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On classification of parabolic reflection groups in $SU(n, 1)$

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§ 0. Introduction.

To classify reflection groups is one of the most important matter in the group theory. It attracted many mathematicians. For example, finite reflection subgroups of orthogonal group $O(n)$ are classified by Coxeter [**3**] —— they are called the Coxeter groups ——, those of the unitary group $U(n)$ are classified by Shephard and Todd [**9**] —— they are called the unitary reflection groups ——, discrete cocompact reflection subgroups of the complex motion group are classified by Popov [**8**] —— they are called the crystallographic reflection groups ——, and discrete reflection subgroups of the parabolic subgroup of the special unitary group $SU(n, 1)$ of signature $(n, 1)$ are partially classified by Yoshida-Hattori [**14**] and Yoshida [**12**] —— they are called the parabolic reflection groups in $SU(n, 1)$.

This paper is devoted to the complete classification of the parabolic reflection groups in $SU(n, 1)$. The group $SU(n, 1)$ gives rise to the group $\mathrm{Aut}(D)$ of analytic automorphisms of a domain $D=\{{}^t(z, u_1, \cdots, u_m)\in \boldsymbol{C}^{m+1};\ 2\,\mathrm{Im}\, z-\sum|u_j|^2>0\}$, which is projectively equivalent to the complex n-ball $B^n=\{{}^t(z_1, \cdots, z_n)\in \boldsymbol{C}^n;\ \sum|z_j|^2<1\}$. The parabolic subgroup G of $SU(n, 1)$ is identified with a subgroup of $\mathrm{Aut}(D)$ which leaves the point P at infinity fixed. Precisely speaking, reflection groups in question are discrete subgroups of G of locally finite covolume at P.

In § 1, we review the structure of discrete subgroup of G. The main theorem is stated in § 2. Proof is given in § 3.

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§ 1. Parabolic subgroup G.

1.1. A matrix representation of G. Let V be an $(m+1)$-dimensional complex vector space with coordinates $(z, u_1, \cdots, u_m)$. Let D be a domain in V defined as follows

$$D := \left\{(z, u_1, \cdots, u_m) \in V;\ 2\,\mathrm{Im}\, z - \sum_{j=1}^{m} |u_j|^2 > 0\right\},$$

which is analytically equivalent to the unit ball $B^{m+1} = \{{}^t(z_0, \cdots, z_m) \in \boldsymbol{C}^{m+1};\ \sum_{j=0}^{m} |z_j|^2 < 1\}$. The domain D can be considered as a domain in the $(m+1)$-dimensional complex projective space $P^{m+1}(\boldsymbol{C})$. Let us choose homogeneous coordinates ${}^t(v_0, \cdots, v_{m+1})$ of $P^{m+1}(\boldsymbol{C})$ which are related to the coordinates of V by $z = v_0/v_{m+1}$, $u_j = v_j/v_{m+1}$ $(j=1, \cdots, m)$. Then D has the following expression:

$$D = \{v = {}^t(v_0, \cdots, v_{m+1}) \in P^{m+1}(\boldsymbol{C});\ {}^t\bar{v}Hv > 0\},$$

where $H = \begin{pmatrix} & & i \\ & -E_m & \\ -i & & \end{pmatrix}$, ($E_m$ denotes the identity matrix of size m).

Since any complex analytic automorphism of D can be extended to an automorphism of $P^{m+1}(\boldsymbol{C})$, we regard the group $\mathrm{Aut}(D)$ of analytic automorphisms of D as a subgroup of the projective transformation group $PGL(m+2, \boldsymbol{C})$. Consequently, $\mathrm{Aut}(D)$ is identified with the quotient group:

$$\{G \in GL(m+2, \boldsymbol{C});\ \ {}^t\bar{G}HG = kH, \text{ for some positive number } k\}/\boldsymbol{C}^*,$$

where $\boldsymbol{C}^*$ is the multiplicative group of $\boldsymbol{C}$. Hereafter we express an element of $\mathrm{Aut}(D)$ by a suitable matrix of size $m+2$ belonging to the corresponding coset.

We denote by $\bar{D}$ and ∂D the closure and the boundary of D in $P^{m+1}(\boldsymbol{C})$, respectively. Then $\bar{D}$ meets the hyperplane at infinity $v_{m+1}=0$ at the unique point $P = {}^t(1, 0, \cdots, 0)$ on ∂D. We consider the subgroup G of $\mathrm{Aut}(D)$ fixing the boundary point P consisting of the elements of the following form.

$$[U, \beta, \gamma] := \begin{pmatrix} 1 & i\,{}^t\bar{\beta}U & \gamma + \frac{i}{2}\,{}^t\bar{\beta}\beta \\ 0 & U & \beta \\ 0 & 0 & 1 \end{pmatrix},$$

$U \in U(m)$: the unitary matrices of size m, $\beta \in \boldsymbol{C}^m$, $\gamma \in \boldsymbol{R}$.

The group G is called the parabolic subgroup at P. For $g = [w, \beta, \gamma] \in G$, we call w the linear part of g, β the translation part of g, and γ the central part of g, respectively. Note that the law of product of G is given by

$$[w_1, \beta_1, \gamma_1][w_2, \beta_2, \gamma_2] = [w_1 w_2, \beta_1 + w_1\beta_2, \gamma_1 + \gamma_2 - \mathrm{Im}\,{}^t\bar{\beta}_1 w_1 \beta_2].$$

For $N > 0$, we define the subdomain $D(N)$ of D by

$$D(N) := \left\{(z, u_1, \cdots, u_m) \in D;\ 2\,\mathrm{Im}\, z - \sum_{j=1}^{m} |u_j|^2 > N\right\}.$$

DEFINITION. Let Γ be a subgroup of G and let $\mathrm{vol}(D(N)/\Gamma)$ be the volume of the quotient space $D(N)/\Gamma$ with respect to the $\mathrm{Aut}(D)$-invariant measure of D. Γ is said to be *of locally finite covolume* (at P) if $\mathrm{vol}(D(N)/\Gamma)<\infty$ for $N>0$.

1.2. Crystallographic groups and discrete subgroups of G. Let V' be the quotient space of V by a 1-dimensional vector subspace $V_0:=\{(z, 0, \cdots, 0)\in V; z\in \boldsymbol{C}\}$ of V with coordinates $u=(u_1, \cdots, u_m)$ and with the natural Hermitian inner product $(u, v)=\bar{v}^t u$. With this inner product, V' is a complex Euclidean space of dimension m. We denote by $E(V')$ the complex motion group on V'. We express elements of $E(V')$ by $(U|\beta)=\begin{pmatrix} U & \beta \\ 0 & 1 \end{pmatrix}$, where $U\in U(m)$ and $\beta\in \boldsymbol{C}^m$.

DEFINITION. A discrete subgroup G_0 of $E(V')$ is called a *crystallographic group* if the quotient space V'/G_0 is compact.

We define a homomorphism π_* from G to $E(V')$ by $\pi_*([U, \beta, \gamma])=(U|\beta)$. Consider a discrete subgroup Γ of G of locally finite covolume. The image $\pi_*(\Gamma)$ is a crystallographic group on V', and the kernel of π_* is the center $Z(\Gamma)$ of Γ, i.e. the group Γ admits the following exact sequence: $1\to Z(\Gamma)\to \Gamma\to \pi_*(\Gamma)\to 1$. Moreover we have $Z(\Gamma)=\{[E, 0, \gamma]; \gamma\in q(\Gamma)\boldsymbol{Z}\}\cong \boldsymbol{Z}$, where $q(\Gamma)=\inf\{|\gamma|; [E, 0, \gamma]\in\Gamma, \gamma\neq 0\}$, and E stands for E_m.

In view of Bieberbach's theorem, any crystallographic group Γ_* admits the following exact sequence: $1\to L_*\to \Gamma_*\to W\to 1$, where L_* is a lattice in $\boldsymbol{C}^m$ and W is a finite subgroup of $GL(m, \boldsymbol{C})$, which is called *the point group* of Γ_*. Hence $L:=\pi_*(\Gamma)\cap\{(E|\beta)\in E(V'); \beta\in \boldsymbol{C}^m\}$ is a lattice of rank $2m$ and $\pi_*(\Gamma)/L$ is a finite group. We can regard L as a subgroup of $\boldsymbol{C}^m$ by identifying $(E|\beta)$ with β. By computing the commutator of $[E, \beta, \gamma]$ and $[E, \beta', \gamma']$, it is shown that $2\,\mathrm{Im}\,{}^t\bar{\beta}\beta'/q(\Gamma)$ is an integer for any $\beta, \beta'\in L$. Therefore, there is a natural number n such that $q(\Gamma)=q_0/n$, where

$$q_0=\inf(\{|2\,\mathrm{Im}\,{}^t\bar{\beta}\beta'|;\quad \beta, \beta'\in L\}-\{0\}).$$

1.3. Reflections in G. Let us recall the definition of unitary reflections.

DEFINITION. An element $w\in U(m)$ is called a reflection if w is of finite order and has exactly $m-1$ eigenvalues equal to 1.

For a reflection $w\in U(m)$ we denote by $\mu(w)$ the unique eigenvalue of w different from 1 and by $r(w)$ an eigenvector corresponding to $\mu(w)$. We call $r(w)$ a root of w.

DEFINITION. An element of $E(V')$ or G is called a reflection if its order is finite ($\neq 1$) and if it leaves a hyperplane in V' or in D pointwise fixed, respectively.

By a straightforward calculation we have

LEMMA. (i) $(w\,|\,\beta)\in E(V')$ *is a reflection if and only if* $w\in U(m)$ *is a reflection and* $\beta=kr(w)$ *for some* $k\in \boldsymbol{C}$.

(ii) $[w, \beta, \gamma]\in G$ *is a reflection if and only if* $(w\,|\,\beta)\in E(V')$ *is a reflection and*

$$\gamma=\frac{i^{t}\bar{\beta}\beta}{2}\frac{\mu(w)+1}{\mu(w)-1}\,.$$

Notice that if Γ is generated by reflections, then the crystallographic group $\pi_*(\Gamma)$ is also generated by reflections.

§ 2. List of parabolic reflection groups.

2.1. Main theorem. Let Γ be a discrete subgroup of G (see § 1.1) and assume that Γ is of locally finite covolume. Then Γ admits the following commutative diagram of exact sequences:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & Z(\Gamma) \arrow[r] \arrow[d] & L(\Gamma) \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & Z(\Gamma) \arrow[r] \arrow[d] & \Gamma \arrow[r] \arrow[d] & \pi_*(\Gamma) \arrow[r] \arrow[d] & 1 \\
 & 1 \arrow[r] & W \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & 1 \\
 & & 1 & 1 &
\end{tikzcd}$$

where the homomorphisms $\Gamma\to\pi_*(\Gamma)$ and $\Gamma\to W$ are given by $[w, \beta, \gamma]\mapsto(w\,|\,\beta)$ and $[w, \beta, \gamma]\mapsto w$, respectively. We also call the point group W of $\pi_*(\Gamma)$ the point group of Γ. We consider the case where W is an irreducible unitary reflection group that is not a Coxeter group. If Γ is generated by reflections we call Γ a parabolic reflection group of dimension $m+1$.

THEOREM. *Let* Γ *be a parabolic reflection group of dimension* $\geqq 3$. *If the point group* W *is not a Coxeter group then* Γ *is conjugate in* Aut(D) *to one of the groups in the table in* § 2.2.

REMARK. Parabolic reflection groups of dimension 2 and those whose point groups are Coxeter groups are classified in [**14**] and [**12**]. Therefore, with these results, this theorem completes the classification of all parabolic reflection groups.

2.2. Table of parabolic reflection groups. We list up the parabolic reflection groups Γ by giving their generating reflections. Corresponding crystallographic reflection groups $\pi_*(\Gamma)$ are presented only by their point groups W and their lattices L when they are the semi-direct products of W and L, while if it is not the case, we give their generators.

NOTATIONS.

$e_j=(0, \cdots, 0, \overset{j}{1}, 0, \cdots, 0)$, $(j=1, \cdots, m)$: canonical bases of $\boldsymbol{C}^m$.

$\omega=e^{2\pi i/3}=\dfrac{-1+i\sqrt{3}}{2}$.

r_j: the reflection of the point group W of which root is β_j.

dim: the dimension of the parabolic reflection group Γ.

Point Group: the point group W of the crystallographic group Γ_*.

Graph: the graph showing generating reflections of W.

Roots: the roots of the generators of W.

Lattice: the lattice consisting of the translations of Γ_*.

Crystallographic Group: the crystallographic reflection group Γ_* such that $\pi_*(\Gamma)=\Gamma_*$.

Center: the center of the parabolic reflection group Γ.

CONVENTION OF THE GRAPHS. A vertex ⓝ of a graph represents a reflection of order n for which one of its root β $(|\beta|=1)$ is specified. For simplicity we write ○ instead of ②. If two roots β_j and β_k are not orthogonal, we join the corresponding two vertices (which represent r_j and r_k, respectively) by a segment, directed from j to k, attached by the value (β_j, β_k). If the value is real, the direction is omitted and moreover if the value is $-1/2$, the value is also omitted. We follow the naming of the groups used by Shephard-Todd [**9**] and by Popov [8].

dim	Point Group	Graph	Roots
$n+1$	$G(3, 1, n)$ $(n\geqq 2)$	1 2 n ③—○—⋯—○ $\frac{1}{\sqrt{2}}$	$\beta_1=e_1$ $\beta_j=\frac{e_{j-1}-e_j}{\sqrt{2}}$ $(j=2, \cdots, n)$

Lattice	Crystallographic Group
$L_1=(\boldsymbol{Z}+\omega\boldsymbol{Z})\frac{1}{\sqrt{2}}\beta_1+\sum_{j=2}^{n}(\boldsymbol{Z}+\omega\boldsymbol{Z})\beta_j$	$G(3, 1, n)\ltimes L_1$

Center	Generators of Parabolic Reflection Group
$\left[E, 0, \frac{1}{2\sqrt{3}}\boldsymbol{Z}\right]$	$[r_1, 0, 0]$ $\left[r_1, \frac{1}{\sqrt{2}}\beta_1, \frac{1}{4\sqrt{3}}\right]$ $\left[r_1, -\frac{1}{\sqrt{2}}\beta_1, \frac{1}{4\sqrt{3}}\right]$ $[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, \omega\beta_j, 0]$ $(j=2, \cdots, n)$
$\left[E, 0, \frac{\sqrt{3}}{2}\boldsymbol{Z}\right]$	$[r_1, 0, 0]$ $\left[r_1, \frac{1}{\sqrt{2}}\beta_1, \frac{1}{4\sqrt{3}}\right]$ $[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, \omega\beta_j, 0]$ $(j=2, \cdots, n)$
	$\left[r_1, \frac{1}{\sqrt{2}}\beta_1, \frac{1}{4\sqrt{3}}\right]$ $\left[r_1, \frac{1+\omega}{\sqrt{2}}\beta_1, \frac{1}{4\sqrt{3}}\right]$ $[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, \omega\beta_j, 0]$ $(j=2, \cdots, n)$

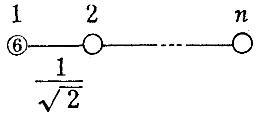
Lattice	Crystallographic Group
$L_2=(\boldsymbol{Z}+\omega\boldsymbol{Z})\frac{1}{\sqrt{2}}\beta_1+\sum_{j=2}^{n}(\boldsymbol{Z}+\omega\boldsymbol{Z})\frac{2+\omega}{3}\beta_j$	$G(3, 1, n)\ltimes L_2$

Center	Generators of Parabolic Reflection Group
$\left[E, 0, \frac{1}{2\sqrt{3}}\boldsymbol{Z}\right]$	$[r_1, 0, 0]$ $\left[r_1, \frac{1}{\sqrt{2}}\beta_1, \frac{1}{4\sqrt{3}}\right]$ $[r_j, 0, 0]$ $\left[r_j, \frac{2+\omega}{3}\beta_j, 0\right]$ $\left[r_j, \frac{-1+\omega}{3}\beta_j, 0\right]$ $(j=2, \cdots, n)$

dim	Point Group	Graph	Roots
$n+1$	$G(4, 1, n)$ $(n\leqq 2)$	1 2 n ④—○—⋯—○ $\frac{1}{\sqrt{2}}$	$\beta_1=e_1$ $\beta_j=\frac{e_{j-1}-e_j}{\sqrt{2}}$ $(j=2, \cdots, n)$

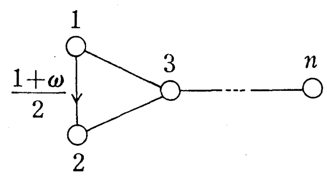
Lattice	Crystallographic Group
$L_1=(\boldsymbol{Z}+i\boldsymbol{Z})\frac{1}{\sqrt{2}}\beta_1+\sum_{j=2}^{n}(\boldsymbol{Z}+i\boldsymbol{Z})\beta_j$	$G(4, 1, n)\ltimes L_1$
Center	**Generators of Parabolic Reflection Group**
$\left[E, 0, \frac{1}{4}\boldsymbol{Z}\right]$	$[r_1, 0, 0]$ $\left[r_1, \frac{1}{\sqrt{2}}\beta_1, \frac{1}{4}\right]$ $\left[r_1^2, \frac{1}{\sqrt{2}}\beta_1, 0\right]$ $[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, i\beta_j, 0]$ $(j=2, \cdots, n)$
$\left[E, 0, \frac{1}{2}\boldsymbol{Z}\right]$	$[r_1, 0, 0]$ $\left[r_1^2, \frac{1}{\sqrt{2}}\beta_1, 0\right]$ $\left[r_1^2, \frac{1+i}{\sqrt{2}}\beta_1, 0\right]$ $[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, i\beta_j, 0]$ $(j=2, \cdots, n)$
	$[r_1, 0, 0]$ $\left[r_1, \frac{1}{\sqrt{2}}\beta_1, \frac{1}{4}\right]$ $\left[r_1, \frac{1+i}{\sqrt{2}}\beta_1, \frac{1}{2}\right]$ $[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, i\beta_j, 0]$ $(j=2, \cdots, n)$
	$[r_1^2, 0, 0]$ $\left[r_1, \frac{1}{\sqrt{2}}\beta_1, \frac{1}{4}\right]$ $\left[r_1^2, \frac{1}{\sqrt{2}}\beta_1, 0\right]$ $[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, i\beta_j, 0]$ $(j=2, \cdots, n)$
$[E, 0, \boldsymbol{Z}]$	$[r_1, 0, 0]$ $\left[r_1^2, \frac{1}{\sqrt{2}}\beta_1, 0\right]$ $[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, i\beta_j, 0]$ $(j=2, \cdots, n)$
	$\left[r_1, \frac{1}{\sqrt{2}}\beta_1, \frac{1}{4}\right]$ $\left[r_1^2, \frac{1}{\sqrt{2}}\beta_1, 0\right]$ $[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, i\beta_j, 0]$ $(j=2, \cdots, n)$

Lattice	Crystallographic Group
$L_2=(\boldsymbol{Z}+i\boldsymbol{Z})\frac{1}{\sqrt{2}}\beta_1+\sum_{j=2}^{n}(\boldsymbol{Z}+i\boldsymbol{Z})\frac{1+i}{2}\beta_j$	$G(4, 1, n)\ltimes L_2$
Center	**Generators of Parabolic Reflection Group**
$\left[E, 0, \frac{1}{4}\boldsymbol{Z}\right]$	$[r_1, 0, 0]$ $\left[r_1, \frac{1}{\sqrt{2}}\beta_1, \frac{1}{4}\right]$ $\left[r_1^2, \frac{1}{\sqrt{2}}\beta_1, 0\right]$ $[r_j, 0, 0]$ $\left[r_j, \frac{1+i}{2}\beta_j, 0\right]$ $\left[r_j, \frac{-1+i}{2}\beta_j, 0\right]$ $(j=2, \cdots, n)$
$\left[E, 0, \frac{1}{2}\boldsymbol{Z}\right]$	$[r_1, 0, 0]$ $\left[r_1, \frac{1}{\sqrt{2}}\beta_1, \frac{1}{4}\right]$ $[r_j, 0, 0]$ $\left[r_j, \frac{1+i}{2}\beta_j, 0\right]$ $\left[r_j, \frac{-1+i}{2}\beta_j, 0\right]$ $(j=2, \cdots, n)$

dim	Point Group	Graph	Roots
$n+1$	$G(6, 1, n)$ $(n \geqq 2)$		$\beta_1=e_1$ $\beta_j=\frac{e_{j-1}-e_j}{\sqrt{2}}$ $(j=2, \cdots, n)$

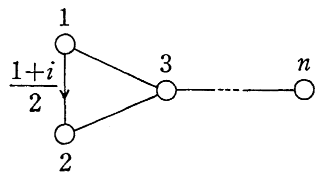
Lattice	Crystallographic Group
$L=(\boldsymbol{Z}+\omega\boldsymbol{Z})\frac{1}{\sqrt{2}}\beta_1+\sum_{j=2}^{n}(\boldsymbol{Z}+\omega\boldsymbol{Z})\beta_j$	$G(6, 1, n)\ltimes L$

Center	Generators of Parabolic Reflection Group
$\left[E, 0, \frac{1}{2\sqrt{3}}\boldsymbol{Z}\right]$	$[r_1, 0, 0]$ $\left[r_1^2, \frac{1}{\sqrt{2}}\beta_1, \frac{1}{4\sqrt{3}}\right]$ $\left[r_1^3, \frac{1}{\sqrt{2}}\beta_1, 0\right]$ $[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, \omega\beta_j, 0]$ $(j=2, \cdots, n)$
$\left[E, 0, \frac{\sqrt{3}}{2}\boldsymbol{Z}\right]$	$[r_1, 0, 0]$ $\left[r_1, \frac{1}{\sqrt{2}}\beta_1, \frac{\sqrt{3}}{4}\right]$ $\left[r_1^3, \frac{1}{\sqrt{2}}\beta_1, 0\right]$ $[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, \omega\beta_j, 0]$ $(j=2, \cdots, n)$

dim	Point Group	Graph	Roots
$n+1$	$G(3, 3, n)$ $(n \geqq 3)$		$\beta_1=\frac{\omega e_1-e_2}{\sqrt{2}}$ $\beta_j=\frac{e_{j-1}-e_j}{\sqrt{2}}$ $(j=2, \cdots, n)$

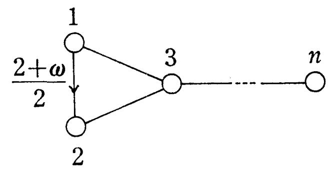
Lattice	Crystallographic Group
$L=\sum_{j=1}^{n}(\boldsymbol{Z}+\omega\boldsymbol{Z})\beta_j$	$G(3, 3, n)\ltimes L$

Center	Generators of Parabolic Reflection Group
$\left[E, 0, \frac{\sqrt{3}}{2}\boldsymbol{Z}\right]$	$[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, \omega\beta_j, 0]$ $(j=1, \cdots, n)$

dim	Point Group	Graph	Roots
$n+1$	$G(4, 4, n)$ $(n \geqq 3)$		$\beta_1=\frac{ie_1-e_2}{\sqrt{2}}$ $\beta_j=\frac{e_{j-1}-e_j}{\sqrt{2}}$ $(j=2, \cdots, n)$

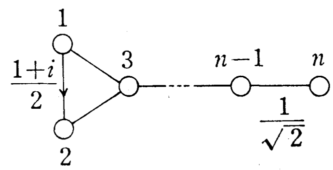
Lattice	Crystallographic Group
$L=\sum_{j=1}^{n}(\boldsymbol{Z}+i\boldsymbol{Z})\beta_j$	$G(4, 4, n)\ltimes L$

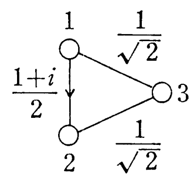
Center	Generators of Parabolic Reflection Group
$[E, 0, \boldsymbol{Z}]$	$[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, i\beta_j, 0]$ $(j=1, \cdots, n)$

dim	Point Group	Graph	Roots
$n+1$	$G(6, 6, n)$ $(n\geqq 3)$		$\beta_1=\frac{(1+\omega)e_1-e_2}{\sqrt{2}}$ $\beta_j=\frac{e_{j-1}-e_j}{\sqrt{2}}$ $(j=2, \cdots, n)$

Lattice	Crystallographic Group
$L=\sum_{j=1}^{n}(\boldsymbol{Z}+\omega\boldsymbol{Z})\beta_j$	$G(6, 6, n)\ltimes L$

Center	Generators of Parabolic Reflection Group
$\left[E, 0, \frac{\sqrt{3}}{2}\boldsymbol{Z}\right]$	$[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, \omega\beta_j, 0]$ $(j=1, \cdots, n)$

dim	Point Group	Graph	Roots
n	$G(4, 2, n-1)$ $(n>3)$		$\beta_1=\frac{ie_1-e_2}{\sqrt{2}}$ $\beta_j=\frac{e_{j-1}-e_j}{\sqrt{2}}$ $(j=2, \cdots, n-1)$ $\beta_n=-e_{n-1}$

dim	Point Group	Graph	Roots
3	$G(4, 2, 2)$		$\beta_1=\frac{ie_1-e_2}{\sqrt{2}}$ $\beta_2=\frac{e_1-e_2}{\sqrt{2}}$ $\beta_3=-e_2$

Lattice	Crystallographic Group
$L_1=\sum_{j=1}^{n-1}(\boldsymbol{Z}+i\boldsymbol{Z})\beta_j$	$G(4, 2, n-1)\ltimes L_1$

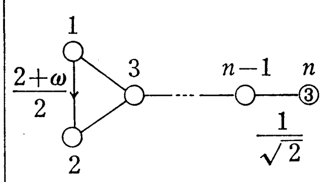
Center	Generators of Parabolic Reflection Group
$\left[E, 0, \frac{1}{2}\boldsymbol{Z}\right]$	$[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, i\beta_j, 0]$ $(j=1, \cdots, n-1)$ $[r_n, 0, 0]$ $\left[r_n, \frac{1+i}{\sqrt{2}}\beta_n, 0\right]$
$[E, 0, \boldsymbol{Z}]$	$[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, i\beta_j, 0]$ $(j=1, \cdots, n-1)$ $[r_n, 0, 0]$
	$[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, i\beta_j, 0]$ $(j=1, \cdots, n-1)$ $\left[r_n, \frac{1+i}{\sqrt{2}}\beta_n, 0\right]$

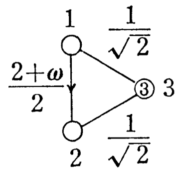
Lattice	Crystallographic Group
$L_2=\sum_{j=1}^{n-1}(\boldsymbol{Z}+i\boldsymbol{Z})\beta_j+(\boldsymbol{Z}+i\boldsymbol{Z})\frac{1}{\sqrt{2}}\beta_n$	$G(4, 2, n-1)\ltimes L_2$

Center	Generators of Parabolic Reflection Group
$\left[E, 0, \frac{1}{2}\boldsymbol{Z}\right]$	$[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, i\beta_j, 0]$ $(j=1, \cdots, n-1)$ $[r_n, 0, 0]$ $\left[r_n, \frac{1+i}{\sqrt{2}}\beta_n, 0\right]$ $\left[r_n, \frac{1}{\sqrt{2}}\beta_n, 0\right]$ $\left[r_n, \frac{i}{\sqrt{2}}\beta_n, 0\right]$
$[E, 0, \boldsymbol{Z}]$	$[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, i\beta_j, 0]$ $(j=1, \cdots, n-1)$ $[r_n, 0, 0]$ $\left[r_n, \frac{1}{\sqrt{2}}\beta_n, 0\right]$ $\left[r_n, \frac{i}{\sqrt{2}}\beta_n, 0\right]$
	$[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, i\beta_j, 0]$ $(j=1, \cdots, n-1)$ $\left[r_n, \frac{1}{\sqrt{2}}\beta_n, 0\right]$ $\left[r_n, \frac{i}{\sqrt{2}}\beta_n, 0\right]$ $\left[r_n, \frac{1+i}{\sqrt{2}}\beta_n, 0\right]$

Lattice	Crystallographic Group
$L_3=(\boldsymbol{Z}+i\boldsymbol{Z})\beta_1+(\boldsymbol{Z}+i\boldsymbol{Z})(1+i)\beta_2$	$G(4, 2, 2)\ltimes L_3$

Center	Generators of Parabolic Reflection Group
$[E, 0, \boldsymbol{Z}]$	$[r_1, 0, 0]$ $[r_1, \beta_1, 0]$ $[r_1, i\beta_1, 0]$ $[r_2, 0, 0]$ $[r_2, (1+i)\beta_2, 0]$ $[r_2, (-1+i)\beta_2, 0]$ $[r_3, 0, 0]$ $[r_3, \sqrt{2}\beta_3, 0]$ $[r_3, \sqrt{2}i\beta_3, 0]$

dim	Point Group	Graph	Roots
n	$G(6, 2, n-1)$ $(n>3)$		$\beta_1=\frac{(1+\omega)e_1-e_2}{\sqrt{2}}$ $\beta_j=\frac{e_{j-1}-e_j}{\sqrt{2}}$ $(j=2, \cdots, n-1)$ $\beta_n=-e_{n-1}$

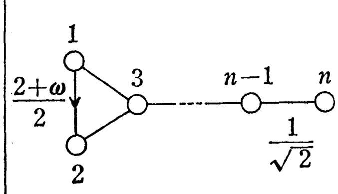
dim	Point Group	Graph	Roots
3	$G(6, 2, 2)$		$\beta_1=\frac{(1+\omega)e_1-e_2}{\sqrt{2}}$ $\beta_2=\frac{e_1-e_2}{\sqrt{2}}$ $\beta_3=-e_2$

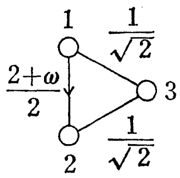
Lattice	Crystallographic Group
$L_1=\sum_{j=1}^{n-1}(\boldsymbol{Z}+\omega\boldsymbol{Z})\beta_j$	$G(6, 2, n-1)\ltimes L_1$

Center	Generators of Parabolic Reflection Group
$\left[E, 0, \frac{1}{2\sqrt{3}}\boldsymbol{Z}\right]$	$[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, \omega\beta_j, 0]$ $(j=1, \cdots, n-1)$ $[r_n, 0, 0]$ $\left[r_n, \frac{1}{\sqrt{2}}\beta_n, \frac{1}{4\sqrt{3}}\right]$
$\left[E, 0, \frac{\sqrt{3}}{2}\boldsymbol{Z}\right]$	$[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, \omega\beta_j, 0]$ $(j=1, \cdots, n-1)$ $[r_n, 0, 0]$
	$[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, \omega\beta_j, 0]$ $(j=1, \cdots, n-1)$ $\left[r_n, \frac{1}{\sqrt{2}}\beta_n, \frac{1}{4\sqrt{3}}\right]$

Lattice	Crystallographic Group
$L_2=(\boldsymbol{Z}+\omega\boldsymbol{Z})\beta_1+(\boldsymbol{Z}+\omega\boldsymbol{Z})(2+\omega)\beta_2$	$G(6, 2, 2)\ltimes L_2$

Center	Generators of Parabolic Reflection Group
$\left[E, 0, \frac{\sqrt{3}}{2}\boldsymbol{Z}\right]$	$[r_1, 0, 0]$ $[r_1, \beta_1, 0]$ $[r_1, \omega\beta_1, 0]$ $[r_2, 0, 0]$ $[r_2, (2+\omega)\beta_2, 0]$ $[r_2, (-1+\omega)\beta_2, 0]$ $[r_3, 0, 0]$ $\left[r_3, \frac{2+\omega}{\sqrt{2}}\beta_3, \frac{\sqrt{3}}{4}\right]$

dim	Point Group	Graph	Roots
n	$G(6, 3, n-1)$ $(n>3)$		$\beta_1=\frac{(1+\omega)e_1-e_2}{\sqrt{2}}$ $\beta_j=\frac{e_{j-1}-e_j}{\sqrt{2}}$ $(j=2, \cdots, n-1)$ $\beta_n=-e_{n-1}$

dim	Point Group	Graph	Roots
3	$G(6, 3, 2)$		$\beta_1=\frac{(1+\omega)e_1-e_2}{\sqrt{2}}$ $\beta_2=\frac{e_1-e_2}{\sqrt{2}}$ $\beta_3=-e_2$

Lattice	Crystallographic Group
$L_1=\sum_{j=1}^{n-1}(\boldsymbol{Z}+\omega\boldsymbol{Z})\beta_j$	$G(6, 3, n-1)\ltimes L_1$

Center	Generators of Parabolic Reflection Group
$\left[E, 0, \frac{\sqrt{3}}{4}\boldsymbol{Z}\right]$	$[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, \omega\beta_j, 0]$ $(j=1, \cdots, n-1)$ $[r_n, 0, 0]$ $\left[r_n, \frac{1}{\sqrt{2}}\beta_n, 0\right]$
$\left[E, 0, \frac{\sqrt{3}}{2}\boldsymbol{Z}\right]$	$[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, \omega\beta_j, 0]$ $(j=1, \cdots, n-1)$ $[r_n, 0, 0]$
	$[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, \omega\beta_j, 0]$ $(j=1, \cdots, n-1)$ $\left[r_n, \frac{1}{\sqrt{2}}\beta_n, 0\right]$

Lattice	Crystallographic Group
$L_2=(\boldsymbol{Z}+2\omega\boldsymbol{Z})\beta_1+(2\boldsymbol{Z}+\omega\boldsymbol{Z})\beta_2$	$G(6, 3, 2)\ltimes L_2$

Center	Generators of Parabolic Reflection Group
$[E, 0, \sqrt{3}\boldsymbol{Z}]$	$[r_1, 0, 0]$ $[r_1, \beta_1, 0]$ $[r_1, 2\omega\beta_1, 0]$ $[r_2, 0, 0]$ $[r_2, 2\beta_2, 0]$ $[r_2, \omega\beta_2, 0]$ $[r_3, 0, 0]$ $[r_3, \sqrt{2}\beta_3, 0]$ $[r_3, \sqrt{2}\omega\beta_3, 0]$

dim	Point Group	Graph	Roots
3	K4	$\overset{1}{\textcircled{3}} \xrightarrow[\frac{2+\omega}{3}]{} \overset{2}{\textcircled{3}}$	$\beta_1=e_1$ $\beta_2=\frac{1-\omega}{3}(e_1+e_2+e_3)$

Lattice	Crystallographic Group
$L=(\boldsymbol{Z}+\omega\boldsymbol{Z})\beta_1+(\boldsymbol{Z}+\omega\boldsymbol{Z})\beta_2$	$\mathrm{K}4\ltimes L$

Center	Generators of Parabolic Reflection Group
$\left[E, 0, \frac{1}{\sqrt{3}}\boldsymbol{Z}\right]$	$[r_1, 0, 0]$ $\left[r_1, \beta_1, \frac{1}{2\sqrt{3}}\right]$ $[r_2, 0, 0]$ $\left[r_2, \beta_2, \frac{1}{2\sqrt{3}}\right]$

dim	Point Group	Graph	Roots
3	K5	$\overset{1}{\textcircled{3}} \xrightarrow[\frac{\sqrt{2}(2+\omega)}{3}]{} \overset{2}{\textcircled{3}}$	$\beta_1=e_1$ $\beta_2=\frac{1-\omega}{3}(\sqrt{2}e_1+e_2)$

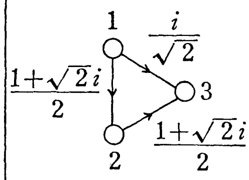
Lattice	Crystallographic Group
$L=(\boldsymbol{Z}+\omega\boldsymbol{Z})\beta_1+(\boldsymbol{Z}+\omega\boldsymbol{Z})\sqrt{2}\beta_2$	$\mathrm{K}4\ltimes L$

Center	Generators of Parabolic Reflection Group
$\left[E, 0, \frac{1}{\sqrt{3}}\boldsymbol{Z}\right]$	$[r_1, 0, 0]$ $\left[r_1, \beta_1, \frac{1}{2\sqrt{3}}\right]$ $[r_2, 0, 0]$ $\left[r_2, \sqrt{2}\beta_2, \frac{1}{\sqrt{3}}\right]$

dim	Point Group	Graph	Roots
3	K8	$\overset{1}{\textcircled{4}} \xrightarrow[\frac{1+i}{2}]{} \overset{2}{\textcircled{4}}$	$\beta_1=e_1$ $\beta_2=\frac{1-i}{2}(e_1-e_2)$

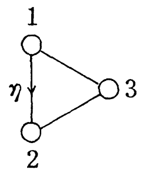
Lattice	Crystallographic Group
$L=(\boldsymbol{Z}+i\boldsymbol{Z})\beta_1+(\boldsymbol{Z}+i\boldsymbol{Z})\beta_2$	$\mathrm{K}8\ltimes L$

Center	Generators of Parabolic Reflection Group
$\left[E, 0, \frac{1}{2}\boldsymbol{Z}\right]$	$[r_1, 0, 0]$ $\left[r_1, \beta_1, \frac{1}{2}\right]$ $[r_1^2, \beta_1, 0]$ $[r_2, 0, 0]$ $\left[r_2, \beta_2, \frac{1}{2}\right]$ $[r_2^2, \beta_2, 0]$
$[E, 0, \boldsymbol{Z}]$	$[r_1, 0, 0]$ $\left[r_1, \beta_1, \frac{1}{2}\right]$ $[r_2, 0, 0]$ $\left[r_2, \beta_2, \frac{1}{2}\right]$

dim	Point Group	Graph	Roots
3	K12		$\beta_1=\frac{1}{\sqrt{2}}e_1+\frac{1+i}{2}e_2$ $\beta_2=\frac{\sqrt{2}+(\sqrt{2}-2)i}{4}e_1+\frac{2+\sqrt{2}-\sqrt{2}i}{4}e_2$ $\beta_3=\frac{1-i}{2}e_1-\frac{i}{\sqrt{2}}e_2$

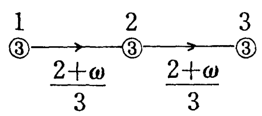
Lattice	Crystallographic Group
$L=(\boldsymbol{Z}+\sqrt{2}i\boldsymbol{Z})\beta_1+(\boldsymbol{Z}+\sqrt{2}i\boldsymbol{Z})\beta_2$	$\mathrm{K}12\ltimes L$

Center	Generators of Parabolic Reflection Group
$\left[E, 0, \frac{1}{2}\boldsymbol{Z}\right]$	$[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, \sqrt{2}i\beta_j, 0]$ $(j=1, 2, 3)$

dim	Point Group	Graph	Roots
4	K24 $\left(\eta=\frac{1+\sqrt{7}i}{2}\right)$		$\beta_1=e_2$ $\beta_2=\frac{1-\eta}{2}(e_2+e_3)$ $\beta_3=-\frac{1}{2}(e_1+e_2-\eta e_3)$

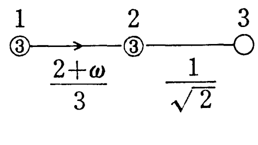
Lattice	Crystallographic Group
$L=\sum_{j=1}^{3}(\boldsymbol{Z}+\eta\boldsymbol{Z})\beta_j$	$\mathrm{K}24\ltimes L$

Center	Generators of Parabolic Reflection Group
$\left[E, 0, \frac{\sqrt{7}}{2}\boldsymbol{Z}\right]$	$[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, \eta\beta_j, 0]$ $(j=1, 2, 3)$

dim	Point Group	Graph	Roots
4	K25		$\beta_1=e_1$ $\beta_2=\frac{1-\omega}{3}(e_1+e_2+e_3)$ $\beta_3=-\omega e_2$

Lattice	Crystallographic Group
$L=\sum_{j=1}^{3}(\boldsymbol{Z}+\omega\boldsymbol{Z})\beta_j$	$\mathrm{K}25\ltimes L$

Center	Generators of Parabolic Reflection Group
$\left[E, 0, \frac{1}{\sqrt{3}}\mathbf{Z}\right]$	$[r_j, 0, 0]$ $\left[r_j, \beta_j, \frac{1}{2\sqrt{3}}\right]$ $(j=1, 2, 3)$

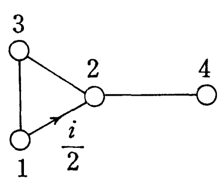
dim	Point Group	Graph	Roots
4	K26		$\beta_1=\frac{2+\omega}{3}(e_1+e_2+e_3)$ $\beta_2=e_2$ $\beta_3=\frac{e_2-e_3}{\sqrt{2}}$

Lattice	Crystallographic Group
$L_1=\sum_{j=1}^{2}(\mathbf{Z}+\omega\mathbf{Z})\frac{1}{\sqrt{2}}\beta_j+(\mathbf{Z}+\omega\mathbf{Z})\beta_3$	$\mathrm{K26}\ltimes L_1$

Center	Generators of Parabolic Reflection Group
$\left[E, 0, \frac{1}{2\sqrt{3}}\mathbf{Z}\right]$	$[r_j, 0, 0]$ $\left[r_j, \frac{1}{\sqrt{2}}\beta_j, \frac{1}{4\sqrt{3}}\right]$ $(j=1, 2)$ $[r_3, 0, 0]$ $[r_3, \beta_3, 0]$ $[r_3, \omega\beta_3, 0]$

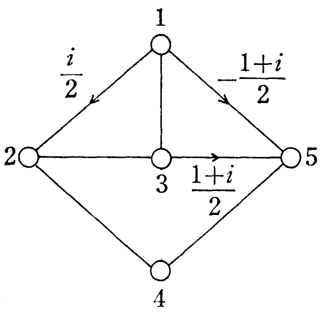
Lattice	Crystallographic Group
$L_2=\sum_{j=1}^{2}(\mathbf{Z}+\omega\mathbf{Z})\frac{1}{\sqrt{2}}\beta_j+(\mathbf{Z}+\omega\mathbf{Z})\frac{2+\omega}{3}\beta_3$	$\mathrm{K26}\ltimes L_2$

Center	Generators of Parabolic Reflection Group
$\left[E, 0, \frac{1}{2\sqrt{3}}\mathbf{Z}\right]$	$[r_j, 0, 0]$ $\left[r_j, \frac{1}{\sqrt{2}}\beta_j, \frac{1}{4\sqrt{3}}\right]$ $(j=1, 2)$ $[r_3, 0, 0]$ $\left[r_3, \frac{2+\omega}{3}\beta_3, 0\right]$ $\left[r_3, \frac{-1+\omega}{3}\beta_3, 0\right]$

dim	Point Group	Graph	Roots
5	K29		$\beta_1=\frac{e_2-e_4}{\sqrt{2}}$ $\beta_2=\frac{-ie_2-e_3}{\sqrt{2}}$ $\beta_3=\frac{-e_3+e_4}{\sqrt{2}}$ $\beta_4=\frac{-1+i}{2\sqrt{2}}\sum_{j=1}^{4}e_j$

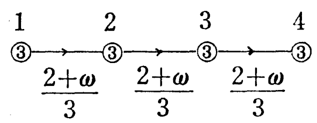
Lattice	Crystallographic Group
$L=\sum_{j=1}^{4}(\boldsymbol{Z}+i\boldsymbol{Z})\beta_j$	$\mathrm{K29}\ltimes L$

Center	Generators of Parabolic Reflection Group
$[E, 0, \boldsymbol{Z}]$	$[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, i\beta_j, 0]$ $(j=1, 2, 3, 4)$

dim	Point Group	Graph	Roots
5	K31		$\beta_1=\frac{e_2-e_4}{\sqrt{2}}$ $\beta_2=\frac{-ie_2-e_3}{\sqrt{2}}$ $\beta_3=\frac{-e_3+e_4}{\sqrt{2}}$ $\beta_4=\frac{-1+i}{2\sqrt{2}}\sum_{j=1}^{4}e_j$ $\beta_5=\frac{1-i}{\sqrt{2}}e_4$

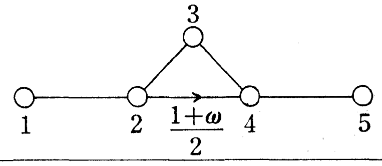
Lattice	Crystallographic Group
$L=\sum_{j=1}^{4}(\boldsymbol{Z}+i\boldsymbol{Z})\beta_j$	$\mathrm{K31}\ltimes L$

Center	Generators of Parabolic Reflection Group
$\left[E, 0, \frac{1}{2}\boldsymbol{Z}\right]$	$[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, i\beta_j, 0]$ $(j=1, 2, 3, 4, 5)$

dim	Point Group	Graph	Roots
5	K32		$\beta_1=e_3$ $\beta_2=\frac{1-\omega}{3}(e_1+e_2+e_3)$ $\beta_3=-\omega e_2$ $\beta_4=\frac{1+2\omega}{3}(e_1-e_2-e_4)$

Lattice	Crystallographic Group
$L=\sum_{j=1}^{4}(\boldsymbol{Z}+\omega\boldsymbol{Z})\beta_j$	$\mathrm{K32}\ltimes L$

Center	Generators of Parabolic Reflection Group
$\left[E, 0, \frac{1}{\sqrt{3}}\boldsymbol{Z}\right]$	$[r_j, 0, 0]$ $\left[r_j, \beta_j, \frac{1}{2\sqrt{3}}\right]$ $(j=1, 2, 3, 4)$

dim	Point Group	Graph
6	K33	

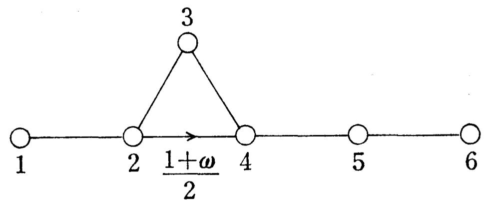
Roots

$$\beta_1=\frac{\omega}{\sqrt{2}}(e_5+e_6) \qquad \beta_k=\frac{e_{k-2}-e_{k-1}}{\sqrt{2}} \quad (k=3, 4, 5)$$

$$\beta_2=\frac{\omega}{2\sqrt{2}}(e_1-(1+2\omega)e_2-e_3-e_4-e_5-e_6)$$

Lattice	Crystallographic Group
$L=\sum_{j=1}^{5}(\boldsymbol{Z}+\omega\boldsymbol{Z})\beta_j$	$\mathrm{K33}\ltimes L$

Center	Generators of Parabolic Reflection Group
$\left[E, 0, \frac{\sqrt{3}}{2}\boldsymbol{Z}\right]$	$[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, \omega\beta_j, 0]$ $(j=1, 2, 3, 4, 5)$

dim	Point Group	Graph
7	K34	

Roots

$$\beta_1=\frac{\omega}{\sqrt{2}}(e_5+e_6)$$

$$\beta_2=\frac{\omega}{2\sqrt{2}}(e_1-(1+2\omega)e_2-e_3-e_4-e_5-e_6)$$

$$\beta_k=\frac{e_{k-2}-e_{k-1}}{\sqrt{2}} \qquad (k=3, 4, 5)$$

$$\beta_6=-\frac{1+\omega}{2\sqrt{2}}(e_1+e_2+e_3+(1+2\omega)e_4+e_5-e_6)$$

Lattice	Crystallographic Group
$L=\sum_{j=1}^{6}(\boldsymbol{Z}+\omega\boldsymbol{Z})\beta_j$	$\mathrm{K34}\ltimes L$

Center	Generators of Parabolic Reflection Group
$\left[E, 0, \frac{\sqrt{3}}{2}\boldsymbol{Z}\right]$	$[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, \omega\beta_j, 0]$ $(j=1, 2, 3, 4, 5, 6)$

dim	Point Group	Lattice
n	$G(4, 2, n-1)$ $(n\geqq 3)$	$L_1=\sum_{j=1}^{n-1}(\boldsymbol{Z}+i\boldsymbol{Z})\beta_j$

Crystallographic Group	Generators of Crystallographic Group
$[G(4, 2, n-1), L_1]^*$	$(r_j \mid 0)$ $(r_j \mid \beta_j)$ $(r_j \mid i\beta_j)$ $(j=1, \cdots, n-1)$ $\left(r_n \middle\vert \frac{1}{\sqrt{2}}\beta_n\right)$

Center	Generators of Parabolic Reflection Group
$[E, 0, \boldsymbol{Z}]$	$[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, i\beta_j, 0]$ $(j=1, \cdots, n-1)$ $\left[r_n, \frac{1}{\sqrt{2}}\beta_n, 0\right]$ $\left[r_n, \frac{i}{\sqrt{2}}\beta_n, 0\right]$ $\left[r_n, -\frac{1}{\sqrt{2}}\beta_n, 0\right]$

dim	Point Group	Lattice
3	K12	$L=(\boldsymbol{Z}+\sqrt{2}i\boldsymbol{Z})\beta_1+(\boldsymbol{Z}+\sqrt{2}i\boldsymbol{Z})\beta_2$

Crystallographic Group	Generators of Crystallographic Group
$[\mathrm{K12}, L]^*$	$(r_1 \mid 0)$ $(r_1 \mid \beta_1)$ $(r_1 \mid \sqrt{2}i\beta_1)$ $(r_2 \mid 0)$ $(r_2 \mid \beta_2)$ $(r_2 \mid \sqrt{2}i\beta_2)$ $\left(r_3 \middle\vert \frac{i}{\sqrt{2}}\beta_3\right)$

Center	Generators of Parabolic Reflection Group
$\left[E, 0, \frac{1}{2}\boldsymbol{Z}\right]$	$[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, \sqrt{2}i\beta_j, 0]$ $(j=1, 2)$ $\left[r_3, \frac{i}{\sqrt{2}}\beta_3, 0\right]$ $\left[r_3, \frac{\sqrt{2}+i}{\sqrt{2}}\beta_3, 0\right]$ $\left[r_3, \frac{3i}{\sqrt{2}}\beta_3, 0\right]$

dim	Point Group	Lattice
5	K31	$L=\sum_{j=1}^{4}(\boldsymbol{Z}+i\boldsymbol{Z})\beta_j$

Crystallographic Group	Generators of Crystallographic Group
$[\mathrm{K31}, L]^*$	$(r_j \mid 0)$ $(r_j \mid \beta_j)$ $(r_j \mid i\beta_j)$ $(j=1, 2, 3, 4)$ $\left(r_5 \middle\vert \frac{1+i}{\sqrt{2}}\beta_5\right)$

Center	Generators of Parabolic Reflection Group
$\left[E, 0, \frac{1}{2}\boldsymbol{Z}\right]$	$[r_j, 0, 0]$ $[r_j, \beta_j, 0]$ $[r_j, i\beta_j, 0]$ $(j=1, 2, 3, 4)$ $\left[r_5, \frac{1+i}{2}\beta_5, 0\right]$ $\left[r_5, \frac{-1+i}{2}\beta_5, 0\right]$ $\left[r_5, \frac{1-i}{2}\beta_5, 0\right]$

2.3. Corollaries. The corollaries below follow immediately from the classification of parabolic reflection groups.

COROLLARY 1. *For any crystallographic reflection group Γ_*, there exists a maximal parabolic reflection group Γ_0 such that $\pi_*(\Gamma_0)=\Gamma_*$, that is, if $\pi_*(\Gamma)=\Gamma_*$ for a parabolic reflection group Γ, then Γ is a subgroup of Γ_0. Moreover, if Γ_* is the semi-direct product of its point group W and the lattice, then Γ_0 has the structure $W \ltimes L(\Gamma_0)$.*

COROLLARY 2. *Let W and W' be the point groups of parabolic reflection groups Γ and Γ', respectively. If $W \subset W'$ then there exists $g \in \mathrm{Aut}(D)$ such that $g^{-1}\Gamma g \subset \Gamma'$.*

REMARK. There are following relations of inclusion among finite unitary reflection groups;

$$\begin{tikzcd}[column sep=small]
 & & G(6,1,n) \arrow[dl, dash, "2"'] \arrow[dr, dash, "3"] & \\
 & G(6,2,n) \arrow[dl, dash, "2^{n-1}"'] \arrow[dr, dash, "3"] & & G(6,3,n) \arrow[dl, dash, "2"'] \\
G(3,1,n) \arrow[dr, dash, "3"'] & & G(6,6,n) \arrow[dl, dash, "2^{n-1}"] & \\
 & G(3,3,n) & &
\end{tikzcd}
\qquad
\begin{tikzcd}
G(4,1,n) \arrow[d, dash, "2"] \\
G(4,2,n) \arrow[d, dash, "2"] \\
G(4,4,n)
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
K32 \arrow[d, dash, "240"] & & \\
K25 \arrow[dr, dash, "27"'] & & K26 \arrow[dl, dash, "54"] \\
 & K4 &
\end{tikzcd}
\qquad
\begin{tikzcd}
K31 \arrow[d, dash, "6"] \\
K29
\end{tikzcd}
\qquad
\begin{tikzcd}
K34 \arrow[d, dash, "756"] \\
K33
\end{tikzcd}$$

The diagram $\begin{array}{c} G \\ |\,k \\ H \end{array}$ means $G \supset H$, $[G:H]=k$.

§ 3. Proof of the theorem.

Any discrete subgroup Γ of G of locally finite covolume admits the exact sequence:

$$1 \longrightarrow Z(\Gamma) \longrightarrow \Gamma \longrightarrow \pi_*(\Gamma) \longrightarrow 1 .$$

If Γ is generated by reflections $\pi_*(\Gamma)$ is a crystallographic reflection group. Conversely, starting from crystallographic reflection groups, we shall construct parabolic reflection groups as extensions of the given crystallographic groups.

3.1. Structure of the center. In this section, we study the center of Γ, and give a necessary condition that Γ is generated by reflections.

PROPOSITION. *Let Γ_* be a crystallographic reflection group and Γ be a parabolic reflection group such that $\pi_*(\Gamma)=\Gamma_*$. There exists the smallest natural number $\tilde{\nu}$ which is determined by Γ_* such that the center of Γ is generated by $[E, 0, q_0/n]$, where q_0 is the number defined in § 1.2 and n is a divisor of $\tilde{\nu}$.*

PROOF. We assume first that Γ_* is the semi-direct product of the point group and the lattice. Let $g=[w, \beta, \gamma]$ be a reflection of order 2, 3, 4 or 6, then by § 1.3 Lemma, the central part γ is equal to 0, $\pm 1/(2\sqrt{3})^t\bar{\beta}\beta$, $\pm 1/2^t\bar{\beta}\beta$ or $\pm\sqrt{3}/2^t\bar{\beta}\beta$, respectively. Hence if Γ is generated by reflections of order 2, then for all $g=[w, \beta, \gamma]\in\Gamma$, the central part γ belongs to $(q_0/2)\boldsymbol{Z}$. Following the same idea for remaining cases, we have the assertion.

Next we study the case that Γ_* is not the semi-direct product of the point group W and the lattice L. Let us consider the diagram in § 2.1 for two parabolic reflection groups Γ and Γ' such that $\pi_*(\Gamma)=\pi_*(\Gamma')=\Gamma_*$. If $W=W'$ and $L=L'$ (these conditions do not necessarily imply $\pi_*(\Gamma)=\pi_*(\Gamma')$), then we have $L(\Gamma)=L(\Gamma')$. Furthermore the center of $L(\Gamma)$ is equal to the center of Γ. Hence by changing the crystallographic group Γ_* into the semi-direct product of W and L and applying the argument of the former case, we get the assertion. □

3.2. Method of constructing parabolic reflection groups. We fix a crystallographic reflection group Γ_* and a divisor ν of $\tilde{\nu}$, where $\tilde{\nu}$ is the number determined by Γ_* in § 3.1. In this section, we study the exact sequence:

$$1 \longrightarrow Z(\Gamma) \longrightarrow \Gamma \longrightarrow \pi_*(\Gamma) = \Gamma_* \longrightarrow 1$$

more closely. Let us assume that the center $Z(\Gamma)$ is

$$Z(\Gamma) = \{[E, 0, \gamma];\ \ \gamma\in\nu\tilde{q}\boldsymbol{Z}\},$$

where $\tilde{q}=q_0/\tilde{\nu}$ (for the definition of q_0, see § 1.2). In view of the coset decomposition of Γ by $Z(\Gamma)$:

$$\Gamma=\bigcup_{(w\mid\beta)\in\Gamma_*}\{[w,\beta,\gamma(w,\beta)+\gamma'];\ \ \gamma'\in\nu\tilde{q}\boldsymbol{Z}\},$$

where $\gamma(w,\beta)\in\tilde{q}\boldsymbol{Z}$, the reflection group Γ can be regarded as a subgroup of the group $\tilde{\Gamma}$:

$$\tilde{\Gamma}=\{[w,\beta,\gamma];\ \ (w\mid\beta)\in\Gamma_*,\ \gamma\in\tilde{q}\boldsymbol{Z}\}.$$

We want to define a map $s:\Gamma_*\to\boldsymbol{Z}$ which satisfies the following condition $(*)$.

$(*)$ *The center of the group generated by the set* $G(s):=\{[w,\beta,s(w\mid\beta)\tilde{q}+\gamma];\ \ (w\mid\beta)\in\Gamma_*,\ \gamma\in\nu\tilde{q}\boldsymbol{Z}\}$ is $\{[E,0,\gamma]:\ \gamma\in\nu\tilde{q}\boldsymbol{Z}\}$.

A map s satisfies the condition $(*)$ if and only if $G(s)$ itself forms a group. If we can define such a map s and if the group $G(s)$ is generated by reflections, then we obtain a parabolic reflection group Γ by putting $\Gamma=G(s)=\{[w,\beta,s(w\mid\beta)\tilde{q}+\gamma];\ (w\mid\beta)\in\Gamma_*,\ \gamma\in\nu\tilde{q}\boldsymbol{Z}\}$. Note that we can modify s without changing $G(s)$ so that the image of $s\subset\{0,1,\cdots,\nu-1\}$, namely we can regard s as a map from Γ_* to $\boldsymbol{Z}_\nu=\boldsymbol{Z}/\nu\boldsymbol{Z}$, and we can determine s by the values of the generators of Γ_*. We construct Γ for a crystallographic group Γ_* and an abelian group $\{[E,0,\gamma];\ \gamma\in\nu\tilde{q}\boldsymbol{Z}\}$ by searching values of the map s such that $G(s)$ is generated by reflections.

3.3. Some lemmas. In this section we prove some lemmas preparing for the classification of the parabolic reflection groups Γ.

LEMMA 1. *Let V be a $2m$-dimensional vector space over $\boldsymbol{R}$, and $e_1,\cdots,e_m$, $e_1',\cdots,e_m'$ be a set of bases of V. We denote by L the lattice $\boldsymbol{Z}e_1+\cdots+\boldsymbol{Z}e_m+\boldsymbol{Z}e_1'+\cdots+\boldsymbol{Z}e_m'$ and let B be a nondegenerate alternating form such that $B(L,L)=q\boldsymbol{Z}$ $(q>0)$. We define a product on $N:=V\oplus\boldsymbol{R}$ as follows:*

$$(b,r)(b',r'):=\left(b+b',\ r+r'+\frac{1}{2}B(b,b')\right)$$

for $(b,r),(b',r')\in N$. Let $\tilde{L}$ denote the subgroup of N generated by $(e_1,0),\cdots,(e_m,0),(e_1',0),\cdots,(e_m',0)$, then $Z(\tilde{L})=\tilde{L}\cap\{(0,r);\ r\in\boldsymbol{R}\}=[\tilde{L},\tilde{L}]=\{(0,r);\ r\in q\boldsymbol{Z}\}$, where $Z(\tilde{L})$ is the center of $\tilde{L}$ and $[\tilde{L},\tilde{L}]$ is the commutator subgroup.

PROOF. Since $[(b,r),(b',r')]=(0,B(b,b'))\in Z(\tilde{L})$, we have $Z(\tilde{L})\supset[\tilde{L},\tilde{L}]=\{(0,r);\ r\in q\boldsymbol{Z}\}$. Let us consider an expression of an element $(0,r)$ of $Z(\tilde{L})$ by generators $(e_1,0),\cdots,(e_m,0),(e_1',0),\cdots,(e_m',0)$ and their inverses. Then the number of $(e_j,0)$ appearing in the expression is equal to that of $(e_j,0)^{-1}=(-e_j,0)$ for all j. The same assertion is also valid for $(e_j',0)$. Since $\tilde{L}/[\tilde{L},\tilde{L}]$

is a commutative group we have $r\equiv 0 \pmod{q\boldsymbol{Z}}$, i.e. $Z(\tilde{L})\subset[\tilde{L}, \tilde{L}]$. $\square$

The set of bases $\beta_1, \cdots, \beta_m$ of the lattice L in the table is called the root basis. For a root α, let w denote the reflection of the highest order which has α as its root. When the order of w is p, we call α the root of order p.

LEMMA 2. *Assume that the order of any reflection of the point group is* 2 *or* 3 *and the lattice is of type* $(\boldsymbol{Z}+i\boldsymbol{Z})$ *or* $(\boldsymbol{Z}+\omega\boldsymbol{Z})$. *Let* α *be a root of order* 2. *If* β_j *is a root base of order* 2 *then* $2(\alpha, \beta_j)$ *belongs to* $\boldsymbol{Z}[i]$ *or* $\boldsymbol{Z}[\omega]$ *according to the type of the lattice. If* β_j *is a root of order* 3 *then* $\sqrt{2}(\alpha, \beta_j)\in\boldsymbol{Z}[\omega]$.

PROOF. It is easy to see that the lemma holds if α is a root base. Let r_k denote the reflection of order 3 which has β_k as a root, and r_m denote the reflection of order 2 which has β_m as a root. Put $\alpha':=r_k(\alpha)=\alpha-(1-\omega)(\alpha, \beta_k)\beta_k$, and $\alpha'':=r_m(\alpha)=\alpha-2(\alpha, \beta_m)\beta_m$. It is sufficient to prove that if the lemma holds for α then it also holds for α' and α''. If β_j is a root base of order 2 then $(\beta_k, \beta_j)=\pm 1/\sqrt{2}$ or 0, i.e. $2(\beta_k, \beta_j)(\alpha, \beta_k)\in\boldsymbol{Z}[\omega]$ and $2(\beta_m, \beta_j)\in\boldsymbol{Z}[i]$ or $\boldsymbol{Z}[\omega]$. Hence, $2(\alpha', \beta_j)=2(\alpha, \beta_j)-2(1-\omega)(\alpha, \beta_k)(\beta_k, \beta_j)\in\boldsymbol{Z}[\omega]$, and $2(\alpha'', \beta_j)=2(\alpha, \beta_j)-4(\alpha, \beta_m)(\beta_m, \beta_j)\in\boldsymbol{Z}[i]$ or $\boldsymbol{Z}[\omega]$. If β_j is a root base of order 3 then $(\beta_k, \beta_j)=(1-\omega)/3$, $(1-\bar{\omega})/3$, 1, or 0, i.e. $(1-\omega)(\beta_k, \beta_j)\in\boldsymbol{Z}[\omega]$ and $(\beta_m, \beta_j)=\pm 1/\sqrt{2}$ or 0. Consequently, $\sqrt{2}(\alpha', \beta_j)$, $\sqrt{2}(\alpha'', \beta_j)\in\boldsymbol{Z}[\omega]$. $\square$

We can prove the following lemma in a similar manner.

LEMMA 2′. *Under the assumption of Lemma* 2, *for a root* α *of order* 3, *if* β_j *is a root of order* 3 *or* 2 *then*

$$(1-\omega)(\alpha, \beta_j),\quad (1-\bar{\omega})(\alpha, \beta_j),\quad 3(\alpha, \beta_j)\in\boldsymbol{Z}[\omega]\quad \textit{or}$$

$$\sqrt{2}(\alpha, \beta_j)\in\boldsymbol{Z}[\omega],$$

respectively.

LEMMA 3. *Let* Γ *be a parabolic reflection group. Assume that the crystallographic group* $\pi_*(\Gamma)$ *is the semi-direct product* $W\ltimes L$ *of the point group* W *and the lattice* L. *If the graph of* W *contains a subgraph* $\underset{j}{\bigcirc}\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\underset{k}{\bigcirc}$ *and*

$$L\cap(\boldsymbol{C}\beta_j+\boldsymbol{C}\beta_k)=(\boldsymbol{Z}+i\boldsymbol{Z})\beta_j+(\boldsymbol{Z}+i\boldsymbol{Z})\beta_k\quad \textit{or}\quad (\boldsymbol{Z}+\omega\boldsymbol{Z})\beta_j+(\boldsymbol{Z}+\omega\boldsymbol{Z})\beta_k$$

then Γ *contains the elements*

$$[E, (m+in)\beta_j, 0],\quad [E, (m+in)\beta_k, 0]\quad \textit{or}$$

$$[E, (m+\omega n)\beta_j, 0],\quad [E, (m+\omega n)\beta_k, 0],$$

for all integers m, n.

REMARK. Lemma 3 implies that the map $s:\pi_*(\Gamma)\to \boldsymbol{Z}_\nu$ defined in § 3.2 vanishes on $L\cap(\boldsymbol{C}\beta_j+\boldsymbol{C}\beta_k)$.

PROOF. Put the center $Z(\Gamma):=\{[E, 0, \gamma]; \gamma\in\nu\tilde{q}\boldsymbol{Z}\}$, where $\tilde{q}$ is the number defined in § 3.2. We define the map $s_p: \boldsymbol{Z}\times\boldsymbol{Z}\to\tilde{q}\boldsymbol{Z}$ by $s_p(m, n)=s((E\,|\,(m+\zeta n)\beta_p))\tilde{q}$ ($\zeta=i$ or ω). We have

$$[r_j, 0, \gamma]^{-1}[E, (m+\zeta n)\beta_k, s_k(m, n)][r_j, 0, \gamma]$$
$$=[E, (m+\zeta n)(\beta_k+\beta_j), s_k(m, n)]\,.$$

On the other hand,

$$[E, (m+\zeta n)\beta_k, s_k(m, n)][E, (m+\zeta n)\beta_j, s_j(m, n)]$$
$$=[E, (m+\zeta n)(\beta_k+\beta_j), s_k(m, n)+s_j(m, n)]\,.$$

Hence $[E, 0, s_j(m, n)]\in\Gamma$, that is $s_j(m, n)\equiv 0 \pmod{\nu\tilde{q}\boldsymbol{Z}}$. Therefore $s((E\,|\,(m+\zeta n)\beta_j))=0$ for all m, n. □

LEMMA 3′. *Under the assumption of Lemma* 3, *if the graph of the point group W has a subgraph* $③\xrightarrow[(2+\omega)/3]{}③$ *and the sublattice of L spanned by β_j and β_k is $(\boldsymbol{Z}+\omega\boldsymbol{Z})\beta_j+(\boldsymbol{Z}+\omega\boldsymbol{Z})\beta_k$ then we have*

$$\left[E, (m+\omega n)\beta_j, \frac{1}{2\sqrt{3}}\,|m+\omega n|^2\right],$$
$$\left[E, (m+\omega n)\beta_k, \frac{1}{2\sqrt{3}}\,|m+\omega n|^2\right]\in\Gamma$$

for all integers m, n.

PROOF. Similar to that of Lemma 3. □

3.4. Construction. Let Γ_* be a crystallographic reflection group and let W be its point group. By using the map $s:\Gamma_*\to\boldsymbol{Z}_\nu$, we construct a parabolic reflection group Γ such that $\pi_*(\Gamma)=\Gamma_*$. Assume first that the number of the generators of W is equal to dimension of V' (see § 1.2). Notice that in these cases the crystallographic group Γ_* is the semi-direct product of W and the lattice.

Case 1. Any reflection of W is of order 2. Namely, the case when W is one of the following: $G(m, m, s)$ ($m=3, 4, 6$, $s=3, 4, \cdots$), K24, K29, K33, K34.

The number $\tilde{\nu}$ (see § 3.1) is 2. By Lemma 3, the map s must vanish on all of the generators. Let Γ be a subgroup of the group $\tilde{\Gamma}$ which is defined in § 3.2. Using Lemma 1 and Lemma 2, we can show that if Γ is generated by reflections then, the center of Γ is generated by $[E, 0, q_0]$, where q_0 is the number defined in §1.2. Consequently we obtain the result in the table.

Case 1′. Any reflection of W is of order 3. Namely, the case when W is K4, K5, K25 or K32.

The number $\tilde{\mathfrak{v}}$ is also 2. By Lemma 3′, the reflection group $\tilde{\Gamma}$ contains all reflections of Γ. Thinking over Lemma 2′, we see that the center of Γ is generated by $[E, 0, q_0]$.

Case 2. Each reflection in W is of order 2 or 3. Namely, the case when W is one of the following: $G(3, 1, s)\,(s=2, 3, \cdots)$, K26.

For each of these finite reflection groups, there are two kinds of invariant lattices. $\tilde{\mathfrak{v}}$ is equal to 6 for the case $\Gamma_*=G(3, 1, s)\ltimes L_1$, and is equal to 2 for the remaining ones. In the case when $\tilde{\mathfrak{v}}$ is 2, by Lemma 2 and 2′, the center of Γ is generated by $[E, 0, q_0]$. The map s is determined by Lemma 3 and the similar calculation as in the proof of Lemma 3 for the subgraph $\textcircled{3}\underset{1/\sqrt{2}}{\text{———}}\bigcirc$.

In the case when $\Gamma_*=G(3, 1, s)\ltimes L_1$, for each possible value of the map s, we must check whether Γ is generated by reflections. Let us assume first that Γ contains all reflections of $\tilde{\Gamma}$. By Lemma 2 and Lemma 2′, if Γ is generated by reflections then the center of Γ is contained in $\{[E, 0, q];\ q\in 1/(2\sqrt{3})\boldsymbol{Z}\}$. Meanwhile, the center always contains the set $\{[E, 0, q];\ q\in\sqrt{3}/2\boldsymbol{Z}\}$. Hence, the center must be generated by $[E, 0, 1/(2\sqrt{3})]$ or $[E, 0, \sqrt{3}/2]$. Let β_1 be the root of order 3 in the table and r_1 be a reflection which has β_1 as a root. Put $s_0:=s((r_1|0))$ and $s_1:=s((r_1|1/\sqrt{2}\beta_1))$. By computing the third powers of $[r_1, 0, s_0/(4\sqrt{3})]$ and $[r_1, 1/\sqrt{2}\beta_1, s_1/(4\sqrt{3})]$, we see that $s_0\equiv 0$, $s_1\equiv 1 \pmod 2$. Hence, if the center is generated by $[E, 0, 1/(2\sqrt{3})]$, then s is determined by $s_0=0$, $s_1=1$. In the case when the center is generated by $[E, 0, \sqrt{3}/2]$, there exist reflections of the form $[r_1, (m+\omega n)/\sqrt{2}\beta_1, *]$ in Γ, if and only if

$$\frac{1}{2\sqrt{3}}\{m(s_1-s_0)+n(2s_0+s_1)-3mn+s_0\}\equiv\frac{1}{2\sqrt{3}}|m+\omega n|^2 \pmod{\sqrt{3}\boldsymbol{Z}},$$

$$\text{i.e.}\quad (m+n)(m+n+s_0-s_1)-(3n+1)s_1\equiv 0 \pmod 6.$$

On the other hand, any reflections of the form $[r_j, (m+n\omega)\beta_j, 0]$ $(j\geqq 2)$ belong to Γ. Let L_r denote the lattice generated by the set of all the translation parts of reflections in Γ. By computation, we can show that when s is determined by $(s_0, s_1)=(0, 1)$, $(0, 5)$ or $(4, 1)$, L_r coincides with the lattice L, i.e. Γ is generated by reflections. And the group constructed by the values of $s:(s_0, s_1)=(0, 5)$ is conjugate to the group constructed by $(s_0, s_1)=(0, 1)$. In other cases Γ has all reflections in $\tilde{\Gamma}$ and we have the result in the table.

Case 3. There exists a reflection of which the order is not a prime. Namely the case when W is $G(4, 1, s)$, $G(6, 1, s)$ $(s=2, 3, \cdots)$ or K8.

There exist reflections of order 4 in $G(4, 1, s)$ and K8, and of order 6 in $G(6, 1, s)$. Note that these groups have reflections which are powers of the reflections of order 4 or 6. For example, we study the case that the point group is $G(4, 1, s)$. Let β_1 be a root of order 4 and r_1 be a reflection which has β_1 as its root. Put $s_0:=s((r_1|0))$ and $s_1:=s((r_1|1/\sqrt{2}\beta_1))$ as in Case 2.

We consider first the case the lattice is L_1. Since $q_0=1$ and $\tilde{\nu}=4$, the center is generated by $[E, 0, 1/4]$, $[E, 0, 1/2]$ or $[E, 0, 1]$. By the contribution of the reflection $[r_1^2, 1/\sqrt{2}\beta_1, 0]$, the element $[E, 0, 1/4]$ is written by a product of reflections. If the center is generated by $[E, 0, 1/4]$, we see that $(s_0, s_1)=(0, 0)$. In the case that the center is generated by $[E, 0, 1/2]$, the condition that there exist reflections of the form $[r_1, (m+in)/\sqrt{2}\beta_1, *]$ is

$$(m+n)(m+n+s_0-s_1)-s_0 \equiv 0 \pmod 2 .$$

Hence $(s_0, s_1)=(0, 0)$, $(0, 1)$ or $(1, 1)$. And L_r coincides with L in each case. In the cases of $(s_0, s_1)=(0, 0)$ and $(1, 1)$, we remark the contribution of the reflections $[r_1^2, *, 0]$. In the case when the center is generated by $[E, 0, 1]$, it must be $2(s_0-s_1)\equiv 0 \pmod 4$, and there exist reflections of the form $[r_1, (m+in)/\sqrt{2}\beta_1, *]$ if and only if

$$(m+n)(m+n+s_0-s_1)-s_0 \equiv 0 \pmod 4 .$$

Therefore $(s_0, s_1)=(0, 0)$, $(0, 2)$, $(1, 1)$ or $(3, 1)$. In the cases of $(s_0, s_1)=(0, 0)$ and $(3, 1)$, L_r coincides with L by the contribution of the reflections $[r_1^2, *, 0]$.

Next we consider the case the lattice is L_2. Since $q_0=1/2$ and $\tilde{\nu}=2$, we can choose $[E, 0, 1/4]$ or $[E, 0, 1/2]$ as a generator of the center. If the center is generated by $[E, 0, 1/4]$, then $(s_0, s_1)=(0, 0)$. By the contribution of the reflection $[r_1^2, 1/\sqrt{2}\beta_1, 0]$, the element $[E, 0, 1/4]$ is written by a product of reflections. In the case when the center is generated by $[E, 0, 1]$, the condition that there exist reflections of the form $[r_1, (m+in)/\sqrt{2}\beta_1, *]$ is

$$(m+n)(m+n+s_0-s_1)-s_0 \equiv 0 \pmod 2 .$$

But, this condition is not useful in this case. For L_r coincides with L in each case by the contribution of the reflections $[r_1^2, *, 0]$. Hence we employ the same method as Lemma 3 for the graph $\textcircled{4}\underset{1/\sqrt{2}}{\text{———}}\bigcirc$. Then we have $[E, (m+in)/\sqrt{2}\beta_1, (1/2)|m+in|^2]\in\Gamma$. This condition is satisfied only in the case $(s_0, s_1)=(0, 1)$.

Now we study the case that the number of the generators of W is greater than dimension of V'.

Let β_s be the root base which does not appear in the expression of the lattice in the table except for L_2 of $G(4, 2, s-1)$. We call it an excessive root base. The map s must vanish on the generators $(r_j|*)$ $(j=1, \cdots, s-1)$, so we study the value of s for $(r_s|*)$.

We have a following list of sublattices $L\cap \boldsymbol{C}\beta_s$ of L.

Finite Reflection Group	Lattice L	$L\cap \boldsymbol{C}\beta_s$
$G(4, 2, s-1)$	L_1	$(\boldsymbol{Z}+i\boldsymbol{Z})\frac{1+i}{\sqrt{2}}\beta_s$
$G(4, 2, s-1)$	L_2	$(\boldsymbol{Z}+i\boldsymbol{Z})\frac{1}{\sqrt{2}}\beta_s$
$G(4, 2, 2)$	L_3	$(\boldsymbol{Z}+i\boldsymbol{Z})\sqrt{2}\beta_s$
$G(6, 2, s-1)$	L_1	$(\boldsymbol{Z}+\omega\boldsymbol{Z})\frac{1}{\sqrt{2}}\beta_s$
$G(6, 3, s-1)$	L_1	$(\boldsymbol{Z}+\omega\boldsymbol{Z})\frac{1}{\sqrt{2}}\beta_s$
$G(6, 2, 2)$	L_2	$(\boldsymbol{Z}+\omega\boldsymbol{Z})\frac{2+\omega}{\sqrt{2}}\beta_s$
$G(6, 3, 2)$	L_2	$(\boldsymbol{Z}+\omega\boldsymbol{Z})\sqrt{2}\beta_3$
K12	L	$(\boldsymbol{Z}+i\sqrt{2}\boldsymbol{Z})\beta_3$
K31	L	$(\boldsymbol{Z}+i\boldsymbol{Z})\beta_5$

Case 4. Crystallographic group is the semi-direct product of W and L. Namely the case when W is one of the following: $G(4, 2, s)$, $G(6, 2, s)$, $G(6, 3, s)$ $(s=3, 4, \cdots)$, K12, K31.

Put $s_0:=s((r_s|0))$, $s_1:=s((r_s|x\beta_s))$ and $s_2:=s((r_s|x'\beta_s))$, where $x=(1+i)/2$ and $x'=ix$ etc. We consider the case all the reflections of Γ are of order 2. Note that $\tilde{\nu}=2$ in these cases. Let us assume that Γ have all reflections in $\tilde{\Gamma}$. By the contribution of the reflections with a root parallel to the excessive root base, $[E, 0, q_0/2]$ is written by a product of reflections, and the map s is given by $(s_0, s_1, s_2)=(0, 0, 0)$.

Next we must study the case $\nu=1$, that is the center is generated by $[E, 0, q_0]$. In the case when the point group is K12 or K31, if Γ is generated by reflections, the center is necessarily $\{[E, 0, q]\in\Gamma;\ q\in q_0/2\boldsymbol{Z}\}$. If the crystallographic group is $G(4, 2, s-1)\ltimes L_1$, $\sqrt{2}\beta_s=-2(\beta_{s-1}+\cdots+\beta_3)+(1+i)(i\beta_2-\beta_1)$, hence $[E, (1+i)/\sqrt{2}\beta_s, 1/2]$, $[E, (1-i)/\sqrt{2}\beta_s, 1/2]\in\Gamma$. Therefore $s_0+1=s((r_s|0))+1\equiv s_1=s((r_s|(1+i)/\sqrt{2}\beta_s))\equiv s_2=s((r_s|(1-i)/\sqrt{2}\beta_s)) \pmod 2$. Consequently, $(s_0, s_1, s_2)=(0, 1, 1)$ or $(1, 0, 0)$, and in each case the center is actually $\{[E, 0, q]\in\Gamma;\ q\in q_0\boldsymbol{Z}\}$. In the case that the point group is $G(6, 2, s-1)$, there exist reflections of order 3. If the lattice is L_1, then $\tilde{\nu}=6$. Assume that Γ

contains all reflections of $\tilde{\Gamma}$, then we see that the center is $\{[E, 0, q] \in \Gamma; q \in q_0/3\boldsymbol{Z}\}$. Hence the center is generated by $[E, 0, q_0/3]$ or $[E, 0, q_0]$. In the case when the center is $\{[E, 0, q] \in \Gamma; q \in q_0/3\boldsymbol{Z}\}$, $s_0 = s((r_s|0)) = 0$ and $s_1 = s((r_s|1/\sqrt{2}\beta_s)) = 1$. If the center is generated by $[E, 0, q_0]$, then $s_0 \equiv 0$, $s_1 \equiv 1 \pmod 2$ and $s_0 - s_1 \equiv 3 \pmod 6$. There exist reflections of the form $[r_s, (m+\omega n)/\sqrt{2}\beta_s, *]$ if and only if

$$(m+n)(m+n+s_0-s_1) \equiv s_0 \pmod 6,$$

hence $s_0 = 0$ or 4. Therefore $(s_0, s_1) = (0, 3)$ or $(4, 1)$.

Case 5. Crystallographic group is not a semi-direct product of W and L. Namely the case when W is $G(4, 2, s)$ $(s = 3, 4, \cdots)$, K12 or K31.

All the reflections in Γ are of order 2, and $\tilde{\nu} = 2$. Let us assume that Γ contains all reflections of $\tilde{\Gamma}$. The center is generated by $[E, 0, q_0]$ for the case that the point group W is $G(4, 2, s-1)$. In the case when $W = $K12 or K31, the center is generated by $[E, 0, q_0/2]$. Moreover $[E, 0, q_0/2] \in \Gamma$, even if we assume the center is $\{[E, 0, q] \in \Gamma; q \in q_0\boldsymbol{Z}\}$. Hence the center is necessarily $\{[E, 0, q] \in \Gamma; q \in q_0/2\boldsymbol{Z}\}$, and the map s vanishes.

We complete the classification.

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