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Finite subgroups of mapping class groups of geometric 3-manifolds

Dedicated to Professor Itiro Tamura on his 60th birthday

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Introduction.

In Ohshika [4], we introduced the concept of Teichmüller spaces of Seifert fibered manifolds and studied its properties. In some aspects, they are analogous to Teichmüller spaces of surfaces. Let G be a finite subgroup of the mapping class group of a surface. G acts on the Teichmüller space by pulling back of metrics. In Kerckhoff [2] it is proved that G has a fixed point in the Teichmüller space. We prove that the same theorem holds for Seifert fibered manifolds under some assumptions (Theorem 2.3).

This theorem has an application similar to that of Kerckhoff [2]. It is so-called Nielsen realization problem. For a manifold M, the Nielsen realization problem asks when a finite subgroup G of $\pi_0\mathrm{Diff}^+(M)$ is realized by a group of diffeomorphisms. In dimension 3 there are results of Zimmermann and Zieschang [10], [11], [12] which reduces the problem to algebra on G. On the other hand, for a hyperbolic surface it is proved by Kerckhoff [2] that G can be realized by a group of isometries with respect to some hyperbolic structure. Also for a Haken hyperbolic 3-manifold, G can be realized by a group of isometries, which is proved easily using Mostow's rigidity theorem.

In this paper we deal with Seifert fibered manifolds whose base orbifolds are either hyperbolic or Euclidean. They have geometric structures modelled on one of $H^2\times\boldsymbol{R}$, $\widetilde{SL_2}$, E^3, Nil. We ask if G can be realized by a group of isometries with respect to some geometric structure. We do not use the results of [10], [11], [12]. In §3 we characterize an isometry with respect to a geometric structure which is isotopic to the identity. Using Theorem 2.3 and a proposition in §3, we can solve the realization problem under some assumptions on M and G.

Throughout this paper we work in C^∞ category. All 3-manifolds are assumed to be compact orientable. $\mathrm{Diff}^+(M)$ denotes the group of all orientation preserving diffeomorphisms of M. $\pi_1^{\mathrm{orb}}(O)$ denotes the fundamental group of O as an orbifold. $\mathcal{T}(O)$ and $\mathcal{T}(M)$ denote Teichmüller spaces defined in [**4**]. The terminology about geometric structures follows Scott [**7**].

After completing this paper, it was pointed out that Theorem 2.3 was implicitly proved earlier by Lee and Raymond [**13**] and that Theorem 3.3 had been proved by Raymond [**14**] using a different method.

§ 1. Preliminaries.

In this section we study some theorems on 2-orbifolds including Nielsen realization theorem and isotopies of Seifert fibered manifolds. Throughout this section orbifolds are possibly nonorientable or with geodesic boundary. For a homeomorphism $f: O \to O$, we define $f_*: \pi_1{}^{\mathrm{orb}}(O) \to \pi_1{}^{\mathrm{orb}}(O)$ as follows. Let $\tilde{f}$ be a lift of f to the universal cover $\tilde{O}$ of O. Let $f_*(g) = \tilde{f} g \tilde{f}^{-1}$ for $g \in \pi_1{}^{\mathrm{orb}}(O)$. It depends on the choice of $\tilde{f}$, but up to conjugacy it is well-defined. The first lemma is probably well-known but has not been mentioned explicitly anywhere.

LEMMA 1.1. *Let O be a hyperbolic or Euclidean 2-orbifold. Let f be an orbifold homeomorphism from O to itself such that f_* acts on $\pi_1{}^{\mathrm{orb}}(O)$ by an inner automorphism. Then f is isotopic to the identity as an orbifold homeomorphism.*

PROOF. For simplicity, we assume O is closed. As f_* acts on $\pi_1{}^{\mathrm{orb}}(O)$ by an inner automorphism, f fixes all cone points on O. We choose a basepoint and a system of essential simple closed curves which divide O into disks possibly with a cone point. By the method used in Epstein [**1**], we can isotope f so that f fixes every simple closed curve of the system. Since the mapping class groups of the disks relative to their boundaries are trivial, we see that f can be isotoped to the identity. ‖

LEMMA 1.2. *Let O be a hyperbolic 2-orbifold, and let $f: O \to O$ be an isometry such that f_* acts on $\pi_1{}^{\mathrm{orb}}(O)$ by an inner automorphism. Then f is equal to the identity.*

PROOF. By assumption we can choose $\tilde{f} \in \mathrm{Isom}(H^2)$ so that $\tilde{f} g \tilde{f}^{-1} = g$ for every $g \in \pi_1{}^{\mathrm{orb}}(O)$. Since $\tilde{f}$ commutes with every element of $\pi_1{}^{\mathrm{orb}}(O)$ and $\pi_1{}^{\mathrm{orb}}(O)$ is nonelementary, $\tilde{f}$ must be the identity. ‖

LEMMA 1.3. *Let O be a Euclidean orbifold which is neither a torus, a Klein bottle, an annulus nor Möbius band. Let $f: O \to O$ be an isometry which acts on $\pi_1{}^{\mathrm{orb}}(O)$ by an inner automorphism. Then f is equal to the identity.*

PROOF. Similarly to the proof of Lemma 1.2, we can choose $\tilde{f}$ so that $\tilde{f}$ commutes every element of $\pi_1{}^{\mathrm{orb}}(O)$. Since $\pi_1{}^{\mathrm{orb}}(O)$ contains infinitely many rotations, $\tilde{f}$ must be the identity. ‖

The following theorem is a generalization of Kerckhoff's realization theorem. In the main theorem of Kerckhoff [2], he deals with only closed orientable hyperbolic surfaces but mentions that the same theorem holds for hyperbolic 2-orbifolds (possibly nonorientable with boundary). Although the definition of Teichmüller spaces of 2-orbifolds with boundaries of [2] differs from ours, it makes no difference in the theorem.

THEOREM 1.4. *Let O be a hyperbolic or Euclidean 2-orbifold and let G be a finite subgroup of $\pi_0\mathrm{Diff}(O)$. G acts naturally on $\mathcal{T}(O)$. Then G has a fixed point in $\mathcal{T}(O)$.*

PROOF. We only need to prove the theorem in the case when O is Euclidean. It is classically known that an extension of a crystallographic group H by a finite subgroup of $\mathrm{Out}(H)$ is also a crystallographic group. As G can be regarded as a subgroup of $\mathrm{Out}(\pi_1{}^{\mathrm{orb}}(O))$ by Lemma 1.1, G can be realized by a group of isometries with respect to some Euclidean structure. ‖

The following proposition is a generalization of a theorem in Vogt [8].

PROPOSITION 1.5. *Let M be a Seifert fibered manifold whose base orbifold O is either hyperbolic or Euclidean. Fix a fibration of M. Let f_0 and f_1 be isotopic autohomeomorphism of M both of which are fiber preserving. Then there exists a fiber preserving isotopy from f_0 to f_1.*

PROOF. We can assume that f_1 is the identity without loss of generality. M has a geometric structure modelled on E which is equal to one of $H^2\times\boldsymbol{R}$, $\widetilde{SL_2}$, E^3, Nil. Let B be a base geometry of E on which the geometric structure of O is modelled. (B is either H^2 or E^2.) f_0 lifts to $\tilde{f}_0: E\to E$ which induces $\bar{f}_0: B\to B$. As f_0 is isotopic to the identity, for $g\in\pi_1(M)$, $\tilde{f}_0 g\tilde{f}_0^{-1}=g$ as an action on E. Hence for $g\in\pi_0{}^{\mathrm{orb}}(O)$ $\bar{f}_0 g\bar{f}_0^{-1}=g$ as an action on B. By Lemma 1.1, there exists an isotopy $F_t: B\to B$ such that $F_0=\bar{f}_0$, $F_1=\mathrm{id}$, $F_t gF_t^{-1}=g$ for all $g\in\pi_1{}^{\mathrm{orb}}(O)$, $t\in[0, 1]$. We then have a covering isotopy $\tilde{F}_t: E\to E$ such that $\tilde{F}_0=\tilde{f}_0$, $\tilde{F}_t g\tilde{F}_t^{-1}=g$ for all $g\in\pi_1(M)$ and $t\in[0, 1]$. As $\tilde{F}_1$ acts on B trivially, there is an isotopy G_t moving to fiber direction such that $G_0=\tilde{F}_1$, $G_1=\mathrm{id}_E$. Connecting F and G we obtain a fiber preserving isotopy from f_0 to f_1. ‖

§ 2. Finite group action on Teichmüller spaces.

Let M be a geometric 3-manifold. The Teichmüller space of M, denoted by $\mathcal{T}(M)$, is the set of all geometric structures on M factored by isotopy. $\mathcal{T}(M)$ has a natural topology induced from C^∞-topology. $\pi_0\mathrm{Diff}(M)$ acts on $\mathcal{T}(M)$ by pulling back of geometric structures.

Throughout this section we assume that M is a Seifert fibered manifold which admits a geometric structure modelled on one of $H^2\times\boldsymbol{R}$, $\widetilde{SL_2}$, E^3, Nil. $p: M\to O$ denotes a Seifert fibration where O is a base orbifold of the fibration. By assumption on M, O is hyperbolic or Euclidean.

Fix a geometric structure on M. Then O has a geometric structure induced by p. Let E be a geometry space on which geometric structures of M are modelled, and B its base geometry (H^2 or E^2). Each element $m\in\mathcal{T}(M)$ determines a faithful discrete representation of $\pi_1(M)$ to $\mathrm{Isom}^+(M)$ up to conjugacy. We denote this representation by ρ_m. There is a natural projection $r_*: \mathrm{Isom}^+(E)\to\mathrm{Isom}(B)$. If $r_*\rho_{m_1}=r_*\rho_{m_2}$, $\rho_{m_1}(\rho_{m_2})^{-1}$ (where the multiplication and the inverse mean those of $\mathrm{Isom}^+(E)$) is a representation of $\pi_1(M)$ to $\mathrm{Isom}^+(\boldsymbol{R})=\boldsymbol{R}$. We denote this representation by $\delta(m_1, m_2): \pi_1(M)\to\boldsymbol{R}$. We note that if both $r_*(a)$ and $r_*(b)$ are orientation preserving, $\delta(m_1, m_2)(ab)=\delta(m_1, m_2)(a)+\delta(m_1, m_2)(b)$. Let $\{a_1, b_1, \cdots, a_g, b_g, d_1, \cdots, d_{b-1}\}$ (when O is orientable) or $\{a_1, \cdots, a_g, d_1, \cdots, d_b\}$ (when O is nonorientable) be a subset of $\pi_1(M)$ whose projection to O generates $\pi_1(X_0)$ so that $p_*(a_i)$ and $p_*(b_i)$ represent a handle and $p_*(d_i)$ represents a boundary component, where X_0 denotes the underlying space of O. Let Γ be $\{a_1, b_1, \cdots, a_g, b_g, d_1, \cdots, d_{b-1}\}$ if O is orientable and $\{a_1a_2, a_1a_3, \cdots, a_1a_g, d_1, \cdots, d_{b-1}\}$ if O is nonorientable. The following proposition was proved in [4].

PROPOSITION 2.1. *There is a natural map $q: \mathcal{T}(M)\to\mathcal{T}(O)\times\boldsymbol{R}$ such that the first factor is the projection of geometric structures to the base orbifold, and that the second factor is the length of the geometric regular fiber. Then for every $\mu\in\mathrm{im}(q)$ and a fixed $m_0\in q^{-1}(\mu)$, there is a homeomorphism from $q^{-1}(\mu)$ to $\boldsymbol{R}^\Gamma$ which maps m to $(\delta(m, m_0)\gamma)_{\gamma\in\Gamma}$.*

REMARK. In [4] the proposition above is proved only when M is not Euclidean. But the proposition holds even in the case when M has E^3-structure. Fix one foliation by straight lines on E^3 and one Seifert fibration on M whose base orbifold is O. This foliation may be assumed to cover any geometric Seifert fibration $M\to O$ by conjugating a corresponding representation in $\mathrm{Isom}^+(E^3)$. Then we can use the same argument as in the unique fibration case to prove the proposition. ||

The proof of the following lemma is trivial.

LEMMA 2.2. *Let E be one of $H^2\times\boldsymbol{R}$, $\widetilde{SL_2}$, E^3, Nil, and $r: E\to B$ a projection to the base geometry (H^2 or E^2). Then for every geodesic λ on B, $r^{-1}(\lambda)$ with the metric induced from E is isometric to E^2.*

DEFINITION. Let γ be an element of $\pi_1(M)$ which projects to a torsion free element of $\pi_1{}^{\mathrm{orb}}(O)$. For $m\in\mathcal{T}(M)$, $r_*\rho_m(\gamma)\in\mathrm{Isom}(B)$ has an axis λ. $l_m(\gamma)$ denotes parallel translation length of γ on $r^{-1}(\lambda)$ with respect to its Euclidean metric induced form E. For a finite subgroup $G\subset\pi_0\mathrm{Diff}^+(M)$, $G\Gamma$ denotes a collection of $g\gamma$ where $g\in G$, $\gamma\in\Gamma$ allowing repetition. Let $L_{G\Gamma}$ be a function on $\mathcal{T}(M)$ such that for $m\in\mathcal{T}(M)$ $L_{G\Gamma}(m)=\sum_{\gamma\in G\Gamma}l_m(\gamma)^2$.

THEOREM 2.3. *Assume that M is not $S^1\times S^1\times I$, and O is neither $S^2(2, 3, r)$ nor $S^2(3, 3, r)$. Let G be a finite subgroup of $\pi_0\mathrm{Diff}^+(M)$. In the case when M is $S^1\times S^1\times S^1$, assume moreover that G is cyclic. G acts on $\mathcal{T}(M)$ by pulling back of metrics. Then the action of G has a fixed point in $\mathcal{T}(M)$.*

PROOF. First we show that the action of G leaves some isotopy class of Seifert fibration invariant. This follows from Satz 10.1 in Waldhausen [**9**] if M is Haken and does not have E^3-structure, and follows from Corollary 2.3 in [**4**] if O is $S^2(p, q, r)$. If O is either $S^2(2, 2, 2, 2)$ or $D^2(2, 2)$, $\pi_1{}^{\mathrm{orb}}(O)$ has trivial center. Then regular fibers of M must be preserved up to isotopy. If O is $P^2(2, 2)$, there is one-sided Klein bottle in M which is unique up to isotopy. So it is reduced to the case when O is $D^2(2, 2)$. If M is $S^1\times S^1\times S^1$, by conjugating a corresponding matrix in $SL_3(\boldsymbol{Z})$, we can see that there is at least one fibration which is preserved up to isotopy.

Fix one of such fibrations $p: M\to O$. Then the action of G on $\mathcal{T}(M)$ descends to the action of $\mathcal{T}(O)$ through the natural projection by Proposition 1.5. By Theorem 1.4, G has a fixed point $\mu\in\mathcal{T}(O)$. Since G acts fiber-preservingly, the fiber length is invariant. Let $Q\subset\mathcal{T}(M)$ be $q^{-1}(\mu\times\{s\})$ which is not empty. We only need to show that G has a fixed point in Q. For that we will show that $L_{G\Gamma}$ has a unique minimum in Q.

Fix an element $m_0\in Q$. By Proposition 2.1, Q is parametrized by $(\delta(m, m_0)\gamma)_{q\in\Gamma}$. Let $\pi_1^*(M)$ be a subgroup of $\pi_1(M)$ which consists of all elements preserving fiber orientation. If O is orientable, $\pi_1^*(M)=\pi_1(M)=\langle\Gamma, c_1, \cdots, c_k, h\rangle$. If O is nonorientable, $|\pi_1(M):\pi_1^*(M)|=2$ and $\pi_1^*(M)=\langle\Gamma, a_1^2, \cdots, a_g^2, c_1, \cdots, c_k, h\rangle$ ($\langle\,\rangle$ denotes normal closure). In both cases c_i denotes an element around a singular fiber, and h denotes the homotopy class of regular fibers. For $\zeta\in\pi_1^*(M)$, we fix a word in terms of above generators representing ζ. For $\gamma\in\Gamma$, $\mathrm{ind}(\zeta, \gamma)$ denotes the sum of exponents of γ in the word representing ζ. (This definition is independent of word representation because it is determined by the homology class of ζ.)

We see that $l_m(\zeta)^2=(\sum_{\gamma\in\Gamma}\mathrm{ind}(\zeta, \gamma)\delta(m, m_0)(\gamma)+m_f(\zeta))^2+m_b(\zeta)^2$ where $m_f(\zeta)$ and $m_b(\zeta)$ denotes respectively fiber coordinate and base coordinate of $\rho_{m_0}(\zeta)$ with respect to product structure of totally geodesic surface preserved by $\rho_{m_0}(\zeta)$. This is a consequence of additivity of $\delta(m, m_0)$ and Lemma 2.2. The equation above holds for $\zeta\in G\Gamma$ because it is contained in $\pi_1^*(M)$. We denote $\delta(m, m_0)\gamma$ by x_γ. Then $Q\cong\{(x_\gamma)_{\gamma\in\Gamma}; x_\gamma\in\boldsymbol{R}\}$. Hence $L_{G\Gamma}(m)=\sum_{\gamma\in G\Gamma}l_m(\gamma)^2$ is a quadratic function of $\{x_\gamma\}$ whose quadratic coefficients are nonnegative. Hence $L_{G\Gamma}$ has either a unique minimum or a linear space with dimension at least one in which $L_{G\Gamma}$ is constant. From the definition, $\mathrm{ind}(\gamma, \gamma)=1$ and $\mathrm{ind}(\gamma_1, \gamma_2)=0$ if $\gamma_i \in \Gamma$ and $\gamma_1 \neq \gamma_2$. So $L_{G\Gamma}$ is greater than $\sum_{\gamma\in\Gamma}(x_\gamma+m_f(\gamma))^2+m_b(\gamma)^2$ for each $\gamma\in\Gamma$. Hence if x_γ tends to infinity, $L_{G\Gamma}$ goes to infinity. Thus the latter case does not occur.

Let $n\in q^{-1}(m)$ be the unique minimum of $L_{G\Gamma}$. As $L_{G\Gamma}$ is G-invariant function on $q^{-1}(m)$, n is fixed by G action. This is what we want. ||

REMARK. The theorem above does not hold for $S^2\times\boldsymbol{R}$ geometry. For example $\boldsymbol{Z}_2\subset\pi_0\mathrm{Diff}(S^2\times S^1)$ which is generated by the element corresponding to the generator of $\pi_1(SO(3))\cong\boldsymbol{Z}_2$ acts on $\mathcal{T}(S^2\times S^1)\cong S^3\times\boldsymbol{R}$ as a free involution on the first factor.

§ 3. Realizing a finite subgroup by isometries.

Throughout this section M is assumed to be a Seifert fibered manifold whose base orbifold O is either Euclidean or hyperbolic and O is neither $S^2(2, 3, r)$ nor $S^2(3, 3, r)$.

PROPOSITION 3.1. *Assume O is neither a torus, a Klein bottle, an annulus nor a Möbius band. Let g be a geometric structure on M modelled on either of* $H^2\times\boldsymbol{R}$, $\widetilde{SL_2}$, E^3, Nil. *Let $f: M\to M$ be an isometry of (M, g) isotopic to the identity. Then*

i) *if O is nonorientable, f is equal to the identity,*

ii) *if O is orientable, f is a rotation along geometric fibers.*

PROOF. As f is an isometry, f preserves the geometric fibration of M. Hence f induces $\bar{f}: O\to O$ such that $p\circ f=\bar{f}\circ p$. $\bar{f}$ is isotopic to the identity by Proposition 1.5. By Lemma 1.2, $\bar{f}$ must be the identity. Hence f is locally rotation along fibers. As f is isometry, rotation distance is constant all over M. If O is nonorientable, there is no rotation along fiber on M except the half rotation which is not isotopic to the identity, because the fibration structure on M is twisted. ||

Now we can prove the realization theorem.

THEOREM 3.2. *Let G be a finite subgroup of $\pi_0\mathrm{Diff}^+(M)$. Assume that O is nonorientable, and neither a Klein bottle nor a Möbius band. Then there is a geometric structure on M with respect to which G can be realized by isometries.*

PROOF. By Theorem 2.3, there exists a geometric structure on M such that for each element $x\in G$ there exists $\varphi_x\in\mathrm{Diff}^+(M)$ representing x which is an isometry with respect to m. The set $\{\varphi_x\}$ makes a group isomorphic to G because every relation of G is preserved in $\{\varphi_x\}$ by Proposition 3.1. Hence G

can be realized by isometries with respect to m. ∥

THEOREM 3.3. *Let G be a finite cyclic subgroup of $\pi_0\mathrm{Diff}(M)$ which acts on the center of $\pi_1(M)$ trivially. Then there is a geometric structure on M with respect to which G can be realized by isometries.*

PROOF. If M is $S^1\times S^1\times I$, the theorem is easy because $\pi_0\mathrm{Diff}^+(M)\cong GL_2(\boldsymbol{Z})$. So we assume that M is not $S^1\times S^1\times I$. Let x be a generator of G. By Theorem 2.3, there exist a geometric structure m and $\varphi_x\in\mathrm{Isom}^+(M, m)$ representing x. The only relation of G is $x^n=1$. φ_x^n is an isometry of (M, m) isotopic to the identity. Hence if O is neither a torus, an annulus, a Klein bottle nor a Möbius band, φ_x^n is a rotation along fiber. Assume that the rotation distance is r. Let φ_x' be an isometry composing φ_x and $(-r/n)$-rotation along fibers. By assumption φ_x preserves the orientation of fibers. Hence $\varphi_x'^n=\mathrm{id}$. As φ_x' is isotopic to φ_x, this is a realization of G.

If O is one of a torus, an annulus, a Klein bottle, a Möbius band, φ_x^n may induce a transformation on O which is an element of S^1 action. So similarly to the above case there is an isometry φ_x' which is isotopic to φ_x such that $\varphi_x'^n=\mathrm{id}$. ∥

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Added in proof. Recently it was announced that Boileau and Otal proved that homotopy implies isotopy in Seifert fibered manifolds with infinite π_1 whose base orbifolds are either $S^2(2, 3, r)$ or $S^2(3, 3, r)$. This makes the assumption that O is neither $S^2(2, 3, r)$ nor $S^2(3, 3, r)$ in Theorem 2.3 and § 3 unnecessary.