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On some properties on span

Dedicated to Professor Yukihiro Kodama on his 60th birthday

By Kazuhiro KAWAMURA

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1. Introduction.

A compact metric space is called a compactum and a continuum means a connected compactum. All maps in this paper are continuous. Lelek [$\mathbf{L}_1$] defined the *span* of compactum X as the following formula.

$$\sigma(X)=\operatorname{Sup}\left\{\begin{array}{l} c\geqq 0 \mid \text{there exists a continuum } Z \text{ in } X\times X \text{ such that} \\ \quad p_1(Z)=p_2(Z) \text{ and } d(x, y)\geqq c \text{ for each } (x, y)\in Z \end{array}\right\}$$

$$=\operatorname{Sup}\left\{\begin{array}{l} c\geqq 0 \mid \text{there exist a continuum } C \text{ and maps } f, g: C\to X \text{ such} \\ \quad \text{that } f(C)=g(C) \text{ and } d(f(p), g(p))\geqq c \text{ for each } p\in C \end{array}\right\},$$

where p_i denotes the projection $X\times X\to X$ to the i-th factor, $i=1, 2$. A continuum is called *chainable*, if it is represented as an inverse limit of closed intervals. Each chainable continuum has span zero, but it is not known whether the converse implication holds or not. So it is natural to study the following general question due to Duda and Lelek (Continuum theory problems edited by Lewis [**Lw**], Problem 162).

QUESTION. *To what extent does span zero parallel chainability?*

Several results in this direction has been obtained by several authors. In particular, Duda asked, in the above question, whether any open image of a continuum of span zero has span zero. Oversteegen has proved that open maps between hereditarily indecomposable continua preserve span zero [**O**].

The purpose of Section 2 of this paper is to answer the above question in the affirmative. Our main tool is 'Whyburn's section theorem' and (the method of proof of) an extension theorem of open maps due to Mačkowiak and Tymchatyn [**M-T**].

In Section 3, we will define two properties of continua, using span, which are weaker than the property of having span zero. After some simple results, we will study whether these properties are preserved by maps which preserve span zero.

2. Open maps preserve span zero.

The theorem which we are going to prove is as follows.

THEOREM 1. *Let $f: X \twoheadrightarrow Y$ be an open map from a compactum X onto a compactum Y. If $\sigma(X)=0$, then $\sigma(Y)=0$.*

REMARK. Rosenholtz showed that open images of chainable continua are chainable [**R**].

We need some preparations for the proof. The following observation is useful later. Let X be a compactum, then $\sigma(X)=0$ if and only if each component of X has span zero.

An onto map $f: X \twoheadrightarrow Y$ between continua X and Y is called *confluent* if for each subcontinuum K of Y and each component C of $f^{-1}(K)$, $f(C)=K$. In this case, $f|C: C \twoheadrightarrow K$ is also confluent [**Ch**]. Each open map is confluent.

THEOREM 2 ([$\mathbf{L_1}$, $\mathbf{L_3}$]). a) *Let $(X_i)_{i=1}^{\infty}$ be a sequence of compacta in a metric space S. Then* $\limsup \sigma(X_i) \leqq \sigma(\limsup X_i)$.

b) *Let X and X_i's be compacta in a metric space S. Suppose that* $\lim X_i = X$ *and there exists an onto ε_i-translation $p_i: X \twoheadrightarrow X_i$ (that is, $d(p_i(x), x) < \varepsilon_i$ for each $x \in X$) for each i, where $\varepsilon_i \to 0$ as $i \to \infty$. Then* $\sigma(X) = \lim \sigma(X_i)$.

THEOREM 3 ([**W**], p. 188). *Let $f: X \twoheadrightarrow Y$ be an onto light open map between compacta X and Y. For each dendrite D in Y (a dendrite is a Peano continuum which does not contain any simple closed curves), there exists a dendrite D_1 in X such that $f(D_1)=D$ and $f|D_1: D_1 \twoheadrightarrow D$ is a homeomorphism onto D.*

LEMMA 4. *Let $f: X \to Y$ be a map between compacta. There exist metrics d_X and d_Y on X and Y respectively, which are compatible with the topologies of X and Y and satisfy*

$$(*) \quad d_Y(f(p), f(q)) \leqq d_X(p, q) \qquad \text{for each } p, q \text{ in } X.$$

PROOF. Let D_X and D_Y be some metrics of X and Y respectively. Then d_X and d_Y are simply defined by

$$d_Y = D_Y, \qquad d_X(p, q) = D_X(p, q) + D_Y(f(p), f(q)).$$

We say that an onto map $f: X \twoheadrightarrow Y$ is *approximatively right invertible* (abbreviated to *ARI*), if for each $\varepsilon>0$, there exists a map $g: Y \to X$ which satisfies $d(fg(p), p) < \varepsilon$ for each $p \in Y$.

PROPOSITION 5. *Let $f: X \twoheadrightarrow Y$ be an ARI map between compacta. Suppose metrics d_X and d_Y on X and Y satisfy condition $(*)$ in Lemma 4. Then $\sigma(X, d_X) \geqq \sigma(Y, d_Y)$.*

The proof of this proposition is direct from the definitions of span and ARI map.

LEMMA 6. *Let* $f: X \twoheadrightarrow Y$ *be an open onto map from a compactum* X *to a continuum* Y *both of which are contained in* Q, *the Hilbert cube. Suppose that* Y *is tree-like, then there exist sequences of graphs* X_n's *and* Y_n's *in* $Q \times Q$ *and onto open maps* $f_n: X_n \twoheadrightarrow Y_n$ *which satisfy the following conditions.*

1) $X \cup \bigcup_{n=1}^{\infty} X_n$, $Y \cup \bigcup_{n=1}^{\infty} Y_n$ *(denoted by* A *and* B *respectively) are compact.* $X \cap X_n = \emptyset = X_m \cap X_n$ *for each distinct* m *and* n, $Y \cap Y_n = \emptyset = Y_m \cap Y_n$ *for each distinct* m *and* n.

2) $\lim X_n = X$, $\lim Y_n = Y$.

3) *All* Y_n's *are trees. There exists an* ε_n-*translation (not necessarily onto)* $p_n: Y \to Y_n$ *for each* n, *where* $\varepsilon_n \to 0$ *as* $n \to \infty$.

4) *If we define* $F: A \to B$ *by* $F|X = f$, $F|X_n = f_n$, *then* F *is well defined and continuous.*

The proof of Lemma 6 is almost the same as ([**M-T**], Theorem 1), so we will omit it.

We are now ready to prove Theorem 1.

PROOF OF THEOREM 1. Let $f: X \twoheadrightarrow Y$ be an open map from a compactum X onto a compactum Y and assume that $\sigma(X)=0$. To prove that $\sigma(Y)=0$, we only have to show that each component C of Y has span zero. The restriction $f|f^{-1}(C)$ is also an open map onto C, so we may assume that Y is a continuum.

As $\sigma(X)=0$, each component K of X has span zero and hence is tree-like by [**L**2, **O-T**2]. From openess of f and the remark before Theorem 2, $f|K: K \twoheadrightarrow Y$ is confluent. By ([**Mc**], Cor. 2.2), Y is tree-like.

We take graphs X_n's, Y_n's and open maps $f_n: X_n \twoheadrightarrow Y_n$ as in Lemma 6 and let T_n be $p_n(Y)$. Then clearly T_n is a tree. Note that any open map from a graph onto a graph is light.

From Lemma 4 and Condition 4) of Lemma 6, we may assume that the metrics d_A and d_B on A and B (see Lemma 6) satisfy

$$(*) \quad d_B(F(p), F(q)) \leqq d_A(p, q) \qquad \text{for each } p, q \in A\,.$$

Applying Theorem 3 to $f_n: X_n \twoheadrightarrow Y_n$ and $T_n \subset Y_n$, there exists a dendrite (hence a tree) $D_n \subset X_n$ such that $f_n(D_n) = T_n$ and $f|D_n: D_n \twoheadrightarrow T_n$ is a homeomorphism. Let $g_n = (f_n|D_n)^{-1}$, then $f_n g_n = \mathrm{Id}_{T_n}$. Applying Proposition 5, we have $\sigma(D_n) \geqq \sigma(T_n)$. We may assume, taking a subsequence if necessary, D_n converges to a continuum D and $\lim \sigma(D_n)$ exists. By the condition 2) of Lemma 6, $D \subset X$ and hence $\sigma(D) \leqq \sigma(X) = 0$.

Applying Theorem 2. a), b) and Condition 3) of Lemma 6, we have

$$\sigma(Y)=\lim \sigma(T_n) \leqq \lim \sigma(D_n) \leqq \sigma(D)=0\,,$$

as desired.

3. Two properties concerning span.

In this section, we will consider the following properties of a continuum X.

(A) Each proper subcontinuum of X has span zero.

(B) There exists a positive number $a>0$ such that each subcontinuum Y of X whose diameter is less than a has span zero.

PROPOSITION 7. 1) *For continua,*

$$\text{span zero} \underset{\not\leftarrow}{\rightarrow} \text{Property (A)} \underset{\not\leftarrow}{\rightarrow} \text{Property (B)} \underset{\not\leftarrow}{\rightarrow} \dim =1\,.$$

$$\text{Property (A)} \;\; \downarrow \!\!\not\uparrow \;\; \text{atriodic}$$

2) *Suppose that X is a Peano continuum. Then,*

$$X \text{ has Property (A)} \longleftrightarrow X \text{ has Property (B)}$$
$$\longleftrightarrow X \text{ is an arc or a simple closed curve.}$$

PROOF. 1) We show first: Property (B) $\rightarrow \dim=1$. Suppose X has (B) and let a be a positive number determined by (B). Take a finite open cover $\{U_1, \cdots, U_n\}$ of X whose mesh is less than $a/2$. It suffices to prove $\dim \mathrm{cl}\, U_i \leqq 1$. If $\dim \mathrm{cl} U_i \geqq 2$, there exist a continuum $C \subset \mathrm{cl} U_i$ and a map $f: C \rightarrow S^1$ which is essential. Since $\sigma(C)=0$, C is tree-like. This is a contradiction.

The simple closed curve has Property (A) but does not have zero span.

Let $X=A_1 \cup A_2 \cup A_3$ be a triod such that each A_i is $\sin(1/x)$-curve and $A_1 \cap A_2 \cap A_3$ is exactly the limit arc. X has Property (B) but does not have Property (A).

There exists a 2-dimensional atriodic continuum. It does not have Property (A).

Other implications are trivial.

2) It can be shown that any Peano continuum which does not contain simple triods is an arc or a simple closed curve. Using this fact, 2) is easily proved.

REMARK. The following theorem completely determines the difference between zero span and Property (A).

THEOREM 8 ([**O-T2**], Theorem 15). *Let X be a weakly chainable continuum which is in class W. Then,*

$$\sigma(X)=0 \quad \textit{if and only if } X \textit{ has Property (A)}.$$

Having span zero is equivalent to having semispan zero, for continua [D]. *So the above statement is the same as the original one.*

Several classes of mappings are known to preserve span zero. They are the classes of: local homeomorphisms, open maps, monotone maps, refinable maps and ARI maps. See [**I**], [**K**] and Theorem 1.

Here we consider whether these classes of maps preserve Property (A) and (B). The list is as follows. + means the statement holds and − means it does not hold.

List. Let $f: X \twoheadrightarrow Y$ be an onto map between continua.

	X has A $\to Y$ has A	Y has A $\to X$ has A	X has B $\to Y$ has B	Y has B $\to X$ has B
local homeomorphism	1. +	2. ?	3. +	4. +
open	5. +	6. −	7. − + if light	8. −
monotone	9. +	10. −	11. −	12. −
refinable	13. +	14. +	15. ? + if light	16. +
ARI	17. +	18. −	19. − + if light	20. −

In the rest of this section, we will give proofs, comments and examples to the above list.

1 and 5 easily follow from Theorem 1. 3, 4 and 9 are trivial.

6, 8, 10, 12, 18, 20. The projection $p: S^1 \times S^1 \twoheadrightarrow S^1$ gives a counterexample.

7 and 11. Let A be the arc of pseudo-arcs by Bing and Jones [**B-J**]. It is a chainable continuum such that

a) There exists a monotone open map $p: A \twoheadrightarrow [0, 1]$ such that each fibre is homeomorphic to a pseudo-arc.

b) Let B be another chainable continuum satisfying a) and $q: B \twoheadrightarrow [0, 1]$ be the required map. For each homeomorphism $h: p^{-1}(0)\cup p^{-1}(1)\twoheadrightarrow q^{-1}(0)\cup q^{-1}(1)$ with $h(p^{-1}(i))=q^{-1}(i)$, $i=0, 1$, there exists a fibre preserving homeomorphic extension $H: A\twoheadrightarrow B$ of h.

Let X_1, X_2, X_3 be three topological copies of the arc of pseudo-arcs and $p_i: X_i\twoheadrightarrow[0, 1]$ be the required maps and P be the pseudo-arc. Let X be the continuum obtained from the sum $X_1\oplus X_2\oplus X_3$ identifying $p_i^{-1}(0)$ with P by homeomorphisms h_i, and T be a simple triod obtained from the sum of three $[0, 1]$'s identifying 0's. p_i $(i=1, 2, 3)$ induce a natural map $p: X\twoheadrightarrow T$.

It can be proved that p is monotone and open, and that $P=p_1^{-1}(0)=p_2^{-1}(0)=p_3^{-1}(0)$ is terminal in X (i.e. each subcontinuum of X which meets both P and $X-P$ contains P). Notice that each $p_i^{-1}(0)$ is terminal in X_i. From this fact, X has Property (B) (take $a<(\operatorname{diam} P)/2$), but T does not.

Using Theorem 1, it is easy to see that light open maps preserve Property (B).

15. Assume that $r: X\twoheadrightarrow Y$ is a light refinable map and X has Property (B). Suppose that there exists a sequence of proper subcontinua (Y_n) of Y such that Y_n converges to a point $p\in Y$ and $\sigma(Y_n)>0$ for each n. Let $r_i: X\twoheadrightarrow Y$ be a $1/i$-refinement of r $(i=1, 2, \cdots)$. For each n, taking a subsequence if necessary, we assume $K_n=\lim_i r_i^{-1}(Y_n)$ exists. It is easy to see that $\sigma(K_n)>0$ and $r(K_n)=Y_n$. By taking a subsequence, K_n converges to a continuum K. But $r(K)=p$, so K is a point. This is a contradiction.

The general case of 15 is not known. See also 19.

We will omit the proofs of 13, 14, 16 and 17, because they are similar to 15.

19. The proof that light ARI maps preserve Property (B) is the same as 17 and 15.

Finally, we will give an example which indicates that ARI maps do not always preserve Property (B). We need some notation below.

Let A be the continuum of 'double $\sin(1/x)$-curve' as in Figure 1. We call each of the two segments of A a 'limit arc'. Let B be the continuum as drawn in Figure 2. Countably many A's converge to a point p. We call this point the 'limit point of B'. The segment at the right end of Figure 2 is called the "initial arc of B" and the segments which are limit arcs of A's are called 'internal arcs". Let C be the continuum as in Figure 3. Countably many A's converge to an arc. We call this arc the "limit arc of C". We also use the terminology "initial arc" and "internal arc" in the same meaning as in B.

First we describe a continuum $X\subset R^3$. It looks like Figure 4. All x_n's $(-\infty\leqq n\leqq\infty)$ are arcs, and all p_n's $(-\infty\leqq n\leqq\infty)$ are points. $[x_n, x_{n+1}]$ (=the irreducible continuum which contains both x_n and x_{n+1}) and $[x_{-n}, x_{-n-1}]$ are homeomorphic copies of A. x_n and x_{n+1}, x_{-n} and x_{-n-1} are limit arcs of them. $x_\infty \cup \bigcup_n [x_n, x_{n+1}]$ and $x_{-\infty} \cup \bigcup_n [x_{-n}, x_{-n-1}]$ are homeomorphic copies of C. $[x_n, p_n]$ and $[x_{-n}, p_n]$ are also copies of B ($n \leqq \infty$). p_n's are their limit points and x_n, x_{-n} are their initial arcs. We can construct so that the union of them is compact.

Next we define a continuum Y as in Figure 5. All y_n's ($-\infty \leqq n \leqq \infty$) are arcs and all q_n's ($-\infty \leqq n \leqq \infty$) are points. $[y_{-1}, q_\infty]$ and $[q_\infty, y_1]$ are homeomorphic copies of B's. q_∞ is their limit point and y_{-1}, y_1 are initial arcs and other y_n's are internal arcs. $[y_{-n}, q_n]$ and $[q_n, y_n]$ are also homeomorphic copies of B's. q_n's are limit points of them and y_n's are initial arcs. We can construct so that the union of them is compact.

A map $f: X \twoheadrightarrow Y$ is defined by "shrinking $[x_{-\infty}, p_\infty] \cup [p_\infty, x_\infty]$ to the point q_∞". It can be verified that f is certainly ARI. By the construction, X has Property (B) but Y does not.

REMARK TO 15. We can easily construct *into* maps $f_n: X \to Y$ which are $1/n$-near to f and $\operatorname{diam} f_n^{-1} f_n(x) < 1/n$ for each $x \in X$ ($n=1, 2, \cdots$).

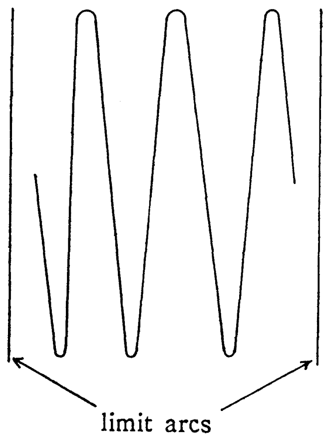

Figure 1. A.

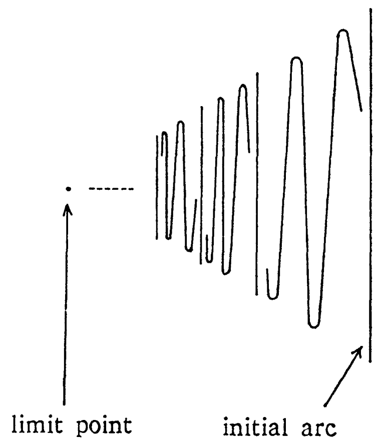

Figure 2. B.

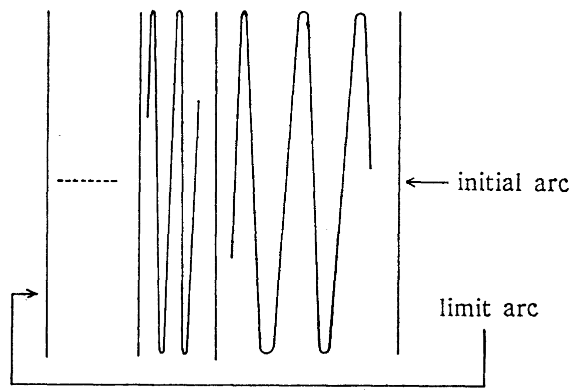

Figure 3. C.

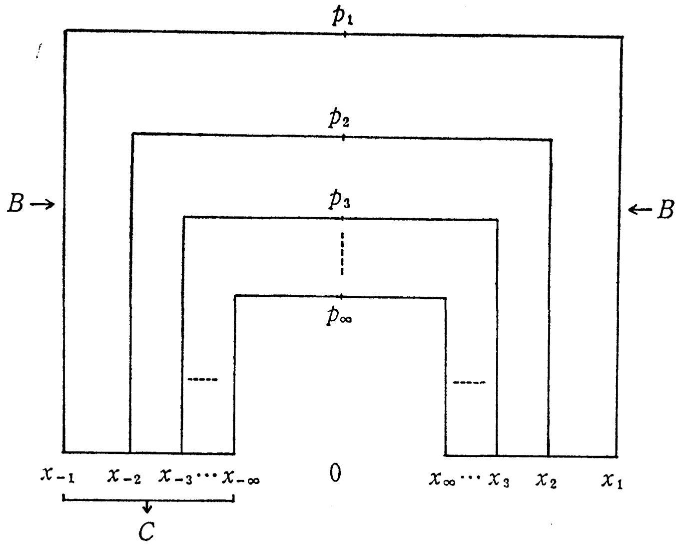

Figure 4. Continuum X.

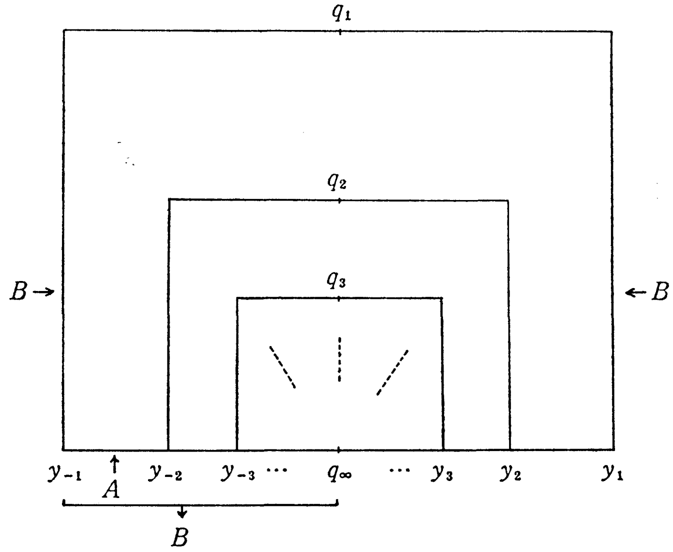

Figure 5. Continuum Y.

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Kazuhiro KAWAMURA
Institute of Mathematics
University of Tsukuba
Ibaraki 305
Japan