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Minimal 2-spheres with constant curvature in $P_n(\boldsymbol{C})$

By Shigetoshi BANDO and Yoshihiro OHNITA

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Introduction.

Minimal surfaces with constant curvature in real space forms have been classified completely (cf. [**5**], [**9**], [**2**]). A next interesting problem is to classify minimal surfaces with constant curvature in complex space forms. The purpose of this peper is to classify minimal 2-spheres with constant curvature in complex projective spaces.

Now let $S^2(c)$ be a 2-dimensional sphere with constant curvature c and $P_n(\boldsymbol{C})$ an n-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature 1. There are two typical classes of minimal isometric immersions of $S^2(c)$ into $P_n(\boldsymbol{C})$.

One is a class of holomorphic isometric imbeddings of $P_1(\boldsymbol{C})$ into $P_n(\boldsymbol{C})$ given by Calabi [**4**];

$$\psi_n : P_1(\boldsymbol{C}) = S^2(1/n) \longrightarrow P_n(\boldsymbol{C})$$

$$(z_0, z_1) \longrightarrow (\sqrt{n!/(l!(n-l)!)}z_0^l z_1^{n-l})_{l=0,\cdots,n},$$

where (z_0, z_1) is the homogeneous coordinate system of $P_1(\boldsymbol{C})$. ψ_n is called the n-th Veronese imbedding of $P_1(\boldsymbol{C})$.

The other is a class of totally real minimal isometric immersions obtained by composing a Borůvka sphere $S^2(1/2k(k+1)) \to S^{2k}(1/4)$ (cf. [**1**]), a natural covering $S^{2k}(1/4) \to P_{2k}(\boldsymbol{R})$ and a totally real totally geodesic imbedding $P_{2k}(\boldsymbol{R}) \to P_{2k}(\boldsymbol{C})$;

$$\mu_k : S^2(1/2k(k+1)) \longrightarrow P_{2k}(\boldsymbol{C}).$$

In this paper we give a family of minimal isometric immersions of 2-spheres with constant curvature into $P_n(\boldsymbol{C})$ which are not always holomorphic or totally real, using the theory of unitary representations of $SU(2)$. For $n \geqq 3$, we get examples of minimal 2-spheres with constant curvature in $P_n(\boldsymbol{C})$ which are neither holomorphic nor totally real. We will get the following:

THEOREM 1. *For any nonnegative integers n and k with* $0 \leqq k \leqq n$, *there exists an* $SU(2)$*-equivariant minimal isometric immersion*

$$\psi_{n,k} : S^2(c) \longrightarrow P_n(\boldsymbol{C}),$$

where $c=1/(2k(n-k)+n)$ *and* $\psi_{n,k}(S^2(c))$ *is not contained in any totally geodesic complex submanifold of* $P_n(\boldsymbol{C})$. *Furthermore* $\{\psi_{n,k}\}$ *satisfy the following statements:*

(1) *If* $k=0$ *or* $k=n$, *then* $\psi_{n,k}$ *is holomorphic (with respect to a suitable fixed complex structure of* $S^2(c)$) *and* $\psi_{n,k}$ *is congruent to* ψ_n.

(2) *If* n *is even and* $k=n/2$, *then* $\psi_{n,k}$ *is totally real and* $\psi_{n,k}$ *is congruent to* μ_k.

(3) *If* n *and* k *are otherwise (necessarily,* $n\geqq 3$), *then* $\psi_{n,k}$ *is neither holomorphic nor totally real.*

Moreover we will show the following rigidity theorem, using the twistor construction of harmonic maps of a 2-sphere into $P_n(\boldsymbol{C})$ (cf. [**3**], [**6**], [**8**], [**7**], [**11**]).

THEOREM 2. *Let* $\psi : S^2(c)\rightarrow P_n(\boldsymbol{C})$ *be a minimal isometric immersion and assume that* $\psi(S^2(c))$ *is not contained in any totally geodesic complex submanifold in* $P_n(\boldsymbol{C})$. *Then there exists an integer* k *with* $0\leqq k\leqq n$ *such that* c *is equal to* $1/(2k(n-k)+n)$, *and* ψ *is congruent to* $\psi_{n,k}$.

Recently Professor Kenmotsu showed that a minimal surface with constant curvature in $P_2(\boldsymbol{C})$ is holomorphic or totally real. Dr. N. Ejiri (Tokyo Metropolitan Univ.) also found independently examples in Theorem 1 in a manner different from ours.

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1. Preliminaries.

We begin by giving a description of the geometry of $P_n(\boldsymbol{C})$. For $X, Y\in \boldsymbol{C}^{n+1}$ the usual Hermitian inner pruduct is defined by

$$(1.1)\qquad (X, Y)=\textstyle\sum_\alpha x_\alpha \bar{y}_\alpha, \qquad X=(x_0, \cdots, x_n), \qquad Y=(y_0, \cdots, y_n),$$

where we employ the index ranges $0\leqq\alpha, \beta, \cdots\leqq n$, $1\leqq i, j, \cdots\leqq n$. The unitary group $U(n+1)$ is the group of all linear transformations on $\boldsymbol{C}^{n+1}$ leaving the Hermitian product (1.1) invariant. $P_n(\boldsymbol{C})$ is the orbit space of $\boldsymbol{C}^{n+1}-\{0\}$ under the action of the group $\boldsymbol{C}^*=\boldsymbol{C}-\{0\}$; $Z\rightarrow\lambda Z$ $(\lambda\in\boldsymbol{C}^*)$. Let $\pi:\boldsymbol{C}^{n+1}-\{0\}\rightarrow P_n(\boldsymbol{C})$ be the natural projection. For a point $x\in P_n(\boldsymbol{C})$ a vector $Z\in\pi^{-1}(x)$ is called a homogeneous coordinate vector of x. We put $Z_0=Z/(Z, Z)^{1/2}$ so that $(Z_0, Z_0)=1$. The Fubini-Study metric on $P_n(\boldsymbol{C})$ with constant holomorphic sectional curvature c is defined by

$$(1.2)\qquad ds^2=(4/c)((dZ_0, dZ_0)-(dZ_0, Z_0)(Z_0, dZ_0)).$$

The Kaehler form of the Fubini-Study metric (1.2) is given by

$$\omega = -(4/c)\sqrt{-1}\,\partial\bar{\partial}\log|Z|^2 .$$

Now let Z_α be a unitary frame in C^{n+1} so that $(Z_\alpha, Z_\beta)=\delta_{\alpha\beta}$. In the bundle of all unitary frames on C^{n+1} we have

$$dZ_\alpha = \sum_\beta \theta_\alpha^\beta Z_\beta , \tag{1.3}$$

where $\theta_\alpha^\beta=-\bar{\theta}_\beta^\alpha=(dZ_\alpha, Z_\beta)$ is a 1-form. The θ_α^β are the Maurer-Cartan forms of the group $U(n+1)$ and so satisfy the Maurer-Cartan structure equations

$$d\theta_\beta^\alpha = -\sum_\gamma \theta_\gamma^\alpha \wedge \theta_\beta^\gamma . \tag{1.4}$$

By (1.2) and (1.3) the Fubini-Study metric can be written as

$$ds^2 = (4/c)\sum_i \theta_0^i \bar{\theta}_0^i .$$

If we set $\phi^i=(2/\sqrt{c})\theta_0^i$ and $\phi_j^i=\theta_j^i-\delta_j^i\theta_0^0$, then these forms satisfy the structure equations

$$d\phi^i = -\sum_j \phi_j^i \wedge \phi^j, \quad \phi_j^i+\bar{\phi}_i^j = 0$$

and

$$d\phi_j^i = -\sum_k \phi_k^i \wedge \phi_j^k + \Psi_j^i ,$$

where $\Psi_j^i=\theta_0^i\wedge\bar{\theta}_0^j+\delta_j^i\sum_k \theta_0^k\wedge\bar{\theta}_0^k$. Therefore ϕ_j^i are the connection forms of the Fubini-Study metric and Ψ_j^i are its curvature forms.

Let M be a Riemann surface. A *full* map of M into $P_n(C)$ is one whose image lies in no proper totally geodesic complex submanifold of $P_n(C)$. We should note that a map of a compact Riemann surface of genus zero into a Riemannian manifold is harmonic if and only if it is a branched minimal immersion.

Next we review results on irreducible unitary representations of the 3-dimensional special unitary group $SU(2)$.

$SU(2)$ is defined by

$$SU(2) = \left\{g=\begin{pmatrix} a & b \\ -\bar{b} & \bar{a}\end{pmatrix};\ a, b\in C,\ |a|^2+|b|^2=1\right\} .$$

The Lie algebra $\mathfrak{su}(2)$ of $SU(2)$ is given by

$$\mathfrak{su}(2) = \left\{X=\begin{pmatrix} \sqrt{-1}x & y \\ -\bar{y} & -\sqrt{-1}x\end{pmatrix};\ x, y', y''\in R,\ y=y'+\sqrt{-1}y''\right\} .$$

We define a basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ of $\mathfrak{su}(2)$ by

$$\varepsilon_1 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1}\end{pmatrix}, \quad \varepsilon_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} \text{ and } \varepsilon_3 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0\end{pmatrix} .$$

Let V_n be an $(n+1)$-dimensional complex vector space of all complex homogeneous polynomials of degree n with respect to z_0, z_1. We define a Hermitian inner product $(\ ,\ )$ on V_n such that

$$\{u_k^{(n)}=z_0^k z_1^{n-k}/\sqrt{k!(n-k)!}\ ;\ 0\leqq k\leqq n\}$$

is a unitary basis for V_n. We define a real inner product by $\langle\ ,\ \rangle=\mathrm{Re}(\ ,\ )$. A unitary representation ρ_n of $SU(2)$ on V_n is defined by

$$\rho_n(g)f(z_0, z_1)=f(az_0-\bar{b}z_1, bz_0+\bar{a}z_1)$$

for $g\in SU(2)$ and $f\in V_n$. Then the action of $\mathfrak{su}(2)$ on V_n is described as follows;

$$\rho_n(X)(u_i^{(n)})=(i-(n-i))x\sqrt{-1}\,u_i^{(n)} -\sqrt{i(n-i+1)}\,\bar{y}u_{i-1}^{(n)}+\sqrt{(i+1)(n-i)}\,yu_{i+1}^{(n)} \tag{1.5}$$

for $0\leqq i\leqq n$ and any element X of $\mathfrak{su}(2)$.

Let $D(SU(2))$ be the set of all inequivalent irreducible unitary representations of $SU(2)$. Then it is well known that $D(SU(2))=\{(V_n, \rho_n)\ ;\ n=0, 1, 2, \cdots\}$.

We denote by $({}_RV_n, {}_R\rho_n)$ an orthogonal representation of $SU(2)$ induced by the scalar restriction of V_n. Then the following proposition is well known:

PROPOSITION 1.1. (1) *If n is odd, then $({}_RV_n, {}_R\rho_n)$ is irreducible.*

(2) *If n is even, then we have an orthogonal direct sum ${}_RV_n=W_l+\sqrt{-1}\,W_l$, where $n=2l$ and W_l is the ${}_R\rho_{2l}(SU(2))$-invariant irreducible real subspace of ${}_RV_{2l}$ spanned by*

$$\{u_l^{(2l)}, (\sqrt{-1})^j(u_{l+j}^{(2l)}+u_{l-j}^{(2l)}), (\sqrt{-1})^{j+1}(u_{l+j}^{(2l)}-u_{l-j}^{(2l)})\ ;\ 1\leqq j\leqq l\}.$$

Put $T=\{\exp(t\varepsilon_1)\in SU(2)\ ;\ t\in\boldsymbol{R}\}$ and we have $S^2=SU(2)/T$. We identify the tangent space at $\{T\}\in S^2=SU(2)/T$ with a subspace $\mathrm{span}_{\boldsymbol{R}}\{\varepsilon_2, \varepsilon_3\}$ of $\mathfrak{su}(2)$. We fix a complex structure on S^2 so that $\varepsilon_2+\sqrt{-1}\varepsilon_3$ is a vector of type (1, 0). Note that for any $SU(2)$-invariant Riemannian metric g on S^2 there is a positive real number a such that $\{a\varepsilon_2, a\varepsilon_3\}$ is an orthonormal basis with respect to g and (S^2, g) has the constant curvature $4a^2$.

2. Construction of homogeneous minimal 2-spheres in $P_n(\boldsymbol{C})$.

Let (V_n, ρ_n) be an irreducible unitary representation of $SU(2)$. We define the usual complex structure of V_n by $J(v)=\sqrt{-1}v$, for $v\in V_n$. Put $S^{2n+1}=\{v\in V_n\ ;\ \langle v, v\rangle=4\}$ and define the usual S^1-action on S^{2n+1} by $\exp(\sqrt{-1}\theta)v$, for $\exp(\sqrt{-1}\theta)\in S^1$ and $v\in S^{2m+1}$. Let $\pi: S^{2n+1}\to P_n(\boldsymbol{C})$ be the natural Riemannian submersion. We also denote by J the complex structure of $P_n(\boldsymbol{C})$. The action of $\rho_n(SU(2))$ on S^{2n+1} induces the action on $P_n(\boldsymbol{C})$ through π.

First we determine all orbits of $SU(2)$ on $P_n(\boldsymbol{C})$ which are 2-dimensional spheres immersed in $P_n(\boldsymbol{C})$.

LEMMA 2.1. *An orbit M of $SU(2)$ on $P_n(\boldsymbol{C})$ is a 2-dimensional sphere immersed in $P_n(\boldsymbol{C})$ if and only if $M=\pi(\rho_n(SU(2))2u_k^{(n)})$ for some integer k with $0\leqq k\leqq n$.*

PROOF Assume that $M=\pi(\rho_n(SU(2))w)$ for some $w\in S^{2m+1}$ and M is a 2-dimensional sphere immersed in $P_n(\boldsymbol{C})$. Put $N=\rho_n(SU(2))w$. Then the dimension of N is 2 or 3. Suppose that the dimension of N is 3. Since $\pi^{-1}(M)$ is a 3-dimensional compact submanifold of S^{2m+1}, we have $N=\pi^{-1}(M)$. Hence N is invariant by the S^1-action. Thus there is an element X of $\mathfrak{su}(2)$ such that $\rho_n(X)w=\sqrt{-1}w$. Since we can write $X=\mathrm{Ad}(g)(x\varepsilon_1)$ for some element $g\in SU(2)$ and a nonzero real number x, we have $\rho_n(x\varepsilon_1)v=\sqrt{-1}v$, where $v=\rho_n(g^{-1})w$. We put $v=2\sum_{i=0}^n v^i u_i^{(n)}$, where $v^i\in \boldsymbol{C}$ and $\sum_{i=0}^n|v^i|^2=1$. By (1.5) we get

$$\rho_n(x\varepsilon_1)v=2x\textstyle\sum_{i=0}^n v^i(i-(n-i))\sqrt{-1}\,u_i^{(n)}=\sqrt{-1}\,v\,.$$

Hence we have $v^i\{(2i-n)x-1\}=0$ for $i=0, 1, \cdots, n$. Since some v^k with $k\neq n/2$ is nonzero, we have $x=1/(2k-n)$ and $v^i=0$ for $i\neq k$. Hence $v=2v^k u_k^{(n)}$, where $|v^k|=1$. Thus we obtain $M=\pi(\rho_n(SU(2))2u_k^{(n)})$. Next suppose that the dimension of N is 2. Then there is an element X of $\mathfrak{su}(2)$ such that $\rho_n(X)w=0$. By the argument similar to the former case we may put $X=x\varepsilon_1$ for some nonzero real number x. Write $w=2\sum_{i=0}^n w^i u_i^{(n)}$, where $w^i\in\boldsymbol{C}$ and $\sum_{i=1}^n|w^i|^2=1$. By (1.5) we get

$$\rho_n(X)w=2x\textstyle\sum_{i=0}^n w^i(i-(n-i))\sqrt{-1}u_i^{(n)}=0\,.$$

Hence $w^i(2i-n)=0$ for $i=0, 1, \cdots, n$. Thus n is even. Put $k=n/2$, and we have $w=2w^k u_k^{(2k)}$. Hence $|w^k|=1$. So we obtain $M=\pi(\rho_n(SU(2))2u_k^{(2k)})$.

Conversely suppose that $v=2w^k u_k^{(n)}\in S^{2n+1}$ and $M=\pi(\rho_n(SU(2))v)$. (1.5) gives that

$$\begin{aligned}\rho_n(X)v=&(2k-n)x\sqrt{-1}v\\&+2y'(-\sqrt{k(n-k+1)}u_{k-1}^{(n)}+\sqrt{(k+1)(n-k)}u_{k+1}^{(n)})\\&+2y''(\sqrt{k(n-k+1)}\sqrt{-1}u_{k-1}^{(n)}+\sqrt{(k+1)(n-k)}\sqrt{-1}u_{k+1}^{(n)})\,,\end{aligned}\tag{2.1}$$

for any element $X\in\mathfrak{su}(2)$. This implies immediately that M is a 2-dimensional sphere immersed in $P_n(\boldsymbol{C})$. q. e. d.

Now for any nonnegative integers n and k with $0\leqq k\leqq n$ we denote by $\psi_{n,k}$ the $SU(2)$-equivariant isometric immersion of a Riemann sphere $S^2(c)$ with constant curvature c into $P_n(\boldsymbol{C})$ given by the orbit $\pi(\rho_n(SU(2))2u_k^{(n)})$;

$$\begin{array}{rcl} \psi_{n,k} : S^2(c) = SU(2)/T & \longrightarrow & P_n(\boldsymbol{C}) \\ \text{\rotatebox{90}{$\in$}} & & \text{\rotatebox{90}{$\in$}} \\ gT & \longmapsto & \pi(\rho_n(g)2u_k^{(n)}) . \end{array}$$

Here c depends on n and k. We show the following.

PROPOSITION 2.2. (1) *$\psi_{n,k}$ is full.*

(2) *c is equal to $1/(2k(n-k)+n)$.*

(3) *$\psi_{n,k}$ is a minimal immersion.*

(4) (a) *If $k=0$ (resp. $k=n$), then $\psi_{n,k}$ is holomorphic (resp. anti-holomorphic).* (b) *If n is even and $k=n/2$, then $\psi_{2k,k}$ is totally real and $\psi_{2k,k}(S^2(c))$ is contained in a totally geodesic totally real submanifold $P_{2k}(\boldsymbol{R})$ of $P_{2k}(\boldsymbol{C})$.* (c) *If n and k are otherwise, then $\psi_{n,k}$ is neither holomorphic, anti-holomorphic nor totally real.*

(5) *$\psi_{n,k}(S^2(c))=\psi_{n,n-k}(S^2(c))$.*

PROOF. From the irreducibility of (V_n, ρ_n), (1) is clear. We put $v=2u_k^{(n)}$. By (2.1) we have

$$\langle\rho_n(X)v, \rho_n(X)v\rangle = (2k-n)^2x^2+4\{2k(n-k)+n\}(y'^2+y''^2), \tag{2.2}$$

for any element X of $\mathfrak{su}(2)$. We define two elements e_2 and e_3 of $\mathfrak{su}(2)$ by

$$e_i = (1/(2\sqrt{2k(n-k)+n}))\varepsilon_i , \qquad \text{for } i=2, 3 . \tag{2.3}$$

Then by (2.2) $\{\pi_*(\rho_n(e_2)v), \pi_*(\rho_n(e_3)v)\}$ is an orthonormal basis at $\pi(v)$ on $\psi_{n,k}(S^2(c))=\pi(\rho_n(SU(2))v)$. Hence we get (2). By (1.5) and (2.3) simple computations give

$$\begin{aligned} 4\rho_n(e_2)\rho_n(e_2)v = & -v+2/(2k(n-k)+n) \\ & \times\{\sqrt{(k-1)k(n-k+1)(n-k+2)}u_{k-2}^{(n)}+\sqrt{(k+1)(k+2)(n-k-1)(n-k)}u_{k+2}^{(n)}\}, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} 4\rho_n(e_3)\rho_n(e_3)v = & -v-2/(2k(n-k)+n) \\ & \times\{\sqrt{(k-1)k(n-k+1)(n-k+2)}u_{k-2}^{(n)}+\sqrt{(k+1)(k+2)(n-k+1)(n-k)}u_{k+2}^{(n)}\}. \end{aligned} \tag{2.5}$$

From (2.4) and (2.5) we get

$$\rho_n(e_2)\rho_n(e_2)v+\rho_n(e_3)\rho_n(e_3)v = (-1/2)v .$$

Hence the mean curvature vector of $\pi(\rho_n(SU(2))2u_k^{(n)})$ in $P_n(\boldsymbol{C})$ vanishes. Thus we get (3). (4) is easily showed from (2.1). When m is even and $k=n/2$, by (2) of Proposition 1.1 the orbit $\rho_{2k}(SU(2))v$ is contained in W_k. Hence $\psi_{2k,k}(S^2(c))$ is contained in a totally geodesic totally real submanifold $P_{2k}(\boldsymbol{R})$ of $P_{2k}(\boldsymbol{C})$. But $\psi_{2k,k}(S^2(c))$ is not contained in any totally geodesic submanifold of $P_{2k}(\boldsymbol{R})$ because of the irreducibility of W_k. By simple computations we have

$$\rho_n\left(\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}\right)u_k^{(n)} = (-\sqrt{-1})^n \rho_n\left(\begin{pmatrix} \sqrt{-1}b & \sqrt{-1}a \\ \sqrt{-1}\bar{a} & -\sqrt{-1}\bar{b} \end{pmatrix}\right)u_{n-k}^{(n)}.$$

This implies (5). q. e. d.

By the rigidity theorems of Calabi [4], [5], we have $\psi_{n,0}=\psi_n$ and $\psi_{2k,k}=\mu_k$. Thus we obtain Theorem 1.

REMARK 2.3. By simple computations the Brouwer degree and the square length σ of the second fundamental form of $\psi_{n,k}$ are given as follows:

(i) $\deg \psi_{n,k}=n-2k$,

(ii) $\sigma=1/2+\{n(3n-4)-20k(n-k)\}/\{2(2k(n-k)+n)\}$.

REMARK 2.4. In [10] Kenmotsu showed the following:
Let $\psi: M^2\to P_n(C)$ be a minimal isometric immersion of a 2-dimensional compact Riemannian manifold M^2 into $P_n(C)$. If the square length σ of the second fundamental form of ψ satisfies $\sigma\leqq 1/2$, then (1) M^2 is homeomorphic to a 2-sphere and ψ is superminimal, or (2) M^2 is a flat torus and is totally real.

For any (n, k) with $(5n-\sqrt{10n(n+2)})/10\leqq k\leqq(5n+\sqrt{10n(n+2)})/10$, $\psi_{n,k}$ satisfies $\sigma\leqq 1/2$.

3. Twistor construction of harmonic maps into $P_n(C)$.

In this section we review the classification theorem of harmonic maps of a Riemann sphere M_0 into $P_n(C)$.

THEOREM 3.1 (Burns [3], Din-Zakrzewski [6], Glaser-Stora [8]). *There is a bijective correspondence between full harmonic maps $\psi: M_0\to P_n(C)$ and pairs (f, r), where $f: M_0\to P_n(C)$ is a full holomorphic map and r is an integer with $0\leqq r\leqq n$.*

(f, r) is called the *directrix* of ψ.

We outline the construction of harmonic maps from holomorphic maps, following the papers of Eells-Wood [7] and Wolfson [11].

Let $f: M_0\to P_n(C)$ be a full holomorphic map. Choose a coordinate neighborhood (U, ζ) in M_0. In terms of homogeneous coordinates on $P_n(C)$, f is given locally by a holomorphic vector valued function $Z(\zeta)=(z_0(\zeta), \cdots, z_n(\zeta))$. The fullness of f means that

$$Z\wedge(\partial Z/\partial\zeta)\wedge\cdots\wedge(\partial^n Z/\partial\zeta^n)\neq 0 \tag{3.1}$$

except perhaps at isolated points. As Z and its derivatives are all holomorphic functions of ζ, any zeros of (3.1) are removable. This enables us to define a field of unitary frames along f which is intimately related to the osculating spaces of f.

Set $Z_0=Z/(Z, Z)^{1/2}$ and choose $Z_l: U\subset M_0\to C^{n+1}-\{0\}$ such that $\{Z_0(x), \cdots, Z_l(x)\}$ forms a unitary basis for the vector space spanned by $Z(x)$, $(\partial Z/\partial\zeta)(x)$, $\cdots$, $(\partial^l Z/\partial\zeta^l)(x)$ (the l-th osculating space of f at x) for each $l=1, \cdots, n$ and $x\in U$. $\{Z_0, \cdots, Z_n\}$ is a field of unitary frames along f which satisfies

$$\begin{aligned} dZ_0 &= \theta_0^0 Z_0+\theta_0^1 Z_1, \\ dZ_i &= \theta_i^{i-1}Z_{i-1}+\theta_i^i Z_i+\theta_i^{i+1}Z_{i+1}, \qquad 1\leqq i\leqq n-1, \\ dZ_n &= \theta_n^{n-1}Z_{n-1}+\theta_n^n Z_n, \end{aligned} \tag{3.2}$$

where θ_i^{i+1} is a form of type (1, 0) for $0\leqq i\leqq n-1$ and θ_i^{i-1} is a form of type (0, 1) for $1\leqq i\leqq n$.

For an integer r with $0\leqq r\leqq n$, let $G_{r+1}(C^{n+1})$ be the Grassmann manifold of all $(r+1)$-dimensional complex subspaces of C^{n+1}. By the Plücker imbedding $G_{r+1}(C^{n+1})$ is realized as a complex submanifold in the complex projective space $P(\Lambda^{r+1}C^{n+1})$. We define $f_r: U\to G_{r+1}(C^{n+1})$ by $f_r(x)=[Z_0\wedge\cdots\wedge Z_r]$ for $x\in U$, where $[Z_0\wedge\cdots\wedge Z_r]$ denotes an $(r+1)$-dimensional complex subspace of C^{n+1} spanned by $Z_0, \cdots, Z_r$. f_r extends uniquely to a holomorphic map of M_0 into $G_{r+1}(C^{n+1})$ and is called the *r-th associated curve* of f. We put

$$\mathcal{H}_{r,n-r}=\{(V, W)\in G_r(C^{n+1})\times G_{r+1}(C^{n+1})\ ;\ V\subset W\}.$$

Here $G_r(C^{n+1})\times G_{r+1}(C^{n+1})$ has the Kaehler structure induced by $P(\Lambda^r C^{n+1})\times P(\Lambda^{r+1}C^{n+1})$, and $P(\Lambda^r C^{n+1})$ and $P(\Lambda^{r+1}C^{n+1})$ are equipped with the Fubini-Study metrics of the same constant holomorphic sectional curvature. $\mathcal{H}_{r,n-r}$ is a flag manifold $U(n+1)/U(r)\times U(1)\times U(n-r)$ and we have a Riemannian submersion $\pi_r: \mathcal{H}_{r,n-r}\to P_n(C)=U(n+1)/U(n)\times U(1)$.

Now we fix an integer r with $0\leqq r\leqq n$. We define a map $\Phi_r: M_0\to\mathcal{H}_{r,n-r}$ by $\Phi_r(x)=(f_{r-1}(x), f_r(x))$ for $x\in M_0$. Then Φ_r is holomorphic with respect to the Kaehler structure on $\mathcal{H}_{r,n-r}$ induced from $G_r(C^{n+1})\times G_{r+1}(C^{n+1})$, and Φ_r is horizontal with respect to the Riemannian submersion $\pi_r: \mathcal{H}_{r,n-r}\to P_n(C)$. Thus $\phi_r=\pi_r\circ\Phi_r$ is a full harmonic map. ϕ_r is an extension of a map $\pi\circ Z_r: U\to P_n(C)$.

Conversely every full harmonic map of M_0 into $P_n(C)$ is manufactured in the above manner from a unique pair (f, r).

Let $\{\psi_{n,k}\}$ be a family of full minimal immersions of S^2 into $P_n(C)$ constructed in Section 2. Then we have the following:

PROPOSITION 3.2. *The directrix of* $\psi_{n,k}$ *is* (ψ_n, k).

PROOF. By (1.5) we have

$$\rho_n(\varepsilon_2-\sqrt{-1}\varepsilon_3)^k v\in C\cdot u_k^{(n)} \tag{3.3}$$

for each integer k with $0 \leqq k \leqq n$. Since $\pi_*(\rho_n(\varepsilon_2 - \sqrt{-1}\varepsilon_3)v)$ is a vector of type (1, 0) with respect to the complex structure defined on $S^2(c)$, from (3.3) it is easy to see that $\phi_r = \psi_{n,r}$ for $f = \psi_n$. q. e. d.

4. Rigidity.

In this section we show that the minimal 2-spheres $\{\psi_{n,k}\}$ constructed in Section 2 exhaust all minimal 2-spheres with constant curvature in $P_n(\boldsymbol{C})$, using the twistor construction of harmonic maps explained in Section 3.

From (3.2) it follows that

$$d\theta_i^{i-1} = -(\theta_{i-1}^{i-1} - \theta_i^i) \wedge \theta_i^{i-1}, \tag{4.1}$$

$$d\theta_i^i = -\theta_i^{i-1} \wedge \bar{\theta}_i^{i-1} - \theta_i^{i+1} \wedge \bar{\theta}_i^{i+1}, \tag{4.2}$$

for $0 \leqq i \leqq n$, where $\theta_0^{-1} = \theta_0^{n+1} = 0$.

PROPOSITION 4.1. *Let ψ be a full minimal isometric immersion of a 2-sphere with constant curvature into $P_n(\boldsymbol{C})$ and (f, r) the directrix of ψ. Then f is congruent to ψ_n.*

Combining Theorem 3.1, Propositions 3.2 and 4.1, we obtain Theorem 2. We use the following lemma to prove Proposition 4.1.

LEMMA 4.2. *Let $f: P_n(\boldsymbol{C}) \to P_l(\boldsymbol{C})$ and $h: P_n(\boldsymbol{C}) \to P_m(\boldsymbol{C})$ be two holomorphic maps, where $P_l(\boldsymbol{C})$ and $P_m(\boldsymbol{C})$ are equipped with the Fubini-Study metrics of the same constant holomorphic sectional curvature c, and define a holomorphic map $F = (f, h): P_n(\boldsymbol{C}) \to P_l(\boldsymbol{C}) \times P_m(\boldsymbol{C})$ by $F(x) = (f(x), h(x))$. If the metric on $P_n(\boldsymbol{C})$ induced by F is a Kaehler metric of constant holomorphic sectional curvature, then the metrics induced by f and h are Kaehler metrics of constant holomorphic sectional curvature, and they are homothetic.*

PROOF OF PROPOSITION 4.1. We use the same notation as in Section 3. Suppose that $\psi = \phi_r$ is a full minimal isometric immersion of a 2-sphere S^2 with constant curvature into $P_n(\boldsymbol{C})$. We note that the metric induced by ϕ_r is congruent to the metric induced by Φ_r. By (1.2) and (3.2), the metric on S^2 induced by $f_l: S^2 \to G_{l+1}(\boldsymbol{C}^{n+1}) \subset P(\Lambda^{l+1}\boldsymbol{C}^{n+1})$ is given by

$$(4/c)\theta_l^{l+1}\bar{\theta}_l^{l+1}. \tag{4.3}$$

Hence the metric induced by ψ is given by

$$(4/c)(\theta_{r-1}^r \bar{\theta}_{r-1}^r + \theta_r^{r+1}\bar{\theta}_r^{r+1}). \tag{4.4}$$

By virtue of Lemma 4.2, $\theta_{r-1}^r\bar{\theta}_{r-1}^r$ and $\theta_r^{r+1}\bar{\theta}_r^{r+1}$ are metrics of constant curvature and are homothetic. From (4.1) the connection form of the Kaehler metric

$\theta^r_{r-1}\bar{\theta}^r_{r-1}$ is $\theta^{r-1}_{r-1}-\theta^r_r$. By (4.2) the curvature form of the Kaehler metric $\theta^r_{r-1}\bar{\theta}^r_{r-1}$ becomes

(4.5) $$d(\theta^{r-1}_{r-1}-\theta^r_r)=\theta^{r-1}_{r-2}\wedge\bar{\theta}^{r-1}_{r-2}-2\theta^r_{r-1}\wedge\bar{\theta}^r_{r-1}+\theta^{r+1}_r\wedge\bar{\theta}^{r+1}_r\,.$$

Since the Kaehler metric $\theta^r_{r-1}\bar{\theta}^r_{r-1}$ has constant curvature, (4.5) is a constant multiple of $\theta^r_{r-1}\wedge\bar{\theta}^r_{r-1}$. Hence $\theta^{r-1}_{r-2}\wedge\bar{\theta}^{r-1}_{r-2}$ is homothetic to $\theta^r_{r-1}\wedge\bar{\theta}^r_{r-1}$. Since the metric on S^2 induced by ϕ_{r-1} is $(4/c)(\theta^{r-1}_{r-2}\bar{\theta}^{r-1}_{r-2}+\theta^r_{r-1}\bar{\theta}^r_{r-1})$, it is a metric of constant curvature. By the induction we conclude that the metric induced by $f=\phi_0$ is a metric of constant curvature. By the rigidity theorem of Calabi for holomorphic isometric imbeddings, f is congruent to the n-th Veronese imbedding ϕ_n. q. e. d.

PROOF OF LEMMA 4.2. In terms of homogeneous coordinates, we express f and h as $f(z)=(f_0(z), \cdots, f_l(z))$ and $h(z)=(h_0(z), \cdots, h_m(z))$, where f_i $(i=0, \cdots, l)$ (resp. h_j $(j=0, \cdots, m)$) are homogeneous polynomials of degree d_1 (resp. d_2) with respect to $z=(z_0, \cdots, z_n)$, which have no common zeros. The Kaehler form induced by f (resp. h) is given by

$$-(4/c)\sqrt{-1}\partial\bar{\partial}\log|f|^2 \qquad (\text{resp. } -(4/c)\sqrt{-1}\partial\bar{\partial}\log|h|^2)\,.$$

Let $\tilde{F}$ be the composite of $F=(f, h): P_n(\boldsymbol{C})\to P_l(\boldsymbol{C})\times P_m(\boldsymbol{C})$ and the Segre imbedding $P_l(\boldsymbol{C})\times P_m(\boldsymbol{C})\to P_{lm+l+m}(\boldsymbol{C})$;

$$\tilde{F}: P_n(\boldsymbol{C}) \longrightarrow P_{lm+l+m}(\boldsymbol{C})$$
$$z \longrightarrow (f_i(z)h_j(z))_{i,j}\,.$$

Then by the assumption we have $\partial\bar{\partial}\log|\tilde{F}|^2=ac\partial\bar{\partial}\log|z|^2$ for some $a>0$. On the other hand, let $\tilde{\omega}$ and ω be the generators of $H^2(P_{lm+l+m}(\boldsymbol{C});\boldsymbol{Z})$ and $H^2(P_n(\boldsymbol{C});\boldsymbol{Z})$, respectively. Then we have $\tilde{F}^*\tilde{\omega}=(d_1+d_2)\omega$. Hence we have $ac=d_1+d_2$. Thus we get $\partial\bar{\partial}\log(|\tilde{F}|^2/|z|^{2ac})=0$. Since $\log(|\tilde{F}|^2/|z|^{2ac})$ is a harmonic function on $P_n(\boldsymbol{C})$, it is constant. Hence we have $|\tilde{F}|^2=b|z|^{2ac}$ for some $b>0$. Thus we have $|f|^2|h|^2=b|z|^{2ac}$. Put $z_i=x_i+\sqrt{-1}y_i$ $(i=0, 1, \cdots, n)$. Since $|z|^2$ is a real irreducible polynomial with respect to x_i and y_i we have $|f|^2=a_1|z|^{2d_1}$ and $|h|^2=a_2|z|^{2d_2}$ for some $a_1, a_2>0$. Therefore we get $\partial\bar{\partial}\log|f|^2=\partial\bar{\partial}\log a_1|z|^{2d_1}=d_1\partial\bar{\partial}\log|z|^2$ and $\partial\bar{\partial}\log|h|^2=d_2\partial\bar{\partial}\log|z|^2$. q. e. d.

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Shigetoshi BANDO
Mathematical Institute
Tohoku University
Sendai 980
Japan

Yoshihiro OHNITA
Department of Mathematics
Tokyo Metropolitan University
Fukasawa, Setagaya-ku
Tokyo 158
Japan