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On homotopy representations with the same dimension function

By Ikumitsu NAGASAKI

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§ 0. Introduction.

This paper is concerned with classification of homotopy representations (up to G-homotopy). Let G be a finite group. A *homotopy representation* X of G is a finite dimensional G-CW-complex such that for each subgroup H of G, the H-fixed point set X^H is homotopy equivalent to a sphere S^m of dimension m which is equal to $\dim X^H$, or the empty set. T. tom Dieck and T. Petrie first introduced homotopy representations in [6] and studied their stable theory. Recently E. Laitinen studied the unstable theory of homotopy representations in [8] and showed that two homotopy representations X and Y are G-homotopy equivalent if and only if their dimension functions are equal and a certain invariant $D_n(X, Y)$ in the unstable Picard group $\mathrm{Pic}_n(G)$ vanishes, where $n=\mathrm{Dim}\, X=\mathrm{Dim}\, Y$.

T. tom Dieck studied the dimension functions of homotopy representations in [2]. In particular, he showed that the dimension function of a homotopy representation of a p-group G is equal to that of some linear G-sphere. (See also [7].) This result implies that the dimension functions of homotopy representations of a p-group are classified by the representation theory.

The purpose of this paper is to investigate the number $\mathrm{Num}(G, n)$ of G-homotopy types of homotopy representations with the same dimension function n.

In Section 1, we show that the number $\mathrm{Num}(G, n)$ is at most the order of $\mathrm{Pic}_n(G)$ (Proposition 1.7). In Section 2, we show that the number $\mathrm{Num}(G, n)$ is equal to the order of $\mathrm{Pic}_n(G)$ under certain hypotheses (Theorem 2.1). In particular, if G is a nilpotent group of odd order, then the number $\mathrm{Num}(G, n)$ is equal to the order of $\mathrm{Pic}_n(G)$ (Corollary 2.7). In Section 3, we compute the order of $\mathrm{Pic}_n(G)$ in general (Theorem 3.6). If a homotopy representation X has a G-homotopy type of a finite G-CW-complex, we call it *finite*. In Section 4, we discuss a similar problem for finite homotopy representations. However it seems difficult to compute the number $\mathrm{Num}_f(G, n)$ of G-homotopy types of finite homotopy representations with the same dimension function n because of complexity of the finiteness obstruction. When G is an abelian group of odd order, the number $\mathrm{Num}_f(G, n)$ is described by using the Swan homomorphisms

(Corollary 4.10).

This paper is based on [6] and [8], which we review in Section 1.

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§1. Preliminaries.

We first collect the various notations and results from [6] and [8]. Let X be a homotopy representation of G and $\mathcal{S}(G)$ the set of subgroups of G. We define an integer-valued function $\operatorname{Dim} X$ on $\mathcal{S}(G)$ by $(\operatorname{Dim} X)(H)=\dim X^H+1$ for $H\in\mathcal{S}(G)$ and we call it the *dimension function* of X. If X^H is empty, we set $\dim X^H=-1$. (We adopt tom Dieck and Petrie's definition for $\operatorname{Dim} X$.) Let $\operatorname{Dim}(G)$ denote the set of the dimension functions of homotopy representations of G. E. Laitinen introduced an essential isotropy group of X. Its definition is based on the following lemma.

LEMMA 1.1 ([8, Lemma 2.1]). *Let n be the dimension function of a homotopy representation X.*

(1) *If $n(H)=n(K)$ and $H\leqq K$, then the inclusion $X^K\subset X^H$ is a homotopy equivalence.*

(2) *Each subgroup H is contained in a unique maximal subgroup $\bar{H}$ with $n(H)=n(\bar{H})$.*

An isotropy group $H\in\operatorname{Iso}(X)$ with $H=\bar{H}$ is called an *essential* isotropy group. The set of essential isotropy groups is denoted by $\operatorname{EssIso}(X)$. By Lemma 1.1, $\operatorname{EssIso}(X)$ depends only on the dimension function. Let $\phi(G)$ be the set of conjugacy classes of subgroups of G and $C(G)$ the set of integer-valued functions on $\phi(G)$. We note that the dimension function can be regarded as an integer-valued function on $\phi(G)$. Let $A(G)$ be the Burnside ring of G. A ring homomorphism $\chi: A(G)\to C(G)$ is defined by $(\chi(x))(H)=|S^H|-|T^H|$ for $x=[S]-[T]$ and $(H)\in\phi(G)$, where S and T are finite G-sets. It is well-known that χ is injective. We regard $A(G)$ as the subring of $C(G)$ via χ. We recall the *Picard group* introduced in [6]. We abbreviate $A(G)$ and $C(G)$ to A and C respectively. We put $\bar{C}=C/|G|C$ and $\bar{A}=A/|G|C$, which make rings. Let C^* be the unit group of C. The Picard group of G is defined by

$$\operatorname{Pic}(G)=\bar{C}^*/\bar{A}^*C^*.$$

Laitinen introduced in [8] the unstable Picard group in order to establish the unstable theory of homotopy representations. The unstable Picard group for $n=\operatorname{Dim} X$ is defined as follows. We denote by C_n (resp. A_n, C_n^*) the subset of functions $d\in C$ (resp. A, C^*) satisfying the following unstability conditions for n.

(1.2) (*Unstability conditions* [8]).

(1) $d(H)=1$ *when* $n(H)=0$.

(2) $d(H)=1, 0, -1$ *when* $n(H)=1$.

(3) $d(H)=d(\bar{H})$ *for any* $(H)\in\phi(G)$.

We call $d\in C$ *invertible* if $d(H)$ is prime to $|G|$ for any $(H)\in\phi(G)$. We denote by $\bar{C}_n^*$ (resp. $\bar{A}_n^*$) the subset of elements of $\bar{C}^*$ (resp. $\bar{A}^*$) represented by invertible functions in C_n (resp. A_n). We note that $\bar{C}_n^*$ makes a subgroup of $\bar{C}^*$ and also $\bar{A}_n^*$ makes a subgroup of $\bar{C}_n^*$. (Note that C_n and A_n are not rings in general.) The *unstable Picard group* of G (for $n\in\mathrm{Dim}(G)$) is defined by

$$\mathrm{Pic}_n(G)=\bar{C}_n^*/\bar{A}_n^*C_n^*.$$

(Note that $\mathrm{Pic}_n(G)$ is a finite abelian group.)

Let X and Y be homotopy representations with the same dimension function n, and $f: Y\to X$ a G-map. If we orient X and Y in the sense of Laitinen, the *degree function* $d(f)$ in C is defined by $d(f)(H)=\deg f^H$, and satisfies the unstability conditions for n. Using the equivariant obstruction theory, Laitinen showed that there exists a G-map $f: Y\to X$ such that $d(f)$ is invertible. Further, he defined an unstable invariant by

$$D_n(X, Y)=[d(f)]\in\mathrm{Pic}_n(G),$$

whose definition is independent of choices of f and orientation [8]. The following is an important result for the classification of homotopy representations.

PROPOSITION 1.3 ([8, Theorem 5]). *Let X and Y be homotopy representations of G with the same dimension function n. Then X is G-homotopy equivalent to Y if and only if $D_n(X, Y)$ vanishes.*

The equivariant obstruction theory played an important role in [6] and [8]. Let X and Y be as above. Let $\mathcal{S}$ be a closed family of $\mathcal{S}(G)$, where $\mathcal{S}$ is called *closed* if $\mathcal{S}$ satisfies the conditions: (a) if $H\in\mathcal{S}$ and $H<K$, then $K\in\mathcal{S}$, and (b) if $H\in\mathcal{S}$, then $gHg^{-1}\in\mathcal{S}$ for any $g\in G$. Then $X(\mathcal{S})=\bigcup_{H\in\mathcal{S}}X^H$ is a G-CW-subcomplex of X. Let $f_{\mathcal{S}}: X(\mathcal{S})\to Y$ be a G-map. The following holds from the equivariant obstruction theory.

PROPOSITION 1.4 ([8, Proposition 3.3], [5, Chap. 8]). *Under the above situation,*

(1) *There exists a G-map $f: X\to Y$ extending $f_{\mathcal{S}}$.*

(2) *Let H be a maximal subgroup in $\mathcal{S}(G)\setminus\mathcal{S}$. We assume that $H\in\mathrm{EssIso}(X)$ and $\dim X^H\geqq 1$. We put a closed family $\mathcal{T}=\mathcal{S}\cup(H)$, where (H) is the conjugacy class of H. Then, for any integer k, there exists a G-map $f_{\mathcal{T}}: X(\mathcal{T})\to Y$ extending $f_{\mathcal{S}}$ such that $\deg f_{\mathcal{T}}^H=\deg f^H+k|WH|$. Here $WH=NH/H$ and NH is the normalizer of H in G.*

Smith theory is also useful for the theory of homotopy representations. The following result will be used in later sections.

LEMMA 1.5. *Let* $f: X \rightarrow Y$ *be a* G*-map of homotopy representations with the same dimension function. Suppose that* $\deg f^H$ *is prime to* $|G|$ *for any* $H \neq 1$. *Then* $\deg f$ *is also prime to* $|G|$.

PROOF. Let p be a prime divisor of $|G|$. Then G has a subgroup $\boldsymbol{Z}/p$ of order p. Since $\deg f^{\boldsymbol{Z}/p}$ is prime to p, $\deg f$ is also prime to p for any prime divisor p of $|G|$. (See [8].) It follows that $\deg f$ is prime to $|G|$.

Let X, Y, Z be homotopy representations with the same dimension function n. Then

LEMMA 1.6.
(1) $D_n(X, Z) = D_n(X, Y) \cdot D_n(Y, Z)$.
(2) $D_n(X, Y)^{-1} = D_n(Y, X)$.

PROOF. Let $g: Z \rightarrow Y$ and $f: Y \rightarrow X$ be G-maps with invertible degree functions. The composition $f \circ g: Z \rightarrow X$ also has an invertible degree function $d(f \circ g) = d(f) \cdot d(g)$. Hence $D_n(X, Z) = [d(f \circ g)] = [d(f)][d(g)] = D_n(X, Y) \cdot D_n(Y, Z)$. In particular $D_n(X, X) = D_n(X, Y) \cdot D_n(Y, X)$. Since $D_n(X, X) = 1$ by Proposition 1.3, it follows that $D_n(X, Y)^{-1} = D_n(Y, X)$.

Let $\mathrm{Num}(G, n)$ denote the number of G-homotopy types of homotopy representations of G with the same dimension function n. The following is our starting point.

PROPOSITION 1.7. $\mathrm{Num}(G, n) \leqq |\mathrm{Pic}_n(G)| (<\infty)$.

PROOF. Let $X_1, X_2, \cdots, X_r$ be homotopy representations with $\mathrm{Dim}\, X_i = n$ such that X_i and X_j are not G-homotopy equivalent for $i \neq j$. If $r > |\mathrm{Pic}_n(G)|$, then there exist X_s and X_t $(s \neq t)$ such that $D_n(X_1, X_s) = D_n(X_1, X_t)$. We see that $D_n(X_s, X_t) = 1$ by Lemma 1.6. Hence X_s and X_t are G-homotopy equivalent by Proposition 1.3. This is a contradiction.

§2. Realization of invertible functions.

Let X be a homotopy representation of G and n the dimension function of X. For any homotopy representation Y with $\mathrm{Dim}\, Y = n$, there exists a G-map $f: Y \rightarrow X$ such that the degree function $d(f)$ is invertible and satisfies the unstability conditions. We shall discuss the converse.

QUESTION. *Let* d *be any invertible function satisfying the unstability conditions. Do there exist a homotopy representation* Y *with* $\mathrm{Dim}\, Y = n$ *and a* G*-map* $f: Y \rightarrow X$ *such that the degree function* $d(f)$ *is equal to* d*?*

From the proof of Proposition 1.7, if this question has an affirmative answer, we see that the number $\mathrm{Num}(G, n)$ is equal to the order of $\mathrm{Pic}_n(G)$. In [**6**, Theorem 6.3], one can find an affirmative answer under certain technical hypotheses. We shall adopt the following hypotheses instead of tom Dieck and Petrie's in order to make their answer sharper.

HYPOTHESIS I. $\mathrm{Ess\,Iso}(X)$ *is closed under intersection.*

We put $\mathcal{S}_0=\{H\in\mathcal{S}(G)\,|\,n(H)\leqq 3\}$.

HYPOTHESIS II. *For any invertible function d satisfying the unstability conditions, there exist a G-CW-complex $Y(\mathcal{S}_0)$ and a G-map $f_{\mathcal{S}_0}: Y(\mathcal{S}_0)\to X$ such that*

(1) $\mathrm{Iso}(Y(\mathcal{S}_0))=\mathcal{S}_0\cap\mathrm{Ess\,Iso}(X)$,

(2) $\dim Y(\mathcal{S}_0)^H=\dim X^H$ *and* $Y(\mathcal{S}_0)^H$ *is homotopy equivalent to* $S^{n(H)-1}$ *for* $H\in\mathcal{S}_0$,

(3) $\deg f_{\mathcal{S}_0}^H=d(H)$ *for* $H\in\mathcal{S}_0$.

(We permit $Y(\mathcal{S}_0)$ to be empty.)

We shall show the following by using tom Dieck and Petrie's argument.

THEOREM 2.1. *We assume that X satisfies Hypotheses* I *and* II. *Then, for any invertible function d satisfying the unstability conditions, there exist a homotopy representation Y and a G-map $f: Y\to X$ such that* $\mathrm{Dim}\,Y=\mathrm{Dim}\,X$ *and* $d(f)=d$. *In particular* $\mathrm{Num}(G, n)=|\mathrm{Pic}_n(G)|$.

We need the following result for the proof of Theorem 2.1.

PROPOSITION 2.2. *Let Z be a homotopy representation with* $\dim Z\geqq 3$. *We assume that* $1\in\mathcal{S}(G)$ *is an essential isotropy group of Z. Then for any integer k which is prime to $|G|$, there exist a homotopy representation B with* $\mathrm{Dim}B=\mathrm{Dim}Z$ *and a G-map $\phi: B\to Z$ such that* $\deg\phi=k$ *and* $\deg\phi^H=1$ *for $H\neq 1$ and further B and ϕ satisfy the following conditions:*

(1) *B is obtained by attaching cells of types $G\times D^{m-1}$ and $G\times D^m$ to the $(m-1)$-skeleton Z_{m-1} of Z, where* $m=\dim Z$,

(2) $\phi|Z_{m-1}=\mathrm{id}$.

PROOF. Since 1 is an essential isotropy group of Z, $\dim Z^H<m$ for $H\neq 1$ (Lemma 1.1). Hence G acts freely on $Z\setminus Z_{m-1}$. Since $\dim Z\geqq 3$, we can apply [**6**, Proposition 6.4] to Z. Then we obtain a G-CW-complex B and a G-map $\phi: B\to Z$ such that B is homotopy equivalent to S^m and $\deg\phi=k$. From the construction in the proof of [**6**, Proposition 6.4], we see that B and ϕ satisfy the conditions (1) and (2). For $H\neq 1$, $B^H=Z^H$ and $\phi^H=\mathrm{id}$ by the conditions (1) and (2). Therefore B and ϕ are the desired homotopy representation and G-map.

PROOF OF THEOREM 2.1. The proof of Theorem 2.1 is similar to that of

[6, Theorem 6.3]. We shall give the detailed proof. We prove it inductively. Let $\mathcal{S}\subset\mathcal{S}(G)$ be a closed family containing $\mathcal{S}_0$. (For the definition of a closed family, see Section 1.) We note $\mathcal{S}_0$ is a closed family. We consider the following statement.

($\mathcal{S}$) *There exist a G-CW-complex $Y(\mathcal{S})$ and a G-map $f_S\colon Y(\mathcal{S})\to X$ such that*

(a, $\mathcal{S}$) $\mathrm{Iso}(Y(\mathcal{S})) = \mathcal{S}\cap\mathrm{EssIso}(X)$,

(b, $\mathcal{S}$) $\dim Y(\mathcal{S})^K = \dim X^K$ *and* $Y(\mathcal{S})^K$ *is homotopy equivalent to* $S^{n(K)-1}$ *for* $K\in\mathcal{S}$,

(c, $\mathcal{S}$) $\deg f_S^K = d(K)$ *for* $K\in\mathcal{S}$.

We note that ($\mathcal{S}_0$) is our assumption of Theorem 2.1. Let H be a maximal subgroup in $\mathcal{S}(G)\setminus\mathcal{S}$. We put $\mathcal{T}=\mathcal{S}\cup(H)$. Then $\mathcal{T}$ is a closed family. As the induction step, we show that ($\mathcal{T}$) holds if ($\mathcal{S}$) holds. If H is not an essential isotropy group, then we put $Y(\mathcal{T})=Y(\mathcal{S})$ and $f_{\mathcal{T}}=f_S$. Then (a, $\mathcal{T}$) is clearly satisfied. For (b, $\mathcal{T}$) and (c, $\mathcal{T}$), we may consider those when $K=H$. Take any $x\in Y(\mathcal{T})^H$. Then the isotropy group G_x belongs to $\mathcal{T}\cap\mathrm{EssIso}(X)$ by (a, $\mathcal{T}$), and $G_x>H$ since $H\notin\mathcal{S}\cap\mathrm{EssIso}(X)$. Since $\mathrm{EssIso}(X)$ is closed under intersection (Hypothesis I), $G_x=\bar{G}_x\geqq\bar{H}>H$ by [8, Lemma 2.9] and hence $\bar{H}\in\mathcal{S}\cap\mathrm{EssIso}(X)$ $=\mathrm{Iso}(Y(\mathcal{S}))$. Hence $Y(\mathcal{T})^H=Y(\mathcal{S})^{\bar{H}}$ and also $Y(\mathcal{S})^{\bar{H}}$ is homotopy equivalent to $S^{n(\bar{H})-1}=S^{n(H)-1}$ by (b, $\mathcal{S}$). Since the inclusion $X^{\bar{H}}\subset X^H$ is a homotopy equivalence (Lemma 1.1), $\deg f_T^H=\deg f_S^{\bar{H}}=d(\bar{H})$. Since d satisfies the unstability conditions, $d(\bar{H})=d(H)$ and hence $\deg f_T^H=d(H)$.

When H is an essential isotropy group of X, we apply [6, Proposition 5.9] to a WH-map $f_S^H\colon Y(\mathcal{S})^H\to X^H$. Then we obtain a homotopy representation $Y'(\mathcal{T})$ ($\supset Y(\mathcal{S})^H$) of WH and a WH-map $f'\colon Y'(\mathcal{T})\to X^H$ such that $\mathrm{Dim}(Y'(\mathcal{T}))=\mathrm{Dim}\,X^H$ and $f'|Y(\mathcal{S})^H=f_S^H$. By the construction in [6, Proposition 5.9], WH acts freely on $Y'(\mathcal{T})\setminus Y(\mathcal{S})^H$. By Lemma 1.5, $\deg f'$ is prime to $|WH|$. Choose an integer k such that $k\cdot\deg f'\equiv d(H) \bmod |WH|$. We apply Proposition 2.2 to $Y'(\mathcal{T})$. Then we obtain a homotopy representation B of WH and a WH-map $\phi\colon B\to Y'(\mathcal{T})$ such that $\deg\phi=k$ and $\deg\phi^K=1$ for $1\neq K\leqq WH$. From the construction of B and $Y'(T)$, we see that

$$Y(\mathcal{S})^H\subset Y'(\mathcal{T})_{b-1}\subset B \quad (b=\dim B), \quad \text{and}$$
$$f'\circ\phi\,|\,Y(\mathcal{S})^H = f_S^H.$$

The degree of $f'\circ\phi\colon B\to X^H$ is $d(H)$ modulo $|WH|$. By Proposition 1.4, we get a WH-map $f''\colon B\to X^H$ such that $\deg f''=d(H)$ and $f''|Y(\mathcal{S})^H=f'\circ\phi|Y(\mathcal{S})^H=f_S^H$. We apply [6, Lemma 4.11] and then we obtain a G-CW-complex $Y(\mathcal{T})$ and a G-map $f_{\mathcal{T}}\colon Y(\mathcal{T})\to X$ such that $Y(\mathcal{T})\supset Y(\mathcal{S})\cup_{Y(S)^H}B$, $f_T|Y(\mathcal{S})=f_S$, $f_{\mathcal{T}}|B=f''$, and $Y(\mathcal{T})/Y(\mathcal{S})=G\times_{NH}B/G\times_{NH}Y(\mathcal{S})^H$. From the proof of [6, Lemma 4.11], we see that

$$Y(\mathcal{T})^H = B\,, \quad Y(\mathcal{T})^K = Y(\mathcal{S})^K \qquad \text{for} \quad K \in \mathcal{S}\,,$$

and also

$$f_{\mathcal{T}}^H = f''\,, \quad f_{\mathcal{T}}^K = f_{\mathcal{S}}^K \qquad \text{for} \quad K \in \mathcal{S}\,.$$

It is not difficult to show that $Y(\mathcal{T})$ and $f_{\mathcal{T}}$ satisfy $(a, \mathcal{T})$, $(b, \mathcal{T})$ and $(c, \mathcal{T})$. Therefore $(\mathcal{T})$ holds if $(\mathcal{S})$ holds.

In the end of induction, we obtain a homotopy representation $Y(\mathcal{S}(G))$ and a G-map $f_{S(G)}$ which are desired. Thus the proof is complete.

REMARK. tom Dieck and Petrie showed that $D: v(G, h^\infty) \to \mathrm{Pic}(G)$ is an isomorphism [**6**, Theorem 6.3], where $v(G, h^\infty)$ is the torsion subgroup of the homotopy representation group of G. Laitinen's result Proposition 1.3 is considered as the unstable version of injectivity of D. Our result is considered as that of surjectivity of D.

We shall discuss Hypotheses I and II. If a homotopy representation X is a smooth G-manifold, then it is well-known that $\mathrm{Ess\,Iso}(X) = \mathrm{Iso}(X)$ and X satisfies Hypothesis I. In particular, a linear G-sphere, which is an unit sphere of a real representation of G, satisfies Hypothesis I. Clearly Hypothesis I depends only on the dimension function of a homotopy representation. Therefore, if a homotopy representation X has the dimension function of a linear G-sphere, then X also satisfies Hypothesis I. A homotopy representation of a p-group always has the dimension function of a linear G-sphere ([**2**], [**7**]). In the sequal we see that a homotopy representation of a p-group satisfies Hypothesis I. More generally Laitinen showed the following.

PROPOSITION 2.3 ([**8**, Proposition 2.12]). *Let G be a finite nilpotent group. Then a homotopy representation of G satisfies Hypothesis* I.

Next, we consider Hypothesis II. We show the following.

PROPOSITION 2.4. *Let X be a homotopy representation with the dimension function n. If $n(H) \equiv n(G) \bmod 2$ for any $H \in S_0$, then X satisfies Hypothesis* II.

We need the following result in [**6**].

PROPOSITION 2.5 ([**6**, Proposition 12.1]). *Let X and Y be homotopy representations of G such that* $(\mathrm{Dim}X)(1) \neq (\mathrm{Dim}Y)(1)$ *and* $(\mathrm{Dim}X)(H) = (\mathrm{Dim}Y)(H)$ *for any $H \neq 1$. Then*

(1) *G has periodic cohomology.*

We denote by $p(G)$ its minimal period.

(2) *$p(G)$ divides* $(\mathrm{Dim}X)(1) - (\mathrm{Dim}Y)(1)$ *except for $G = \boldsymbol{Z}/2$.*

PROOF OF PROPOSITION 2.4. If $n(G) > 3$, we need not give the proof.

Case (1): $n(G)=3$. It is easy to see that $\mathcal{S}_0\cap\mathrm{EssIso}(X)=\{G\}$. Since X^G is homotopy equivalent to S^2, there is a map $f:S^2\to X^G$ of degree $d(G)$. We put $Y(\mathcal{S}_0)=S^2$, which has trivial G-action. Then the composition of f with the inclusion $X^G\subset X$ is the desired G-map $f_{\mathcal{S}_0}$.

Case (2): $n(G)=2$. By the assumption for n, we see $\mathcal{S}_0\cap\mathrm{EssIso}(X)=\{G\}$. Hence the above argument is still valid.

Case (3): $n(G)=1$. If $\mathcal{S}_0\cap\mathrm{EssIso}(X)=\{G\}$, then the proof is similar to that of Case (1). If $H\neq G$ and $H\in\mathcal{S}_0\cap\mathrm{EssIso}(X)$, then $n(H)=3$ by the assumption. A WH-space X^H is a homotopy representation of WH, and $\dim(X^H)^{H/H}=2$ and $\dim(X^H)^{K/H}=0$ for $H<K\leqq NH$. We apply Proposition 2.5 to X^H and S^0 (with trivial WH-action). Then we see that WH has periodic cohomology and its period $p(WH)\leqq 2$. Hence WH is cyclic. (See [**1**, p. 159].)

ASSERTION. *There exist a free reprentation V_H of WH and a WH-map $f_H:S(V_H\oplus\boldsymbol{R})=S(V_H)*S^0\to X^H$ such that $\deg f_H=d(H)$, $f_H(S^0)=X^G$ and the degree of $f_H|S^0:S^0\to X^G$ is equal to $d(G)$ $(=\pm 1)$. Here a representation V of G is called free if G acts freely on $V\setminus\{0\}$ and $*$ means the join.*

We assume this for a while. Let $G, H_1, \cdots, H_r$ be representatives of conjugacy classes of subgroups in $\mathcal{S}_0\cap\mathrm{EssIso}(X)$. We put $Y=\coprod_{i=1}^r G\times_{NH_i}S(V_{H_i}\oplus\boldsymbol{R})$, where $S(V_{H_i}\oplus\boldsymbol{R})$ is regarded as a NH_i-space via the projection $NH_i\to WH_i=NH_i/H_i$. Also f_{H_i} is regarded as a NH_i-map. We define a G-map $G\times_{NH_i}f_{H_i}:G\times_{NH_i}S(V_{H_i}\oplus\boldsymbol{R})\to X$ by $[g, x]\to gf_{H_i}(x)$ for $g\in G$ and $x\in S(V_{H_i}\oplus\boldsymbol{R})$. We put $f=\coprod_{i=1}^r G\times_{NH_i}f_{H_i}:Y\to X$. We also put $S^0=\{p^+, p^-\}\subset S(V_{H_i}\oplus\boldsymbol{R})$. Collapse $\coprod_{i=1}^r G\times_{NH_i}\{p^+\}$ (resp. $\coprod_{i=1}^r G\times_{NH_i}\{p^-\}$) to a point. We denote by $Y(\mathcal{S}_0)$ the resulting G-CW-complex, and by $f_{\mathcal{S}_0}:Y(\mathcal{S}_0)\to X$ the G-map induced from f. It is not difficult to check the conditions in Hypothesis II.

Case (4): $n(H)=0$. We omit the proof because it is similar to the proof of Case (3).

PROOF OF ASSERTION. If $WH=1$, then one can easily see it. We assume $WH\neq 1$. Let $g:S^0\to X^G$ be a map of degree $d(G)$. Let α be a generator of the cyclic group WH. Let W_j be a 2-dimensional representation of WH such that the generator α acts on W_j by rotation of $2\pi j/|WH|$. If j is prime to $|WH|$, then W_j is a free representation. By the equivariant obstruction theory, or Proposition 1.4, there exists a WH-map $h:S(W_j\oplus\boldsymbol{R})\to X^H$ such that $h^{WH}=g$. It follows from Lemma 1.5 that $\deg h$ is prime to $|WH|$. Put $\deg h=l$. We choose an integer s with $l\cdot s\equiv d(H) \bmod |WH|$ and also choose an integer t with $t\cdot s\equiv 1 \bmod |WH|$. Then there exists a WH-map $k:S(W_t\oplus\boldsymbol{R})\to S(W_1\oplus\boldsymbol{R})$ such that $\deg k\equiv s \bmod |WH|$ and $\deg k^{WH}=1$. Hence $\deg h\circ k\equiv d(H) \bmod |WH|$ and $\deg(h\circ k)^{WH}=1$. By the equivariant obstruction theory, or Proposition 1.4, we obtain the desired WH-map f_H.

We give an example satisfying the assumption in Proposition 2.4.

LEMMA 2.6. *Let X be a homotopy representation of G with the dimension function n. If G has an odd order, then $n(H)\equiv n(G) \bmod 2$ for any $H\in\mathcal{S}(G)$.*

PROOF. This is the well-known result. We give its short proof. We define $d\in C(G)$ by $d(H)=\chi(X^H)-1=(-1)^{n(H)-1}$, where χ denotes the Euler characteristic. Then d belongs to the unit group $A(G)^*$ of the Burnside ring [**5**]. Since G has an odd order, $A(G)^*$ consists of ± 1 [**5**]. Hence $d(H)=d(G)$ and so $n(H)\equiv n(G) \bmod 2$.

We obtain from the above results:

COROLLARY 2.7. *Let G be a nilpotent group of odd order. Let n be the dimension function of a homotopy representation. Then the number* $\mathrm{Num}(G, n)$ *is equal to the order of* $\mathrm{Pic}_n(G)$.

§ 3. The order of the unstable Picard group.

Let n be the dimension function of a homotopy representation X of G. We first compute the order of $\mathrm{Pic}_n(G)$ when $n(G)\geqq 2$. We assume $n(G)\geqq 2$ for a while. Then we note that C_n is a subring of $C=C(G)$ and also A_n is a subring of $A=A(G)$. Since $|G|C_n$ is an ideal of C_n, we can define a ring $\bar{C}_n=C_n/|G|C_n$. We can also define a ring $\bar{A}_n=A_n/|G|C_n$ since $|G|C_n$ is an ideal of A_n. We consider the following pullback diagram of rings (cf. [**5**]).

$$\begin{array}{ccc} A_n & \longrightarrow & C_n \\ \downarrow & & \downarrow \\ \bar{A}_n & \longrightarrow & \bar{C}_n . \end{array}$$

Here the horizontal maps are the natural inclusions and the vertical maps are the projections. From the Mayer-Vietoris sequence of Picard groups of rings [**5**], we obtain the exact sequence:

$$1 \longrightarrow A_n^* \longrightarrow C_n^*\times\bar{A}_n^* \longrightarrow \bar{C}_n^* \longrightarrow \mathrm{Pic}\, A_n \longrightarrow \mathrm{Pic}\, C_n\times\mathrm{Pic}\,\bar{A}_n,$$

where $*$ indicates the unit group of a ring. We note that $\bar{A}_n^*$ and $\bar{C}_n^*$ in the sequence are isomorphic to $\bar{A}_n^*$ and $\bar{C}_n^*$ defined in Section 1 respectively. It follows from the same argument as in [**5**, Proposition 10.3.6] that $\mathrm{Pic}\, C_n=1$ and $\mathrm{Pic}\,\bar{A}_n=1$. Hence we obtain the exact sequence:

$$1 \longrightarrow C_n^*\bar{A}_n^* \longrightarrow \bar{C}_n^* \longrightarrow \mathrm{Pic}\, A_n \longrightarrow 1,$$

since $C_n^*\bar{A}_n^*$ is the image of $C_n^*\times\bar{A}_n^*$. Hence $\mathrm{Pic}\, A_n$ is isomorphic to $\mathrm{Pic}_n(G)$. We shall compute the order of $\mathrm{Pic}\, A_n$. We still assume $n(G)\geqq 2$. We define an isomorphism of rings $i: C \to \prod_{(H)\in\phi(G)} \boldsymbol{Z}$ by $i(d)=(d(H))$. Then the image of A coincides to the subring:

$$\{(d_H)\in \prod_{(H)\in\phi(G)} \boldsymbol{Z} \mid \text{ congruences } (*)_H \text{ for } (H)\in\phi(G)\},$$

where

$$(*)_H: \quad d_H \equiv -\sum n_{H,K} d_K \mod |WH|,$$

and $n_{H,K}$ are certain integers. The sum is taken over G-cojugacy classes (K) such that $H \lhd K$, $H \neq K$ and K/H is cyclic. (For the detail, see [**5**].)

We put $\mathcal{F}_n$ (abbr. $\mathcal{F}$) $=\{(H)\in\phi(G) \mid H=\bar{H}\}$. From the unstability conditions and $n(G)\geqq 2$, we can also define an isomorphism of rings $j: C_n \to \prod_{(H)\in\mathcal{F}} \boldsymbol{Z}$ by $j(d)=(d(H))$. We put

$$B=\{(d_H)\in \prod_{(H)\in\mathcal{F}} \boldsymbol{Z} \mid \text{ congruences } (**)_H \text{ for } (H)\in\mathcal{F}\},$$

where

$$(**)_H: \; d_H \equiv -\sum n_{H,K} d_{\bar{K}} \mod |WH|.$$

The sum is taken over G-conjugacy classes (K) such that $H \lhd K$, $H \neq K$ and K/H is cyclic.

LEMMA 3.1. *$j(A_n)=B$.*

PROOF. Take $d\in A_n$. Then $i(d)$ satisfies the congruences $(*)_H$ for $(H)\in\phi(G)$, and $d(K)=d(\bar{K})$ for $(K)\in\phi(G)$. Hence $j(A_n)\subset B$. Take $(d_H)\in B$. From the proof of [8, Theorem 2], we see that there exists a G-map $f: X\to X$ of a homotopy representation X with $\operatorname{Dim}X=n$ such that $d(f)(H)=d_H$ for $(H)\in\mathcal{F}$. Since $d(f)\in A_n$ by [8], it follows that $(d_H)\in j(A_n)$. Therefore $j(A_n)=B$.

We may compute the order of $\operatorname{Pic} B$. The order of $\operatorname{Pic} B$ can be computed by the same way as in [**10**] or [4]. Then we see that

$$|\operatorname{Pic} B| = 2^{-f} |B^*| \prod_{(H)\in\mathcal{F}} \varphi(|WH|),$$

where $f=|\mathcal{F}|$ and φ is the Euler function. We have obtained

PROPOSITION 3.2. *If $n(G)\geqq 2$, then*

$$|\operatorname{Pic}_n(G)| = 2^{-f} |A_n^*| \prod_{(H)\in\mathcal{F}} \varphi(|WH|).$$

From now we consider the general case. We put $n=\operatorname{Dim} X$ and $n'=\operatorname{Dim} X*S^1$. Note that $n'=n+2\geqq 2$ and $\mathcal{F}_n=\mathcal{F}_{n'}$. We put $\mathcal{F}(i)=\{(H)\in\mathcal{F} \mid n(H)=i\}$ and $\mathcal{F}(\geqq i)=\{(H)\in\mathcal{F} \mid n(H)\geqq i\}$. We define a homomorphism

$$k: \; \bar{C}_{n'}^* \longrightarrow \prod_{(H)\in\mathcal{F}(1)} \boldsymbol{Z}/|WH|^*/\pm 1$$

by $k([d])=(d(H)\cdot d(G)^{-1})$, where d is an invertible function satisfying the unstability conditions for n'.

LEMMA 3.3. *The homomorphism k is surjective and* $\mathrm{Ker}\, k=\bar{C}_n^*\bar{A}_{n'}^*$.

PROOF. The surjectivity is obvious. Take $[d]\in\bar{C}_n^*$. Since $d(H)=\pm 1$ when $n(H)=1$, it follows that $k([d])=1$. Take $[d]\in\bar{A}_{n'}^*$. Then d satisfies $(**)_H$. If $(H)\in\mathcal{T}(1)$, then $d(H)\equiv -\sum n_{H,K}d(\bar{K})=-\sum n_{H,K}d(G)\equiv d(G) \mod |WH|$. Hence $k([d])=1$, and so $\bar{C}_n^*\bar{A}_{n'}^*\subset\mathrm{Ker} k$. Take $[d]\in\mathrm{Ker} k$. We choose $c_1\in C_n^*$ such that $(c_1d)(H)\cdot(c_1d)(H)^{-1}=1$ in $\boldsymbol{Z}/|WH|^*$ for $(H)\in\mathcal{T}(1)$. Then c_1d satisfies $(**)_H$ for $(H)\in\mathcal{T}$ with $n(H)\leqq 1$. Indeed if $(H)\in\mathcal{T}(0)$, then $H=G$ and so $WH=1$. Hence $(**)_H$ is satisfied. If $(H)\in\mathcal{T}(1)$, then $-\sum n_{H,K}(c_1d)(\bar{K})=-\sum n_{H,K}(c_1d)(G)$ $\equiv(c_1d)(G)\equiv(c_1d)(H) \mod |WH|$.

ASSERTION. *There exists $e\in A_{n'}$ such that e is invertible and $e(H)=(c_1d)(H)$ when $n(H)\leqq 1$.*

We assume this for a while. Choose an integer $e'(H)$ such that $e'(H)e(H)$ $\equiv 1 \mod |G|$ for $(H)\in\mathcal{T}$. We define $c_2\in C_n$ which is invertible by $c_2(H)=$ $d(H)e'(\bar{H})$ if $n(H)\geqq 2$, and $c_2(H)=c_1(H)$ if $n(H)\leqq 1$. Then $(c_2e)(H)=d(H)e'(\bar{H})e(H)$ $=d(H)e'(\bar{H})e(\bar{H})\equiv d(H) \mod |G|$ for $n(H)\geqq 2$, and $(c_2e)(H)=c_1(H)c_1(H)d(H)=d(H)$ for $n(H)\leqq 1$. Therefore $[c_2][e]=[d]$ in $\bar{C}_{n'}^*$ and $\mathrm{Ker} k=\bar{C}_n^*\bar{A}_{n'}^*$.

PROOF OF ASSERTION. We inductively choose an integer e_H satisfying $(**)_H$ for $(H)\in\mathcal{T}$ and $e_H=(c_1d)(H)$ when $n(H)\leqq 1$. (Here the integers e_H need not be prime to $|G|$.) We define $\bar{e}\in C_{n'}$ by $\bar{e}(H)=e_{\bar{H}}$. Then $\bar{e}\in A_{n'}$ by Lemma 3.1. It follows that there exists a G-map $f: Y\to Y$ such that $d(f)=\bar{e}$ [8], where $Y=$ $X*S^1$. We construct the desired e inductively. Suppose that there exists a G-map $f_S: Y\to Y$ such that $d(f_S)(H)$ is prime to $|G|$ for $(H)\in\mathcal{S}$, and $d(f_S)(H)=$ $(c_1d)(H)$ when $n(H)\leqq 1$. Here $\mathcal{S}$ is a subfamily of $\phi(G)$ such that if $(H)<(K)$ and $(H)\in\mathcal{S}$, then $(K)\in\mathcal{S}$. Let (H) be a maximal element of $\phi(G)\setminus\mathcal{S}$. We put $\mathcal{T}=\mathcal{S}\cup\{(H)\}$. If $H\neq\bar{H}$, then $(\bar{H})\in\mathcal{S}$. Hence $d(f_S)(H)=d(f_S)(\bar{H})$ is prime to $|G|$. If $H=\bar{H}$, then $d(f_S)(H)=\deg f_S^H$ is prime to $|WH|$ by Lemma 1.5. We can choose an integer m such that $\deg f_S^H+m|WH|$ is prime to $|G|$. By Proposition 1.4, we obtain a G-map $f_{\mathcal{T}}: Y\to Y$ such that $\deg f_{\mathcal{T}}^H=\deg f_S^H+m|WH|$ and $\deg f_{\mathcal{T}}^K$ $=\deg f_S^K$ for $(K)\in\mathcal{S}$. In the end of this process, we obtain a G-map $h: Y\to Y$ such that $\deg h^K$ is prime to $|G|$ for $(K)\in\phi(G)$ and $\deg h^K=(c_1d)(K)$ when $n(K)\leqq 1$. Then $d(h)$ is the desired $e\in A_{n'}$.

We put $\mathrm{Inv}_n(G)=\bar{C}_n^*/\bar{A}_n^*$. It is easy to see that the following sequences are exact (cf. [4]).

(3.4) $$1\longrightarrow A_n^*\longrightarrow C_n^*\longrightarrow \mathrm{Inv}_n(G)\longrightarrow \mathrm{Pic}_n(G)\longrightarrow 1$$

(3.5) $$1\longrightarrow \mathrm{Inv}_n(G)\longrightarrow \mathrm{Inv}_{n'}(G)\longrightarrow \bar{C}_{n'}^*/\bar{C}_n^*\bar{A}_{n'}^*\longrightarrow 1.$$

Here the maps are defined by the obvious way and $n'=n+2$, and A_n^* denotes the subgroup of elements of A^* satisfying the unstability conditions for n.

By Proposition 3.2 and (3.4), we have

$$|\mathrm{Inv}_{n'}(G)| = \prod_{(H)\in\mathcal{F}} \varphi(|WH|).$$

By Lemma 3.3 and (3.5), we have

$$|\mathrm{Inv}_n(G)| = 2^s \times \prod_{(H)\in\mathcal{F}(\geqq 2)} \varphi(|WH|),$$

where $s=|\{(H)\in\mathcal{F}\,|\,n(H)=1,\ |WH|\geqq 3\}|$. By (3.4), we obtain

$$|\mathrm{Pic}_n(G)| = 2^s |A_n^*|\,|C_n^*|^{-1} \prod_{(H)\in\mathcal{F}(\geqq 2)} \varphi(|WH|).$$

It can easily be seen that

$$|A_n^*| = \begin{cases} |A_{n'}^*| & \text{if } n(G)\geqq 1 \\ |A_{n'}^*|/2 & \text{if } n(G)=0 \end{cases}$$

and

$$|C_n^*| = \begin{cases} 2^f & \text{if } n(G)\geqq 1 \\ 2^{f-1} & \text{if } n(G)=0 \quad (f=|\mathcal{F}|) \end{cases}$$

Hence $|A_n^*|\,|C_n^*|^{-1}=2^{-f}|A_{n'}^*|$. Therefore we have obtained

THEOREM 3.6. *For any* $n\in\mathrm{Dim}(G)$,

$$\begin{aligned} |\mathrm{Pic}_n(G)| &= 2^s |A_n^*|\,|C_n^*|^{-1} \prod_{(H)\in\mathcal{F}(\geqq 2)} \varphi(|WH|) \\ &= 2^{s-f} |A_{n'}^*| \prod_{(H)\in\mathcal{F}(\geqq 2)} \varphi(|WH|), \end{aligned}$$

where $f=|\mathcal{F}|$, $s=|\{(H)\in\mathcal{F}\,|\,n(H)=1,\ |WH|\geqq 3\}|$.

COROLLARY 3.7. *Let G be a finite group of odd order. Then*

$$|\mathrm{Pic}_n(G)| = 2^{1-f} \prod_{(H)\in\mathcal{F}(\geqq 2)} \varphi(|WH|).$$

PROOF. Since G has an odd order, A^* consists of ± 1. (See [**5**].) On the other hand it is clear that $\pm 1\in A_{n'}^*\subset A^*$. Hence $|A_{n'}^*|=2$. If $(H)\in\mathcal{F}$ and $n(H)=1$, then $H=G$ by Lemma 2.6. Hence $s=0$.

§ 4. Finite homotopy representations.

A homotopy representation X which has a G-homotopy type of a finite G-CW-complex is called a *finite* homotopy representation. In [**6**], tom Dieck and Petrie defined finiteness obstruction $\rho(X)\in\kappa(G)=\bigoplus_{(H)}\tilde{K}_0(\boldsymbol{Z}WH)$, where $\tilde{K}_0(\boldsymbol{Z}WH)$ is the reduced projective class group of $\boldsymbol{Z}WH$. tom Dieck and Petrie's finiteness obstruction has the following properties.

PROPOSITION 4.1 ([**6**]).

(1) *For homotopy representations* X, Y, $\rho(X*Y)=\rho(X)+\rho(Y)$.

(2) *If* X *is a finite homotopy representation, then* $\rho(X)=0$.

(3) *Assume that* $\bigcup_{H\in\mathcal{S}_0} X^H$ *is a finite* G-CW-*complex, where* $\mathcal{S}_0=\{H\in\mathcal{S}(G) \mid \dim X^H\leqq 2\}$. *Then* X *is a finite homotopy representation if* $\rho(X)=0$.

From the proof of [**6**, Proposition 7.24], we can replace the assumption of (3) by the following:

HYPOTHESIS. *There exist a finite* G-CW-*complex* $Y(\mathcal{S}_0)$ *and a* G-*map* $f_{\mathcal{S}_0}$: $Y(\mathcal{S}_0)\to X$ *such that* $\dim Y(\mathcal{S}_0)^H=\dim X^H$, *and* $Y(S_0)^H$ *is homotopy equivalent to* $S^{n(H)-1}$ *and* $\deg f_{\mathcal{S}_0}=\pm 1$ *for any* $H\in\mathcal{S}_0$.

From the proof of Proposition 2.4, we see that if $n=\mathrm{Dim}X$ satisfies that $n(H)\equiv n(G) \mod 2$ for any $H\in\mathcal{S}_0$, then Hypothesis is satisfied for X. In particular, if G has an odd order, then Hypothesis is satisfied for any homotopy representation. tom Dieck and Petrie also defined the homomorphism

$$\rho:\ \mathrm{Pic}(G) \longrightarrow \kappa(G).$$

Let X and Y be homotopy representations with the same dimension function n and $f: Y\to X$ any G-map with an invertible degree function $d(f)$. Then it is known that $\rho([d(f)])=\rho(X)-\rho(Y)$ [**6**]. We denote the composition of ρ with the natural homomorphism $\mathrm{Pic}_n(G)\to\mathrm{Pic}(G)$ by

$$\rho_n:\ \mathrm{Pic}_n(G) \longrightarrow \kappa(G).$$

Let $\mathrm{Num}_f(G, n)$ denote the number of G-homotopy types of finite homotopy representations with the same dimension function $n\in\mathrm{Dim}_f(G)$, where $\mathrm{Dim}_f(G)$ denotes the set of dimention functions of finite homotopy representations of G.

PROPOSITION 4.2.

(1) $\mathrm{Num}_f(G, n)\leqq |\mathrm{Ker}\,\rho_n|$.

(2) *If* G *is a nilpotent group of odd order, then* $\mathrm{Num}_f(G, n)=|\mathrm{Ker}\,\rho_n|$.

PROOF. Let $X_1, X_2, \cdots, X_r$ be finite homotopy representations such that X_i and X_j are not G-homotopy equivalent for $i\neq j$. Let $f_i: X_i\to X_1$ be a G-map with an invertible degree function. Since X_i and X_1 are finite, $\rho([d(f_i)])=0$ and hence $\rho_n([d(f_i)])=0$. By the similar argument as in Proposition 1.7, one can see (1). Assume G is a nilpotent group of odd order. For any $[d]\in\mathrm{Ker}\,\rho_n$, there exist a homotopy representation Y and a G-map $f: Y\to X_1$ such that $\mathrm{Dim}\,Y=n$ and $d(f)=d$ by Theorem 2.1. Then $\rho(Y)=\rho(Y)-\rho(X_1)=-\rho_n([d(f)])=0$. It follows from Proposition 4.1 that Y is a finite homotopy representation. Therefore (2) follows. (See Section 2.)

It seems difficult to compute $\operatorname{Ker}\rho_n$, or its order in general (cf. [9]). If G is abelian, then $\operatorname{Ker}\rho_n$ is described by using Swan homomorphisms. We shall explain this. Let G be an abelian group. We define $e_H: \bar{C}^* \to \boldsymbol{Z}/|G|^*$ by $e_H([d])=d(H)$. We inductively define $u_H: \bar{C}^* \to \boldsymbol{Z}/|G|^*$ by $u_G=e_G$ and $u_H=e_H \prod_{K>H} u_K^{-1}$. Let denote by $\tilde{u}_H$ the composition of u_H with the projection $\boldsymbol{Z}/|G|^* \to \boldsymbol{Z}/|G/H|^*$.

PROPOSITION 4.3 ([3], [6]). *The sequences*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \bar{A}^* & \longrightarrow & \bar{C}^* & \xrightarrow{u} & \prod_H \boldsymbol{Z}/|G/H|^* & \longrightarrow & 1 \\ & & & & & & \downarrow p & & \\ 1 & \longrightarrow & \bar{A}^*C^* & \longrightarrow & \bar{C}^* & \xrightarrow{p\circ u} & \prod_H \boldsymbol{Z}/|G/H|^*/\pm 1 & \longrightarrow & 1 \end{array}$$

are exact, where $u=(\tilde{u}_H)$ and p is the projection. In particular,

$$\operatorname{Inv}(G) = \bar{C}^*/\bar{A}^* \cong \prod_H \boldsymbol{Z}/|G/H|^*$$

$$\operatorname{Pic}(G) = \bar{C}^*/\bar{A}^*C^* \cong \prod_H \boldsymbol{Z}/|G/H|^*/\pm 1 .$$

We compute $\operatorname{Pic}_n(G)$ of an abelian group G. It is easy to see the next two lemmas.

LEMMA 4.4. *If $H=\bar{H}$ and $n(H)=0$, then $u_H=1$ on $\bar{C}_n^*$.*

LEMMA 4.5. *If $H=\bar{H}$ and $n(H)=1$, then $u_H=\pm 1$ on $\bar{C}_n^*$.*

We put $\mathcal{F}=\{H \leqq G \mid H=\bar{H}\}$.

LEMMA 4.6. *If $H \in \mathcal{F}^c=\mathcal{S}(G)\setminus\mathcal{F}$, then $u_H=1$ on $\bar{C}_n^*$.*

PROOF. We prove it inductively. Let $\mathcal{F}_1$ be a subset of $\mathcal{F}^c$. We assume that $u_K=1$ on $\bar{C}_n^*$ if $K \in \mathcal{F}_1$. Let H be a maximal subgroup in $\mathcal{F}^c \setminus \mathcal{F}_1$. We put $\mathcal{F}_2=\mathcal{F}_1 \cup \{H\}$. Then on $\bar{C}_n^*$

$$\begin{aligned} u_H &= e_H \prod_{K>H} u_K{}^{-1} \\ &= e_{\bar{H}} \prod_{\substack{K>H \\ K=\bar{K}}} u_K{}^{-1} && \text{(By the assumption.)} \\ &= e_{\bar{H}} \Big(\prod_{\substack{K>\bar{H} \\ K=\bar{K}}} u_K{}^{-1} \Big) u_{\bar{H}}{}^{-1} && \text{(Because if } K>H,\ K=\bar{K}, \text{ then } K \geqq \bar{H} \text{ [8, Lemma 2.9].)} \\ &= \Big(e_{\bar{H}} \prod_{K>\bar{H}} u_K{}^{-1} \Big) u_{\bar{H}}{}^{-1} && \text{(By the assumption.)} \\ &= u_{\bar{H}} \cdot u_{\bar{H}}{}^{-1} \\ &= 1 . \end{aligned}$$

Hence $u_K=1$ on $\bar{C}_n^*$ if $K \in \mathcal{F}_2$. In the end of this process, we see that $u_K=1$

on $\bar{C}_n^*$ if $K\in\mathfrak{F}^c$.

We obtain the commutative diagram:

$$\begin{array}{ccc}
\bar{C}^* & \xrightarrow{u} & \prod_{H\in S(G)} Z/|G/H|^* \\
\cup & & \cup \\
\bar{C}_n^* & \xrightarrow{u_n} & \prod_{H\in\mathfrak{F}(\geq 2)} Z/|G/H|^*\times\prod_{H\in\mathfrak{F}(1)} \mathrm{Im}\tilde{u}_H \\
\| & & \downarrow p \\
\bar{C}_n^* & \xrightarrow{p\circ u_n} & \prod_{H\in\mathfrak{F}(\geq 2)} Z/|G/H|^*/\pm 1,
\end{array}$$

where $u_n=u|\bar{C}_n^*$. It is not difficult to show that u_n and $p\circ u_n$ are surjective. We also note $u_n(C_n^*)=\mathrm{Ker}\,p$. We show $\mathrm{Ker}\,p\circ u_n=\bar{A}_n^*C_n^*$. Since $\mathrm{Ker}\,u_n=\bar{C}_n^*\cap\mathrm{Ker}\,u=\bar{C}_n^*\cap\bar{A}^*=\bar{A}_n^*$, it follows that $\bar{A}_n^*\subset\mathrm{Ker}\,p\circ u_n$. Since $C_n^*\subset\mathrm{Ker}\,p\circ u_n$, it follows that $\bar{A}_n^*C_n^*\subset\mathrm{Ker}\,p\circ u_n$. Take $x\in\mathrm{Ker}\,p\circ u_n$. There exists $c\in C_n^*$ such that $u_n(c)=u_n(x)$ since $u_n(x)$ is in $\mathrm{Ker}\,p$. Hence $xc^{-1}\in\mathrm{Ker}\,u_n=\bar{A}_n^*$ and so $x\in\bar{A}_n^*C_n^*$. Therefore $\mathrm{Ker}\,p\circ u_n=\bar{A}_n^*C_n^*$.

We have proved

PROPOSITION 4.7. *The following diagram commutes and the horizontal maps are isomorphisms, which are induced from $p\circ u_n$ and $p\circ u$.*

$$\begin{array}{ccc}
\mathrm{Pic}_n(G) & \xrightarrow[\cong]{} & \prod_{H\in\mathfrak{F}(\geq 2)} Z/|G/H|^*/\pm 1 \\
\downarrow & & \cap \\
\mathrm{Pic}(G) & \xrightarrow[\cong]{} & \prod_{H\in S(G)} Z/|G/H|^*/\pm 1.
\end{array}$$

tom Dieck and Petrie proved that the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Pic}(G) \arrow[r, "\cong"'] \arrow[dr, "\rho"'] & \prod_{H\in S(G)} Z/|G/H|^*/\pm 1 \arrow[d, "\prod S_{G/H}"] \\
& \kappa(G)
\end{tikzcd} \tag{4.8}$$

Here $S_{G/H}: Z/|G/H|^*/\pm 1\to\tilde{K}_0(Z[G/H])$ is the Swan homomorphism of G/H. (For the Swan homomorphism, see [**11**] and also [**9**].) From Proposition 4.7 and (4.8), we have

PROPOSITION 4.9. *If G is abelian, then*

$$\mathrm{Ker}\,\rho_n\cong\prod_{H\in\mathfrak{F}(\geq 2)}\mathrm{Ker}\,S_{G/H}.$$

COROLLARY 4.10. *If an abelian group G has an odd order, then* $\mathrm{Num}_f(G, n)=|\prod_{H\in\mathfrak{F}(\geq 2)}\mathrm{Ker}\,S_{G/H}|$, *where* $n\in\mathrm{Dim}_f(G)$.

We shall give an example lastly. Let G be $\boldsymbol{Z}/p\times\boldsymbol{Z}/p$ (p: an odd prime), and $H_1, H_2, \cdots, H_{p+1}$ all subgroups of order p. The dimension function n of a homotopy representation is that of a linear G-sphere. Since a linear G-sphere has a finite G-CW-complex structure, the dimension function n is in $\mathrm{Dim}_f(G)$. We define functions n_i ($i=0, 1, \cdots, p+1$) as follows.

$$n_0(K)=1 \quad \text{for} \quad H\in\mathcal{S}(G),$$

and for $i\geqq 1$,

$$n_i(K)=\begin{cases} 2 & \text{if} \quad K=H_i \text{ or } 1 \\ 0 & \text{if} \quad K=H_j \ (j\neq i) \text{ or } G. \end{cases}$$

Note that n_i are the dimension functions of unit spheres of irreducible real representations of G. From the representation theory, the dimension function $n\in\mathrm{Dim}_f(G)=\mathrm{Dim}(G)$ is uniquely described as $n=\alpha_0 n_0+\sum_{i=1}^{p+1}\beta_i n_i$, where α_0 and β_i are non-negative integers. We put $P(n)=|\{i\,|\,\beta_i>0\}|$. We note that $|\mathrm{Ker}\, S_{Z/p}|=|\mathrm{Ker}\, S_G|=(p-1)/2$ [11]. We have the following.

Case	$\mathrm{Num}(G, n)$	$\mathrm{Num}_f(G, n)$
$P(n)=0$	1	1
$P(n)=1$	$(p-1)/2$	$(p-1)/2$
$P(n)=k\geqq 2$	$p((p-1)/2)^{k+1}$	$((p-1)/2)^{k+1}$

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Ikumitsu NAGASAKI
Department of Mathematics
Faculty of Science
Osaka University
Toyonaka, Osaka 560
Japan