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Weak expectations in C^*-dynamical systems

By Charles J. K. BATTY and Masaharu KUSUDA

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1. Introduction.

Attempts to extend a factorial state φ on a C^*-algebra B to a factorial state on a larger C^*-algebra A mainly centred around searches for solutions of a tensor product problem, or equivalently for weak expectations for the GNS representation π_φ, that is, linear contractions P of A into $\pi_\varphi(B)''$ such that $P|_B=\pi_\varphi$ (see [1] and the references cited therein). The eventual solutions of the problem [7, 9] were variants of this method.

In the case when there is an action α of an amenable group G on A leaving B invariant, an analogous problem is to consider an α-invariant state φ of B which is centrally ergodic in the sense that

$$\pi_\varphi(B)''\cap\pi_\varphi(B)'\cap u_\varphi(G)' = \boldsymbol{C}\cdot 1,$$

where (π_φ, u_φ) is the associated covariant representation of (B, G, α), and to try to find an extension to a centrally ergodic state of A. It was shown in [3] that this can be done by the method of [1] if B is (semi)nuclear, but the von Neumann algebra theory developed in [7, 9] is not sufficient to provide a general solution. A corollary of a successful solution is that if A is separable and G-central (and B is nuclear), then B is also G-central.

The purpose of this paper is to clarify the covariant situation. Firstly, in Section 2, we consider the problem lifted to the C^*-crossed products. Thus the existence of a weak expectation $\hat{Q}$ for the representation $\pi_\varphi\times u_\varphi$ of $A\times_\alpha G$ (with respect to the subalgebra $B\times_\alpha G$) is seen to be equivalent to the existence of a (covariant) completely positive contraction Q of A into $(\pi_\varphi(B)\cup u_\varphi(G))''$ such that $Q|_B=\pi_\varphi$. Under these circumstances, one may apply the results of [1] to the crossed products. Secondly, in Section 3, it is observed that, if A is G-central, then $\hat{Q}$ and Q always exist. Thus the question of G-centrality of B is reduced to the problem of arranging that Q maps A into $\pi_\varphi(B)''$.

For the theory of crossed products, the reader is referred to [8, Chapter 7]; for the basic theory of invariant states, to [4, 4.3].

2. Covariant weak expectations.

Let (A, G, α) be a C^*-dynamical system, and B be an α-invariant C^*-subalgebra of A. Let $(\mathcal{H}, \pi, u)$ be a covariant representation of (B, G, α) and $\mathcal{M}=(\pi(B)\cup u(G))''$. A *covariant weak expectation* for $(\mathcal{H}, \pi, u)$ is a completely positive linear contraction $Q: A\to\mathcal{M}$ such that $Q|_B=\pi$ and $Q(\alpha_t(a))=u_tQ(a)u_t^*$ $(a\in A,\ t\in G)$.

We may also consider the C^*-crossed product $A\times_\alpha G$, which is the completion of $L^1(G; A)$ in a suitable norm, and the C^*-subalgebra B_G of $A\times_\alpha G$ generated by $L^1(G; B)$. A *weak expectation* for $(\mathcal{H}, \pi\times u)$ is a linear contraction $\hat{Q}: A\times_\alpha G\to\mathcal{M}$ such that $\hat{Q}(y)=(\pi\times u)(y)$ $(y\in L^1(G; B))$. Note that this definition is not quite covered by the definition of weak expectations in [**1**], since there is no reason, a priori, why it is automatically possible to embed $B\times_\alpha G$ in $A\times_\alpha G$, or to factor $\pi\times u$ through B_G. (In general, B_G is a quotient of $B\times_\alpha G$; the algebras coincide if G is amenable.)

PROPOSITION 1. *There is a bijective correspondence between covariant weak expectations $Q: A\to\mathcal{M}$ for $(\mathcal{H}, \pi, u)$ and weak expectations $\hat{Q}: A\times_\alpha G\to\mathcal{M}$ for $(\mathcal{H}, \pi\times u)$.*

PROOF. Suppose that $Q: A\to\mathcal{M}$ is a covariant weak expectation for $(\mathcal{H}, \pi, u)$. Define $\hat{Q}: L^1(G; A)\to\mathcal{M}$ by

$$\hat{Q}(x)=\int_G Q(x(t))u_t dt.$$

Then

$$\begin{aligned}\hat{Q}(x^*)&=\int_G \Delta(t)^{-1}Q(\alpha_t(x(t^{-1})^*))u_t dt\\ &=\int_G \Delta(t)^{-1}u_tQ(x(t^{-1}))^*dt\\ &=\int_G u_t^*Q(x(t))^*dt\\ &=\hat{Q}(x)^*.\end{aligned}$$

For y in $L^1(G; B)$,

$$\hat{Q}(y)=\int_G Q(y(t))u_t dt=\int_G \pi(y(t))u_t dt=(\pi\times u)(y).$$

Let ξ be a unit vector in $\mathcal{H}$. Consider the map $\Psi: G\to A^*$ defined by

$$\Psi(t)(a)=\langle Q(a)u_t\xi, \xi\rangle.$$

For t_i in G and a_i in A,

$$\sum_{i,j=1}^n \Psi(t_i^{-1}t_j)(\alpha_{t_i^{-1}}(a_i^*a_j))=\sum_{i,j=1}^n \langle u_{t_i}^* Q(a_i^*a_j)u_{t_i}u_{t_i}^*u_{t_j}\xi, \xi\rangle \geqq 0$$

by [**10**, IV. 3.4]. Thus Ψ is positive-definite. Also $\Psi(e)(a)=\langle Q(a)\xi, \xi\rangle$, so $\Psi(e)$ is a state of A. By [**8**, 7.6.8], there is a state ω_ξ of $A\times_\alpha G$ such that

$$\Psi(t)(a) = \omega_\xi(a\lambda_t)$$

where the same symbols are used to denote the canonical extension of ω_ξ to the multiplier algebra $M(A\times_\alpha G)$, A is embedded in $M(A\times_\alpha G)$, and λ is the unitary representation of G in $M(A\times_\alpha G)$. For $x=x^*$ in $L^1(G; A)$,

$$\omega_\xi(x) = \int_G \omega_\xi(x(t)\lambda_t)dt = \int_G \langle Q(x(t))u_t\xi, \xi\rangle dt = \langle \hat{Q}(x)\xi, \xi\rangle .$$

Thus

$$|\langle \hat{Q}(x)\xi, \xi\rangle| \leqq \|x\|_{A\times_\alpha G}.$$

Since $\hat{Q}(x)^*=\hat{Q}(x^*)=\hat{Q}(x)$, $\|\hat{Q}(x)\|\leqq\|x\|_{A\times_\alpha G}$. Hence $\hat{Q}$ extends by continuity to a bounded self-adjoint linear map, also denoted by $\hat{Q}$, of $A\times_\alpha G$ into $\mathcal{M}$ which is a contraction on the self-adjoint part. Then $\hat{Q}$ extends to an ultraweakly continuous linear map, also denoted by $\hat{Q}$, of $(A\times_\alpha G)^{**}$ into $\mathcal{M}$ which is a contraction between the self-adjoint parts. Furthermore, $\pi\times u=\hat{Q}\circ\Phi$ where $\Phi: B\times_\alpha G\rightarrow B_G$ is the canonical $*$-homomorphism, so this identity remains valid for the ultraweakly continuous extensions. Since $\pi\times u$ is non-degenerate, $\hat{Q}(\hat{e})=I_{\mathcal{H}}$, where $\hat{e}$ is the identity of $B_G{}^{**}$, so $\hat{e}$ is a projection in $(A\times_\alpha G)^{**}$. Now, if $\hat{1}$ is the identity of $(A\times_\alpha G)^{**}$,

$$\|I_{\mathcal{H}}\pm\hat{Q}(\hat{1}-\hat{e})\| = \|Q(\hat{e}\pm(\hat{1}-\hat{e}))\| \leqq \|\hat{e}\pm(\hat{1}-\hat{e})\| = 1.$$

Hence $\hat{Q}(\hat{1}-\hat{e})=0$ so $\hat{Q}(\hat{1})=I_{\mathcal{H}}$. For x in $(A\times_\alpha G)^{**}$ with $0\leqq x\leqq\hat{1}$,

$$\|I_{\mathcal{H}}-\hat{Q}(x)\| \leqq \|\hat{1}-x\| \leqq 1.$$

Since $\hat{Q}(x)$ is self-adjoint, $\hat{Q}(x)\geqq 0$. Thus $\hat{Q}$ is positive. Since $\hat{Q}(\hat{1})=I_{\mathcal{H}}$, $\hat{Q}$ is a contraction on $(A\times_\alpha G)^{**}$ and hence on $A\times_\alpha G$ [**4**, 3.2.6].

Let (f_i) be an approximate unit for $L^1(G)$. For a in A, put $(a\otimes f_i)(t)=f_i(t)a$, so $a\otimes f_i\in L^1(G; A)$ and $a\otimes f_i\rightarrow a$ ultraweakly in $(A\times_\alpha G)^{**}$. Then

$$Q(a) = \lim\Big(\int_G f_i(t)u_t dt\Big)Q(a) = \lim \hat{Q}(a\otimes f_i) = \hat{Q}(a),$$

the limits being in the ultraweak topology.

Conversely, let $\hat{Q}: A\times_\alpha G\rightarrow\mathcal{M}$ be a weak expectation for $(\mathcal{H}, \pi\times u)$. Then $\hat{Q}$ extends to an ultraweakly continuous mapping, also denoted by $\hat{Q}$, of $(A\times_\alpha G)^{**}$ into $\mathcal{M}$. Furthermore, the kernel of Φ is contained in the kernel of $\pi\times u$, so there is a representation ρ of B_G such that $\pi\times u=\rho\circ\Phi$ and $\hat{Q}$ is a weak expectation for ρ in the sense of [**1**]. By [**1**, 2.1], $\hat{Q}$ is completely positive, and satisfies the module property:

$$\hat{Q}(y_1 x y_2) = \rho(y_1)\hat{Q}(x)\rho(y_2) \qquad (y_1, y_2\in B_G{}^{**};\ \ x\in(A\times_\alpha G)^{**}).$$

Identifying A with its image in $M(A\times_\alpha G)$, put $Q=\hat{Q}|_A$. Then Q is a completely positive contraction of A into $\mathcal{M}$,

$$Q(b)=\hat{Q}(b)=\rho(b)=\pi(b) \qquad (b\in B)$$

$$Q(\alpha_t(a))=\hat{Q}(\lambda_t a\lambda_t^*)=\rho(\lambda_t)\hat{Q}(a)\rho(\lambda_t^*)=u_t Q(a)u_t^* \qquad (a\in A).$$

Thus Q is a covariant weak expectation.

For x in $L^1(G;A)$, $x=\int_G x(t)\lambda_t dt$, the integral being ultraweakly convergent in $(A\times_\alpha G)^{**}$. Hence

$$\hat{Q}(x)=\int_G \hat{Q}(x(t)\lambda_t)dt=\int_G \hat{Q}(x(t))\rho(\lambda_t)dt=\int_G Q(x(t))u_t dt.$$

This establishes the bijective correspondence.

REMARKS. 1. From the proof of Proposition 1, we see that a covariant weak expectation Q satisfies the module property

$$Q(b_1ab_2)=\pi(b_1)Q(a)\pi(b_2) \qquad (a\in A;\, b_1,\, b_2\in B).$$

This may also be deduced from Stinespring's theorem for any completely positive mapping $Q: A\to\mathcal{M}$ such that $Q|_B=\pi$.

2. There is a standard argument to show that any linear contraction $Q: A\to\mathcal{M}$, such that $Q|_B=\pi$, is positive. Moreover, Q is completely positive if it satisfies any one of the following additional properties:

(i) Q is a complete contraction,

(ii) Q maps A into $\pi(B)''$ [**1**, 2.1],

(iii) Q is covariant, and for t_i in G and a_i in A,

$$\sum_{i,j=1}^{n} u_{t_i}^* Q(a_i^* a_j)u_{t_j}\geqq 0$$

(see the proof of Proposition 1).

However, in general, Q may not be completely positive, even if it is covariant. For example, let A be the C^*-algebra M_2 of 2×2 complex matrices, B be the subalgebra of diagonal matrices, $G=\{0, 1\}$, $\alpha_1=\mathrm{Ad}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, π be the identity representation of B on $\boldsymbol{C}^2$, $u_1=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, and Q be the transpose map.

3. A covariant weak expectation Q may fail to map A into $\pi(B)''$. For example, let $A=M_2\otimes M_2$, $B=M_2\otimes I_2$, $G=U(2)$, $\alpha_t=\mathrm{Ad}(t\otimes\bar{t})$, $\mathcal{H}=\boldsymbol{C}^2\otimes\boldsymbol{C}^2$, $\pi(b\otimes I_2)=b\otimes I_2$ $(b\in M_2)$, $u_t=t\otimes\bar{t}$. Then $(\mathcal{H}, \pi, u)$ is a covariant representation of (B, G, α) with $u(G)$-invariant cyclic vector $(1/\sqrt{2})((1, 0)\otimes(1, 0)+(0, 1)\otimes(0, 1))$, and $\pi(B)''=\pi(B)=M_2\otimes I_2$, $\mathcal{M}=M_2\otimes M_2$. The identity representation $Q=\pi_0$ of A is a covariant weak expectation, mapping A onto $\mathcal{M}$. Here $Q=\pi_0\times u$.

4. Suppose that G is amenable, and let m be an invariant mean on $L^{\infty}(G)$. Suppose that there is a completely positive contraction $P: A \rightarrow \mathcal{M}$ such that $P|_B=\pi$. Then there is a covariant weak expectation $Q: A \rightarrow \mathcal{M}$ given by

$$\langle Q(a)\xi, \eta\rangle = m(t \rightarrow \langle u_t^* P(\alpha_t(a))u_t\xi, \eta\rangle) \qquad (\xi, \eta \in \mathcal{H}).$$

In particular, if there is an injective von Neumann algebra $\mathcal{N}$ such that $\pi(B)'' \subseteqq \mathcal{N} \subseteqq \mathcal{M}$, then there is a weak expectation $\hat{Q}: A \times_\alpha G \rightarrow \mathcal{M}$. If B is nuclear, one may take $\mathcal{N}=\pi(B)''$ or $\mathcal{N}=\mathcal{M}$ since $B \times_\alpha G$ is nuclear [5]. If B is semi-nuclear [6], there is a weak expectation $P: A \rightarrow \pi(B)''$ and hence a covariant weak expectation $Q: A \rightarrow \mathcal{M}$.

Recall that there is an affine homeomorphism between α-invariant states φ of B and states $\tilde{\varphi}$ of $B \times_\alpha G$ with $\tilde{\varphi}(\lambda_t)=1$ for all t in G, given by

$$\tilde{\varphi}(y) = \int_G \varphi(y(t))dt \qquad (y \in L^1(G; B))$$

(see, for example, [2, 4.1]). The GNS representation of $\tilde{\varphi}$ is $(\mathcal{H}_\varphi, \pi_\varphi \times u_\varphi)$.

THEOREM 2. *Let φ be an α-invariant state of B with associated covariant representation $(\mathcal{H}_\varphi, \pi_\varphi, u_\varphi)$ of (B, G, α), and let $\mathcal{M}_\varphi$ be the von Neumann algebra generated by $\pi_\varphi(B) \cup u_\varphi(G)$. There are bijective correspondences between:*

(i) *$(\alpha \otimes 1)$-invariant states ω of $A \otimes_{\max} \mathcal{M}_\varphi'$ such that*

$$(*) \qquad \omega(b \otimes d) = \langle \pi_\varphi(b)d\xi_\varphi, \xi_\varphi\rangle \qquad (b \in B,\ d \in \mathcal{M}_\varphi'),$$

(ii) *covariant weak expectations $Q: A \rightarrow \mathcal{M}_\varphi$ for $(\mathcal{H}_\varphi, \pi_\varphi, u_\varphi)$,*

(iii) *α-invariant states ψ of A such that $\psi|_B=\varphi$ and $E_\psi \pi_\psi(A) E_\psi \subseteqq \mathcal{M}_\varphi$, where E_ψ is the projection of $\mathcal{H}_\psi$ onto $\mathcal{H}_\varphi$,*

(iv) *weak expectations $\hat{Q}: A \times_\alpha G \rightarrow \mathcal{M}_\varphi$ for $(\mathcal{H}_\varphi, \pi_\varphi \times u_\varphi)$,*

(v) *states $\tilde{\omega}$ of $(A \times_\alpha G) \otimes_{\max} \mathcal{M}_\varphi'$ such that*

$$(**) \qquad \tilde{\omega}(x \otimes d) = \int_G \langle \pi_\varphi(x(t))d\xi_\varphi, \xi_\varphi\rangle dt \qquad (x \in L^1(G; B)),$$

(vi) *states $\tilde{\psi}$ of $A \times_\alpha G$ such that $\tilde{\psi} \circ \Phi = \tilde{\varphi}$ and $E_{\tilde{\psi}} \pi_{\tilde{\psi}}(A \times_\alpha G) E_{\tilde{\psi}} \subseteqq \mathcal{M}_\varphi$, where Φ is the $*$-homomorphism of $B \times_\alpha G$ onto $B_{G'}$ and $E_{\tilde{\psi}}$ is the projection of $\mathcal{H}_{\tilde{\psi}}$ onto $[\pi_{\tilde{\psi}}(B)\xi_{\tilde{\psi}}]$.*

PROOF. The proof of [1, 2.3] shows that there is a correspondence between states ω of $A \otimes_{\max} \mathcal{M}_\varphi'$ satisfying $(*)$ and completely positive contractions $Q: A \rightarrow \mathcal{M}_\varphi$ such that $Q|_B = \pi_\varphi$, given by

$$\omega(a \otimes d) = \langle Q(a)d\xi_\varphi, \xi_\varphi\rangle \qquad (a \in A,\ d \in \mathcal{M}_\varphi').$$

(The proof in [1] did not use the assumption that the C^*-subalgebra D is ultraweakly dense in $\pi_\varphi(B)'$ except to show that $Q(A) \subseteqq \pi_\varphi(B)'$ $(=D'')$. Now taking $D=D''=\mathcal{M}'_{\varphi}$, the same proof gives the present result.) Furthermore,

Q is covariant

$$\Longleftrightarrow \quad \langle Q(\alpha_t(a))\pi_{\varphi}(b_1)d\xi_{\varphi},\ \pi_{\varphi}(b_2)\xi_{\varphi}\rangle = \langle u_{\varphi}(t)Q(a)u_{\varphi}(t)^*\pi_{\varphi}(b_1)d\xi_{\varphi},\ \pi_{\varphi}(b_2)\xi_{\varphi}\rangle$$
$$(a\in A\,;\ b_1,\ b_2\in B\,;\ t\in G\,;\ d\in\mathcal{M}'_{\varphi})$$

$$\Longleftrightarrow \quad \langle Q(b_2^*\alpha_t(a)b_1)d\xi_{\varphi},\ \xi_{\varphi}\rangle = \langle Q(\alpha_{t^{-1}}(b_2^*)a\alpha_{t^{-1}}(b_1))u_{\varphi}(t)^*du_{\varphi}(t)\xi_{\varphi},\ \xi_{\varphi}\rangle$$
$$(a\in A\,;\ b_1,\ b_2\in B\,;\ t\in G\,;\ d\in\mathcal{M}'_{\varphi})$$

$$\Longleftrightarrow \quad \omega(b_2^*\alpha_t(a)b_1\otimes d) = \omega(\alpha_{t^{-1}}(b_2^*)a\alpha_{t^{-1}}(b_1)\otimes d)$$
$$(a\in A\,;\ b_1,\ b_2\in B\,;\ t\in G\,;\ d\in\mathcal{M}'_{\varphi})$$

$$\Longleftrightarrow \quad \omega(\alpha_t(a)\otimes d) = \omega(a\otimes d) \qquad (a\in A\,;\ t\in G\,;\ d\in\mathcal{M}'_{\varphi})$$

$$\Longleftrightarrow \quad \omega \text{ is } (\alpha\otimes 1)\text{-invariant}.$$

This establishes the correspondence between (i) and (ii).

It was also shown in [**1**, 2.3] that the restriction map of the state space of $A\otimes_{\max}\mathcal{M}'_{\varphi}$ into the state space of A gives an affine homeomorphism between states ω satisfying (*) and states ϕ of A with $\phi|_B=\varphi$ and $E_{\phi}\pi_{\phi}(A)E_{\phi}\subseteq\mathcal{M}_{\varphi}$. Clearly, if ω is $(\alpha\otimes 1)$-invariant, ϕ is α-invariant. On the other hand, if ϕ is α-invariant, then it follows, for example by the uniqueness of ω, that ω is $(\alpha\otimes 1)$-invariant. This establishes the correspondence between (i) and (iii).

The correspondence between (ii) and (iv) is immediate from Proposition 1, while the correspondences between (iv), (v) and (vi) again follow from [**1**]. One merely has to observe that the condition (**) is equivalent to the requirement that

$$\tilde{\omega}(x\otimes d) = \langle(\pi_{\varphi}\times u_{\varphi})(x)d\xi_{\varphi},\ \xi_{\varphi}\rangle,$$

and that if $\tilde{\omega}$ exists, then $\tilde{\omega}(y\otimes 1)=\tilde{\varphi}(y)$ $(y\in L^1(G\,;B))$, so $\tilde{\varphi}$ factors through B_G, $\pi_{\varphi}\times u_{\varphi}$ induces a representation ρ_{φ} of B_G and the weak expectations $\hat{Q}$ for $(\mathcal{H}_{\varphi},\ \pi_{\varphi}\times u_{\varphi})$ correspond to the weak expectations for the representation $(\mathcal{H}_{\varphi},\ \rho_{\varphi})$ of the C^*-subalgebra B_G.

REMARKS. 1. The correspondences of Theorem 2 are all affine homeomorphisms in the weak* and point-ultraweak topologies. The correspondence between (iii) and (vi) is the canonical correspondence between α-invariant states ϕ of A and states $\tilde{\phi}$ of $A\times_{\alpha}G$ with $\tilde{\phi}(\lambda_t)=1$ $(t\in G)$.

2. This is an opportunity to correct an error of detail in the proof of Theorem 1 of [**3**]. Instead of working with $A\otimes_{\max}\pi_{\varphi}(B)'$, one should consider $A\otimes_{\max}D$, where D is an ultraweakly dense C^*-subalgebra of $\pi_{\varphi}(B)'$ and the action $\mathrm{Ad}\,u_{\varphi}$ of G leaves D invariant and is strongly continuous on D. This

ensures that one can apply an invariant mean to a measurable (even, continuous) function to obtain a G-invariant extension of $\tilde{\varphi}$ to $A\otimes_{\max} D$.

3. G-centrality.

Recall that an α-invariant state ψ of A is said to be *G-abelian* if, for each a, b in A and u_ψ-invariant vector η in $\mathcal{H}_\psi$,

$$\inf |\langle \pi_\psi(a'b-ba')\eta, \eta\rangle| = 0$$

where the infimum is taken over all a' in the convex hull of $\{\alpha_t(a): t\in G\}$. Moreover, A is said to be *G-abelian* if every α-invariant state ψ is G-abelian; equivalently, for each ψ, $\mathcal{M}'_\psi(=\pi_\psi(A)'\cap u_\psi(G)')$ is abelian; equivalently, the α-invariant states of A form a Choquet simplex [4, 4.3.11].

PROPOSITION 3. *Suppose that G is amenable, and A is G-abelian. For each α-invariant state φ of B, there is a covariant weak expectation for $(\mathcal{H}_\varphi, \pi_\varphi, u_\varphi)$.*

PROOF. The first step is to note that B is G-abelian. This is well known, but for completeness we give the proof. We have to show that for each α-invariant φ, and a, b in B,

$$(*) \qquad \inf |\varphi(a'b-ba')| = 0.$$

Since G is amenable, there is an α-invariant state ψ of A extending φ, and then $(*)$ follows from the G-abelianness of ψ.

Now $\mathcal{M}'_\varphi(=\pi_\varphi(A)'\cap u_\varphi(G)')$ is abelian, so $\mathcal{M}_\varphi$ is of type I, hence injective, and the existence of a weak expectation $\hat{Q}: A\times_\alpha G\to\mathcal{M}_\varphi$ follows, since $B_G\cong B\times_\alpha G$.

Recall also that an α-invariant state ψ of A is said to be *G-central* if, for each a, b in A and u_ψ-invariant vector η in $\mathcal{H}_\psi$, and x in $\pi_\varphi(A)'$,

$$\inf |\langle \pi_\psi(a'b-ba')x\eta, \eta\rangle| = 0$$

where the infimum is taken over all a' in the convex hull of $\{\alpha_t(a): t\in G\}$. Moreover, A is said to be *G-central* if every α-invariant state ψ is G-central; equivalently, A is G-central if $\pi_\psi(A)'\cap u_\psi(G)'\subseteqq\pi_\psi(A)''$ for each ψ; equivalently, the α-invariant states of A form a Choquet simplex whose boundary measures are subcentral [4, 4.3.14].

In [3], attention was given to the question whether B is G-central, assuming that A is G-central and G is amenable. In separable cases, it is enough to show that every centrally ergodic state φ of B is compressible in A (that is, there is a weak expectation $P: A\to\pi_\varphi(B)''$ for π_φ). Proposition 3 shows that there exist covariant expectations $Q: A\to\mathcal{M}_\varphi$, but in general there is no reason to suppose that φ is compressible.

One non-amenable instance when the existence of Q implies the existence of P is described in the following result.

PROPOSITION 4. *Let G be the unitary group of the C^*-algebra $\tilde{B}$ spanned by B and a unit of A (adjointed to A if necessary), and let α be the inner action of G on A. Let φ be a trace (α-invariant state) of B. Any covariant weak expectation $Q: A \to \mathcal{M}_\varphi$ maps A into $\pi_\varphi(B)''$. Conversely, any weak expectation $P: A \to \pi_\varphi(B)''$ is convariant.*

PROOF. It is possible to prove the first statement directly, but we give an alternative proof using the correspondences developed above. Let ψ be the α-invariant state of A corresponding to Q given by Theorem 2. The α-invariance means that ψ is B-central ($\psi(ab)=\psi(ba)$ for a in A, b in B), and by [1, 3.1] ψ corresponds to a weak expectation $P: A \to \pi_\varphi(B)''$. Since the correspondences are the same and one-one, $P=Q$.

Conversely, the covariance of P follows from the identity:

$$P(\alpha_v(a)) = P(vav^*) = \pi_\varphi(v)P(a)\pi_\varphi(v^*) = u_\varphi(v)P(a)u_\varphi(v)^*$$

for a in A, unitary v in $\tilde{B}$.

Various examples where φ is a trace were given in [1, Section 4].

References

[1] R. J. Archbold and C. J. K. Batty, Extensions of factorial states of C^*-algebras, J. Funct. Anal., **63** (1985), 86-100.
[2] C. J. K. Batty, Simplexes of states of C^*-algebras, J. Operator Theory, **4** (1980), 3-23.
[3] C. J. K. Batty, G-central subalgebras, and extensions of KMS states, J. Funct. Anal., **66** (1986), 11-20.
[4] O. Bratteli and D. W. Robinson, Operator Algebras and Quantum Statistical Mechanics I, Springer-Verlag, Berlin-Heidelberg-New York, 1979.
[5] P. Green, The local structure of twisted covariance algebras, Acta Math., **140** (1978), 191-250.
[6] C. Lance, Tensor products and nuclear C^*-algebras, Proc. Symp. Pure Math., **38**, Part 1, Amer. Math. Soc., Providence, R. I., 1982, pp. 379-399.
[7] R. Longo, Solution of the factorial Stone-Weierstrass conjecture. An application of standard W^*-inclusions, Invent. Math., **76** (1984), 145-155.
[8] G. K. Pedersen, C^*-Algebras and their Automorphism Groups, Academic Press, London, 1979.
[9] S. Popa, Constructing semiregular maximal abelian subalgebras in factors, Invent. Math., **76** (1984), 157-161.
[10] M. Takesaki, Theory of Operator Algebras I, Springer-Verlag, Berlin-Heidelberg-New York, 1979.

Charles J. K. BATTY
St. John's College
Oxford OX1 3JP
England

Masaharu KUSUDA
Department of Applied Mathematics
Faculty of Engineering Science
Osaka University
Toyonaka, Osaka 560
Japan