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Integral arithmetically Buchsbaum curves in $\mathbf{P}^3$

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Introduction.

When a curve X (not assumed to be smooth nor reduced) in $\mathbf{P}^3$ has the property that its deficiency module $\oplus_n H^1(\mathcal{J}_X(n))$ is annihilated by the homogeneous coordinates x_1, x_2, x_3, x_4 of $\mathbf{P}^3$, it is called an arithmetically Buchsbaum curve. In [**1**], we defined a numerical invariant "basic sequence" of a curve in $\mathbf{P}^3$ (see [**1**; Definition 1.4]) and classified arithmetically Buchsbaum curves with nontrivial deficiency modules in terms of their basic sequences. But there, an important problem was left unconsidered; to find a necessary and sufficient condition for the existence of integral arithmetically Buchsbaum curves with a given basic sequence. The aim of this paper is to give a complete answer to this problem in the case where the base field has characteristic zero. The existence theorems for some special cases, e.g. [**1**; Theorem 4.4], [**2**; Corollary 2.6], [**3**; Proposition 4.7] and [**4**; pp. 125-126], are now corollaries to our general theorem.

NOTATION AND CONVENTION. The base field k is algebraically closed. We do not assume that $\mathrm{char}(k)=0$ except in the main theorem. The word "curve" means an equidimensional complete scheme over k of dimension one without any embedded points. Given a matrix Φ, $\Phi\binom{i}{j}$ denotes the matrix obtained by deleting the i-th row and the j-th column from Φ. We say that a sequence of integers $z_1, \cdots, z_n$ is connected if $z_i \leqq z_{i+1} \leqq z_i+1$ for all $1 \leqq i \leqq n-1$ or $n=0$ (i.e. the sequence is empty). The ideal sheaf of a curve X in $\mathbf{P}^3$ is denoted by $\mathcal{J}_X$ and we set $I_{X,n}=H^0(\mathcal{J}_X(n))$, $I_X=\oplus_n I_{X,n} \subset R$, where $R=k[x_1, x_2, x_3, x_4]$. For simplicity we abbreviate "arithmetically Buchsbaum" to "a. B.".

§1. Preliminaries.

Given a curve X in $\mathbf{P}^3$, we define the basic sequence of X to be the sequence of positive integers $(a; \nu_1, \cdots, \nu_a; \nu_{a+1}, \cdots, \nu_{a+b})$ $(b \geqq 0)$ which satisfies the conditions (1.1), (1.2), (1.3) below and denote it by $B(X)$ (see [**1**; §§1, 2]). Let x_1, x_2, x_3, x_4 be generic homogeneous coordinates of $\mathbf{P}^3$ and set $R'=k[x_1, x_2, x_3]$, $R''=k[x_3, x_4]$.

(1.1) $a \leqq \nu_1 \leqq \cdots \leqq \nu_a$, $\nu_1 \leqq \nu_{a+1} \leqq \cdots \leqq \nu_{a+b}$, *where* $(\nu_{a+1}, \cdots, \nu_{a+b})$ *is empty if* $b=0$.

(1.2) *There are generators* $f_0, f_1, \cdots, f_a, f_{a+1}, \cdots, f_{a+b}$ *of* I_X *such that* $\deg(f_0)=a$, $\deg(f_i)=\nu_i$ $(1 \leqq i \leqq a+b)$ *and*

$$I_X = Rf_0 \oplus \bigoplus_{i=1}^{a} R'f_i \oplus \bigoplus_{j=1}^{b} R''f_{a+j}.$$

(1.3) *The deficiency module* $M(X):=\oplus_n H^1(\mathcal{J}_X(n))$ *has a minimal free resolution of the form*

$$0 \longrightarrow \bigoplus_{j=1}^{b} R''(-\nu_{a+j}) \longrightarrow \bigoplus_{i=1}^{r_1} R''(-\varepsilon_i^1) \longrightarrow \bigoplus_{i=1}^{r_0} R''(-\varepsilon_i^0) \longrightarrow M(X) \longrightarrow 0$$

as an R''*-module, where* ε_i^j $(1 \leqq i \leqq r_j,\ j=0, 1)$ *are integers.*

The basic sequences of a. B. curves have some special properties. First of all, a sequence $(a; \nu_1, \cdots, \nu_a; \nu_{a+1}, \cdots, \nu_{a+b})$ of positive integers satisfying (1.1) is the basic sequence of an a. B. curve if and only if $a \geqq 2b$ and there are $(m_1, \cdots, m_{a-2b})$, $(n_1, \cdots, n_b)$ $(m_1 \leqq \cdots \leqq m_{a-2b},\ n_1 \leqq \cdots \leqq n_b)$ such that $(\nu_{a+1}, \cdots, \nu_{a+b}) = (n_1, \cdots, n_b)$, $(\nu_1, \cdots, \nu_a) = (m_1, \cdots, m_{a-2b}, n_1, \cdots, n_b, n_1, \cdots, n_b)$ up to permutation (see [**1**; Theorem 3.1, Lemma 4.2]). Furthermore a. B. curves of the same basic sequence are parameterized by a Zariski open subset of an affine space over k. Let X be an a. B. curve. With the notation above the sequence $B_{sh}(X):=(a; m_1, \cdots, m_{a-2b}; n_1, \cdots, n_b)$ is called the short basic sequence of X in [1] (cf. [**1**; Corollary 3.3, (4.1.4)]). It follows from (1.3) and the definition of a. B. curves that

$$M(X) \cong \bigoplus_{j=1}^{b} k(-n_j+2). \tag{1.4}$$

Besides, examining the relation between I_X and $M(X)$ closely, we find that the R-module $\tilde{R}_X:=\oplus_n H^0(\mathcal{O}_X(n))$ has a free resolution of the form

$$\begin{aligned} 0 &\longrightarrow \bigoplus_{i=1}^{a-2b} R(-m_i-1) \oplus \Big(\bigoplus_{j=1}^{b} R(-n_j)\Big)^3 \\ &\xrightarrow{\tau} R(-a) \oplus \bigoplus_{i=1}^{a-2b} R(-m_i) \oplus \Big(\bigoplus_{j=1}^{b} R(-n_j+1)\Big)^4 \\ &\xrightarrow{\sigma} R \oplus \Big(\bigoplus_{r=1}^{b} R(-n_j+2)\Big) \xrightarrow{\rho} \tilde{R}_X \longrightarrow 0, \end{aligned} \tag{1.5}$$

where

$$\sigma = \left(\begin{array}{c|c} \multicolumn{2}{c}{*} \\ \hline 0 & x_1 1_b,\ x_2 1_b,\ x_3 1_b,\ x_4 1_b \end{array}\right)$$

with a $b \times b$ unit matrix 1_b (see [**1**; (3.4.1)]).

§ 2. Short basic sequences of integral a.B. curves.

In the following argument the results concerning a. B. curves will be stated in the language of their short basic sequences.

Let F and G be vector bundles on $\mathbf{P}^3$ of rank p and q respectively ($p>1$, $q>0$) and let X be a curve in $\mathbf{P}^3$ whose ideal sheaf $\mathcal{I}_X$ has a locally free resolution of the form

$$(2.1)\qquad 0 \longrightarrow \bigoplus_{i=1}^{p+q-1} \mathcal{O}_{\mathbf{P}^3}(-d_i) \xrightarrow{\ v\ } F\oplus G \xrightarrow{\ w\ } \mathcal{I}_X \longrightarrow 0$$

(cf. [**6**; Lemma 1.1]). Here the map v is defined by the multiplication by global sections $(v_i^F, v_i^G)\in H^0((F\oplus G)\otimes\mathcal{O}_{\mathbf{P}^3}(d_i))$ $(v_i^F\in H^0(F(d_i)),\ v_i^G\in H^0(G(d_i)),\ 1\leqq i\leqq p+q-1)$ and locally it is represented by the $(p+q)\times(p+q-1)$-matrix $\boldsymbol{v}=\begin{pmatrix}\boldsymbol{v}^F\\ \boldsymbol{v}^G\end{pmatrix}$, where $\boldsymbol{v}^F=(v_1^F, \cdots, v_{p+q-1}^F)$ and $\boldsymbol{v}^G=(v_1^G, \cdots, v_{p+q-1}^G)$.

LEMMA 1. *Suppose that $v_i^G=0$ for $1\leqq i\leqq p-1$ and that X is integral. Then*

$$F\cong \mathcal{O}_{\mathbf{P}^3}\Big(c_1(F)+\sum_{i=1}^{p-1} d_i\Big)\oplus\bigoplus_{i=1}^{p-1}\mathcal{O}_{\mathbf{P}^3}(-d_i) \qquad \text{or} \qquad G\cong\bigoplus_{i=p}^{p+q-1}\mathcal{O}_{\mathbf{P}^3}(-d_i).$$

PROOF. Let Y denote the closed subscheme of $\mathbf{P}^3$ defined locally by the maximal minors of $(v_1^F, \cdots, v_{p-1}^F)$. Clearly $Y\subset X$ by the hypothesis $v_i^G=0$ $(1\leqq i\leqq p-1)$, therefore Y is either empty or is a curve and in any case $\mathcal{I}_Y$ has the locally free resolution

$$(2.2)\qquad 0 \longrightarrow \bigoplus_{i=1}^{p-1} \mathcal{O}_{\mathbf{P}^3}(-d_i) \xrightarrow{\ v'\ } F \xrightarrow{\ w'\ } \mathcal{I}_Y(c) \longrightarrow 0$$

with $c=c_1(F)+\sum_{i=1}^{p-1}d_i$ (cf. [**1**; (2.10.5)]) where v' and w' are defined by $(v_1^F, \cdots, v_{p-1}^F)$ in the same way as above. If Y is empty, then $\mathcal{I}_Y=\mathcal{O}_{\mathbf{P}^3}$ so that (2.2) splits and we have $F\cong\mathcal{O}_{\mathbf{P}^3}(c)\oplus\bigoplus_{i=1}^{p-1}\mathcal{O}_{\mathbf{P}^3}(-d_i)$. Now suppose that Y is a curve. In this case $Y=X$, since X is integral. Let ζ be an element of $H^0(\bigwedge^q G(\sum_{i=p}^{p+q-1}d_i))$ given by $\det(v_p^G, \cdots, v_{p+q-1}^G)$ and let D denote the zero locus of ζ. If $\zeta=0$ or D is a divisor of positive degree, we take a point $x\in X\cap D$ and consider the stalk of $\mathcal{I}_X$ at x. Set $\boldsymbol{v}'=(v_1^F, \cdots, v_{p-1}^F)$, $\boldsymbol{u}=(v_p^G, \cdots, v_{p+q-1}^G)$, $h=\det(\boldsymbol{u})$, $g_i'=(-1)^{i-1}\det(\boldsymbol{v}'\binom{i}{\ })$ $(1\leqq i\leqq p)$ and $g_i=(-1)^{i-1}\det(\boldsymbol{v}\binom{i}{\ })$ $(1\leqq i\leqq p+q)$. Then it follows from (2.1) and (2.2) that

$$(2.3)\qquad \mathcal{I}_{X,x}=(g_1', \cdots, g_p')\mathcal{O}_{\mathbf{P}^3,x}=(g_1, \cdots, g_{p+q})\mathcal{O}_{\mathbf{P}^3,x}.$$

Since h vanishes at x, the rank r of $\boldsymbol{u}$ at x is smaller than q. We may assume therefore that $\boldsymbol{v}$ is of the form

$$\left[\begin{array}{c:c:c} \boldsymbol{v}' & * & 0 \\ \hdashline & \boldsymbol{u}' & \\ \hdashline 0 & 0 & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} \end{array}\right]\Big\}\, r$$

up to multiplication by $GL(p+q, \mathcal{O}_{\mathbf{P}^3,x})$ on the left and $GL(p+q-1, \mathcal{O}_{\mathbf{P}^3,x})$ on the right, where all the components of $\boldsymbol{u}'$ are contained in the maximal ideal $\mathfrak{m}_x$ of x. Consequently,

$$\mathcal{I}_{X,x} \subset (g_1', \cdots, g_p')\mathfrak{m}_x + g_{p+1}\mathcal{O}_{\mathbf{P}^3,x}$$

by (2.3), which implies that $\mathcal{I}_{X,x} = g_{p+1}\mathcal{O}_{\mathbf{P}^3,x}$ by Nakayama's lemma. This contradicts the fact that X is a curve passing through x. Hence $D=\emptyset$ and $G \cong \bigoplus_{i=p}^{p+q-1}\mathcal{O}_{\mathbf{P}^3}(-d_i)$. Q.E.D.

Let A be a finitely generated regular k-algebra, n ($n \geqq 3$) an integer and $s=\{s_{ij} \mid 1 \leqq i \leqq \min(j+2, n), 1 \leqq j \leqq n-1\}$ a set of indeterminates over A. We denote by S the matrix of size $n \times (n-1)$ whose (i, j)-component is s_{ij} if $1 \leqq i \leqq j+2$ and 0 otherwise. Given a $n \times (n-1)$-matrix $H=(h_{ij})$ with components in $A[s]$ such that

(2.4) *all the components of $H-S$ lie in A,*

let $Q(H)$ denote the closed subscheme of $V:=\mathrm{Spec}(A[s])$ determined by the maximal minors of H.

LEMMA 2. *There is a closed subscheme Z of codimension larger than or equal to* 5 *in V such that $Q(H) \setminus Z$ is smooth over k.*

PROOF. We first consider the case $n=3$. Let U_{ij} be the complement of the divisor $h_{ij}=0$ for each (i, j) ($1 \leqq i \leqq 3$, $1 \leqq j \leqq 2$). Since

$$h_{1j}\det(H\binom{1}{\ }) - h_{2j}\det(H\binom{2}{\ }) + h_{3j}\det(H\binom{3}{\ }) = 0$$

for $j=1, 2$, the scheme $Q(H) \cap U_{11}$ is isomorphic to

$$\mathrm{Spec}(A[s]_{h_{11}}/(h_{32}-h_{12}h_{31}/h_{11},\ h_{22}-h_{12}h_{21}/h_{11})),$$

which is of codimension 2 in U_{11} and smooth over k. The same thing holds also for the other $Q(H) \cap U_{ij}$'s. Therefore $Q(H)$ is smooth over k in the outside of the closed subscheme $V \setminus (\bigcup_{i,j} U_{ij})$ of codimension 6 in V. Now suppose that $n \geqq 4$ and that the assertion holds for $n-1$. Let U_i ($1 \leqq i \leqq 5$) be the complements of the divisors $h_{i1}=0$ ($1 \leqq i \leqq 3$), $h_{n,n-6+i}=0$ ($4 \leqq i \leqq 5$) respectively. On each open set U_i ($1 \leqq i \leqq 5$), there are matrices $K_i \in GL(n, k[U_i])$ and $K_i' \in GL(n-1, k[U_i])$ such that K_iHK_i' takes the form $\begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & H_i' \\ 0 & \end{pmatrix}$, where H_i' satisfies the condition (2.4) with A and S replaced by $A[\{s_{lm} \mid l=i \text{ or } m=1\}]_{h_{i1}}$ and $S\binom{i}{1}$ ($1 \leqq i \leqq 3$) or $A[\{s_{lm} \mid l=n \text{ or } m=n-6+i\}]_{h_{n,n-6+i}}$ and $S\binom{n}{n-6+i}$ ($4 \leqq i \leqq 5$). By the induction hypothesis there are closed subschemes Z_i ($1 \leqq i \leqq 5$) of U_i such that $\mathrm{codim}_{U_i}(Z_i) \geqq 5$ and $Q(H_i') \setminus Z_i$ are smooth over k. We have only to put $Z=(V \setminus \bigcup_{i=1}^5 U_i) \cup \bigcup_{i=1}^5 \bar{Z}_i$. Q.E.D.

Let a, b, m_i $(1 \leqq i \leqq a-2b)$ and n_i $(1 \leqq i \leqq b)$ be positive integers such that $a-2b \geqq 0$, $a \leqq m_1 \leqq m_2 \leqq \cdots \leqq m_{a-2b}$ and $a \leqq n_1 \leqq n_2 \leqq \cdots \leqq n_b$. We set

$$B_{sh} = (a\,;\, m_1, \cdots, m_{a-2b}\,;\, n_1, \cdots, n_b).$$

One knows that there exists an a. B. curve X in $\mathbf{P}^3$ with short basic sequence B_{sh} (see Section 1). For each integer $n \geqq 0$ we put

$$\begin{cases} e_n = \#\{i \mid m_i = n,\ 1 \leqq i \leqq a-2b\}, & e'_n = \#\{i \mid n_i = n,\ 1 \leqq i \leqq b\}, \\ \alpha = \min(m_1, n_1 - 1), & \beta = \max(m_{a-2b}, n_b - 1), \end{cases}$$

where $\#$ denotes the number of the elements and $\alpha = n_1 - 1$, $\beta = n_b - 1$ if $a-2b=0$. Let E denote the vector bundle of rank 3 on $\mathbf{P}^3$ defined by the exact sequence

$$0 \longrightarrow E \longrightarrow \mathcal{O}_{\mathbf{P}^3}(1)^4 \xrightarrow{(x_1,\, x_2,\, x_3,\, x_4)} \mathcal{O}_{\mathbf{P}^3}(2) \longrightarrow 0,$$

namely $E = \Omega_{\mathbf{P}^3/k}(2)$. One sees that $h^0(E(n))=0$ for $n<0$ and that E is generated over $\mathcal{O}_{\mathbf{P}^3}$ by its global sections. Set

$$F_m = \mathcal{O}_{\mathbf{P}^3}(-a) \oplus \bigoplus_{n=\alpha}^{m} (\mathcal{O}_{\mathbf{P}^3}(-n)^{e_n} \oplus E(-n-1)^{e'_{n+1}}),$$

$$G_m = \bigoplus_{n=m+1}^{\beta} (\mathcal{O}_{\mathbf{P}^3}(-n)^{e_n} \oplus E(-n-1)^{e'_{n+1}}),$$

$$L_m = \bigoplus_{n=\alpha}^{m} \mathcal{O}_{\mathbf{P}^3}(-n-1)^{e_{n+3}e'_{n+1}},$$

for $\alpha \leqq m \leqq \beta$. It follows from [**1**; (2.10.5) and (3.4.1)] that $\mathcal{I}_X$ has a locally free resolution of the form

$$0 \longrightarrow L_\beta \xrightarrow{v} F_\beta \xrightarrow{w} \mathcal{I}_X \longrightarrow 0. \tag{2.5}$$

LEMMA 3. *Suppose that X is reduced. Then X is connected if and only if $n_1 \geqq 3$.*

PROOF. Since X is connected if and only if $H^1(\mathcal{I}_X)=0$, the assertion follows from (1.4).

LEMMA 4. *Let X' be another* a. B. *curve whose ideal sheaf $\mathcal{I}_{X'}$ has a locally free resolution of the form* (2.5) *with the same L_β and F_β. Then the short basic sequence of X' coincides with B_{sh}.*

PROOF. Since $M(X') \cong M(X)$, $h^0(\mathcal{I}_{X'}(n)) = h^0(\mathcal{I}_X(n))$ for all $n \geqq 0$ by (2.5), it follows from (1.1), (1.2), (1.3) and (1.4) that $B_{sh}(X') = B_{sh}$.

THEOREM. i) *If there is an integral* a. B. *curve in $\mathbf{P}^3$ with short basic sequence B_{sh}, then one of the following two conditions is satisfied.*

$$a = 2,\quad b = 1,\quad n_1 \geqq 3, \tag{2.6.1}$$

(2.6.2) $\quad a \geqq 3, \quad a-2b \geqq n_b - n_1, \quad m_1 \leqq n_1, \quad n_b - 1 \leqq m_{a-2b}$

and $m_1, \cdots, m_{a-2b}$ *is connected.*

ii) *In the case* char$(k)=0$, *both these conditions are sufficient for the existenc of an integral* a. B. *curve with short basic sequence* B_{sh}.

PROOF. If the condition (2.6.1) or (2.6.2) is fulfilled, we have

(2.7) $\quad \alpha=\beta$ *or* $e_{n+1}\neq 0$ *for every integer* n $(\alpha\leqq n\leqq\beta-1)$.

Conversely if (2.7) is satisfied, we have (2.6.1), (2.6.2) or

(2.8) $\quad a=2, \quad b=1, \quad n_1=2.$

Let X be an integral a. B. curve in $\mathbf{P}^3$ with short basic sequence B_{sh} and assume that neither (2.6.1) nor (2.6.2) is satisfied. Then, since the case (2.8) cannot occur by Lemma 3, we have $\alpha<\beta$ and there is an integer l $(\alpha\leqq l\leqq\beta-1)$ such that $e_{l+1}=0$ by the remark above. One sees $H^0(G_l\otimes L_l^\vee)=0$, $F_\beta=F_l\oplus G_l$, rank$(F_l)=$rank$(L_l)+1>1$ and rank$(G_l)>0$, therefore (2.5) satisfies the conditions of Lemma 1 with $F=F_l$ and $G=G_l$. Consequently $F_l\cong\mathcal{O}_{\mathbf{P}^3}(c_1(F_l)-c_1(L_l))\oplus L_l$ or $G_l\cong L_\beta/L_l$. In the first case, since $h^1(E(-2))\neq 0$, one has $l+1<n_1$, $F_l=\mathcal{O}_{\mathbf{P}^3}(-a)\oplus\bigoplus_{n=\alpha}^{l}\mathcal{O}_{\mathbf{P}^3}(-n)^{e_n}$ and $L_l=\bigoplus_{n=\alpha}^{l}\mathcal{O}_{\mathbf{P}^3}(-n-1)^{e_n}$. Moreover, $c_1(F_l)-c_1(L_l)=-a+\text{rank}(L_l)>-a\geqq\min\{-n\mid e_n\neq 0\ (\alpha\leqq n\leqq l)\}>\min\{-n-1\mid e_n\neq 0\ (\alpha\leqq n\leqq l)\}$. Since the splitting of a vector bundle on $\mathbf{P}^3$ as the direct sum of line bundles is unique, if it exists, this cannot happen. In the second case, one has $l+2>n_b$, $G_l=\bigoplus_{n=l+1}^{\beta}\mathcal{O}_{\mathbf{P}^3}(-n)^{e_n}$ and $L_\beta/L_l=\bigoplus_{n=l+1}^{\beta}\mathcal{O}_{\mathbf{P}^3}(-n-1)^{e_n}$ by the same reason as above, and again we are led to a contradiction. This proves i).

Now suppose that B_{sh} satisfies (2.7). Let X be an arbitrary a. B. curve with short basic sequence B_{sh}. Let $H_1, \cdots, H_\gamma$ be the basis of $H^0(F_\beta\otimes L_\beta^\vee)$, $t=\{t_i\mid 1\leqq i\leqq\gamma\}$ be a set of indeterminates over R and $T:=\text{Spec}(k[t])$. Set $\tilde{H}=\sum_{i=1}^{\gamma}t_iH_i$. Since $c_1(F_\beta)-c_1(L_\beta)=0$ by (2.5), we can construct the deformation of the complex (2.5)

$$(2.9)\qquad 0\longrightarrow L_\beta\otimes_k\mathcal{O}_T\xrightarrow{\ \tilde{v}\ }F_\beta\otimes_k\mathcal{O}_T\xrightarrow{\ \tilde{w}\ }\tilde{\mathcal{J}}\longrightarrow 0$$

in a natural way, where $\tilde{v}$ is defined by $\tilde{H}$ and $\tilde{\mathcal{J}}$ is the ideal sheaf in $\mathcal{O}_{\mathbf{P}^3_T}$ generated locally by the maximal minors of $\tilde{H}$. Let $\tilde{X}$ denote the closed subscheme of $\mathbf{P}^3_T$ determined by $\tilde{\mathcal{J}}$ and $\pi:\mathbf{P}^3_T\to T$ the natural projection. Since $e_{m+1}\neq 0$ $(\alpha\leqq m\leqq\beta-1)$ and $(F_m\oplus\mathcal{O}_{\mathbf{P}^3}(-m-1)^{e_{m+1}})\otimes\mathcal{O}_{\mathbf{P}^3}(m+1)$ is generated by its global sections for every m $(\alpha\leqq m\leqq\beta)$, each point of $\mathbf{P}^3_T$ has a neighborhood on which $\tilde{H}$ satisfies the condition (2.4) with suitable A and S. Here, observe that A is the quotient ring of a polynomial ring over k with respect to a multiplicative set of the form $\{\varphi^j\mid j\geqq 0\}$. There exists therefore by Lemma 2 a closed subscheme Z of $\mathbf{P}^3_T$ such that $\text{codim}_{\mathbf{P}^3_T}(Z)\geqq 5$ and $\tilde{X}\setminus Z$ is smooth over k. Since $\dim(\pi(Z))\leqq\dim(T)-2$, general fibers of $\pi_{|\tilde{X}}$ are smooth curves if char(k)=0. Besides, the restriction of the complex (2.9) to a general point of T is exact. Let $\pi^{-1}(o):=X_o$ ($o\in T$) be one of the general fibers of $\pi_{|\tilde{X}}$. Since the restriction of (2.9) to o is exact, we see by Lemma 4 that the short basic sequence of X_o is B_{sh}, and X_o is connected except in the case (2.8) by Lemma 3. Therefore if char(k)=0 and B_{sh} fulfills (2.6.1) or (2.6.2), it is realized by smooth irreducible a.B. curves in $\mathbf{P}^3$. Q.E.D.

REMARK 1. One can deduce the necessity of (2.6.1) or (2.6.2) also from [**2**; Corollary 1.3], taking into account the explicit form of the matrix of relations among the generators of I_X associated with the basic sequence of X (see [**1**; (4.1.1), 2), 3) and 4)]).

COROLLARY 1. *All the integral* a.B. *curves in* $\mathbf{P}^3$ *with the same short basic sequence are parameterized by a smooth rational variety and the general members are smooth in the case* char(k)=0.

PROOF. See [**1**; Remark 5.3].

COROLLARY 2 (cf. [**1**; Theorem 3.1]). *Let X be an integral* a.B. *curve with short basic sequence B_{sh}. Then $a\geqq 2b+n_b-n_1$, with equality if and only if $a-2b=n_b-n_1=0$ or $a-2b=n_b-n_1>0$, $m_1=n_1$, $m_i=m_{i-1}+1$ $(2\leqq i\leqq a-2b)$ and $m_{a-2b}=n_b-1$.*

COROLLARY 3. *Let X be as above. Put $\nu=\min(m_1, n_1)$ and $\delta=\min\{m \mid I_{X,m}$ generates $\bigoplus_{n\geqq m} I_{X,n}$ over $R\}$. Then $\delta\leqq\max(a-2b+\nu-2, n_b)$.*

PROOF. Let $B(X)=(a; \nu_1, \cdots, \nu_a; \nu_{a+1}, \cdots, \nu_{a+b})$ be the basic sequence of X and $(f_0; f_1, \cdots, f_a; f_{a+1}, \cdots, f_{a+b})$ the generators of I_X associated with $B(X)$, where $\deg(f_0)=a$, $\deg(f_i)=\nu_i$ $(1\leqq i\leqq a+b)$. Then $(\nu_1, \cdots, \nu_a)=(m_1, \cdots, m_{a-2b}, n_1, \cdots, n_b, n_1, \cdots, n_b)$ up to permutation and $(\nu_{a+1}, \cdots, \nu_{a+b})=(n_1, \cdots, n_b)$ (see Section 1). By definition $\nu_1=\nu$ and $\nu_a=\max(m_{a-2b}, n_b)$. Clearly one sees

$$\delta\leqq\nu_a. \tag{2.10}$$

If $a-2b=0$, then $B(X)=(2b; \nu^{2b}; \nu^b)$ and $\nu=n_b$ by (2.6.1) or (2.6.2), which implies the assertion. In the case $a-2b>0$, one has $\nu=m_1$, $n_b-1\leqq m_{a-2b}\leqq m_1+(a-2b-1)$ by (2.6.2). If $m_{a-2b}\leqq n_b$, then $\delta\leqq n_b\leqq\max(a-2b+\nu-2, n_b)$ by (2.10). Now suppose $m_{a-2b}>n_b$. Then $\nu_a=m_{a-2b}$, $\delta\leqq m_{a-2b}\leqq m_1+(a-2b-1)$. Since $\max(a-2b+\nu-2, n_b)=m_1+(a-2b-2)$, we have only to show that the case $\delta=m_{a-2b}=m_1+(a-2b-1)$ does not occur. If $m_{a-2b}=m_1+(a-2b-1)$, then $m_i=m_1+(i-1)$ for all $1\leqq i\leqq a-2b$ by (2.6.2). This implies that $\#\{i \mid \nu_i=\nu_a\ (1\leqq i\leqq a)\}=1$, $a<\nu_a$ and $\nu_i<\nu_a$ for all i distinct from a, therefore we find by [**2**; Corollary 1.3] that $f_a\in I_{X,\nu_a-1}\cdot R$. Consequently $\delta<\nu_a$ and the assertion follows.

Q.E.D.

REMARK 2. 1) Note that $a=\min\{n\mid h^0(\mathcal{J}_X(n))\neq 0\}$, $\nu=\min\{n\mid (I_X/(f_0))_n\neq 0\}$, $b=\sum_{n\in N}h^1(\mathcal{J}_X(n))$, $n_1=\min(N)+2$, $n_b=\max(N)+2$, where $N=\{n\mid h^1(\mathcal{J}_X(n))\neq 0\}$.

2) The inequality $a\geqq 2b+n_b-n_1$ is proved in [**3**; Theorem 2.12] by a different method.

3) Since $\max(a-2b+\nu-2,\ n_b)\leqq a-2b+\nu$, one has $\delta\leqq a-2b+\nu$. This inequality is proved in [**5**; Theorems 5.4 and 5.6] by a different method.

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Note added in proof. At the proofreading stage, the author made a minor change in the choice of the open sets U_i appearing in the proof of Lemma 2 and raised the lower bound of the codimension of Z by one for future application.