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Singularities of the scattering kernel for two balls

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§ 1. Introduction.

Let $\mathcal{O}$ be a compact obstacle in $\boldsymbol{R}^n$ $(n\geqq 2)$ with a C^∞ boundary $\partial\Omega$, and assume that $\Omega=\boldsymbol{R}^n-\mathcal{O}$ is connected. Let us consider the scattering by $\mathcal{O}$ expressed by the equation

$$(1.1)\qquad \begin{cases} \square u(t, x)=0 & \text{in } \boldsymbol{R}^1\times\Omega \quad (\square=\partial_t^2-\Delta_x), \\ u(t, x')=0 & \text{in } \boldsymbol{R}^1\times\partial\Omega, \\ u(0, x)=f_1(x) & \text{on } \Omega, \\ \partial_t u(0, x)=f_2(x) & \text{on } \Omega. \end{cases}$$

We denote by $k_-(s, \omega)$ $(k_+(s, \omega))\in L^2(\boldsymbol{R}^1\times S^{n-1})$ the incoming (outgoing) translation representation of the initial data $f=(f_1, f_2)$. The scattering operator $S: k_-\rightarrow k_+$ becomes a unitary operator from $L^2(\boldsymbol{R}^1\times S^{n-1})$ to $L^2(\boldsymbol{R}^1\times S^{n-1})$ (cf. Lax and Phillips [5], [6]), and is represented with a distribution kernel $S(s, \theta, \omega)$:

$$(Sk_-)(s, \omega)=\iint S(s-t, \theta, \omega)k_-(t, \omega)dtd\omega.$$

$S(s, \theta, \omega)$ is called the scattering kernel. Lax and Phillips in [5] showed that the scattering operator S determined the obstacle $\mathcal{O}$ uniquely (cf. Theorem 5.6 of Ch. V in [5]). But, it was not made clear how the analytical properties of S were connected with the geometrical properties of $\mathcal{O}$.

Recently some authors have examined the relation between $\mathcal{O}$ and $S(s, \theta, \omega)$. Majda in [7] has obtained the following results in the case of $n=3$:

$$(1.2)\qquad \operatorname{supp} S(\cdot, -\omega, \omega)\subset(-\infty, -2r(\omega)],$$

$$(1.3)\qquad -2r(\omega)\in \operatorname{sing\,supp} S(\cdot, -\omega, \omega),$$

where $r(\omega)=\min_{x\in\mathcal{O}} x\cdot\omega$. The above results are proved also in the case of $n\geqq 2$ by Soga [12]. Soga [11] and Yamamoto [14] have characterized the convexity of $\mathcal{O}$ with the singularities of $S(s, -\omega, \omega)$:

(1.4) *$\mathcal{O}$ is convex if and only if* $\operatorname{sing\,supp} S(\cdot, -\omega, \omega)$ *has only one point for any* $\omega \in S^{n-1}$.

In the present paper we shall examine $\operatorname{sing\,supp} S(\cdot, -\omega, \omega)$ precisely when $\mathcal{O}$ consists of two balls $\mathcal{O}_1$ and $\mathcal{O}_2 \subset \boldsymbol{R}^2$ or $\boldsymbol{R}^3$. In this case, by the above results (1.2)$\sim$(1.4), the right end point of $\operatorname{sing\,supp} S(\cdot, -\omega, \omega)$ is $-2r(\omega)$, and furthermore there exist other points of $\operatorname{sing\,supp} S(\cdot, -\omega, \omega)$ in $(-\infty, -2r(\omega))$ for some $\omega \in S^{n-1}$.

Let d_i be the radius of $\mathcal{O}_i$ and $r_i(\omega) = \min_{x \in \mathcal{O}_i} x \cdot \omega$ $(i=1, 2)$. Suppose that $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$. The first main result is the following theorem:

THEOREM 1. *Let ω be any vector in S^{n-1} $(n=2, 3)$ such that every line parallel to ω does not intersect both $\mathcal{O}_1$ and $\mathcal{O}_2$. Then we have*

$$\operatorname{sing\,supp} S(\cdot, -\omega, \omega) \cap [\min_{i=1,2}(-2r_i(\omega)), +\infty) = \{-2r_i(\omega)\}_{i=1,2}.$$

For more restricted ω, we can know whole distribution of $\operatorname{sing\,supp} S(\cdot, -\omega, \omega)$ completely. Let $x_0 \in P = \{x : x \cdot \omega = \min_{i=1,2} r_i(\omega) - 1\}$, and consider the broken ray starting at x_0 in the direction ω according to the law of geometrical optics. Then we suppose that this ray is reflected m times at the points $x_1, \cdots, x_m$ of the boundary and returns to the point x_{m+1} of P in the direction $-\omega$. Set

$$s_m^i = \sum_{j=1}^{m+1} |x_{j-1} - x_j| - 2 \qquad \text{when } x_1 \in \partial \mathcal{O}_i \quad (i=1, 2). \tag{1.5}$$

THEOREM 2. *Assume that*

$$\operatorname{dist}(\mathcal{O}_1, \mathcal{O}_2) > 13 \max_{i=1,2} d_i,$$

and let ω satisfy

$$|r_1(\omega) - r_2(\omega)| < \max_{i=1,2} d_i.$$

Then there exist the broken rays associated with (1.5) *for any positive integer m, and we have*

(i) $\operatorname{sing\,supp} S(\cdot, -\omega, \omega) = \{-2 \min_{j=1,2} r_j(\omega) - s_m^i\}_{\substack{i=1,2\\ m=1,2,\cdots}}$,

(ii) $\lim_{m \to +\infty} (s_{m+2}^i - s_m^i) = 2 \operatorname{dist}(\mathcal{O}_1, \mathcal{O}_2)$ $(i=1, 2)$,

(iii) $\lim_{m \to +\infty} \left\{ s_{2m}^i - \dfrac{(s_{2m-1}^1 + s_{2m-1}^2)}{2} \right\} = \operatorname{dist}(\mathcal{O}_1, \mathcal{O}_2)$ $(i=1, 2)$.

By Theorem 1, shifting the direction ω, we can know the radius of $\mathcal{O}_1$ and $\mathcal{O}_2$ from the right end point and the next point of $\operatorname{sing\,supp} S(\cdot, -\omega, \omega)$. Furthermore in the same way, we can look for the direction ω satisfying the condition in Theorem 2.

To analyze the singularities of $S(\cdot, -\omega, \omega)$, we use the following representation:

$$(1.6)\quad S(s, \theta, \omega)=\int_{\partial\Omega}\{\nu\cdot\theta\partial_t^{n-1}v(x\cdot\theta-s, x;\omega)-\partial_t^{n-2}\partial_\nu v(x\cdot\theta-s, x;\omega)\}dS_x \quad (\theta\neq\omega).$$

Here, ν denotes the unit inner vector normal to the boundary $\partial\Omega$, and $v(t, x;\omega)$ is the solution of the equation

$$(1.7)\quad \begin{cases} \Box v(t, x;\omega)=0 & \text{in } \boldsymbol{R}^1\times\Omega, \\ v=-2^{-1}(-2\pi i)^{1-n}\delta(t-x\cdot\omega) & \text{on } \boldsymbol{R}^1\times\partial\Omega, \\ v=0 & \text{for } t<r(\omega). \end{cases}$$

The representation (1.6) was proved by Majda [**7**] in the case of $n=3$, and by Soga [**11**] in the case of $n\geqq 2$. In §3 we prove Theorem 1 and Theorem 2 by examining how the singularities of v influence $\operatorname{sing\,supp} S(\cdot, -\omega, \omega)$ through (1.6) by the same procedures as in [**7**], [**11**], etc. In view of Guillemin [**1**], Petkov [**9**], etc., we expect that $\operatorname{sing\,supp} S(\cdot, -\omega, \omega)$ is contributed by only the broken rays associated with (1.5). The main tasks in the proof of Theorem 2 are to show that there exist actually such rays for any m (cf. Theorem 2.1) and to investigate those properties precisely (cf. Theorems 2.2 and 2.3).

§2. Properties of the broken rays.

At first, we define precisely the broken rays stated in Introduction. Denote by $\nu(x)$ the unit inner vector normal to the boundary $\partial\Omega$ at $x\in\partial\Omega$. We suppose that $\{x=x_0+l\xi_0;\ l>0\}\cap\partial\Omega\neq\emptyset$ for $x_0\in\Omega$ and $\xi_0\in S^{n-1}$, and define l_{j-1}, x_j and ξ_j successively for $j=1, 2, \cdots$ by

$$l_{j-1}=\inf\{l>0;\quad x_{j-1}+l\xi_{j-1}\in\partial\Omega\},$$
$$x_j=x_{j-1}+l_{j-1}\xi_{j-1},$$
$$\xi_j=\xi_{j-1}-2(\xi_{j-1}\cdot\nu(x_j))\nu(x_j),$$

where $l_{j-1}=\infty$ when $x_{j-1}+l\xi_{j-1}\notin\partial\Omega$ for any $l>0$. Assuming that these $\{l_j\}$, $\{x_j\}$ and $\{\xi_j\}$ are well-defined, we call the set

$$L(x_0, \xi_0)=\bigcup_j\{x=x_j+l\xi_j;\quad 0\leqq l<l_j\}$$

the broken ray starting at x_0 in the direction ξ_0, and $\{x_j\}$ the reflection points. When there exists an integer $m\geqq 1$ such that $\{x=x_m+l\xi_m;\ l>0\}\cap\partial\Omega=\emptyset$, we set

$$^{\#}\mathrm{ref}\, L(x_0, \xi_0)=m, \qquad \mathrm{dir}_\infty L(x_0, \xi_0)=\xi_m.$$

One of the main purposes in this section is to show the following theorem, which plays a fundamental role on the proof of Theorem 2 in Introduction.

THEOREM 2.1. *Let ω be any vector in S^{n-1} ($n=2, 3$) such that every line parallel to ω does not intersect both $\mathcal{O}_1$ and $\mathcal{O}_2$. Then, for any positive integer m there exists a broken ray $L^i(x_0, \omega)$ uniquely such that*

(i) *x_0 is on the plane $P=\{x : x\cdot\omega=\min_{i=1,2}r_i(\omega)-1\}$,*

(ii) *the first reflection point x_1 belongs to $\mathcal{O}_i$,*

(iii) $^{\#}\mathrm{ref}\, L^i(x_0, \omega)=m$,

(iv) $\mathrm{dir}_\infty L^i(x_0, \omega)=-\omega$.

Before proving this theorem, we explain a key lemma for the proof. The proof in the case of $n=3$ will be reduced to that in the case of $n=2$, and so we consider only the case of $n=2$ for a while.

Let the assumption in Theorem 2.1 be satisfied. Then we can assume without loss of generality that $\mathcal{O}_1\subset\{x=(x^1, x^2)\,;\, x^1<0\}$, $\mathcal{O}_2\subset\{x\,;\, x^1>0\}$ and $\omega=(0, 1)$. We employ the following mappings Φ, $\tilde{\Phi}$, Φ_1 and Φ_2 from $(-\pi, \pi]$ to the circles S^1, S^1, $\partial\mathcal{O}_1$ and $\partial\mathcal{O}_2\subset\boldsymbol{R}_x^2$ respectively:

$$\begin{aligned}\Phi(\theta) &= (\cos\theta, \sin\theta)\,,\\ \tilde{\Phi}(\theta) &= (\cos(\theta+\pi), \sin(\theta+\pi))\,,\\ \Phi_1(\theta) &= c_1+d_1(\cos\theta, \sin\theta)\,,\\ \Phi_2(\theta) &= c_2+d_2(\cos(\theta+\pi), \sin(\theta+\pi))\,.\end{aligned}$$

Note that Φ, $\tilde{\Phi}$ and Φ_i ($i=1, 2$) are diffeomorfic on $(-\pi, \pi)$ and have the inverse mappings Φ^{-1}, $\tilde{\Phi}^{-1}$ and Φ_i^{-1} respectively. For a S^1-valued smooth function $\xi(y)$ on $\partial\mathcal{O}_1$ (or an arc in $\partial\mathcal{O}_1$) consider the line $\{x=y+l\xi(y)\,;\, l>0\}$. We suppose that this line intersects $\partial\mathcal{O}_2$, and set

$$l(y)=\inf\{l>0\,;\ y+l\xi(y)\in\partial\mathcal{O}_2\},\qquad \tilde{y}(y)=y+l(y)\xi(y)\qquad (\in\partial\mathcal{O}_2)\,,$$

$$\tilde{\xi}(y)=\xi(y)-2\{\xi(y)\cdot\nu(\tilde{y}(y))\}\nu(\tilde{y}(y))\qquad (\in S^1)\,. \tag{2.1}$$

On these notations, we have

LEMMA 2.1. *Let $\xi(y)$ be a S^1-valued C^1 function on an arc $\{y=\Phi_1(\theta)\}_{\theta_1<\theta<\theta_2}\subset\partial\mathcal{O}_1$ satisfying $\xi(y)\cdot\nu(y)>0$. Assume that the function $\phi(\theta)=\Phi^{-1}\xi(\Phi_1(\theta))$ satisfies*

$$\frac{d\phi}{d\theta}(\theta)>0\qquad on\ (\theta_1, \theta_2)\,,\qquad (-\pi/2, \pi/2)\subset\phi((\theta_1, \theta_2))\,.$$

Then the line $\{\Phi_1(\theta)+l\xi(\Phi_1(\theta))\}_{l>0}$ intersects $\partial\mathcal{O}_2$ for any θ in some interval $(\theta_3, \theta_4)(\subset(\theta_1, \theta_2))$, and the mapping: $y\to\tilde{y}(y)$ is a diffeomorphism from $\{\Phi_1(\theta)\}_{\theta_3<\theta<\theta_4}$ to $\{\tilde{y}=\Phi_2(\mu)\}_{\mu_1<\mu<\mu_2}$. Furthermore $\eta(\tilde{y})=\tilde{\xi}(y(\tilde{y}))$ has the same properties as $\xi(y)$ (where $y(\tilde{y})$ is the inverse mapping of $\tilde{y}(y)$); that is, $\eta(\tilde{y})\cdot\nu(\tilde{y})>0$ holds, and the function $\tilde{\phi}(\mu)=\tilde{\Phi}^{-1}\eta(\Phi_2(\mu))$ satisfies

$$\frac{d\tilde{\phi}}{d\mu}(\mu) > 0 \qquad on\ (\mu_1, \mu_2), \tag{2.2}$$

$$[-\pi/2, \pi/2] \subset \tilde{\phi}((\mu_1, \mu_2)).$$

Ikawa [**3**] shows the diffeomorphicity of the mapping $\tilde{y}: y \to \tilde{y}(y)$ locally (see Lemma 3.2 of [**3**]). But we need the further properties of this mapping.

REMARK 2.1. Lemma 2.1 is valid also when $\mathcal{O}_1$ and $\mathcal{O}_2$ are exchanged each other.

PROOF OF LEMMA 2.1. From the assumptions, it follows that the mapping $T: (\theta, l) \to \Phi_1(\theta) + l\xi(\Phi_1(\theta))$ is diffeomorphic on $M = (\theta_1, \theta_2) \times (0, \infty)$ and that TM contains $\{x : x^1 > 0\}$. Set

$$\theta_3 = \inf\{\theta \in (\theta_1, \theta_2);\ T(\theta, l) \in \partial\mathcal{O}_2 \text{ for some } l > 0\},$$

$$\theta_4 = \sup\{\theta \in (\theta_1, \theta_2);\ T(\theta, l) \in \partial\mathcal{O}_2 \text{ for some } l > 0\}.$$

Then, as is easily seen, for any $\theta \in (\theta_3, \theta_4)$ the line $\{T(\theta, l)\}_{l>0}$ intersects $\partial\mathcal{O}_2$ transversally at two points. Let $T(\theta, l(\theta))$ be the point closer to $\partial\mathcal{O}_1$, and set $\mu(\theta) = \Phi_2^{-1}T(\theta, l(\theta))$. Note that these $l(\theta)$ and $\mu(\theta)$ are also the implicit functions defined by the equation

$$F(\theta, l, \mu) \equiv \Phi_1(\theta) + l\xi(\Phi_1(\theta)) - \Phi_2(\mu) = 0.$$

These implicit functions are well-defined since $\partial F/\partial(l, \mu) = \det(\xi(\Phi_1(\theta)), -\partial_\mu\Phi_2(\mu)) \neq 0$ (i.e., $\{T(\theta, l)\}_{l>0}$ is transversal to $\partial\mathcal{O}_2$ when $\theta_3 < \theta < \theta_4$). Denote by $\xi^\perp(\theta)$ the unit vector normal to $\xi(\Phi_1(\theta))$ with $\det(\xi, \xi^\perp) > 0$. Then, from the equality $\partial_\theta[F(\theta, l(\theta), \mu(\theta))] \cdot \xi^\perp(\theta) = 0$, we have

$$\frac{d\mu}{d\theta}(\theta)(\partial_\mu\Phi_2(\mu(\theta)) \cdot \xi^\perp(\theta)) = -(\partial_\theta\Phi_1(\theta) + l(\theta)\partial_\theta[\xi(\Phi_1(\theta))]) \cdot \xi^\perp(\theta).$$

It is seen from the assumptions that $\partial_\mu\Phi_2 \cdot \xi^\perp < 0$ and $(\partial_\theta\Phi_1 + l\partial_\theta\xi) \cdot \xi^\perp < 0$ when $\theta_3 < \theta < \theta_4$. Hence we obtain

$$\frac{d\mu}{d\theta}(\theta) < 0 \qquad \text{on } (\theta_3, \theta_4). \tag{2.3}$$

This implies that $\tilde{y}(y) = \Phi_2(\mu(\Phi_1^{-1}(y)))$ is diffeomorphic on $\{\Phi_1(\theta)\}_{\theta_3<\theta<\theta_4}$. From the definition (2.1), the inequality $\eta(\tilde{y}) \cdot \nu(\tilde{y}) > 0$ is obvious. Set

$$\psi(\mu, \sigma) = \tilde{\Phi}^{-1}[\Phi(\sigma) - 2\{\Phi(\sigma) \cdot \nu(\Phi_2(\mu))\}\nu(\Phi_2(\mu))].$$

Then we have

$$\tilde{\phi}(\mu(\theta)) = \psi(\mu(\theta), \phi(\theta)), \quad \theta_3 < \theta < \theta_4.$$

It is easily see that $\tilde{\phi}(\mu(\theta))$ is smooth on (θ_3, θ_4) and satisfies $\tilde{\phi}(\mu(\theta)) > \pi/2$ as

$\theta \to \theta_3$ and $< -\pi/2$ as $\theta \to \theta_4$. This yields that

$$[-\pi/2, \pi/2] \subset \tilde{\phi}((\mu_1, \mu_2)) \qquad (\mu_i = \mu(\theta_{5-i})).$$

When $\Phi(\sigma) \cdot \nu(\Phi_2(\mu)) < 0$, we have

$$\partial_\mu \phi(\mu, \sigma) > 0, \qquad \partial_\sigma \phi(\mu, \sigma) < 0.$$

Therefore it follows that

$$\frac{d\tilde{\phi}}{d\mu} \frac{d\mu}{d\theta} = \partial_\mu \phi \frac{d\mu}{d\theta} + \partial_\sigma \phi \frac{d\phi}{d\theta} < 0,$$

which implies that $d\tilde{\phi}/d\mu > 0$ (see (2.3)). The proof is complete.

PROOF OF THEOREM 2.1. We take the coordinates $x = (x^1, x^2)$ stated below Theorem 2.1, and consider any broken ray $L(x_0, \omega)$ with $x_0 \in P$. Assume that the first reflection point x_1 of $L(x_0, \omega)$ belongs to $\partial\mathcal{O}_1$. The case of $x_1 \in \partial\mathcal{O}_2$ can be treated in the same way. Setting $\theta = \Phi_1^{-1}(x_1)$, from the equality $\xi_1(x_1) = \omega - 2(\omega \cdot \nu(x_1))\nu(x_1)$ we have $\Phi^{-1}\xi_1(\Phi_1(\theta)) = 2\theta + \pi/2$ and $\xi_1(\Phi_1(\theta)) \cdot \nu(\Phi_1(\theta)) > 0$ on $[-\pi/2, 0)$. This yields that $(d/d\theta)[\Phi^{-1}\xi_1(\Phi_1(\theta))] > 0$ on $(-\pi/2, 0)$ and $\Phi^{-1}\xi_1(\Phi_1((-\pi/2, 0))) = (-\pi/2, \pi/2)$. Therefore $\xi_1(x_1)$ satisfies the assumptions in Lemma 2.1. Using Lemma 2.1 inductively (cf. Remark 2.1), for any positive integer m we have broken rays $L(x_0, \omega)$ with $^\#\mathrm{ref}\, L(x_0, \omega) = m$ such that ξ_m is a continuous function of x_0 and that $\Phi^{-1}\xi_m(x_0)$ (or $\tilde{\Phi}^{-1}\xi_m(x_0)$) covers $[-\pi/2, \pi/2]$ when x_0 moves on some open set in P. The uniqueness of this broken ray for each x_0 follows from (2.2) in Lemma 2.1. Therefore we obtain the broken ray $L^1(x_0, \omega)$ with the all required properties. The proof is complete.

THEOREM 2.2. *Assume that*

$$\mathrm{dist}(\mathcal{O}_1, \mathcal{O}_2) > 13 \max_{i=1,2} d_i,$$

and let $\omega \in S^{n-1}$ *satisfy*

$$|r_1(\omega) - r_2(\omega)| < \max_{i=1,2} d_i.$$

Then s_m^i *defined by* (1.5) *satisfies*

$$\min_{i=1,2} s_{m+1}^i > \max_{i=1,2} s_m^i \qquad \text{for } m \geqq 1.$$

For the proof of this theorem, we shall explain some lemmas concerned with the reflection points $x_1, \cdots, x_m$. Let $a_j \in \partial\mathcal{O}_j$, $j=1, 2$ be the points with $|a_1 - a_2| = \mathrm{dist}(\mathcal{O}_1, \mathcal{O}_2)$.

LEMMA 2.2. *Let* $x_1, \cdots, x_l, \cdots$ *be the reflection points of a broken ray. If* $x_j \in \partial\mathcal{O}_1$ *and* $x_{j-1} \in \partial\mathcal{O}_2$ *satisfy* $\Phi_1^{-1}(x_j) \geqq \Phi_1^{-1}(a_1)$ *and* $\Phi_2^{-1}(x_{j-1}) > \Phi_2^{-1}(a_2)$, *then it holds that*

$$\Phi_1^{-1}(x_j) < \Phi_1^{-1}(x_{j+2}) < \Phi_1^{-1}(x_{j+4}) < \cdots\cdots\cdots,$$
$$\Phi_2^{-1}(a_2) > \Phi_2^{-1}(x_{j+1}) > \Phi_2^{-1}(x_{j+3}) > \cdots\cdots\cdots.$$

PROOF. From the assumption and the law of the reflection, it follows that

$$\Phi^{-1}\Big(\frac{x_{j-1}-x_j}{|x_{j-1}-x_j|}\Big) < \Phi^{-1}\Big(\frac{a_2-a_1}{|a_2-a_1|}\Big) < \Phi^{-1}\Big(\frac{x_{j+1}-x_j}{|x_{j+1}-x_j|}\Big).$$

This implies that

$$\Phi_2^{-1}(x_{j-1}) > \Phi_2^{-1}(a_2) > \Phi_2^{-1}(x_{j+1}).$$

In the same way we have $\Phi_1^{-1}(x_j)<\Phi_1^{-1}(x_{j+2})$. Repeating these methods inductively, we obtain the lemma.

The following lemma is concerned with the reflection points of the broken ray L when $^{\#}\mathrm{ref}\,L$ is odd.

LEMMA 2.3. *Let $x_1, \cdots, x_{2m-1}$ ($x_1\in\partial\mathcal{O}_1$) be the reflection points of the broken ray $L^1(x_0, \omega)$ ($^{\#}\mathrm{ref}\,L^1=2m-1$) stated in Theorem* 2.1. *Then the following* (i) *or* (ii) *holds;*

(i) *If $x_m\in\partial\mathcal{O}_1$, then we have*

$$-\pi/2 \leqq \Phi_1^{-1}(x_1) < \Phi_1^{-1}(x_3) < \cdots < \Phi_1^{-1}(x_m) < \Phi_1^{-1}(a_1),$$
$$\pi/2 \geqq \Phi_2^{-1}(x_2) > \Phi_2^{-1}(x_4) > \cdots > \Phi_2^{-1}(x_{m-1}) > \Phi_2^{-1}(a_2),$$
$$x_j = x_{2m-1-(j-1)} \qquad \textit{for } j=1, 2, \cdots, m.$$

(ii) *If $x_m\in\partial\mathcal{O}_2$, then we have*

$$-\pi/2 \leqq \Phi_1^{-1}(x_1) < \Phi_1^{-1}(x_3) < \cdots < \Phi_1^{-1}(x_{m-1}) < \Phi_1^{-1}(a_1),$$
$$\pi/2 \geqq \Phi_2^{-1}(x_2) > \Phi_2^{-1}(x_4) > \cdots > \Phi_2^{-1}(x_m) > \Phi_2^{-1}(a_2),$$
$$x_j = x_{2m-1-(j-1)} \qquad \textit{for } j=1, 2, \cdots, m.$$

PROOF. Let us show only (i). (ii) can be treated in the same way. If $x_1\neq x_{2m-1}$, then $x_2\neq x_{2m-2}$ follows from Lemma 2.1 and $\xi_{2m-1}=-\omega$. Therefore successively we obtain $x_m\neq x_{2m-m}$ ($=x_m$). This is a contradiction. Hence we have

$$x_j = x_{2m-1-(j-1)} \qquad \text{for } j=1, 2, \cdots, m. \tag{2.4}$$

It is obvious that $-\pi/2\leqq\Phi_1^{-1}(x_1)$ and $\pi/2>\Phi_2^{-1}(x_2)$. We obtain $\Phi_1^{-1}(x_{2i-1})<\Phi_1^{-1}(a_1)$ and $\Phi_2^{-1}(x_{2j})>\Phi_2^{-1}(a_2)$ for any i and j ($i=1, 2, \cdots, m$; $j=1, 2, \cdots, m-1$): If not, for some $\tilde{i}$ it holds that $\Phi_1^{-1}(x_{2\tilde{i}+1})\geqq\Phi_1^{-1}(a_1)$ and $\Phi_2^{-1}(x_{2\tilde{i}})>\Phi_2^{-1}(a_2)$, which implies from Lemma 2.2 that $\Phi_1^{-1}(x_{2\tilde{i}+1})<\Phi_1^{-1}(x_{2\tilde{i}+2})<\cdots<\Phi_1^{-1}(x_{2m-1})$; this does not consist with (2.4). Let $\Phi_1^{-1}(x_i)\geqq\Phi_1^{-1}(x_{i+2})$ for an i ($1\leqq i\leqq m-2$). Then, by the same procedures as in the proof of Lemma 2.2, we have

(2.5) $\Phi_1^{-1}(x_{i+2}) > \Phi_1^{-1}(x_{i+4}) > \cdots > \Phi_1^{-1}(x_{m-2}) > \Phi_1^{-1}(x_m) > \Phi_1^{-1}(x_{m+1}) > \cdots .$

However, from (2.4), $\Phi_1^{-1}(x_{m-2})$ is equal to $\Phi_1^{-1}(x_{m+2})$, which does not consist with (2.5). Hence we have

$$\Phi_1^{-1}(x_1) < \Phi_1^{-1}(x_3) < \cdots < \Phi_1^{-1}(x_m).$$

Similarly, we have

$$\Phi_2^{-1}(x_2) > \Phi_2^{-1}(x_4) > \cdots > \Phi_2^{-1}(x_{m-1}).$$

The proof is complete.

When $^{\#}\mathrm{ref}\,L$ is even, the following lemma is obtained by the same procedures as for Lemma 2.3.

LEMMA 2.4. *Let* $x_1, \cdots, x_{2m}$ $(x_1 \in \partial\mathcal{O}_1)$ *be the reflection points of the broken ray* $L^1(x_0, \omega)$ $(^{\#}\mathrm{ref}\,L^1 = 2m)$ *stated in Theorem* 2.1. *Then there exists only one integer* l *such that*

$$\Phi_1^{-1}(x_1) < \Phi_1^{-1}(x_3) < \cdots < \Phi_1^{-1}(x_{2l+1}),$$

$$\Phi_1^{-1}(x_{2l+1}) > \Phi_1^{-1}(x_{2l+3}) > \cdots > \Phi_1^{-1}(x_{2m-1}),$$

$$\Phi_2^{-1}(x_2) > \Phi_2^{-1}(x_4) > \cdots > \Phi_2^{-1}(x_{2l}),$$

$$\Phi_2^{-1}(x_{2l+2}) < \Phi_2^{-1}(x_{2l+4}) < \cdots < \Phi_2^{-1}(x_{2m}),$$

$$-\pi/2 \leqq \Phi_1^{-1}(x_{2j-1}) < \Phi_1^{-1}(a_1) \quad \text{and} \quad \Phi_2^{-1}(a_2) < \Phi_2^{-1}(x_{2j}) \leqq \pi/2$$

$$\text{for } j=1, 2, \cdots, m.$$

REMARK 2.2. We can get the same lemmas as Lemmas 2.3 and 2.4 also when the first reflection point x_1 belongs to $\partial\mathcal{O}_2$.

PROOF OF THEOREM 2.2. Let $y_0, \cdots, y_{m+2}$ and $x_0, \cdots, x_{m+1}$ be the points defining $\min_{i=1,2} s_{m+1}^i$ and $\max_{i=1,2} s_m^i$ (cf. (1.5)) respectively. We have

$$\min_{i=1,2} s_{m+1}^i \geqq |y_0 - y_1| + |y_{m+1} - y_{m+2}| + m\,\mathrm{dist}(\mathcal{O}_1, \mathcal{O}_2) - 2,$$

$$\max_{i=1,2} s_m^i \leqq |x_0 - x_1| + |x_m - x_{m+1}| + 2\sum_{k=1}^{m} |x_k - a(x_k)| + (m-1)\,\mathrm{dist}(\mathcal{O}_1, \mathcal{O}_2) - 2,$$

where $a(x_k) = a_1$ if $x_k \in \partial\mathcal{O}_1$ and $a(x_k) = a_2$ if $x_k \in \partial\mathcal{O}_2$. From the assumption $\mathrm{dist}(\mathcal{O}_1, \mathcal{O}_2) > 13\max\{d_1, d_2\}$ and the law of the reflection, it follows that

$$\text{(2.6)} \quad \begin{aligned} |x_{k+1} - a_2| &< (13)^{-1}|x_k - a_1| \quad \text{if } \Phi_1^{-1}(x_{k+2}) \geqq \Phi_1^{-1}(x_k), \\ |x_k - a_1| &< (13)^{-1}|x_{k+1} - a_2| \quad \text{if } \Phi_1^{-1}(x_{k+2}) \leqq \Phi_1^{-1}(x_k). \end{aligned}$$

Therefore, by Lemma 2.3 and Lemma 2.4 we obtain

$$\sum_{k=1}^{m}|x_k-a(x_k)| < 4\max\{d_1,\ d_2\}\sum_{k=0}^{\infty}(13)^{-k} = \frac{13}{3}\max\{d_1,\ d_2\},$$

which yields that

$$\max_{i=1,2} s_m^i \leqq |x_0-x_1|+|x_m-x_{m+1}|-2+\frac{26}{3}\max\{d_1,\ d_2\}+(m-1)\mathrm{dist}(\mathcal{O}_1,\ \mathcal{O}_2).$$

On the other hand, from $|r_1(\omega)-r_2(\omega)|<\max\{d_1,\ d_2\}$ we have

$$0 \leqq |x_0-x_1|+|x_m-x_{m+1}|-2 \leqq 4\max\{d_1,\ d_2\}.$$

Therefore, noting that $|y_0-y_1|+|y_{m+1}-y_{m+2}|-2\geqq 0$, we obtain

$$\max_{i=1,2} s_m^i < \left(4+\frac{26}{3}\right)\max_{i=1,2}\{d_1,\ d_2\}+(m-1)\,\mathrm{dist}(\mathcal{O}_1,\ \mathcal{O}_2) < \min_{i=1,2} s_{m+1}^i\,.$$

The proof is complete.

The following theorem is concerned with the distribution of s_m^i defined by (1.5) as $m\to+\infty$.

THEOREM 2.3. *Assume that* $\mathrm{dist}(\mathcal{O}_1,\ \mathcal{O}_2)>13\max\{d_1,\ d_2\}$ *and let* $\omega\in S^{n-1}$ *satisfy the assumptions stated in Theorem* 1. *Then we have*

(i) $\lim_{m\to+\infty}(s_{m+1}^i-s_{m-1}^i) = 2\,\mathrm{dist}(\mathcal{O}_1,\ \mathcal{O}_2) \qquad (i=1,\ 2)$,

(ii) $\lim_{m\to+\infty}\left\{s_{2m}^i-\frac{(s_{2m-1}^1+s_{2m-1}^2)}{2}\right\} = \mathrm{dist}(\mathcal{O}_1,\ \mathcal{O}_2) \qquad (i=1,\ 2)$.

We explain some lemmas for the proof.

LEMMA 2.5. *Let* $x_1, \cdots, x_{2m-1}$ *and* $y_1, \cdots, y_{2m}$ $(m\geqq 1)$ *be the reflection points of the broken rays* $L^i(x_0,\ \omega)$ *and* $L^i(y_0,\ \omega)$ $(i=1,\ 2)$ *respectively with the properties stated in Theorem* 2.1. *Then it holds that*

(i) $\Phi_1^{-1}(x_1) < \Phi_1^{-1}(y_1) \qquad$ *if* x_1 *and* $y_1\in\partial\mathcal{O}_1$,

(ii) $\Phi_2^{-1}(x_1) > \Phi_2^{-1}(y_1) \qquad$ *if* x_1 *and* $y_1\in\partial\mathcal{O}_2$.

PROOF. Let us show only (i). (ii) can be treated in the same way. If $\Phi_1^{-1}(x_1)=\Phi_1^{-1}(y_1)$, then we obtain $x_j=y_j$ for $j=0,\ 1, \cdots, 2m-1$. Therefore there cannot exist y_{2m}. This is a contradiction. If $\Phi_1^{-1}(x_1)>\Phi_1^{-1}(y_1)$, then using Lemma 2.1 successively we have $\Phi_2^{-1}(x_2)<\Phi_2^{-1}(y_2)$, $\Phi_1^{-1}(x_3)>\Phi_1^{-1}(y_3)$, $\cdots$, $\Phi_2^{-1}(x_{2m-2})<\Phi_2^{-1}(y_{2m-2})$, $\Phi_1^{-1}(x_{2m-1})<\Phi_1^{-1}(y_{2m-1})$. If $\Phi_1^{-1}(x_{2m-1})<\Phi_1^{-1}(y_{2m-1})$, there cannot exist y_{2m} with $\mathrm{dir}_\infty L^1(x_0,\ \omega)=-\omega$. Hence we obtain this lemma.

LEMMA 2.6 (Lemma 3.3 in Ikawa [**3**]). *Set*

$$\mathcal{L} = \{x : x=ta_1+(1-t)a_2,\ t\in\boldsymbol{R}\}, \qquad U(\delta) = \{x\in\partial\Omega\,;\ \mathrm{dist}(x,\ \mathcal{L})\leqq\delta\},\quad \delta>0\,.$$

Let x_1, x_2, $\cdots$ *be the reflection points of a broken ray* $L(x_0, \xi_0)$, *and assume that* $x_1 \in \partial\Omega - U(\delta)$ *and* $L(x_0, \xi_0) \cap U(\delta) = \emptyset$. *Then there exists a positive constant* C *independent of* δ *such that*

$$^{\#}\mathrm{ref}\, L(x_0, \xi_0) \leqq C\delta^{-2}.$$

PROOF OF THEOREM 2.3. At first, let us show that for any $\varepsilon > 0$

$$\left|\mathrm{dist}(\mathcal{O}_1, \mathcal{O}_2) + \frac{s^i_{2m-1}}{2} - \frac{s^i_{2m+1}}{2}\right| < \varepsilon$$

if m is large enough. Combining this with (ii) in the theorem, we get (i) in the theorem. We take the δ in Lemma 2.6 so that $\delta = \varepsilon$. Let $\{x_j\}_{j=0,\cdots,2m}$ and $\{y_j\}_{j=0,\cdots,2m+2}$ be the points defining s^i_{2m-1} and s^i_{2m+1} respectively (cf. (1.5)). Since the equalities $x_j = x_{2m-1-(j-1)}$ $(j=1, \cdots, m)$ follow from Lemma 2.3, we have

$$\frac{s^i_{2m-1}}{2} = \sum_{j=0}^{m-1} |x_j - x_{j+1}| - 1.$$

From Lemma 2.6, there exists a positive integer $l = l(\varepsilon)$ independent of m such that $j < l$ if $1 \leqq j \leqq m-1$ and $x_j \notin U(\varepsilon)$. We have the same properties for $\{y_j\}_{j=1,\cdots,m+1}$. Hence we obtain

$$\begin{aligned}
&\left|\mathrm{dist}(\mathcal{O}_1, \mathcal{O}_2) + \frac{s^i_{2m-1}}{2} - \frac{s^i_{2m+1}}{2}\right| \\
&\leqq \left|\sum_{j=0}^{l-1} (|x_j - x_{j+1}| - |y_j - y_{j+1}|)\right| \\
&\quad + \left|\sum_{j=l}^{m-1} (|x_j - x_{j+1}| - |y_j - y_{j+1}|)\right| + |\mathrm{dist}(\mathcal{O}_1, \mathcal{O}_2) - |y_m - y_{m+1}|| \\
&\equiv I_1 + I_2 + I_3.
\end{aligned}$$

Taking account of (2.6) and Lemma 2.3, we get

$$\begin{aligned}
\{m-1-(l-1)\}\mathrm{dist}(\mathcal{O}_1, \mathcal{O}_2) &\leqq \sum_{j=l}^{m-1} |x_j - x_{j+1}| \\
&\leqq \{m-1-(l-1)\}\mathrm{dist}(\mathcal{O}_1, \mathcal{O}_2) + 2|x_l - a(x_l)| \cdot \sum_{j=0}^{m-1} (13)^{-j} \\
&\leqq \{m-1-(l-1)\}\mathrm{dist}(\mathcal{O}_1, \mathcal{O}_2) + C(\varepsilon) \cdot \sum_{j=0}^{m-1} (13)^{-j},
\end{aligned}$$

where the constant $C(\varepsilon)$ (>0) does not depend on m and tends to 0 as $\varepsilon \to 0$. The same inequality holds for $\sum_{j=l}^{m-1} |y_j - y_{j+1}|$. Therefore we have

$$I_2 \leqq C(\varepsilon) \cdot \sum_{j=0}^{m-1} (13)^{-j} < 2C(\varepsilon).$$

From Lemmas 2.1, 2.3 and 2.5 we see that each j-th reflection points x_j and y_j for $j\leqq l$ tend to the same point as $m\to+\infty$. Hence we get $I_1<\varepsilon$ for large m. By Lemma 2.6, it holds that $I_3<\varepsilon$ if m is large enough. Therefore the required inequality is obtained.

Next, let us check (ii). Let $\{x_j^i\}_{j=0,\cdots,2m}$ and $\{y_j^i\}_{j=0,\cdots,2m+1}$ be the points defining s_{2m-1}^i and s_{2m}^i $(i=1, 2)$ respectively. The broken ray for s_{2m}^1 coincides with that for s_{2m}^2, and so y_j^1 is equal to $y_{2m-(j-1)}^2$ for $j=1, 2, \cdots, 2m$. Hence we have

$$s_{2m}^1=\sum_{j=1}^{2m+1}|y_{j-1}^1-y_j^1|-2=\sum_{j=1}^{m}|y_{j-1}^1-y_j^1|+|y_m^1-y_{m+1}^1|+\sum_{j=1}^{m}|y_{j-1}^2-y_j^2|-2.$$

Therefore it follows that

$$\begin{aligned}&\left|s_{2m}^1-\left\{\frac{s_{2m-1}^1+s_{2m-1}^2}{2}+\mathrm{dist}(\mathcal{O}_1, \mathcal{O}_2)\right\}\right|\\ &\leqq\left|\sum_{j=1}^{m}(|y_{j-1}^1-y_j^1|-|x_{j-1}^1-x_j^1|)+2^{-1}(|y_m^1-y_{m+1}^1|-\mathrm{dist}(\mathcal{O}_1, \mathcal{O}_2))\right|\\ &\quad+\left|\sum_{j=1}^{m}(|y_{j-1}^2-y_j^2|-|x_{j-1}^2-x_j^2|)+2^{-1}(|y_m^2-y_{m+1}^2|-\mathrm{dist}(\mathcal{O}_1, \mathcal{O}_2))\right|\\ &\equiv\tilde{I}_1+\tilde{I}_2.\end{aligned}$$

By the same procedures as above, we see that $\tilde{I}_i\to0$ $(i=1, 2)$ as $m\to+\infty$. Hence (ii) is obtained. The proof is complete.

Lastly let us prove Theorem 2.1 in the case of $n=3$. Noting that $\mathcal{O}_1$ and $\mathcal{O}_2$ are balls, we see that on the (2 dimensional) plane

$$Q=\{x=t_1\omega+t_2\overrightarrow{a_1a_2}+c_1\,;\,t_1,\ t_2\in\boldsymbol{R}\}$$

there exists the broken ray with the properties stated in Theorem 2.1. Therefore it suffices to show that if the first reflection point x_1 is not on Q then $\mathrm{dir}_\infty L^i(x_0, \omega)$ is different from $-\omega$ for any m. If $x_1\notin Q$, then the half line $\{x_1+l\xi_1\,;\,l\geqq0\}$ does not intersect Q. Furthermore, by induction, we see that $x_j\notin Q$ and $\{x_j+l\xi_j\,;\,l\geqq0\}\cap Q=\emptyset$. This implies that ξ_m cannot be equal to $-\omega$.

§3. Proof of the main theorems.

Fix $\omega\in S^{n-1}$ satisfying the assumptions in Theorem 1 (or Theorem 2). Let $\alpha(s)$ be a C^∞ function such that $0\leqq\alpha(s)\leqq1$ for $s\in\boldsymbol{R}^1$, $\alpha(s)=1$ for $|s|<1/2$ and $\alpha(s)=0$ for $|s|>1$, and set

$$\alpha_\varepsilon(s)=\alpha\left(\frac{s}{2\varepsilon}\right)\qquad(\varepsilon>0).$$

From (1.6) it follows that

$$(3.1)\quad \begin{aligned} &F[\alpha_\varepsilon(s-s_0)S(s, -\omega, \omega)](\sigma) \\ &= -\iint_{R^1\times\partial\Omega} \nu\cdot\omega e^{i\sigma(s+x\cdot\omega)}\alpha_\varepsilon(-x\cdot\omega-s-s_0)\partial_s^{n-1}v(s, x; \omega)dsdS_x \\ &\quad -\iint_{R^1\times\partial\Omega} e^{i\sigma(s+x\cdot\omega)}\alpha_\varepsilon(-x\cdot\omega-s-s_0)\partial_s^{n-2}\partial_\nu v(s, x; \omega)dsdS_x, \end{aligned}$$

where F denotes the Fourier transformation in the variable s and the integral in s is in the sense of the distributions.

We take a partition of unity $\{\chi_{pq}(t, x)\}_{\substack{q=1,2\\ p=1,2,\cdots,l_q}}$ on $R^1\times\partial\Omega$ such that $\mathrm{supp}[\chi_{pq}]\cap(R^1\times\partial\mathcal{O}_{3-q})=\emptyset$ for any $p=1, \cdots, l_q$ $(q=1, 2)$. Let $v_{pq}(t, x; \omega)$ be the solution of the equation

$$(3.2)\quad \begin{cases} \Box v_{pq}\in C^\infty(R^1\times\bar{\Omega}), \\ (v_{pq}+2^{-1}(-2\pi i)^{1-n}\chi_{pq}\delta(t-x\cdot\omega))|_{R^1\times\partial\Omega}\in C^\infty(R^1\times\partial\Omega), \\ v_{pq} \quad \text{smooth} \quad \text{if } t<r(\omega). \end{cases}$$

Then $v(t, x; \omega)$ is equal to $\sum_{q=1}^{2}\sum_{p=1}^{l_q}v_{pq}(t, x; \omega) \bmod C^\infty$, and so by (3.1) we have

$$\begin{aligned} &F[\alpha_\varepsilon(s-s_0)S(s, -\omega, \omega)](\sigma) \\ &= -\sum_{q'=1}^{2}\sum_{q=1}^{2}\sum_{p=1}^{l_q}\Big\{\iint_{R^1\times\partial\mathcal{O}_{q'}} \nu\cdot\omega e^{i\sigma(s+x\cdot\omega)}\alpha_\varepsilon(-x\cdot\omega-s-s_0)\partial_s^{n-1}v_{pq}(s, x; \omega)dsdS_x \\ &\quad +\iint_{R^1\times\partial\mathcal{O}_{q'}} e^{i\sigma(s+x\cdot\omega)}\alpha_\varepsilon(-x\cdot\omega-s-s_0)\partial_s^{n-2}\partial_\nu v_{pq}(s, x; \omega)dsdS_x\Big\}+O(|\sigma|^{-\infty}) \\ &\equiv -\sum_{q'=1}^{2}\sum_{q=1}^{2}\sum_{p=1}^{l_q}\{I_{pqq'}^1(\sigma)+I_{pqq'}^2(\sigma)\}+O(|\sigma|^{-\infty}). \end{aligned}$$

In view of the boundary condition, we have

$$\sum_{q'=1}^{2}\sum_{q=1}^{2}\sum_{p=1}^{l_q}I_{pqq'}^1(\sigma)=\sum_{q'=1}^{2}\sum_{j=0}^{n-1}c_j^1\sigma^{n-1-j}\int_{\partial\mathcal{O}_{q'}}\nu\cdot\omega e^{2i\sigma x\cdot\omega}\alpha_\varepsilon^{(j)}(-2x\cdot\omega-s_0)dS_x,$$

where $c_0^1=-2^{-1}(2\pi)^{1-n}$; furthermore we obtain

$$I_{pqq'}^2(\sigma)=\sum_{j=0}^{n-2}c_j^2\sigma^{n-2-j}\iint_{R^1\times\partial\mathcal{O}_{q'}}e^{i\sigma(s+x\cdot\omega)}\alpha_\varepsilon^{(j)}(-x\cdot\omega-s-s_0)\partial_\nu v_{pq}(s, x; \omega)dS_x$$

where $c_0^2=(-i)^{n-2}$.

The phase function $x\cdot\omega|_{\partial\mathcal{O}_{q'}}$ has two stationary points: The one $x_{q'}^-$ is on $\partial\mathcal{O}_{q'}\cap\{x: x\cdot\omega=\inf_{x\in\partial\mathcal{O}_{q'}}x\cdot\omega\}$ and the other $x_{q'}^+$ on $\partial\mathcal{O}_{q'}\cap\{x: x\cdot\omega=\sup_{x\in\partial\mathcal{O}_{q'}}x\cdot\omega\}$. Therefore, if $s^0\notin\{-2r_{q'}(\omega), -2\tilde{r}_{q'}(\omega)\}$ (where $\tilde{r}_{q'}(\omega)=\sup_{x\in\mathcal{O}_{q'}}x\cdot\omega$), we have

(3.3) $$\int_{\partial\mathcal{O}_{q'}} \nu\cdot\omega e^{2i\sigma x\cdot\omega}\alpha_\varepsilon^{(j)}(-2x\cdot\omega-s_0)dS_x = O(|\sigma|^{-\infty})$$

for sufficiently small ε.

Let $\{x_l\}$ be the reflection points of a broken ray $L(x_0, \omega)$ where $x_0\in P=\{x: x\cdot\omega=\min_{i=1,2}r_i(\omega)-1\}$. And we employ the notations stated in §2 (e.g., ξ_l, $\nu(x_l)$, etc.). Since $\mathcal{O}_i$ is strictly convex, by Taylor [**13**] it is known that

$$\mathrm{WF}[\partial_\nu v_{pq}|_{R^1\times\partial\Omega}] \subset \mathrm{WF}[v_{pq}|_{R^1\times\partial\Omega}]\,,$$

where WF denotes the wave front set (cf. §3 of Ch. 10 in [**4**]). Therefore,

if $(s, x; \mathrm{grad}(s+x\cdot\omega|_{R^1\times\partial\Omega}))$ *does not belong to* $\mathrm{WF}[v_{pq}|_{R^1\times\partial\Omega}]$, *we have*

(3.4) $$\iint_{R^1\times\partial\Omega_{q'}} e^{i\sigma(s+x\cdot\omega)}\alpha_\varepsilon^{(j)}(-x\cdot\omega-s-s_0)\partial_\nu v_{pq}(s, x; \omega)dsdS_x = O(|\sigma|^{-\infty})\,.$$

On the other hand, it is easily seen that

$$\begin{aligned}&\mathrm{WF}[v_{pq}|_{R^1\times\partial\Omega}]\cap\{(s, x; \mathrm{grad}(s+x\cdot\omega|_{R^1\times\partial\Omega})): (s, x)\in \boldsymbol{R}^1\times\partial\Omega\\ &\qquad\qquad\qquad\qquad\cap\operatorname{supp}\alpha_\varepsilon(-x\cdot\omega-s-s_0)\}\\ &\subset \bigcup_{i=1}^{2}\bigcup_{m=1}^{\infty}\{(s_m^i+2\min_{i=1,2}r_i(\omega)-x_m^i,\ x_m^i; 1, \eta): x_m^i \text{ is the last reflection}\\ &\qquad \text{point associated with } s_m^i,\ \eta=-(-\omega-(-\omega\cdot\nu(x_m^i))\nu(x_m^i))\}\\ &\qquad \cup\{(\tilde{r}_i(\omega), x_i^\dagger; 1, 0): x_i^\dagger\in\partial\Omega,\ x_i^\dagger\cdot\omega=\tilde{r}_i(\omega)\} \equiv \bigcup_{i=1}^{2}\bigcup_{m=1}^{\infty}\Lambda_{im}\cup\tilde{\Lambda}_i\,.\end{aligned}$$

Thus we have only to consider the terms of $I_{pqq'}^2(\sigma)$ satisfying $(\bigcup_{i=1}^2\bigcup_{m=1}^\infty\Lambda_{im}\cup\tilde{\Lambda}_i)\cap\mathrm{WF}[v_{pq}|_{R^1\times\partial\Omega}]\neq\emptyset$.

We fix the m arbitrarily, and make the $\{\chi_{pq}\}$ so fine that for only one $p=\tilde{p}$ $\mathrm{supp}[\chi_{pq}]$ contains the first reflection point x_1 associated with s_m^q. Let us consider only the case of $q=2$. The case of $q=1$ can be treated in the same way. We can construct the asymptotic solution of the equation (3.2) with $(p, q)=(\tilde{p}, 2)$ in the same way as in §7 of Ikawa [**3**]. That is of the form

(3.5) $$\sum_{r=1}^{m}\frac{1}{2\pi}\int_{|k|\geq 1}e^{ik(\phi_r(x)-t)}\sum_{j=0}^{N}w_{r,j}(t, x)k^{-j}dk\,.$$

Here the integral is in the sense of oscillatory integral (cf. §6 of Ch. 1 in [**4**]), and ϕ_r and $w_{r,j}$ are the solutions of the following equations:

(3.6) $$\begin{cases}|\nabla\phi_r| = 1 \quad \text{in } \Omega\,,\\ \phi_r|_{\partial\mathcal{O}_l} = \phi_{r-1}|_{\partial\mathcal{O}_l} \quad (\phi_0=x\cdot\omega)\,,\\ \dfrac{\partial\phi_r}{\partial\nu}\Big|_{\partial\mathcal{O}_l} = -\dfrac{\partial\phi_{r-1}}{\partial\nu}\Big|_{\partial\mathcal{O}_l}\,,\end{cases}$$

where $l=1$ for even r and $l=2$ for odd r;

$$(3.7)\qquad \begin{cases} 2\dfrac{\partial w_{r,j}}{\partial t}+2\nabla\phi_r\cdot\nabla w_{r,j}+(\Delta\phi_r)w_{r,j}=-i\Box w_{r,j-1} \qquad (w_{r,-1}=0), \\ w_{r,j}|_{R^1\times\partial\Omega}=-w_{r-1,j}|_{R^1\times\partial\Omega}, \end{cases}$$

where $w_{1,0}|_{R^1\times\partial\Omega}=-2^{-1}(-2\pi i)^{1-n}\chi_{\tilde{p}2}(t,x)|_{R^1\times\partial\Omega}$, $w_{0,0}|_{R^1\times\partial\Omega}=0$ and $w_{1,j}|_{R^1\times\partial\Omega}=-w_{0,j}|_{R^1\times\partial\Omega}=0$ for $j\geqq 1$. By (3.4), the terms

$$v_r(t,x)\equiv\frac{1}{2\pi}\int_{|k|\geq 1}e^{ik(\phi_r-t)}\sum_{j=0}^{N}w_{r,j}k^{-j}dk$$

in (3.5) satisfy

$$\iint_{R^1\times\partial\mathcal{O}_{q'}}e^{i\sigma(s+x\cdot\omega)}\alpha_\varepsilon^{(j)}\partial_\nu v_r ds dS_x=O(|\sigma|^{-\infty}) \qquad \text{if } r\leqq m-2.$$

Therefore we see that

$$(3.8)\qquad \begin{aligned} &\iint_{R^1\times\partial\mathcal{O}_{q'}}e^{i\sigma(s+x\cdot\omega)}\alpha_\varepsilon^{(j)}\partial_\nu v_{\tilde{p}2} ds dS_x \\ &\begin{cases} =2i\sigma\displaystyle\int_{\partial\mathcal{O}_{q'}}e^{i\sigma(x\cdot\omega+\phi_m(x))}\alpha_\varepsilon^{(j)}(-x\cdot\omega-\phi_m(x)-s_0)\frac{\partial\phi_m}{\partial\nu}(x)w_{m,0}(\phi_m(x),x)dS_x \\ \quad +(\text{Similar integrals multiplying smaller power of } \sigma) \\ \quad +O(|\sigma|^{-\infty}) \qquad \text{for even } m \text{ and } q'=1 \text{ or odd } m \text{ and } q'=2, \\ =O(|\sigma|^{-\infty}) \qquad \text{for even } m \text{ and } q'=2 \text{ or odd } m \text{ and } q'=1. \end{cases} \end{aligned}$$

The phase function $(x\cdot\omega+\phi_m(x))|_{\partial\mathcal{O}_{q'}}$ has only one stationary point, which is the last reflection point $x_m^{q'}$; moreover, by Lemma 4.1 in Ikawa [**3**], it is non-degenerate. If s_0 is not equal to $-2\min_{i=1,2}r_i(\omega)-s_m^2$ and $\varepsilon>0$ is small enough, $\alpha_\varepsilon^{(j)}(-x\cdot\omega-\phi_m(x)-s_0)$ vanishes in a neighborhood of the stationary point $x_m^{q'}$, and then we have

$$\iint_{R^1\times\partial\mathcal{O}_{q'}}e^{i\sigma(s+x\cdot\omega)}\alpha_\varepsilon^{(j)}\partial_\nu v_{pq} ds dS_x=O(|\sigma|^{-\infty}).$$

From now on, let us prove Theorem 1 and Theorem 2.

PROOF OF THEOREM 1. Without loss of generality, we may assume that $r_1(\omega)\leqq r_2(\omega)$. Let us consider only the case of $r_1(\omega)<r_2(\omega)$ since the case $r_1(\omega)=r_2(\omega)$ can be treated more easily. By Majda [**7**] and Soga [**12**], it is known that $-2r_1(\omega)$ belongs to $\operatorname{sing\,supp}S(\cdot,-\omega,\omega)$. In the same way as in the proof of Lemma 4.1 in Soga [**11**], we see that v_{pq} with $\operatorname{supp}[\chi_{pq}]\ni(\tilde{r}_i(\omega),x_i^+)$ does not contribute to $\operatorname{sing\,supp}S(\cdot,-\omega,\omega)$. Let $s_0=-2r_2(\omega)$. Then, by the earlier argument, we have seen that only the v_{pq} in (3.2) satisfying $\mathrm{WF}[v_{pq}|_{R^1\times\partial\Omega}]\cap\Lambda_{21}$

$\neq \emptyset$ may influence the singularity of $S(s, -\omega, \omega)$. Therefore, by (3.8) (with $m=1$) we can write for any integer $N>0$

$$F[\alpha_\varepsilon(s+2r_2(\omega))S(s, -\omega, \omega)](\sigma)$$

$$=\sigma^{n-1}\int_{\partial\mathcal{O}_2} e^{2i\sigma x\cdot\omega}\sum_{j=0}^{N}\beta_j(x)\sigma^{-j}dS_x+O(|\sigma|^{n-1-(N+1)}),$$

where $\beta_j(x)\in C^\infty(\partial\mathcal{O}_2)$ and $\beta_0(x_1^2)=(2\pi)^{1-n}$. By means of the stationary phase methods (cf. §4 in [8]), we obtain

$$|F[\alpha_\varepsilon(s+2r_2(\omega))S(s, -\omega, \omega)](\sigma)| \geqq C|\sigma|^{(n-1)/2} \qquad \text{as } |\sigma|\to\infty$$

for a constant $C>0$. This shows that

$$\alpha_\varepsilon(s+2r_2(\omega))S(s, -\omega, \omega) \notin C^\infty(\boldsymbol{R}_s^1).$$

The proof is complete.

PROOF OF THEOREM 2. (ii) and (iii) in Theorem 2 have been proved in Theorem 2.3. From Theorem 1, it suffices to prove (i) in Theorem 2 when $m>1$. At first, we consider the case of $s_m^1\neq s_m^2$. By Theorem 2.2, we see that

$$F[\alpha_\varepsilon(s+s_m^2+2\min_{i=1,2}r_i(\omega))S(s, -\omega, \omega)](\sigma)$$

$$=2ic_0^2\sigma^{n-1}\int_{\partial\mathcal{O}_2} e^{i\sigma(x\cdot\omega+\phi_m(x))}\alpha_\varepsilon(-x\cdot\omega-\phi_m(x)+2\min_{i=1,2}r_i(\omega)+s_m^2)$$

$$\times\frac{\partial\phi_m}{\partial\nu}(x)w_{m,0}(\phi_m(x), x)dS_x$$

$$+(\text{Similar integrals multiplying smaller power of } \sigma)+O(|\sigma|^{-\infty}).$$

Therefore, by the same argument as in the proof of Theorem 1, we have

$$|F[\alpha_\varepsilon(s+s_m^2+2\min_{i=1,2}r_i(\omega))S(s, -\omega, \omega)](\sigma)| \geqq C'|\sigma|^{(n-1)/2} \qquad \text{as } |\sigma|\to\infty$$

for a constant $C'>0$. This implies that

$$\alpha_\varepsilon(s+s_m^2+2\min_{i=1,2}r_i(\omega))S(s, -\omega, \omega) \notin C^\infty. \tag{3.9}$$

In the same way, it is seen that $\alpha_\varepsilon(s+s_m^1+2\min_{i=1,2}r_i(\omega))S(s, -\omega, \omega)\notin C^\infty$.

Let $s_m^1=s_m^2$. We obtain

$$F[\alpha_\varepsilon(s+s_m^2+2\min_{i=1,2}r_i(\omega))S(s, -\omega, \omega)](\sigma)$$

$$=2ic_0^2\sigma^{n-1}\int_{\partial\mathcal{O}_2} e^{i\sigma(x\cdot\omega+\phi_m)}\alpha_\varepsilon(-x\cdot\omega-\phi_m+2\min_{i=1,2}r_i(\omega)+s_m^2)\frac{\partial\phi_m}{\partial\nu}w_{m,0}dS_x$$

$$+2ic_0^2\sigma^{n-1}\int_{\partial\mathcal{O}_1} e^{i\sigma(x\cdot\omega+\tilde{\phi}_m)}\alpha_\varepsilon(-x\cdot\omega-\tilde{\phi}_m+2\min_{i=1,2} r_i(\omega)+s_m^2)\frac{\partial\tilde{\phi}_m}{\partial\nu}\tilde{w}_{m,0}dS_x$$
$$+(\text{Similar integrals multiplying smaller power of } \sigma)+O(|\sigma|^{-\infty}),$$

where $\tilde{\phi}_m$ and $\tilde{w}_{m,0}$ are the solutions of (3.6) and (3.7) when $q=1$. Noting that $w_{m,0}$ and $\tilde{w}_{m,0}$ have the same sign, also when $s_m^1=s_m^2$ we get (3.9) in the same way.

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