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Note on H^p on Riemann surfaces

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The purpose of this note is to prove a theorem which implies the following: Given an arbitrary open Riemann surface R and an arbitrary positive real number p. There exist a holomorphic function f on R and two subregions S and T of R with $S\cup T=R$ such that $f|S$ ($f|T$, resp.) belongs to $H^p(S)$ ($H^p(T)$, resp.) and yet f does not belong to $H^p(R)$.

1. We denote by $H^p(R)$ for a positive real number p the class of holomorphic functions f on an open Riemann surface R such that $|f|^p$ has a harmonic majorant on R. In this note we prove the following

THEOREM. *For an arbitrary holomorphic function f on an arbitrary open Riemann surface R and any positive real number p, there exist two subregions S_f and T_f of R with $S_f\cup T_f=R$ such that $f|S_f$ ($f|T_f$, resp.) belongs to $H^p(S_f)$ ($H^p(T_f)$, resp.).*

This result was originally obtained by Bañuelos and Wolff [1] when R is the unit disk. The proof will be given in nos. 2-7.

Proof of the Theorem.

2. First we fix our basic notation. We take an exhaustion $\{R_n\}_1^\infty$ of R (cf. e.g. [2]) and denote by $\{U_{nj}\}_{j=1}^{\nu_n}$ ($n=1, 2, \cdots$) the connected components of $U_n=R_{2n-1}-\bar{R}_{2n-2}$, where we set $R_0=\emptyset$. We connect U_{nj} ($j=1, \cdots, \nu_n$; $n=2, 3, \cdots$) with R_{2n-3} by a strip $V_{nj}=\phi_{nj}(D_{nj})$ in $R_{2n-2}-\bar{R}_{2n-3}$, i.e. an image of a rectangle

$$D_{nj}=\{x+yi : 0<x<1, 0<y<y_{nj}\}$$

by a conformal mapping ϕ_{nj} of a neighborhood of $\bar{D}_{nj}$ to R. We may assume that

$$\phi_{nj}([0, y_{nj}i])=\partial V_{nj}\cap\partial R_{2n-3},$$

$$\phi_{nj}([1, 1+y_{nj}i])=\partial V_{nj}\cap\partial U_{nj},$$

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$$\bar{V}_{nj} \cap \bar{V}_{nk} = \emptyset \qquad (j \neq k),$$

where $[z_1, z_2]$ means the line segment $\{tz_1+(1-t)z_2 : 0 \leqq t \leqq 1\}$ for points z_1, z_2 in the complex plane. We divide V_{nj} into two strips

$$V_{nj}^- = \phi_{nj}(\{x+yi : 0<x<1/2, 0<y<y_{nj}\}),$$

$$V_{nj}^+ = \phi_{nj}(\{x+yi : 1/2<x<1, 0<y<y_{nj}\}).$$

Then we set $V_1^+ = \emptyset$,

$$V_n = \bigcup_{j=1}^{\nu_n} V_{nj}, \qquad V_n^- = \bigcup_{j=1}^{\nu_n} V_{nj}^-, \qquad V_n^+ = \bigcup_{j=1}^{\nu_n} V_{nj}^+ \qquad (n=2, 3, \cdots),$$

and

$$W_n = \left[\bigcup_{m=1}^{n} (\bar{V}_m^+ \cup U_m \cup \bar{V}_{m+1}^-)\right]^\circ \qquad (n=1, 2, \cdots).$$

Here X° means the interior of the set X.

3. We give complementary 'slits' σ_n in V_n as follows. For every integer n with $n \geqq 2$ and real number r with $0<r<1$, we consider a subset

$$\sigma_n(r) = \bigcup_{j=1}^{\nu_n} \phi_{nj}([1/2, 1/2+ry_{nj}i])$$

of ∂W_{n-1}. Since the harmonic measure $v_n(r; z) = \omega(\sigma_{n+1}(r), W_n; z)$ $(n=1, 2, \cdots)$ of $\sigma_{n+1}(r)$ considered on W_n converges to 0 as $r \to 0$, there exists a real number $a_{n+1} = a_{n+1}(f)$ with $0<a_{n+1}<1$ depending on the positive number

$$M_{n+1}(f) = \max\{|f(z)|^p : z \in \bar{U}_{n+1} \cup \bar{V}_{n+2}\} + 1$$

in such a way that

$$v_n(a_{n+1}; z) \leqq \frac{1}{2M_{n+1}(f)}$$

on

$$\Gamma_{n+1}^- = W_n \cap \partial R_{2n-1}.$$

The first requirement for σ_n is thus given here in this number and the second in Number 5.

4. Now we give slits τ_n in V_n as follows. We fix a sequence $\{r_n\}_2^\infty$ of real numbers r_n with $0<r_n \leqq a_n$ and consider slits

$$\tau_n(r_n) = \bigcup_{j=1}^{\nu_n} \phi_{nj}([1/2+r_n y_{nj} i, 1/2+y_{nj} i])$$

in V_n $(n=2, 3, \cdots)$. Let u be any bounded harmonic function on an unbounded open subset

$$S_n = S_n(\{r_m\}_{n+1}^{\infty}) = \left[\bigcup_{m=n}^{\infty}(\bar{V}_m^+ \cup U_m \cup \bar{V}_{m+1}^-)\right]^{\circ} - \bigcup_{m=n+1}^{\infty}\tau_m(r_m)$$

($n=1, 2, \cdots$) of R with vanishing boundary values on ∂S_n. Since $|u(z)|$ is dominated by

$$\|u\| = \sup_{S_n}|u(z)|$$

on $\sigma_m(r_m)$ ($m=n+1, n+2, \cdots$), we have

$$|u(z)| \leqq \|u\| v_{m-1}(r_m; z) \leqq \|u\| v_{m-1}(a_m; z) \leqq \|u\|/2$$

on Γ_m^- and hence on $\sigma_{m-1}(r_{m-1})$ if $m \geqq n+2$. Then by induction we obtain

$$|u(z)| \leqq \|u\|/2^{m-n}$$

on Γ_{n+1}^- so that $u \equiv 0$. This shows the uniqueness of the solution for the Dirichlet problem on S_n which is equivalent to the maximum principle for S_n: $\sup_{S_n} u = \sup_{\partial S_n - X} u$ for any bounded continuous function u on $\bar{S}_n$ except for a subset X of ∂S_n of logarithmic capacity zero locally and harmonic on S_n.

5. Now the second requirement for complementary slits σ_n is formulated. We fix an integer n with $n \geqq 2$. For a real number r with $0 < r \leqq a_n$, we consider the harmonic measure

$$w_n(r; z) = \omega(\partial S_n(\{a_m\}_{n+1}^{\infty}) - \sigma_n(r), S_n(\{a_m\}_{n+1}^{\infty}); z)$$

of $\partial S_n(\{a_m\}_{n+1}^{\infty}) - \sigma_n(r)$ on $S_n(\{a_m\}_{n+1}^{\infty})$. Since $w_n(r; z)$ converges to 1 as $r \to 0$, there exists a real number $b_n = b_n(f)$ with $0 < b_n \leqq a_n$ such that

$$w_n(b_n; z) \geqq 1/2$$

on

$$\Gamma_n^+ = S_n(\{a_m\}_{n+1}^{\infty}) \cap \partial R_{2n-2}.$$

6. Finally we construct a harmonic majorant for $|f|^p$ on S_f. We set

$$S_f = S_1(\{b_m\}_2^{\infty}).$$

The open set S_f is connected since $\nu_1 = 1$ and $U_{11} = R_1$ is connected. For every integer n with $n \geqq 2$ we consider a positive harmonic function h_n on S_f such that boundary values of h_n is 0 for (Carathéodory) boundary points accessible from $S_f - S_n(\{b_m\}_{n+1}^{\infty})$ and $2M_n$ for boundary points accessible from $S_n(\{b_m\}_{n+1}^{\infty})$. Since h_n is dominated by $2M_n$ on $\sigma_n(b_n)$, we have

$$h_n(z) \leqq 2M_n v_{n-1}(b_n; z) \leqq 2M_n v_{n-1}(a_n; z) \leqq 1$$

on Γ_n^- and hence on $\sigma_{n-1}(b_{n-1})$ if $n \geqq 3$. Then by induction h_n is dominated by $1/2^{n-2}$ on Γ_2^-. On the other hand we have

$$h_n(z) \geqq 2M_n w_n(b_n ; z) \geqq M_n$$

on Γ_n^+ and hence on $S_f - R_{2n-2}$. Therefore we can define a harmonic majorant

$$h_f(z) = M_1(f) + \sum_{n=2}^{\infty} h_n(z)$$

of $|f|^p$ on S_f, where we set $M_1(f) = \max\{|f(z)|^p : z \in \bar{R}_1 \cup \bar{V}_2\}$.

7. We now briefly complete our proof by constructing another subregion T_f of R. We take another exhaustion $\{P_n\}_1^\infty$ of R with

$$\bigcup_{n=1}^{\infty} ((R_{2n-1} - \bar{R}_{2n-2}) \cup (P_{2n-1} - \bar{P}_{2n-2})) = R ,$$

where $P_0 = \emptyset$. We consider a subregion T_f of R by connecting the components $\{P_{2n-1} - \bar{P}_{2n-2}\}_1^\infty$ in the same way as that of S_f. Then $S_f \cup T_f = R$ and $f|S_f$ ($f|T_f$, resp.) belongs to $H^p(S_f)$ ($H^p(T_f)$, resp.). □

8. To derive the statement in the introduction from the theorem we need to construct a holomorphic function on an arbitrary open Riemann surface R which is not in $H^p(R)$ $(0 < p < \infty)$. This follows at once from the Behnke-Stein-Florack existence theorem and the fact that any g in $H^p(R)$ is Lindelöfian (i.e. $\log^+|g|$ admits a superharmonic majorant) by observing special distributions of zeros of Lindelöfian holomorphic functions (cf. e.g. [2], p. 270).

References

[1] R. Bañuelos and T. Wolff, Note on H^p on plane domains, Proc. Amer. Math. Soc., **95** (1985), 217-218.
[2] L. Sario and M. Nakai, Classification Theory of Riemann Surfaces, Springer, 1970.

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