gamma_0(4) \cap \Gamma^0(4).$$

Therefore we may write $j(\gamma, z)$ instead of $j'(\gamma, z)$ for $\gamma \in \Gamma^0(4)$.

Now let k be an odd integer and put $\sigma = k/2$. Put $\boldsymbol{T} = \{t \in \boldsymbol{C} \mid |t| = 1\}$. We denote by $\mathcal{G} = \mathcal{G}_\sigma$ the set of pairs $(\alpha, l(z))$ of $\alpha \in SL_2(\boldsymbol{Q})$ and a holomorphic function $l(z)$ on $\boldsymbol{H}$ satisfying

$$l(z)^2 = t \cdot (cz+d)^k, \qquad t \in \boldsymbol{T}, \ \alpha = \begin{pmatrix} * & * \\ c & d \end{pmatrix}.$$

Then $\mathcal{G}$ is a group by the following group law:

$$(\alpha, l(z))\cdot(\alpha', l'(z)) = (\alpha\alpha', l(\alpha' z)l'(z))$$

For $\xi=(\alpha, l(z))$, we write $\alpha=\mathrm{pr}(\xi)$ and $l(z)=l_\xi(z)$. Then pr is the projection from $\mathcal{G}$ to $SL_2(\boldsymbol{Q})$. We define a subgroup $\mathcal{P}$ of $\mathcal{G}$ by

$$\mathcal{P} = \{\xi\in\mathcal{G} \mid \mathrm{pr}(\xi)\in P\}.$$

For a function $f(z)$ on $\boldsymbol{H}$, we let $\xi=(\alpha, l(z))\in\mathcal{G}$ act on f by

$$(f\|\xi)(z) = f(\alpha z)l(z)^{-1}.$$

We define an injection Λ_σ of $\Gamma_0(4)\cup\Gamma^0(4)$ to $\mathcal{G}_\sigma$ by

$$\Lambda_\sigma(\gamma) = (\gamma, j(\gamma, z)^k), \qquad \gamma\in\Gamma_0(4)\cup\Gamma^0(4).$$

We note that the restrictions of Λ_σ to $\Gamma_0(4)$ and to $\Gamma^0(4)$ are group homomorphisms. For $\gamma\in\Gamma_0(4)\cup\Gamma^0(4)$ and a function f on $\boldsymbol{H}$, we put

$$(f\|_\sigma\gamma)(z) = (f\|\Lambda_\sigma(\gamma))(z) = f(\gamma z)j(\gamma, z)^{-k}.$$

By a congruence subgroup of $\mathcal{G}_\sigma$, we understand a subgroup Δ of $\mathcal{G}_\sigma$ satisfying

(1.1a) Δ contains $\Lambda_\sigma(\Gamma(N))$ with an integer N divisible by 4;

(1.1b) pr induces an isomorphism of Δ onto a congruence modular group.

Now we define a differential operator L^σ on $\boldsymbol{H}$ by

$$L^\sigma = -4y^2\frac{\partial^2}{\partial z\partial\bar{z}}+2i\sigma y\frac{\partial}{\partial\bar{z}}.$$

Let Δ be a congruence subgroup of $\mathcal{G}_\sigma$. For a complex number λ, we denote by $\mathcal{A}(\sigma, \lambda, \Delta)$ the set of all the real analytic functions f on $\boldsymbol{H}$ satisfying the following three conditions:

(1.2a) $f\|\xi = f, \quad \xi\in\Delta$;

(1.2b) $L^\sigma f = \lambda f$;

(1.2c) for every $\eta\in\mathcal{G}_\sigma$, there exist positive constants A, B and c such that

$$y^{\sigma/2}|(f\|\eta)(x+iy)|\leqq Ay^c, \qquad \text{if } y>B.$$

We call elements of $\mathcal{A}(\sigma, \lambda, \Delta)$ automorphic eigenforms of weight σ with respect to Δ. To present the Fourier expansions of automorphic eigenforms, we should introduce Whittaker functions. We denote by $\omega(t;\alpha, \beta)$ the function defined on $\boldsymbol{R}_+\times\boldsymbol{C}\times\boldsymbol{C}$, which is holomorphic in (α, β), real analytic in (t, α, β) and has an expression

$$\omega(t;\alpha, \beta) = t^\beta\Gamma(\beta)^{-1}\int_0^\infty e^{-ut}(1+u)^{\alpha-1}u^{\beta-1}du, \qquad \text{for } \mathrm{Re}(\beta)>0.$$

It satisfies

(1.3a) $\omega(t;1-\beta, 1-\alpha)=\omega(t;\alpha, \beta)$,

(1.3b) $\omega(t;1-\alpha, 0)=1$,

(1.3c) $\lim_{t\to\infty}\omega(t;\alpha, \beta)=1$.

For a complex number λ, we denote by α and β the roots of the quadratic equation

$$X^2-(1-\sigma)X+\lambda=0\,.$$

For $t\in\boldsymbol{R}$ $(t\neq 0)$, we put

$$W(t;\sigma, \lambda)=\begin{cases}\omega(4\pi t;1-\alpha, \beta) & (t>0)\,,\\ \omega(4\pi|t|;\beta, 1-\alpha) & (t<0)\,.\end{cases}$$

By (1.3a), $W(t;\alpha, \beta)$ is independent of the choice of α and β. For any element $\xi\in\mathcal{G}$, there exists a positive number A such that

$$\{\pm 1\}\cdot\mathrm{pr}(\mathcal{P}\cap(\xi^{-1}\Delta\xi))=\left\{\pm\begin{pmatrix}1 & n/A\\ 0 & 0\end{pmatrix}\middle| n\in\boldsymbol{Z}\right\}.$$

Then for $f\in\mathcal{A}(\sigma, \lambda, \Delta)$ and $\xi\in\mathcal{G}$, we have a Fourier expansion

$$(f\|\xi)(z)=a_0(y)+\sum_{n=1}^{\infty}a_nW\Big(\frac{ny}{2A};\sigma, \lambda\Big)\mathrm{e}\Big(\frac{nz}{2A}\Big)+y^{-\sigma}\cdot\sum_{n=1}^{\infty}a_{-n}W\Big(\frac{-ny}{2A};\sigma, \lambda\Big)\mathrm{e}\Big(\frac{-n\bar{z}}{2A}\Big),$$

with a function $a_0(y)$ on $\boldsymbol{R}_+$ and $a_n\in\boldsymbol{C}$ $(n\neq 0)$ ([**9**]). We call $f\in\mathcal{A}(\sigma, \lambda, \Delta)$ a cusp form if the constant term $a_0(y)=0$ for any $\xi\in\mathcal{G}$. We denote by $\mathcal{S}(\sigma, \lambda, \Delta)$ the space of cusp forms in $\mathcal{A}(\sigma, \lambda, \Delta)$.

For two elements f and g of $\mathcal{A}(\sigma, \lambda, \Delta)$, we put

$$\langle f, g\rangle=\mu(\Gamma\backslash\boldsymbol{H})^{-1}\int_{\Gamma\backslash\boldsymbol{H}}\bar{f}gy^{\sigma-1}dxdy\,,$$

where $\Gamma=\mathrm{pr}(\Delta)$ and

$$\mu(\Gamma\backslash\boldsymbol{H})=\int_{\Gamma\backslash\boldsymbol{H}}y^{-2}dxdy\,.$$

Put

$$\mathcal{N}(\sigma, \lambda, \Delta)=\{g\in\mathcal{A}(\sigma, \lambda, \Delta)\mid\langle f, g\rangle=0 \textit{ for any } f\in\mathcal{S}(\sigma, \lambda, \Delta)\}.$$

Then $\mathcal{A}(\sigma, \lambda, \Delta)=\mathcal{S}(\sigma, \lambda, \Delta)\oplus\mathcal{N}(\sigma, \lambda, \Delta)$. Further we put

$$\mathcal{A}(\sigma, \lambda)=\bigcup_{\Delta}\mathcal{A}(\sigma, \lambda, \Delta)\,,\qquad \mathcal{S}(\sigma, \lambda)=\bigcup_{\Delta}\mathcal{S}(\sigma, \lambda, \Delta)\,,$$

where the unions are taken over all congruence subgroups of $\mathcal{G}$. We also put

$$\mathcal{N}(\sigma, \lambda)=\{g\in\mathcal{A}(\sigma, \lambda)\mid\langle f, g\rangle=0 \textit{ for any } f\in\mathcal{S}(\sigma, \lambda)\}.$$

Then we observe

(1.4) $\mathcal{N}(\sigma, \lambda, \Delta)=\{f\in\mathcal{N}(\sigma, \lambda)\mid f\|\xi=f \textit{ for any } \xi\in\Delta\}$,

for any congruence subgroup Δ of $\mathcal{G}$.

For a congruence subgroup $\varDelta$ of $\mathcal{G}$, we call $\mathcal{P}\cap\varDelta$ is regular if $l_\xi(z)=1$ for any $\xi\in\mathcal{P}\cap\varDelta$. We define the Eisenstein series $E(z, s; \varDelta)$ $(z\in \boldsymbol{H}, s\in \boldsymbol{C})$ by

$$E(z, s; \varDelta)=\begin{cases} \sum_{\alpha\in\mathcal{P}\cap\varDelta\backslash\varDelta} y^s\|\alpha & \text{if } \mathcal{P}\cap\varDelta \text{ is regular,} \\ 0 & \text{otherwise.}\end{cases}$$

The series is convergent for $\mathrm{Re}(s)>1-\sigma/2$ and can be continued as a meromorphic function in s to the whole s-plane. If $E(z, s; \varDelta)$ is holomorphic at $s_0\in\boldsymbol{C}$, then

$$E(z, s_0; \varDelta)\in\mathcal{N}(\sigma, \lambda, \varDelta), \qquad \lambda=s_0(1-\sigma-s_0).$$

Further for $\xi\in\mathcal{G}$, we put

$$E(z, s; \varDelta, \xi)=E(z, s; \xi\varDelta\xi^{-1})\|\xi.$$

Let ξ and ξ' be two elements of $\mathcal{G}$ such that $\mathrm{pr}(\xi)=\mathrm{pr}(\xi')$. Then we see easily that $\xi\varDelta\xi^{-1}=\xi'\varDelta\xi'^{-1}$. Therefore

$$E(z, s; \varDelta, \xi')=cE(z, s; \varDelta, \xi)$$

with a constant c. Let α be an element of $SL_2(\boldsymbol{Q})$, and $\xi=(\alpha, l(z))\in\mathcal{G}$. Put $\Gamma=\mathrm{pr}(\varDelta)$. If $\alpha\Gamma\alpha^{-1}=\Gamma$, then $\xi\varDelta\xi^{-1}=\varDelta$ and therefore

$$E(z, s; \varDelta, \xi)=E(z, s; \varDelta)\|\xi.$$

THEOREM 1.2. (1) *If* $\mathrm{Re}(s_0)\geqq(1-\sigma)/2$, *then* $E(z, s; \varDelta)$ *is holomorphic at* $s=s_0$ *except the case when* $s_0=3/4-\sigma/2$ *and* $\sigma-1/2$ *is either an even nonnegative integer or an odd negative integer.*

(2) *For* $\lambda\in\boldsymbol{C}$, *take* $s_0\in\boldsymbol{C}$ *so that*

$$s_0(1-\sigma-s_0)=\lambda \quad and \quad \mathrm{Re}(s_0)\geqq\frac{1}{2}(1-\sigma).$$

If (s_0, σ) *is not in the exceptional case of* (1), *then* $\mathcal{N}(\sigma, \lambda, \varDelta)$ *is generated by* $E(z, s; \varDelta, \xi)$ $(\xi\in\mathcal{G}, \mathrm{pr}(\xi)\in SL_2(\boldsymbol{Z}))$.

COROLLARY 1.3. *If* $\sigma\geqq3/2$, *then* $\mathcal{N}(\sigma, 0, \varDelta)$ *is generated by Eisenstein series* $E(z, 0; \varDelta, \xi)$ $(\xi\in\mathcal{G}, \mathrm{pr}(\xi)\in SL_2(\boldsymbol{Z}))$.

We denote by $\boldsymbol{\mathcal{H}}(\boldsymbol{\sigma}, \varDelta)$ the set of all holomorphic functions on $\boldsymbol{H}$ satisfying (1.2a) and (1.2c). Then

$$\mathcal{H}(\sigma, \varDelta)\subset\mathcal{A}(\sigma, 0, \varDelta).$$

We put

$$\mathcal{N}\mathcal{H}(\sigma, \varDelta)=\mathcal{N}(\sigma, 0, \varDelta)\cap\mathcal{H}(\sigma, \varDelta).$$

Then we have

THEOREM 1.4. *If* $\sigma>3/2$, *then*

$$\mathcal{N}\mathcal{H}(\sigma, \varDelta)=\mathcal{N}(\sigma, 0, \varDelta).$$

§ 2. Eisenstein series.

For a congruence moduar group Γ contained in $\Gamma_0(4)\cup\Gamma^0(4)$, we put

$$\mathfrak{N}(\sigma, \lambda, \Gamma)=\mathfrak{N}(\sigma, \lambda ; \Lambda_\sigma(\Gamma))$$

and

$$E_\sigma(z, s ; \Gamma)=E(z, s ; \Lambda_\sigma(\Gamma)) .$$

Let M be a positive integer and N a positive integer divisible by 4. For integers μ, ν ($0\leqq\mu<M$, $1\leqq\nu<N$, $(\nu, 2)=1$), we put

$$E_\sigma\Big(z, s ; \frac{\mu}{M}, \frac{\nu}{N}\Big)=y^s\varepsilon_\nu^k \sum_{\substack{m\equiv\mu(M)\\ n\equiv\nu(N)}}\Big(\frac{m}{n}\Big)(mz+n)^{-\sigma}|mz+n|^{-2s}, \qquad z\in \boldsymbol{H},\ s\in \boldsymbol{C} .$$

Then $E_\sigma(z, s ; \mu/M, \nu/N)$ is convergent for $\mathrm{Re}(s)>1-\sigma/2$ and continued meromorphically in s to the whole s-plane by Propositions 2.2 and 2.3 below. The following lemma is easily proved ([2], Lemma 7.1.6).

LEMMA 2.1. *Let M be a positive integer* ($\geqq 3$).

(1) *The map $\Gamma(M)\ni\gamma\mapsto(c_\gamma, d_\gamma)\in\boldsymbol{Z}^2$ induces a bijection*

$$(P\cap\Gamma(M))\backslash\Gamma(M)\longrightarrow\Big\{(m, n)\in\boldsymbol{Z}^2\,\Big|\begin{array}{l}m\equiv 0 \pmod M,\\ n\equiv 1 \pmod M,\end{array}\ (m, n)=1\Big\}.$$

(2) *For $\alpha=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL_2(\boldsymbol{Z})$, the correspondence $\Gamma(M)\alpha\ni\gamma\mapsto(c_\gamma, d_\gamma)\in\boldsymbol{Z}^2$ induces a bijection*

$$(P\cap\Gamma(M))\backslash\Gamma(M)\alpha\longrightarrow\Big\{(m, n)\in\boldsymbol{Z}^2\,\Big|\begin{array}{l}m\equiv c \pmod M,\\ n\equiv d \pmod M,\end{array}\ (m, n)=1\Big\}.$$

PROPOSITION 2.2. *Let N be a positive integer divisible by* 4, *and let μ and ν be integers such that $0\leqq\mu, \nu<N$ and $(\nu, 2)=1$. Then*

(1) $E_\sigma\Big(z, s ; \dfrac{0}{N}, \dfrac{1}{N}\Big)=E_\sigma(z, s ; \Gamma(N))$.

(2) $E_\sigma\Big(z, s ; \dfrac{\mu}{N}, \dfrac{\nu}{N}\Big)=0 \qquad$ *if* $(\mu, \nu, N)\neq 1$.

(3) *If $\alpha=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma^0(4)$, then*

$$E_\sigma(z, s ; \Gamma(N), \alpha)=E_\sigma\Big(z, s ; \frac{c'}{N}, \frac{d'}{N}\Big)$$

with $0\leqq c', d'<N$ such that $(c', d')\equiv(c, d) \pmod N$.

(4) *If $\alpha\in\Gamma^0(4)$, then*

$$E_\sigma\Big(z, s ; \frac{\mu}{N}, \frac{\nu}{N}\Big)\Big\|_\sigma\alpha=E_\sigma\Big(z, s ; \frac{\mu'}{N}, \frac{\nu'}{N}\Big) .$$

Here μ', ν' are integers such that $0\leqq\mu', \nu'<N$ and

$$(\mu', \nu') \equiv (\mu, \nu)\alpha \pmod N.$$

PROOF. (1): Put $\Gamma=\Gamma(N)$. Then by Lemma 2.1 (1), we see that

$$E_\sigma\left(z, s; \frac{0}{N}, \frac{1}{N}\right) = y^s \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma, z)^{-k}|j(\gamma, z)|^{-4s} = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} y^s\|_\sigma\gamma$$

$$= E(z, s; \Lambda_\sigma(\Gamma(N))).$$

(2): Assume $(\mu, \nu, N)\neq 1$. Then all pairs of integers (m, n) such that $m\equiv\mu \pmod N$, $n\equiv\nu \pmod N$ are not coprime. Therefore $\left(\frac{m}{n}\right)=0$.

(3), (4): Since Λ_σ is a group homomorphism on $\Gamma^0(4)$, the third assertion is a direct consequence of Lemma 2.1 (2), and the fourth assertion is an easy consequence of (3). □

PROPOSITION 2.3. *Let N, μ and ν be the same as in Proposition 2.2. Assume $(\mu, \nu, N)=1$ and put $(\mu, N)=u$ and $(\nu, N)=v$. We also put*

$$\mu' = \frac{\mu}{u},\quad \nu' = \frac{\nu}{v},\quad M' = \frac{N}{u},\quad N' = \frac{N}{v}.$$

Then

$$E_\sigma\left(z, s; \frac{\mu}{N}, \frac{\nu}{N}\right) = \left(\frac{\varepsilon_\nu}{\varepsilon_{\nu'}}\right)^k\left(\frac{u}{v}\right)\left(\frac{\mu'}{v}\right)\left(\frac{u}{\nu'}\right)v^{-\sigma-s}u^{-s}E_\sigma\left(\frac{u}{v}z, s; \frac{\mu'}{M'}, \frac{\nu'}{N'}\right).$$

PROOF.

$$E_\sigma\left(z, s; \frac{\mu}{N}, \frac{\nu}{N}\right) = y^s\varepsilon_\nu^k \sum_{\substack{m\equiv\mu(N)\\ m\equiv\nu(N)}}\left(\frac{m}{n}\right)(mz+n)^{-\sigma}|mz+n|^{-2s}$$

$$= \left(\frac{u}{v}y\right)^s\left(\frac{\varepsilon_\nu}{\varepsilon_{\nu'}}\right)^k\varepsilon_{\nu'}^k \sum_{\substack{m\equiv\mu'(M')\\ n\equiv\nu'(N')}}\left(\frac{u}{v}\right)\left(\frac{m}{v}\right)\left(\frac{u}{n}\right)\left(\frac{m}{n}\right)v^{-\sigma-s}u^{-s}\left(m\frac{u}{v}z+n\right)^{-\sigma}\left|m\frac{u}{v}z+n\right|^{-2s}.$$

Since u and v are coprime, we see that N' is divisible by u and M' is divisible by v. This implies $\left(\frac{m}{v}\right)=\left(\frac{\mu'}{v}\right)$ and $\left(\frac{u}{n}\right)=\left(\frac{u}{\nu'}\right)$ in the last summation. □

Let χ and ψ be Dirichlet characters defined modulo M and modulo N, respectively. Assume that N is divisible by 4. We put

$$E_\sigma(z, s; \chi, \psi) = y^s \sum_{m,n=-\infty}^{\infty} \chi(m)\psi(n)\varepsilon_n^k\left(\frac{m}{n}\right)(mz+n)^{-\sigma}|mz+n|^{-2s}$$

$$= \sum_{\substack{0\leq\mu<M\\ 0<\nu<N}} \chi(\mu)\psi(\nu)E_\sigma\left(z, s; \frac{\mu}{M}, \frac{\nu}{N}\right).$$

Here we understand that

$$\chi(0) = 1 \quad \textit{if } M=1, \qquad \textit{and} \qquad \chi(0) = 0 \quad \textit{otherwise}.$$

Since N is divisible by 4, $\psi(n)=0$ for even integers n. Then the summation on n is extended only over odd integers.

PROPOSITION 2.4. *Assume $\sigma\geqq 3/2$. Let N be a positive integer divisible by 4.*

(1) *For any integers μ, ν $(0\leqq\mu, \nu<N, (\nu, 2)=1)$, $E_\sigma(z, s; \mu/N, \nu/N)$ is holomorphic at $s=0$ and $E_\sigma(z, 0; \mu/N, \nu/N)$ belongs to $\mathfrak{N}(\sigma, 0, \Gamma(N))$.*

(2) *Let M_1 be a divisor of N, and N_1 a divisor of N divisible by 4. Let χ and ψ be Dirichlet characters defined modulo M_1 and modulo N_1, respectively. Then $E_\sigma(z, s; \chi, \psi)$ is holomorphic at $s=0$ and $E_\sigma(z, 0; \chi, \psi)$ belongs to $\mathfrak{N}(\sigma, 0, \Gamma(N))$.*

PROOF. By Corollary 1.3 and Proposition 2.2, we see that $E_\sigma(z, s; \mu/N, \nu/N)$ is holomorphic at $s=0$ and $E_\sigma(z, 0; \mu/N, \nu/N)\in\mathfrak{N}(\sigma, 0, \Gamma(N))$. Then

$$E_\sigma(z, s; \chi, \psi)=\sum_{\substack{0\leqq\mu<M_1\\ 0<\nu<N_1}}\chi(\mu)\psi(\nu)E_\sigma\left(z, s; \frac{\mu}{M_1}, \frac{\nu}{N_1}\right).$$

Further put $N'=\text{L.C.M.}(M_1, N_1)$. Then

$$E_\sigma\left(z, s; \frac{\mu}{M_1}, \frac{\nu}{N_1}\right)=\sum E_\sigma\left(z, s; \frac{\mu'}{N'}, \frac{\nu'}{N'}\right)$$

where the summation extends over integers μ and ν satisfying

$$0\leqq\mu, \nu<N', \quad \mu'\equiv\mu \pmod{M_1}, \quad \nu'\equiv\nu \pmod{N_1}.$$

Therefore $E_\sigma(z, s; \chi, \psi)$ is holomorphic at $s=0$. Since $E_\sigma(z, 0; \mu'/N', \nu'/N')$ belongs to $\mathfrak{N}(\sigma, 0, \Gamma(N'))$ and $\mathfrak{N}(\sigma, 0, \Gamma(N'))\subset\mathfrak{N}(\sigma, 0, \Gamma(N))$, $E_\sigma(z, s; \chi, \psi)$ also belongs to $\mathfrak{N}(\sigma, 0, \Gamma(N))$. □

The next proposition can be easily proved by using the orthogonal relation of characters.

PROPOSITION 2.5. *Let M be a positive integer, and N a positive integer divisible by 4. If $(\mu, M)=1$ and $(\nu, N)=1$, then*

$$E_\sigma\left(z, s; \frac{\mu}{M}, \frac{\nu}{N}\right)=\frac{1}{\varphi(M)\varphi(N)}\sum_\chi\sum_\psi\bar{\chi}(\mu)\bar{\psi}(\nu)E_\sigma(z, s; \chi, \psi).$$

Here χ (resp. ψ) is taken over all Dirichlet characters defined modulo M (resp. modulo N) and φ is the Euler function.

For the Fourier coefficients of Eisenstein series, we obtain

THEOREM 2.6. *Assume $\sigma\geqq 3/2$. For any $\alpha\in SL_2(\boldsymbol{Z})$, there exists an element $\xi\in\mathcal{G}_\sigma$ such that $\mathrm{pr}(\xi)=\alpha$ and*

$$E_\sigma(z, 0; \Gamma(N))\|\xi=c+a_0\pi^{2-2\sigma}y^{1-\sigma}+\sum_{n=1}^{\infty}a_n e(nz/N)$$

$$+\pi^{1-2\sigma}y^{-\sigma}\sum_{n=1}^{\infty}a_{-n}\omega(-4\pi ny/N;\,0,\,\sigma)\mathrm{e}(-n\bar{z}/N)\,,$$

with a_n $(n\in\boldsymbol{Z})$, $c\in\boldsymbol{Q}_{\mathrm{ab}}$. *Furthermore if* $\sigma>3/2$, *then* $a_{-n}=0$ *for all* $n\geqq 0$.

The proof of Theorem 2.6 will be given in § 3.

For a modular group Γ contained in $\Gamma_0(4)\cup\Gamma^0(4)$, we put

$$\mathcal{H}(\sigma,\,\Gamma)=\mathcal{H}(\sigma,\,\Lambda_\sigma(\Gamma))\,,\qquad \mathcal{NH}(\sigma,\,\Gamma)=\mathcal{NH}(\sigma,\,\Lambda_\sigma(\Gamma))\,.$$

Let $f(z)$ be an element of $\mathcal{H}(\sigma,\,\Gamma)$, and let the Fourier expansion of f be

$$f(z)=\sum_{n=0}^{\infty}a_n\mathrm{e}(nz/A)\,,\qquad a_n\in\boldsymbol{C}\,,$$

with some positive integer A. For a subfield K of $\boldsymbol{C}$, we call $f(z)$ K-rational if $a_n\in K$ for all $n\geqq 0$. We denote by $\mathcal{NH}(\sigma,\,\Gamma,\,K)$ (resp. $\mathcal{H}(\sigma,\,\Gamma,\,K)$) the set of all K-rational elements in $\mathcal{NH}(\sigma,\,\Gamma)$ (resp. $\mathcal{H}(\sigma,\,\Gamma)$). We also denote by $\mathcal{N}(\sigma,\,0,\,\Gamma,\,K)$ the set of all functions $f(z)$ in $\mathcal{N}(\sigma,\,0,\,\Gamma)$ which has the Fourier expansion

$$f(z)=c+a_0\pi^{2-2\sigma}y^{1-\sigma}+\sum_{n=1}^{\infty}a_n\mathrm{e}\Big(\frac{nz}{A}\Big)+\pi^{1-2\sigma}y^{-\sigma}\sum_{n=1}^{\infty}a_{-n}\omega\Big(\frac{4\pi ny}{A}\,;\,0,\,\sigma\Big)\mathrm{e}\Big(\frac{-n\bar{z}}{A}\Big),$$

with a_n $(n\in\boldsymbol{Z})$, $c\in K$ and a positive integer A. Then

$$\mathcal{NH}(\sigma,\,\Gamma,\,K)=\mathcal{N}(\sigma,\,0,\,\Gamma,\,K)\cap\mathcal{H}(\sigma,\,\Gamma)\,.$$

If $\sigma>3/2$, Theorem 2.6 together with Theorem 1.4 implies that

$$\mathcal{N}(\sigma,\,0,\,\Gamma(N),\,\boldsymbol{Q}_{\mathrm{ab}})=\mathcal{NH}(\sigma,\,\Gamma(N),\,\boldsymbol{Q}_{\mathrm{ab}})\,.$$

Now we obtain

THEOREM 2.7. *Assume* $\sigma\geqq 3/2$. *Let* N *be a positive integer divisible by* 4. *Then the space* $\mathcal{NH}(\sigma,\,\Gamma(N))$ *is generated by* $\boldsymbol{Q}_{\mathrm{ab}}$*-rational elements.*

PROOF. By Corollary 1.3, $\mathcal{N}(\sigma,\,0,\,\Gamma(N))$ is generated by $E_\sigma(z,\,0\,;\,\Gamma(N))\|\xi$ $(\xi\in\mathcal{G},\ \mathrm{pr}(\xi)\in SL_2(\boldsymbol{Z}))$. Then by Theorem 2.6, we can take a basis of $\mathcal{N}(\sigma,0,\Gamma(N))$ among the elements in $\mathcal{N}(\sigma,\,0,\,\Gamma(N),\,\boldsymbol{Q}_{\mathrm{ab}})$, which we denote by $f_1(z),\,\cdots,\,f_m(z)$. Write

$$f_j(z)=c^{(j)}+a_0^{(j)}\pi^{2-\sigma}y^{1-\sigma}+\sum_{n=1}^{\infty}a_n^{(j)}\mathrm{e}\Big(\frac{nz}{N}\Big)+\pi^{1-2\sigma}y^{-\sigma}\sum_{n=1}^{\infty}a_{-n}^{(j)}\omega\Big(\frac{4\pi ny}{N}\,;\,0,\,\sigma\Big)\mathrm{e}\Big(\frac{-n\bar{z}}{N}\Big)$$

with $a_n^{(j)}$, $c^{(j)}\in\boldsymbol{Q}_{\mathrm{ab}}$. Let $f(z)\in\mathcal{NH}(\sigma,\,\Gamma(N))$. Then

$$f(z)=p_1f_1(z)+\cdots+p_mf_m(z)\,,\qquad p_l\in\boldsymbol{C}\,.$$

By the uniqueness of the Fourier coefficients, we have infinitely many linear equations

$$(2.1)\quad \begin{aligned} p_1 a_{-n}^{(1)} + \cdots + p_m a_{-n}^{(m)} &= 0 \qquad (n=0, 1, 2, \cdots), \\ p_1 c^{(1)} + \cdots + p_m c^{(m)} &= 0 . \end{aligned}$$

Since the coefficients of the equations belong to $\boldsymbol{Q}_{\mathrm{ab}}$, we can find fundamental solutions in $\boldsymbol{Q}_{\mathrm{ab}}^m$, say $(p_1^{(1)}, \cdots, p_m^{(1)}), \cdots, (p_1^{(r)}, \cdots, p_m^{(r)})$. Put

$$g_t = p_1^{(t)} f_1 + \cdots + p_m^{(t)} f_m \qquad (1 \leqq t \leqq r) .$$

Then g_t is $\boldsymbol{Q}_{\mathrm{ab}}$-rational and $f(z)$ is a linear combination of g_t $(1 \leqq t \leqq r)$. □

COROLLARY 2.8. *Assume* $\sigma \geqq 3/2$. *Let* Δ *be a congruence subgroup of* $\mathcal{G}_\sigma$ *such that* $\mathrm{pr}(\Delta) \subset SL_2(\boldsymbol{Z})$. *Then* $\mathcal{N}\mathcal{H}(\sigma, \Delta)$ *is generated by* $\boldsymbol{Q}_{\mathrm{ab}}$*-rational elements.*

PROOF. Take a principal congruence modular group $\Gamma(N)$ $(4|N)$ so that $\Lambda_\sigma(\Gamma(N)) \subset \Delta$. Let $f_1, \cdots, f_m$ be the same as in the proof of Theorem 2.7. Let $\xi = (\alpha, l(z))$ be an element of Δ and $\xi' = (\alpha, l'(z))$ be the element of $\mathcal{G}_\sigma$ for α in Theorem 2.6. Since some power of ξ belongs to $\Lambda_\sigma(\Gamma(N))$, $l(z) = c l'(z)$ with a root of unity c by the construction of ξ'. Therefore $f_1 \| \xi, \cdots, f_m \| \xi$ are linear combinations of $f_1, \cdots, f_m$ over $\boldsymbol{Q}_{\mathrm{ab}}$; say

$$f_i \| \xi = c(i, 1, \xi) f_1 + \cdots + c(i, m, \xi) f_m , \qquad i=1, \cdots, m .$$

Let f be an element of $\mathcal{N}\mathcal{H}(\sigma, \Delta)$. Then $f = p_1 f_1 + \cdots + p_m f_m$ with $p_i \in \boldsymbol{C}$. Since f is holomorphic, $p_1, \cdots, p_m$ satisfy the linear equations in (2.1). The property that f belongs to $\mathcal{N}(\sigma, 0, \Delta)$ is characterized by the linear equations

$$p_1 c(1, j, \xi) + \cdots + p_m c(m, j, \xi) = p_j$$

for $j=1, \cdots, m$ and the representatives $\{\xi\}$ of $\Lambda_\sigma(\Gamma(N)) \backslash \Delta$. Since all those equations have coefficients in $\boldsymbol{Q}_{\mathrm{ab}}$, we have fundamental solutions in $\boldsymbol{Q}_{\mathrm{ab}}^m$ as in Theorem 2.7 and f is a linear combination of $\boldsymbol{Q}_{\mathrm{ab}}$-rational elements of $\mathcal{N}\mathcal{H}(\sigma, \Delta)$. □

REMARK. The case $\sigma > 3/2$ in Theorem 2.7 is a direct consequence of Corollary 1.3, Theorem 1.4 and Theorem 2.6. We note that even in that case our proof is elementary and different from Shimura [**9**], Proposition 6.2 which uses results of canonical models.

§ 3 Fourier coefficients of Eisenstein series.

Let M be a positive integer, and N a positive integer divisible by 4. Let χ and ψ be Dirichlet characters defined modulo M and N, respectively.

PROPOSITION 3.1. (1) *If* $\chi(-1)\psi(-1) \neq 1$, *then* $E_\sigma(z, s; \chi, \psi) = 0$.

(2) *If* $\chi(-1)\psi(-1)=1$ *then*

$$E_\sigma(z, s; \chi, \psi) = 2y^s\Big\{\chi(0)+\sum_{m=1}^{\infty}\chi(m)\sum_{n=-\infty}^{\infty}\psi(n)\varepsilon_n^k\Big(\frac{m}{n}\Big)(mz+n)^{-\sigma}|mz+n|^{-2s}\Big\}.$$

PROOF. Assume that the term for a pair of integers (m, n) does not vanish. Then $(m, n)=1$, and n is an odd integer. Hence there exists an element $\gamma\in\Gamma^0(4)$ such that $\gamma=\begin{pmatrix} * & * \\ m & n\end{pmatrix}$. Then

$$\varepsilon_n^k\Big(\frac{m}{n}\Big)(mz+n)^{-\sigma} = j(\gamma, z)^{-k} = \Big(\frac{\theta(\delta\gamma z)}{\theta(\delta z)}\Big)^{-k}.$$

Therefore we obtain that

$$\varepsilon_{-n}^k\Big(\frac{-m}{-n}\Big)(-mz-n)^{-\sigma} = j(-\gamma, z)^{-k} = \Big(\frac{\theta(\delta(-\gamma)z)}{\theta(\delta z)}\Big)^{-k}$$

$$= \Big(\frac{\theta(\delta\gamma z)}{\theta(\delta z)}\Big)^{-k} = \varepsilon_n^k\Big(\frac{m}{n}\Big)(mz+n)^{-\sigma}.$$

Since

$$E_\sigma(z, s; \chi, \psi) = y^s\Big\{\chi(0)\psi(1)+\chi(0)\psi(-1)$$

$$+\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}(\chi(m)\psi(n)+\chi(-m)\psi(-n))\varepsilon_n^k\Big(\frac{m}{n}\Big)(mz+n)^{-\sigma}|mz+n|^{-2s}\Big\},$$

we obtain the assertion. □

Now we will calculate the Fourier coefficients of Eisenstein series by generalizing the arguments of Shimura [5], Sturm [10], and Pei [3]. Assume $\chi(-1)\psi(-1)=1$. Then

$$(3.1)\quad E_\sigma(z, s; \chi, \psi) = 2y^s\Big\{\chi(0)+\sum_{m=1}^{\infty}\chi(m)\sum_{n=-\infty}^{\infty}\psi(n)\varepsilon_n^k\Big(\frac{m}{n}\Big)(mz+n)^{-\sigma}|mz+n|^{-2s}\Big\}$$

$$= 2y^s\Big\{\chi(0)+\sum_{m=1}^{\infty}\chi(m)\sum_{n=-\infty}^{\infty}\psi(n)\varepsilon_n^k\Big(\frac{m}{n}\Big)(mz+n)^{-\sigma-s}(m\bar{z}+n)^{-s}\Big\}$$

$$= 2y^s\Big\{\chi(0)+\sum_{m=1}^{\infty}\chi(m)(mN)^{-\sigma-2s}\sum_{a=1}^{mN-1}\psi(a)\varepsilon_a^k\Big(\frac{m}{a}\Big)$$

$$\times\sum_{n=-\infty}^{\infty}\Big(\frac{z}{N}+\frac{a}{mN}+n\Big)^{-\sigma-s}\Big(\frac{\bar{z}}{N}+\frac{a}{mN}+n\Big)^{-s}\Big\}.$$

We put

$$S(z, \alpha, \beta) = \sum_{n=-\infty}^{\infty}(z+n)^{-\alpha}(\bar{z}+n)^{-\beta} \qquad (z\in\boldsymbol{H},\ \alpha, \beta\in\boldsymbol{C}).$$

Then $S(z, \alpha, \beta)$ is absolutely convergent at least when $z\in\boldsymbol{H}$ and $\mathrm{Re}(\alpha+\beta)>1$. It is continued in (α, β) to a meromorphic function on $\boldsymbol{C}\times\boldsymbol{C}$ which is real analytic in z. Using this function, we can write

(3.1) $E_\sigma(z, s; \chi, \phi)$

$$= 2y^s\Big\{\chi(0)+\sum_{m=1}^{\infty}\chi(m)(mN)^{-\sigma-2s}\sum_{a=1}^{mN-1}{}'\phi(a)\varepsilon_a^k\Big(\frac{m}{a}\Big)S\Big(\frac{z}{N}+\frac{a}{mN}, \sigma+s, s\Big)\Big\}.$$

The following Fourier expansion of $S(z, \alpha, \beta)$ is well known ([7], Theorem 6.1, see also [2] Theorem 7.2.8).

LEMMA 3.2.

$$\begin{aligned} S(z, \alpha, \beta) &= i^{\beta-\alpha}(2\pi)^{\alpha+\beta}\Gamma(\alpha)^{-1}\Gamma(\beta)^{-1}\Gamma(\alpha+\beta-1)(4\pi y)^{1-\alpha-\beta} \\ &\quad + i^{\beta-\alpha}(2\pi)^{\alpha}\Gamma(\alpha)^{-1}(2y)^{-\beta}\sum_{n=1}^{\infty}n^{\alpha-1}\omega(4\pi ny; \alpha, \beta)\mathrm{e}(nz) \\ &\quad + i^{\beta-\alpha}(2\pi)^{\beta}\Gamma(\beta)^{-1}(2y)^{-\alpha}\sum_{n=1}^{\infty}n^{\beta-1}\omega(4\pi ny; \beta, \alpha)\mathrm{e}(-n\bar{z}). \end{aligned}$$

Now substituting Lemma 3.2 into (3.1), we obtain

THEOREM 3.3. *Assume* $\chi(-1)\phi(-1)=1$. *Then*

$$\begin{aligned} \frac{1}{2}E_\sigma(z, s; \chi, \phi) &= \chi(0)y^s + C(s)y^{1-\sigma-s}a(0, s) \\ &\quad + A(s)\sum_{n=1}^{\infty}n^{\sigma-s-1}a(n, s)\omega\Big(\frac{4\pi ny}{N}; \sigma+s, s\Big)\mathrm{e}\Big(\frac{nz}{N}\Big) \\ &\quad + B(s)y^{-\sigma}\sum_{n=1}^{\infty}n^{s-1}a(-n, s)\omega\Big(\frac{4\pi ny}{N}; s, \sigma+s\Big)\mathrm{e}\Big(-\frac{n\bar{z}}{N}\Big), \end{aligned}$$

where

$$\begin{aligned} A(s) &= i^{-\sigma}(2\pi)^{\sigma+s}\Gamma(\sigma+s)^{-1}(N/2)^s, \\ B(s) &= i^{-\sigma}(2\pi)^s\Gamma(s)^{-1}(N/2)^{\sigma+s}, \\ C(s) &= i^{-\sigma}(2\pi)\Gamma(\sigma+s)^{-1}\Gamma(s)^{-1}\Gamma(\sigma+2s-1)(N/2)^{\sigma+2s-1}, \end{aligned}$$

and

$$a(n, s) = \sum_{m=1}^{\infty}\chi(m)(mN)^{-\sigma-2s}\sum_{a=1}^{mN-1}{}'\phi(a)\varepsilon_a^k\Big(\frac{m}{a}\Big)\mathrm{e}\Big(\frac{an}{mN}\Big) \qquad (n\in \boldsymbol{Z}).$$

To calculate the Dirichlet series $a(n, s)$, we generalize the Gauss sum. Let L be a positive integer, and ω a Dirichlet character defined modulo L. For an integer b, we put

$$G_b(\omega) = \sum_{a=1}^{L-1}\omega(a)\mathrm{e}\Big(\frac{ab}{L}\Big),$$

which is the usual Gauss sum. When L is divisible by 4, we also put for an integer b and an odd integer k,

$$G_b^{(k)}(\omega) = \sum_{a=1}^{L-1}\varepsilon_a^k\omega(a)\mathrm{e}\Big(\frac{ab}{L}\Big),$$

which we call the twisted Gauss sum. They are related by the following lemma, which can be easily proved.

LEMMA 3.4.

(1) $$G_b^{(k)}(\omega)=\frac{1}{2}\left\{G_b(\omega)+G_b\left(\omega\left(\frac{-1}{*}\right)\right)\right\}+\frac{i^k}{2}\left\{G_b(\omega)-G_b\left(\omega\left(\frac{-1}{*}\right)\right)\right\}.$$

(2) $$G_b^{(-k)}(\omega)=G_b^{(k)}\left(\omega\left(\frac{-1}{*}\right)\right).$$

To calculate the twisted Gauss sum for the product of two Dirichlet characters, we need the well known properties of the Gauss sum for the product of two Dirichlet characters. Let χ and ψ be Dirichlet characters defined modulo M and N, respectively. If M and N are coprime, then

(3.2a) $$G_b(\chi\psi)=\chi(N)\psi(M)G_b(\chi)G_b(\psi).$$

Further let ω be a Dirichlet character defined modulo L. For integers b and l, we have

(3.2b) $$\sum_{a=1}^{lL-1}\omega(a)e\left(\frac{ab}{lL}\right)=0, \qquad \text{if } b \text{ is not divisible by } l.$$

Now we obtain

PROPOSITION 3.5. *Let χ and ψ be Dirichlet characters defined modulo M and N respectively. Assume M and N are coprime and N is divisible by* 4. *Then*

$$G_b^{(k)}(\chi\psi)=\begin{cases}\chi(N)\psi(M)G_b(\chi)G_b^{(k)}(\psi) & \text{if } \left(\frac{-1}{M}\right)=1,\\ i^k\chi(N)\psi(M)G_b(\chi)G_b^{(k)}\left(\psi\left(\frac{-1}{*}\right)\right) & \text{if } \left(\frac{-1}{M}\right)=-1.\end{cases}$$

PROOF. By Lemma 3.4(1), we have

$$\begin{aligned}G_b^{(k)}(\chi\psi)&=\frac{1}{2}\left\{G_b(\chi\psi)+G_b\left(\chi\psi\left(\frac{-1}{*}\right)\right)\right\}+\frac{i^k}{2}\left\{G_b(\chi\psi)-G_b\left(\chi\psi\left(\frac{-1}{*}\right)\right)\right\}\\&=\chi(N)\psi(M)G_b(\chi)T.\end{aligned}$$

where

$$T=\frac{1}{2}\left(G_b(\psi)+\left(\frac{-1}{M}\right)G_b\left(\psi\left(\frac{-1}{*}\right)\right)\right)+\frac{i^k}{2}\left(G_b(\psi)-\left(\frac{-1}{M}\right)G_b\left(\psi\left(\frac{-1}{*}\right)\right)\right).$$

If $\left(\frac{-1}{M}\right)=1$, then

$$T=G_b^{(k)}(\psi).$$

If $\left(\frac{-1}{M}\right)=-1$, then by Lemma 3.4(1)

$$\begin{aligned}T&=\frac{1}{2}\left(G_b(\psi)-G_b\left(\psi\left(\frac{-1}{*}\right)\right)\right)+\frac{i^k}{2}\left(G_b(\psi)+G_b\left(\psi\left(\frac{-1}{*}\right)\right)\right)\\&=i^kG_b^{(-k)}(\psi)=i^kG_b^{(k)}\left(\psi\left(\frac{-1}{*}\right)\right). \quad \square\end{aligned}$$

For a Dirichlet character ω, we denote by $L(s, \omega)$ the Dirichlet L-function. For a positive integer N, $L_N(s, \omega)$ denotes the function obtained from $L(s, \omega)$ by omitting p-Euler factors for all prime components p of N. Now we obtain

PROPOSITION 3.6.

$$a(0, s) = \frac{L_N(4s+2\sigma-2, \chi^2\psi^2)}{L_N(4s+2\sigma-1, \chi^2\psi^2)}\alpha(0, s),$$

$$\alpha(0, s) = \begin{cases} \dfrac{1\pm i^k}{2}\chi(l)\varphi(N)l^{-\sigma-2s+1}N^{-\sigma-2s}\prod_{p\mid N}(1-\chi(p^2)p^{-2\sigma-4s+2})^{-1} \\ \qquad \text{if } \psi \text{ is a character derived from a quadratic} \\ \qquad \text{character } \left(\frac{\pm l}{*}\right) \text{ with a positive square free } l, \\ 0 \qquad \text{if } \psi \text{ is not derived from a quadratic character.} \end{cases}$$

PROOF. By definition, we see that

$$a(0, s) = \sum_{m=1}^{\infty}\chi(m)(mN)^{-\sigma-2s}\sum_{a=1}^{mN-1}\psi(a)\varepsilon_a^k\left(\frac{m}{a}\right) = \sum_{m=1}^{\infty}\chi(m)(mN)^{-\sigma-2s}G_0^{(k)}\left(\psi\left(\frac{m}{*}\right)\right).$$

Here we consider $\psi\left(\frac{m}{*}\right)$ as a Dirichlet character defined modulo Nm. Decompose m as $m=m'd$, a product of an integer m' dividing some power of N (which we express as $m'\mid N^\infty$) and a positive integer d relatively prime to N. Put $m^*=\left(\frac{-1}{d}\right)m'$. Then by the reciprocity law,

$$\left(\frac{m}{a}\right) = \left(\frac{a}{d}\right)\left(\frac{m^*}{a}\right), \qquad \textit{for } a\in \boldsymbol{Z}.$$

Therefore by Proposition 3.5, we obtain

$$\begin{aligned} G_0^{(k)}\left(\psi\left(\frac{m}{*}\right)\right) &= G_0^{(k)}\left(\psi\left(\frac{*}{d}\right)\left(\frac{m^*}{*}\right)\right) \\ &= \begin{cases} \left(\frac{Nm'}{d}\right)\psi(d)\left(\frac{m^*}{d}\right)\cdot G_0\left(\left(\frac{*}{d}\right)\right)\cdot G_0^{(k)}\left(\psi\left(\frac{m^*}{*}\right)\right) & \text{if } \left(\frac{-1}{d}\right)=1 \\ i^k\left(\frac{Nm'}{d}\right)\psi(d)\left(\frac{m^*}{d}\right)\cdot G_0\left(\left(\frac{*}{d}\right)\right)\cdot G_0^{(k)}\left(\psi\left(\frac{m^*}{*}\right)\left(\frac{-1}{*}\right)\right) & \text{if } \left(\frac{-1}{d}\right)=-1 \end{cases} \\ &= \begin{cases} \psi(d)\varphi(d)G_0^{(k)}\left(\psi\left(\frac{m'}{*}\right)\right) & \text{if } d \text{ is a square} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Here φ is the Euler function. Therefore we can write

$$a(0, s) = \sum_{(d,N)=1}\chi(d^2)\psi(d^2)d^{-2(\sigma+2s)}\varphi(d^2)\sum_{m\mid N^\infty}\chi(m)(mN)^{-(\sigma+2s)}G_0^{(k)}\left(\psi\left(\frac{m}{*}\right)\right).$$

Now we see that

$$\sum_{(d,N)=1} \chi(d^2)\psi(d^2)d^{-2(\sigma+2s)}\varphi(d^2)$$

$$= \prod_{p\nmid N}\left\{\sum_{n=0}^{\infty}\chi(p^{2n})\psi(p^{2n})p^{-2n(\sigma+2s)}p^{2n} - \sum_{n=1}^{\infty}\chi(p^{2n})\psi(p^{2n})p^{-2n(\sigma+2s)}p^{2n-1}\right\}$$

$$= \frac{L_N(4s-2\sigma-2,\ \chi^2\psi^2)}{L_N(4s-2\sigma-1,\ \chi^2\psi^2)}.$$

By Lemma 3.4, if neither $\psi\left(\frac{m}{*}\right)$ nor $\psi\left(\frac{m}{*}\right)\left(\frac{-1}{*}\right)$ is trivial, then

$$G_0^{(k)}\left(\psi\left(\frac{m}{*}\right)\right) = 0.$$

Moreover, if $\psi\left(\frac{m}{*}\right)$ $\left(\text{resp. } \psi\left(\frac{-m}{*}\right)\right)$ is trivial, then

$$G_0^{(k)}\left(\psi\left(\frac{m}{*}\right)\right) = \frac{1+i^k}{2}\varphi(mN) \ \left(\text{resp. } \frac{1-i^k}{2}\varphi(mN)\right) \quad \text{and} \quad \varphi(mN) = m\varphi(N).$$

Using this we easily obtain our expression. □

Proposition 3.7. *For $n(\neq 0)\in \boldsymbol{Z}$, we put $n=tr^2$ (t: square free). Then we have*

$$a(n, s) = \frac{L_N\left(2s+\sigma-\frac{1}{2},\ \chi\psi\left(\frac{-1}{*}\right)^{\sigma-1/2}\left(\frac{t}{*}\right)\right)}{L_N(4s+2\sigma-1,\ \chi^2\psi^2)}\alpha(n, s).$$

Here $\alpha(n, s)$ is a finite Dirichet series with coefficients in $\boldsymbol{Q}_{\text{ab}}$.

Proof. We can write

$$a(n, s) = \sum_{m=1}^{\infty}\chi(m)(mN)^{-\sigma-2s}G_n^{(k)}\left(\psi\left(\frac{m}{*}\right)\right).$$

Here we consider the character $\psi\left(\frac{m}{*}\right)$ is defined modulo mN. We decompose m as a product $m=m'd$ ($m'|N^{\infty}$, $(d, N)=1$) and put $m^*=\left(\frac{-1}{d}\right)m'$ as in the proof of Proposition 3.6. Then

$$G_n^{(k)}\left(\psi\left(\frac{m}{*}\right)\right) = G_n^{(k)}\left(\psi\left(\frac{m^*}{*}\right)\left(\frac{*}{d}\right)\right)$$

$$= \begin{cases} \left(\frac{Nm'}{d}\right)\psi(d)\left(\frac{m^*}{d}\right)\cdot G_n\left(\left(\frac{*}{d}\right)\right)\cdot G_n^{(k)}\left(\psi\left(\frac{m^*}{*}\right)\right) & \text{if } \left(\frac{-1}{d}\right)=1 \\ i^k\left(\frac{Nm'}{d}\right)\psi(d)\left(\frac{m^*}{d}\right)\cdot G_n\left(\left(\frac{*}{d}\right)\right)\cdot G_n^{(k)}\left(\psi\left(\frac{m^*}{*}\right)\left(\frac{-1}{*}\right)\right) & \text{if } \left(\frac{-1}{d}\right)=-1 \end{cases}$$

$$= \varepsilon_d^k\psi(d)\left(\frac{-N}{d}\right)G_n\left(\left(\frac{*}{d}\right)\right)G_n^{(k)}\left(\psi\left(\frac{m'}{*}\right)\right).$$

Therefore we obtain

$$a(n, s) = \sum_{(d,N)=1}^{\infty} \chi(d)\psi(d)\Big(\frac{-N}{d}\Big)\varepsilon_d^k G_n\Big(\Big(\frac{*}{d}\Big)\Big)d^{-\sigma-2s}$$

$$\times N^{-\sigma-2s}\sum_{m\mid N^\infty}^{\infty} \chi(m)m^{-\sigma-2s}G_n^{(k)}\Big(\psi\Big(\frac{m}{*}\Big)\Big).$$

Now the Dirichlet series obtained as a summation on d is the one in Shimura [**5**], (3.4) and we have

$$\sum_{(d,N)=1} = \frac{L_N\Big(2s+\sigma-1/2,\ \chi\psi\Big(\frac{-1}{*}\Big)^{\sigma-1/2}\Big(\frac{t}{*}\Big)\Big)}{L_N(4s+2\sigma-1,\ \chi^2\psi^2)}b(n, s),$$

and

$$b(n, s) = \sum\mu(a)\chi(a)\psi(a)\Big(\frac{-1}{a}\Big)^{\sigma-1/2}\Big(\frac{n}{a}\Big)\chi(b)^2\psi(b)^2a^{1/2-\sigma-2s}b^{2-\sigma-s},$$

where the summation extends over all positive integers a, b which are prime to N and satisfy $(ab)^2\mid n$, and μ is the Möbius function. Further we put

$$c(n, s) = N^{-\sigma-2s}\sum_{m\mid N^\infty} \chi(m)m^{-\sigma-2s}G_n^{(k)}\Big(\psi\Big(\frac{m}{*}\Big)\Big).$$

Here we consider $\psi\Big(\frac{m}{*}\Big)$ as a character defined modulo mN. Note that $\psi\Big(\frac{m}{*}\Big)$ and $\psi\Big(\frac{m}{*}\Big)\Big(\frac{-1}{*}\Big)$ are Dirichlet characters defined modulo N. Therefore if n is not divisible by m, then $G_n^{(k)}\Big(\psi\Big(\frac{m}{*}\Big)\Big)=0$ by (3.2b). Hence $c(n, s)$ is a finite sum. By putting

$$a(n, s) = b(n, s)c(n, s),$$

we obtain the assertion. □

THEOREM 3.8. *Let χ and ψ be Dirichlet characters defined modulo divisors of N. If $\sigma\geqq 3/2$, then*

$$E_\sigma(z, 0; \chi, \psi) \in \mathfrak{N}(\sigma, 0, \Gamma(N), \boldsymbol{Q}_{\rm ab}).$$

PROOF. We let $s=0$ in Theorem 3.3. First assume $\sigma>3/2$. Then $a(n, s)$ is finite at $s=0$. Since $B(0)=C(0)=0$, the terms for $-n$ $(n\geqq 0)$ vanishes. In general, for non-zero complex numbers a and b, we write $a\sim b$ if a/b belongs to $\boldsymbol{Q}_{\rm ab}$. Then

$$A(0) \sim \pi^{\sigma-1/2}$$

and for $n>0$,

$$a(n, 0) \sim \frac{L\Big(\sigma-1/2,\ \chi\psi\Big(\frac{-1}{*}\Big)^{\sigma-1/2}\Big(\frac{t}{*}\Big)\Big)}{L(2\sigma-1,\ \chi^2\psi^2)} \sim \pi^{1/2-\sigma},$$

unless $a(n,0)=0$. Hence $A(0)a(n,0)n^{\sigma-1}\in\boldsymbol{Q}_{\rm ab}$. Assume $\sigma=3/2$. First we see easily that $A(0) \sim \pi$. If n is positive, then $\left(\frac{-1}{*}\right)\left(\frac{t}{*}\right)\chi\psi$ is not trivial, since t is positive. Therefore if $a(n, 0) \neq 0$ $(n>0)$, then $a(n, 0) \sim \pi^{-1}$. This implies $A(0)a(n, 0)n^{1/2} \in \boldsymbol{Q}_{\mathrm{ab}}$. We also see that

$$B(s)a(-n, s)|_{s=0} \sim \frac{\zeta(s)}{\Gamma(s)}\bigg|_{s=0} \cdot \frac{1}{L(2\sigma-1, \chi^2\psi^2)} \sim \pi^{-2} \qquad (n>0),$$

$$C(s)a(0, s)|_{s=0} \sim \pi\frac{\zeta(s)}{\Gamma(s)}\bigg|_{s=0} \cdot \frac{1}{\zeta(2)} \sim \pi^{-1},$$

unless they are 0. This implies the theorem. □

Now we are going to prove Theorem 2.6. Put

$$E(z) = E_\sigma(z, 0; \Gamma(N)).$$

Then by Propositions 2.2, 2.3, 2.5 and Theorem 3.8, we have

$$E(z)\|_\sigma \gamma \in \mathcal{M}(\sigma, 0, \Gamma(N), \boldsymbol{Q}_{\mathrm{ab}}), \qquad \text{for any } \gamma \in \Gamma^0(4). \tag{3.3}$$

Now as a complete set of representatives for $\Gamma^0(4) \backslash SL_2(\boldsymbol{Z})$, we have the following six elements:

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}, \begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}, \begin{pmatrix}1 & 3\\0 & 1\end{pmatrix}, \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}, \begin{pmatrix}2 & -1\\1 & 0\end{pmatrix}. \tag{3.4}$$

Put $\Lambda = \Lambda_\sigma$. Let α be one of the first four elements, and put

$$\xi(\alpha) = (\alpha, 1).$$

Then for any $\gamma \in \Gamma^0(4)$, we have $\mathrm{pr}(\Lambda(\gamma)\xi(\alpha)) = \gamma\alpha$ and the action of $\xi(\alpha)$ is nothing but a translation $z \mapsto z+a$ $(a=0, 1, 2, 3)$. Therefore by (3.3), we see that

$$E(z)\|\Lambda(\gamma)\xi(\alpha) \in \mathcal{M}(\sigma, 0, \Gamma(N), \boldsymbol{Q}_{\mathrm{ab}}).$$

Next let $\alpha = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$ and put $\xi = (\alpha, z^\sigma) \in \mathcal{G}$. Since $E\|_\sigma \gamma$ $(\gamma \in \Gamma^0(4))$ is a linear combination of $E_\sigma((u/v)z, 0; \chi, \psi)$ by Propositions 2.2, 2.3, 2.5, we have only to prove $E_\sigma((u/v)z, 0; \chi, \psi)\|\xi \in \mathcal{M}(\sigma, 0, \Gamma(N), \boldsymbol{Q}_{\mathrm{ab}})$. But this can be proved in parallel with Theorem 3.3 and Theorem 3.8 (see also [5] and [10]). Finally let $\alpha = \begin{pmatrix}2 & -1\\1 & 0\end{pmatrix}$ and put $\xi = (\alpha, z^\sigma) \in \mathcal{G}$. Since $E(z)\|_\sigma \gamma$ $(\gamma \in \Gamma^0(4))$ is of the form $E_\sigma(z, 0; \mu/N, \nu/N)$, $(\mu, \nu, N)=1$, we have only to prove

$$E_\sigma\left(z, 0; \frac{\mu}{N}, \frac{\nu}{N}\right)\|\xi \in \mathcal{M}(\sigma, 0, \Gamma(N), \boldsymbol{Q}_{\mathrm{ab}}).$$

Put

$$F(z, s) = E_\sigma\left(z, s; \frac{\mu}{N}, \frac{\nu}{N}\right)\|\xi.$$

Then

$$F(z, s)=\operatorname{Im}\Big(\frac{2z-1}{z}\Big)^{s}\sum_{\substack{m\equiv\mu(N)\\ n\equiv\nu(N)}}\varepsilon_n^k\Big(\frac{m}{n}\Big)\Big(\frac{m(2z-1)}{z}+n\Big)^{-\sigma}\Big|\frac{m(2z-1)}{z}+n\Big|^{-2s}z^{-\sigma}$$

$$= y^{s}\sum_{\substack{m\equiv\mu(N)\\ n\equiv\nu(N)}}\varepsilon_n^k\Big(\frac{m}{n}\Big)\delta\{(2m+n)z-m\}^{-\sigma}|(2m+n)z-m|^{-2s},$$

where

$$\delta=\begin{cases}1 & \text{if } 2m+n\geqq 0, \text{ or } 2m+n<0 \text{ and } m<0,\\ -1 & \text{if } 2m+n<0 \text{ and } m\geqq 0.\end{cases}\tag{3.5}$$

Let μ' be an integer $0\leqq\mu'<N$ satisfying $2\mu+\nu\equiv\mu' \pmod N$. Then by Lemma 3.9 below, we have

$$F(z, s)=y^{s}\sum_{\substack{m\equiv\mu'(N)\\ n\equiv\nu(N)}}\varepsilon_n^k\Big(\frac{2m}{n}\Big)\Big(m\Big(z-\frac{1}{2}\Big)+\frac{n}{2}\Big)^{-\sigma}\Big|m\Big(z-\frac{1}{2}\Big)+\frac{n}{2}\Big|^{-2s}$$

$$=2^{\sigma+2s}y^{s}\sum_{\substack{m\equiv\mu'(N)\\ n\equiv\nu(N)}}\varepsilon_n^k\Big(\frac{2m}{n}\Big)\Big(2m\Big(z-\frac{1}{2}\Big)+n\Big)^{-\sigma}\Big|2m\Big(z-\frac{1}{2}\Big)+n\Big|^{-2s}$$

$$=2^{\sigma+2s}\sum_{\substack{m\equiv 2\mu'(2N)\\ n\equiv\nu(N)}}\varepsilon_n^k\Big(\frac{m}{n}\Big)\Big(m\Big(z-\frac{1}{2}\Big)+n\Big)^{-\sigma}\Big|m\Big(z-\frac{1}{2}\Big)+n\Big|^{-2s}$$

$$=2^{\sigma+2s}\Big\{E_\sigma\Big(z-\frac{1}{2}, s; \frac{2\mu'}{2N}, \frac{\nu}{2N}\Big)+E_\sigma\Big(z-\frac{1}{2}, s; \frac{2\mu'}{2N}, \frac{\nu+N}{2N}\Big)\Big\}.$$

Since both $E_\sigma(z, 0; 2\mu'/2N, \nu/2N)$ and $E_\sigma(z, 0; 2\mu'/2N, (\nu+N)/2N)$ belong to $\mathcal{N}(\sigma, 0, \Gamma(2N), \boldsymbol{Q}_{\mathrm{ab}})$, $F(z, 0)$ has $\boldsymbol{Q}_{\mathrm{ab}}$-rational Fourier coefficients, and therefore it belongs to $\mathcal{N}(\sigma, 0, \Gamma(N), \boldsymbol{Q}_{\mathrm{ab}})$. □

LEMMA 3.9. *Let m and n be integers. Assume $(m, n)=1$ and n is odd. Let δ be ± 1 given by* (3.5). *Put $m'=2m+n$. Then* $\Big(\frac{m}{n}\Big)\delta=\Big(\frac{2m'}{n}\Big)$.

PROOF. Note that $m=(m'-n)/2$.

(1) Assume $m'\geqq 0$. Then $\delta=1$.

(i) If $n>0$, then $\Big(\frac{m}{n}\Big)\delta=\Big(\frac{2}{n}\Big)\Big(\frac{m'-n}{n}\Big)=\Big(\frac{2}{n}\Big)\Big(\frac{m'}{n}\Big)=\Big(\frac{2m'}{n}\Big)$.

(ii) If $n<0$, then $m>0$ and $\Big(\frac{m}{n}\Big)\delta=\Big(\frac{m}{-n}\Big)=\Big(\frac{2}{-n}\Big)\Big(\frac{m'-n}{-n}\Big)=\Big(\frac{2m'}{-n}\Big)=\Big(\frac{2m'}{n}\Big)$.

(2) Assume $m'<0$. If $n>0$, then $m<0$ and $\delta=1$. Therefore

$$\Big(\frac{m}{n}\Big)\delta=\Big(\frac{2}{n}\Big)\Big(\frac{m'-n}{n}\Big)=\Big(\frac{2}{n}\Big)\Big(\frac{m'}{n}\Big)=\Big(\frac{2m'}{n}\Big).$$

Assume $n<0$.

(i) If $m>0$, then $\delta=-1$ and

$$\left(\frac{m}{n}\right)\delta=-\left(\frac{m}{n}\right)=-\left(\frac{m}{-n}\right)=-\left(\frac{2}{-n}\right)\left(\frac{m'-n}{-n}\right)=-\left(\frac{2m'}{-n}\right)=\left(\frac{2m'}{n}\right).$$

(ii) If $m<0$, then $\delta=1$ and

$$\left(\frac{m}{n}\right)\delta=\left(\frac{m}{n}\right)=-\left(\frac{m}{-n}\right)=-\left(\frac{2}{-n}\right)\left(\frac{m'-n}{-n}\right)=-\left(\frac{2m'}{-n}\right)=\left(\frac{2m'}{n}\right).$$

(iii) If $m=0$, then $\delta=-1$. Now we have

$$\left(\frac{m}{n}\right)\delta=-\left(\frac{0}{n}\right)=-1\ (n=-1),\qquad \text{or } 0\ (n<-1),$$

$$\left(\frac{2m'}{n}\right)=\left(\frac{2n}{n}\right)=-1\ (n=-1),\qquad \text{or } 0\ (n<-1).$$

Therefore $\left(\frac{m}{n}\right)\delta=\left(\frac{2m'}{n}\right)$. □

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