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Extensions of nonlinear completely positive maps

By Fumio HIAI and Yoshihiro NAKAMURA

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Introduction.

It has been well recognized that the most appropriate notion of positivity for linear maps between C^*-algebras is the complete positivity. Although there were classical works [**8**, **11**, **12**] on numerical completely positive functions, it was not until the recent papers of Ando and Choi [**1**] and Arveson [**3**] that the nonlinear complete positivity was investigated in the C^*-algebraic framework. According to [**1**], in spite of the extent of nonlinearity, any completely positive map between arbitrary C^*-algebras admits a nice representation as a doubly infinite sum of compressions of completely positive linear maps on certain C^*-tensor products. On the other hand, the essentially similar representation was obtained in [**3**] for bounded completely positive complex-valued functions on the open unit ball of a unital C^*-algebra.

Since Arveson's Hahn-Banach type extension theorem [**2**] for completely positive linear maps, the linear completely positive extension has been discussed especially in connection with injectivity and nuclearity of C^*-algebras (see e.g. [**5**, **7**]). It seems natural to consider the nonlinear counterpart of complete positive extension. The purpose of this paper is to investigate the problem when completely positive maps defined on $\mathcal{A}$ (resp. ball $\mathcal{A}$, the open unit ball of $\mathcal{A}$) can be extended on $\mathcal{B}$ (resp. ball $\mathcal{B}$) given a C^*-subalgebra $\mathcal{A}$ of a C^*-algebra $\mathcal{B}$.

In Section 1 of this paper, on the lines of [**1**] we generalize the representation theorem in [**3**] to bounded completely positive maps on ball $\mathcal{A}$ with values in a von Neumann algebra. In Section 2, we show the local uniform continuity of completely positive maps. In Section 3, we give some completely positive extension theorems in special cases when $\mathcal{B}=\mathcal{A}_{\mathrm{I}}$ or $\mathcal{A}$ is seminuclear. We further characterize pairs $\mathcal{A}\subset\mathcal{B}$ of C^*-algebras having the completely positive extension property. It is proved above all that every completely positive map from $\mathcal{A}$ to $B(\mathcal{H})$ is extended on $\mathcal{B}$ if and only if $\mathcal{A}^{\otimes m}\otimes\bar{\mathcal{A}}^{\otimes n}\subset\mathcal{B}^{\otimes m}\otimes\bar{\mathcal{B}}^{\otimes n}$ for all $m, n\geqq 0$, where $\bar{\mathcal{A}}$ is the C^*-algebra conjugate to $\mathcal{A}$ and $\mathcal{A}^{\otimes m}\otimes\bar{\mathcal{A}}^{\otimes n}$ is the projective C^*-tensor product of m copies of $\mathcal{A}$ and n copies of $\bar{\mathcal{A}}$. Finally in Section 4, we show that any completely positive map from $\mathcal{A}$ to a von Neumann algebra $\mathcal{M}$ can be uniquely extended to a normal completely positive map from the enveloping von Neumann algebra $\mathcal{A}^{**}$ to $\mathcal{M}$.

1. Representation of completely positive maps.

Throughout this paper, let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras which are not necessarily assumed to be unital. Let $\boldsymbol{M}_n$ be the $n\times n$ complex matrix algebra. We denote by $\boldsymbol{M}_n(\mathcal{A})$ the C^*-algebra of $n\times n$ matrices with entries in $\mathcal{A}$, i.e., $\boldsymbol{M}_n(\mathcal{A})=\mathcal{A}\otimes\boldsymbol{M}_n$, the C^*-tensor product. Given a map $\phi:\mathcal{A}\to\mathcal{B}$ and an integer $n\geqq 1$, let $\phi_n:\boldsymbol{M}_n(\mathcal{A})\to\boldsymbol{M}_n(\mathcal{B})$ be defined by $\phi_n([a_{ij}])=[\phi(a_{ij})]$, $[a_{ij}]\in\boldsymbol{M}_n(\mathcal{A})$. In accordance with the case of linear maps, a map $\phi:\mathcal{A}\to\mathcal{B}$ is said to be *completely positive* if, for every $n\geqq 1$, $\phi_n([a_{ij}])$ is positive in $\boldsymbol{M}_n(\mathcal{B})$ whenever $[a_{ij}]$ is positive in $\boldsymbol{M}_n(\mathcal{A})$. In the nonlinear case, it is meaningful to discuss completely positive maps on $\mathrm{ball}\,\mathcal{A}=\{a\in\mathcal{A}:\|a\|<1\}$, the open unit ball of $\mathcal{A}$, as well. We call a map $\phi:\mathrm{ball}\,\mathcal{A}\to\mathcal{B}$ to be completely positive if, for every $n\geqq 1$, $\phi_n([a_{ij}])\geqq 0$ in $\boldsymbol{M}_n(\mathcal{B})$ whenever $[a_{ij}]\geqq 0$ in $\boldsymbol{M}_n(\mathcal{A})$ with $a_{ij}\in\mathrm{ball}\,\mathcal{A}$. Note that a completely positive map ϕ defined on $\{a\in\mathcal{A}:\|a\|<r\}$, $r>0$, is reduced to one on $\mathrm{ball}\,\mathcal{A}$ by a scaling transform $\phi(ra)$.

In contrast to the linear case, the nonlinear complete positivity is far from trivial even when $\mathcal{A}=\mathcal{B}=\boldsymbol{C}$. Concerning numerical complete positive maps, it was earlier known (see [8, 11, 12]) that a complex-valued function f on $\boldsymbol{C}$ (resp. $\boldsymbol{D}=\{z\in\boldsymbol{C}:|z|<1\}$) is completely positive if and only if it admits a (unique) representation:

$$f(z)=\sum_{m,n=0}^{\infty}c_{mn}z^m\bar{z}^n,\qquad \text{absolutely convergent,}\quad z\in\boldsymbol{C}\ \ (\text{resp. } z\in\boldsymbol{D}),$$

with $c_{mn}\geqq 0$. This representation was recently extended by Ando and Choi [1] to completely positive maps between arbitrary C^*-algebras. For integers $m,n\geqq 0$, $\phi:\mathcal{A}\to\mathcal{B}$ is said to be *(m, n)-mixed homogeneous* if

$$\phi(za)=z^m\bar{z}^n\phi(a),\qquad z\in\boldsymbol{C},\quad a\in\mathcal{A},$$

where the (0, 0)-mixed homogenuity means that ϕ is a constant map. Ando and Choi [1] proved the following theorem for ϕ defined on $\mathcal{A}$, but their proof remains valid also for ϕ on $\mathrm{ball}\,\mathcal{A}$ with a slight modification.

THEOREM 1.1. *For every completely positive map $\phi:\mathcal{A}\to\mathcal{B}$ (resp. $\phi:\mathrm{ball}\,\mathcal{A}\to\mathcal{B}$), there exist unique completely positive maps $\phi_{mn}:\mathcal{A}\to\mathcal{B}$, $m,n\geqq 0$, such that ϕ_{mn} is (m, n)-mixed homogeneous and*

$$\phi(a)=\sum_{m,n=0}^{\infty}\phi_{mn}(a),\qquad \textit{absolutely norm convergent,}\quad a\in\mathcal{A}\ \ (\textit{resp. } a\in\mathrm{ball}\,\mathcal{A}).$$

The notion of complete positivity is introduced also for maps of several variables. Let $\mathcal{A}_k$, $k=1, \cdots, n$, be C^*-algebras and $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ the Cartesian product of $\mathcal{A}_1, \cdots, \mathcal{A}_n$. A map $\Phi: \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \to \mathcal{B}$ is called to be completely positive if, for every $l \geqq 1$, $[\Phi(a_{1,ij}, \cdots, a_{n,ij})]_{ij} \geqq 0$ in $\boldsymbol{M}_l(\mathcal{B})$ whenever $[a_{k,ij}]_{ij} \geqq 0$ in $\boldsymbol{M}_l(\mathcal{A}_k)$, $k=1, \cdots, n$. Now let $\Phi: \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \to B(\mathcal{H})$ be a completely positive multilinear map where $B(\mathcal{H})$ is the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. Then Φ is bounded, i.e.,

$$\|\Phi\| = \sup\{\|\Phi(a_1, \cdots, a_n)\| : a_k \in \mathcal{A}_k, \|a_k\| \leqq 1\} < \infty,$$

and, as mentioned in [1], Φ admits the following Stinespring representation. The proof can be done as in the case of completely positive linear maps (see [13], [14, IV. 3]).

THEOREM 1.2. *If $\Phi: \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \to B(\mathcal{H})$ is a completely positive multilinear map, then there exists a triple $\{\mathcal{K}, (\pi_1, \cdots, \pi_n), V\}$ of a Hilbert space $\mathcal{K}$, representations $\pi_k: \mathcal{A}_k \to B(\mathcal{K})$, $k=1, \cdots, n$, and a bounded linear operator $V: \mathcal{H} \to \mathcal{K}$ such that*

(1) $\pi_k(\mathcal{A}_k)$, $k=1, \cdots, n$, *commute mutually,*

(2) $\Phi(a_1, \cdots, a_n) = V^* \cdot \prod_{k=1}^{n} \pi_k(a_k) \cdot V, \qquad a_k \in \mathcal{A}_k,$

(3) $\left[\prod_{k=1}^{n} \pi_k(\mathcal{A}_k) \cdot V\mathcal{H}\right] = \mathcal{K}.$

If $\{\mathcal{K}', (\pi_1', \cdots, \pi_n'), V'\}$ is another such triple, then there exists a unitary operator $U: \mathcal{K} \to \mathcal{K}'$ such that $\pi_k'(a_k) = U\pi_k(a_k)U^*$, $a_k \in \mathcal{A}_k$, $k=1, \cdots, n$, and $V' = UV$. Furthermore $\|\Phi\| = \|V\|^2$ holds.

Let $CP(\mathcal{A}, \mathcal{B})$ denote the set of all completely positive "linear" maps from $\mathcal{A}$ to $\mathcal{B}$. We denote by $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$ the projective C^*-tensor product (i.e., C^*-tensor product with respect to the largest C^*-crossnorm) of C^*-algebras $\mathcal{A}_1, \cdots, \mathcal{A}_n$, which is determined independently of the order of taking tensor products. Because projective C^*-tensor products are considered in this paper rather than injective C^*-tensor products, we use simply the symbol $\otimes$ instead of the usual $\otimes_{\max}$.

COROLLARY 1.3. *For every completely positive multilinear map $\Phi: \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \to \mathcal{B}$, there exists a unique $\rho \in CP(\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n, \mathcal{B})$ such that*

$$\Phi(a_1, \cdots, a_n) = \rho(a_1 \otimes \cdots \otimes a_n), \qquad a_k \in \mathcal{A}_k.$$

Conversely, for every $\rho \in CP(\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n, \mathcal{B})$, Φ defined by the above equation is a completely positive multilinear map and $\|\Phi\| = \|\rho\|$.

PROOF. Assuming $\mathcal{B} \subset B(\mathcal{H})$, we take the Stinespring representation

$\{\mathcal{K}, (\pi_1, \cdots, \pi_n), V\}$ of Φ by Theorem 1.2. Then there exists a representation $\pi: \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \to B(\mathcal{K})$ such that

$$\pi(a_1 \otimes \cdots \otimes a_n) = \prod_{k=1}^{n} \pi_k(a_k), \qquad a_k \in \mathcal{A}_k.$$

If we define $\rho \in CP(\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n, B(\mathcal{H}))$ by

$$\rho(x) = V^* \pi(x) V, \qquad x \in \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n,$$

then ρ satisfies the desired equation, and so $\rho(\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n) \subset \mathcal{B}$. The uniqueness of ρ is clear. Further we have $\|\Phi\| = \|V\|^2 = \|\rho\|$ by Theorem 1.2. The converse part is seen from the fact that the map $(a_1, \cdots, a_n) \mapsto a_1 \otimes \cdots \otimes a_n$ from $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ to $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$ is completely positive and multilinear. □

For each $n \geqq 1$, let $\mathcal{A}^{(n)}$ be the n-fold Cartesian product of $\mathcal{A}$. For $a \in \mathcal{A}$, the element $(a, \cdots, a)$ in $\mathcal{A}^{(n)}$ is denoted by $a^{(n)}$.

The C^*-algebra conjugate to $\mathcal{A}$ is denoted by $\bar{\mathcal{A}}$, which is defined as the same underlying set as $\mathcal{A}$, having the same addition, multiplication, involution and norm, but whose scalar multiplication is conjugated. We denote by $\bar{a}$ the element in $\bar{\mathcal{A}}$ corresponding to $a \in \mathcal{A}$. Then the scalar multiplication in $\bar{\mathcal{A}}$ is defined by $z\bar{a} = \overline{\bar{z}a}$, $z \in \boldsymbol{C}$, $a \in \mathcal{A}$. The map $a \mapsto \bar{a}$ from $\mathcal{A}$ to $\bar{\mathcal{A}}$ is a conjugate-linear isomorphism and so a completely positive map. Note that $\bar{\mathcal{A}}$ is isomorphic to the C^*-algebra $\mathcal{A}^{\mathrm{op}}$ opposite to $\mathcal{A}$.

It was proved in [1] that completely positive mixed homogeneous maps can be extended to completely positive multilinear maps as follows.

THEOREM 1.4. *If $\phi: \mathcal{A} \to \mathcal{B}$ is a completely positive (m, n)-mixed homogeneous map with $m+n>0$, then there exists a completely positive multilinear map $\Phi: \mathcal{A}^{(m)} \times \bar{\mathcal{A}}^{(n)} \to \mathcal{B}$ such that*

$$\phi(a) = \Phi(a^{(m)}, \bar{a}^{(n)}), \qquad a \in \mathcal{A}.$$

We here state the construction of Φ for later use. Let G be the $(m+n)$-fold product group of the multiplicative group $\{1, -1, \sqrt{-1}, -\sqrt{-1}\}$. Define a completely positive map $\Lambda: \mathcal{A}^{(m)} \times \bar{\mathcal{A}}^{(n)} \to \boldsymbol{M}_l(\mathcal{A})$ with $l = 4^{m+n}$ by

$$\Lambda(a_1, \cdots, a_m, \bar{a}_{m+1}, \cdots, \bar{a}_{m+n}) = \sum_{k=1}^{m+n} [\chi_k \bar{\chi}'_k a_k]_{\chi, \chi' \in G}, \qquad a_k \in \mathcal{A},$$

where χ_k stands for the kth component of $\chi \in G$. Also define a completely positive map $\Theta: \boldsymbol{M}_l(\mathcal{B}) \to \mathcal{B}$ by

$$\Theta([b_{\chi\chi'}]) = \frac{1}{l^2 m!\, n!} \sum_{\chi, \chi' \in G} \zeta(\chi) \overline{\zeta(\chi')} b_{\chi\chi'}, \qquad [b_{\chi\chi'}] \in \boldsymbol{M}_l(\mathcal{B}),$$

where

$$\zeta(\chi) = \prod_{k=1}^{m} \bar{\chi}_k \cdot \prod_{k=m+1}^{m+n} \chi_k .$$

Then the desired $\Phi : \mathcal{A}^{(m)} \times \bar{\mathcal{A}}^{(n)} \to \mathcal{B}$ is given by

$$\Phi = \Theta \circ \phi_\iota \circ \Lambda .$$

Ando and Choi's structure theorem for completely positive maps is obtained by Theorems 1.1, 1.2 and 1.4.

For each $n \geqq 1$, let $\mathcal{A}^{\otimes n} = \mathcal{A} \otimes \cdots \otimes \mathcal{A}$ be the n-fold projective C^*-tensor product of $\mathcal{A}$, and $\mathcal{A}^n$ be the C^*-subalgebra of $\mathcal{A}^{\otimes n}$ generated by the elements $a^{\otimes n} = a \otimes \cdots \otimes a$, $a \in \mathcal{A}$. We put the conventions $\mathcal{A}^{\otimes 0} = \mathcal{A}^0 = \boldsymbol{C}$ and $a^{\otimes 0} = 1$, $a \in \mathcal{A}$. Note that the similar notations $\mathcal{A}^{(n)}$, $\mathcal{A}^{\otimes n}$ and $\mathcal{A}^n$ ought to be distinguished. For m, $n \geqq 0$, let $\mathcal{A}^{\otimes m} \otimes \bar{\mathcal{A}}^{\otimes n}$ (resp. $\mathcal{A}^m \otimes \bar{\mathcal{A}}^n$) be the projective C^*-tensor product of $\mathcal{A}^{\otimes m}$ and $\bar{\mathcal{A}}^{\otimes n}$ (resp. $\mathcal{A}^m$ and $\bar{\mathcal{A}}^n$). The permutation group of $\{1, 2, \cdots, n\}$ is denoted by S_n. We put the convention $S_0 = S_1$. The product group $S_m \times S_n$ gives rise to the natural action $\alpha_{\sigma,\tau}$, $(\sigma, \tau) \in S_m \times S_n$, on $\mathcal{A}^{\otimes m} \times \bar{\mathcal{A}}^{\otimes n}$ defined by

$$\begin{aligned} &\alpha_{\sigma,\tau}(a_1 \otimes \cdots \otimes a_m \otimes \bar{a}_{m+1} \otimes \cdots \otimes \bar{a}_{m+n}) \\ &= a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(m)} \otimes \bar{a}_{m+\tau(1)} \otimes \cdots \otimes \bar{a}_{m+\tau(n)}, \qquad a_k \in \mathcal{A}. \end{aligned}$$

We call a map on $\mathcal{A}^{\otimes m} \otimes \bar{\mathcal{A}}^{\otimes n}$ to be *$S_m \times S_n$-invariant* if it is invariant under the action α of $S_m \times S_n$.

Given C^*-algebras $\mathcal{B}_k$ and C^*-subalgebras $\mathcal{A}_k$ of $\mathcal{B}_k$, $k=1, \cdots, n$, we write

$$\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \subset \mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_n$$

if the natural homomorphism θ from $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$ to $\mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_n$ satisfying

$$\theta(a_1 \otimes \cdots \otimes a_n) = a_1 \otimes \cdots \otimes a_n \qquad a_k \in \mathcal{A},$$

is injective. This is not always the case in contrast to the case of injective C^*-tensor products.

LEMMA 1.5. *For each m, $n \geqq 0$, $\mathcal{A}^m \otimes \bar{\mathcal{A}}^n \subset \mathcal{A}^{\otimes m} \otimes \bar{\mathcal{A}}^{\otimes n}$. Indeed $\mathcal{A}^m \otimes \bar{\mathcal{A}}^n$ is the fixed point subalgebra of $\mathcal{A}^{\otimes m} \otimes \bar{\mathcal{A}}^{\otimes n}$ under the action of $S_m \times S_n$.*

PROOF. To show that the natural homomorphism $\theta : \mathcal{A}^m \otimes \bar{\mathcal{A}}^n \to \mathcal{A}^{\otimes m} \otimes \bar{\mathcal{A}}^{\otimes n}$ is injective, suppose that there exists a nonzero positive element w in $\mathcal{A}^m \otimes \bar{\mathcal{A}}^n$ such that $\theta(w) = 0$. Take a positive linear functional f of $\mathcal{A}^m \otimes \bar{\mathcal{A}}^n$ with $f(w) > 0$. Since $\mathcal{A}^m$ (resp. $\bar{\mathcal{A}}^n$) is the fixed point subalgebra of $\mathcal{A}^{\otimes m}$ (resp. $\bar{\mathcal{A}}^{\otimes n}$) under the natural action α of S_m (resp. S_n), we can define a completely positive multilinear function $\Phi : \mathcal{A}^{\otimes m} \otimes \bar{\mathcal{A}}^{\otimes n} \to \boldsymbol{C}$ by

$$\Phi(x, y) = f(\tilde{x} \otimes \tilde{y}), \qquad x \in \mathcal{A}^{\otimes m}, \quad y \in \bar{\mathcal{A}}^{\otimes n},$$

where

$$\tilde{x} = \frac{1}{m!} \sum_{\sigma \in S_m} \alpha_\sigma(x) \quad \text{and} \quad \tilde{y} = \frac{1}{n!} \sum_{\tau \in S_n} \alpha_\tau(y).$$

Then, by Corollary 1.3, there is a positive linear functional ρ of $\mathcal{A}^{\otimes m} \otimes \bar{\mathcal{A}}^{\otimes n}$ such that

$$\Phi(x, y) = \rho(x \otimes y), \qquad x \in \mathcal{A}^{\otimes m}, \quad y \in \bar{\mathcal{A}}^{\otimes n},$$

and so $f = \rho \circ \theta$, contradicting $f(w) > 0$. Hence $\mathcal{A}^m \otimes \bar{\mathcal{A}}^n \subset \mathcal{A}^{\otimes m} \otimes \bar{\mathcal{A}}^{\otimes n}$. Now it is easily seen that any $S_m \times S_n$-invariant bounded linear functional of $\mathcal{A}^{\otimes m} \otimes \bar{\mathcal{A}}^{\otimes n}$ vanishing on $\mathcal{A}^m \otimes \bar{\mathcal{A}}^n$ is identically zero. This shows the last assertion. □

THEOREM 1.6. *Let $\phi : \mathcal{A} \to \mathcal{B}$ be a completely positive (m, n)-mixed homogeneous map with integers $m, n \geqq 0$. Then:*

(1) *There exists a unique $S_m \times S_n$-invariant $\rho \in CP(\mathcal{A}^{\otimes m} \otimes \bar{\mathcal{A}}^{\otimes n}, \mathcal{B})$ such that*

$$\phi(a) = \rho(a^{\otimes m} \otimes \bar{a}^{\otimes n}), \qquad a \in \mathcal{A}.$$

(2) *There exists a unique $\rho' \in CP(\mathcal{A}^m \otimes \bar{\mathcal{A}}^n, \mathcal{B})$ such that*

$$\phi(a) = \rho'(a^{\otimes m} \otimes \bar{a}^{\otimes n}), \qquad a \in \mathcal{A}.$$

Furthermore $\|\phi\| = \|\rho\| = \|\rho'\|$ holds where $\|\phi\| = \sup\{\|\phi(a)\| : \|a\| \leqq 1\}$.

PROOF. (1) Let $\Phi : \mathcal{A}^{(m)} \times \bar{\mathcal{A}}^{(n)} \to \mathcal{B}$ be as in Theorem 1.4. By the construction of Φ, $\Phi(a_1, \cdots, a_m, \bar{a}_{m+1}, \cdots, \bar{a}_{m+n})$ is invariant under permutations of $(a_1, \cdots, a_m)$ and $(\bar{a}_{m+1}, \cdots, \bar{a}_{m+n})$, so that we get a desired $\rho \in CP(\mathcal{A}^{\otimes m} \otimes \bar{\mathcal{A}}^{\otimes n}, \mathcal{B})$ by Corollary 1.3. The uniqueness of ρ is readily checked, since for any ρ as in (1) we have

$$\begin{aligned} &\rho(a_1 \otimes \cdots \otimes a_m \otimes \bar{a}_{m+1} \otimes \cdots \otimes \bar{a}_{m+n}) \\ &= \frac{1}{m!\, n!} \frac{\partial^{m+n}}{\partial z_1 \cdots \partial z_m \partial \bar{z}_{m+1} \cdots \partial \bar{z}_{m+n}} \phi\Big(\sum_{k=1}^{m+n} z_k a_k\Big). \end{aligned}$$

(2) follows from (1) and Lemma 1.5. Finally $\|\phi\| \leqq \|\rho'\| \leqq \|\rho\|$ is clear. If $\{e_\lambda\}$ is an approximate identity of $\mathcal{A}$, then $\{e_\lambda^{\otimes m} \otimes \bar{e}_\lambda^{\otimes n}\}$ becomes an approximate identity of $\mathcal{A}^{\otimes m} \otimes \bar{\mathcal{A}}^{\otimes n}$ and hence

$$\|\rho\| = \lim_\lambda \|\rho(e_\lambda^{\otimes m} \otimes \bar{e}_\lambda^{\otimes n})\| = \lim_\lambda \|\phi(e_\lambda)\| \leqq \|\phi\|. \qquad \square$$

Arveson [**3**] characterized bounded completely positive complex-valued functions defined on ball$\mathcal{A}$ of a unital C^*-algebra $\mathcal{A}$. Here a map $\phi : \text{ball}\mathcal{A} \to \mathcal{B}$ is called to be *bounded* if $\sup\{\|\phi(a)\| : \|a\| < 1\} < \infty$. In the following theorem, we establish this characterization in a more general setup where $\mathcal{A}$ is not necessarily unital and the range algebra is a von Neumann algebra. Let

$e^{\mathcal{A}}=\bigoplus_{n=0}^{\infty}\mathcal{A}^n$ be the direct sum C^*-algebra of $\mathcal{A}^n$, $n\geqq 0$. There is a natural mapping Γ from ball $\mathcal{A}$ to ball $e^{\mathcal{A}}$ defined by $\Gamma(a)=\bigoplus_{n=0}^{\infty}a^{\otimes n}$, $\|a\|<1$. Analogously $e^{\bar{\mathcal{A}}}$ and Γ on ball $\bar{\mathcal{A}}$ are defined. Then the projective C^*-tensor product $e^{\mathcal{A}}\otimes e^{\bar{\mathcal{A}}}$ is identified with the direct sum C^*-algebra of $\mathcal{A}^m\otimes\bar{\mathcal{A}}^n$, $m, n\geqq 0$, i.e.,

$$e^{\mathcal{A}}\otimes e^{\bar{\mathcal{A}}}=\bigoplus_{m,n=0}^{\infty}(\mathcal{A}^m\otimes\bar{\mathcal{A}}^n).$$

Under this identification, we can write

$$\Gamma(a)\otimes\Gamma(\bar{a})=\bigoplus_{m,n=0}^{\infty}(a^{\otimes m}\otimes\bar{a}^{\otimes n}),\qquad \|a\|<1.$$

THEOREM 1.7. *For each map ϕ: ball $\mathcal{A}\to\mathcal{M}$ where $\mathcal{M}$ is a von Neumann algebra, the following conditions are equivalent:*

(i) *ϕ is bounded and completely positive,*

(ii) *there exists a $\rho\in CP(e^{\mathcal{A}}\otimes e^{\bar{\mathcal{A}}}, \mathcal{M})$ such that*

$$\phi(a)=\rho(\Gamma(a)\otimes\Gamma(\bar{a})),\qquad \|a\|<1.$$

In this case, ρ is unique and

$$\|\rho\|=\sup_{\|a\|<1}\|\phi(a)\|.$$

PROOF. (ii)⇒(i). Let ϕ be given as in (ii). Since the map $a\mapsto a^{\otimes m}\otimes\bar{a}^{\otimes n}$ is completely positive for each $m, n\geqq 0$, it follows that the map $a\mapsto\Gamma(a)\otimes\Gamma(\bar{a})$ from ball $\mathcal{A}$ to $e^{\mathcal{A}}\otimes e^{\bar{\mathcal{A}}}$ is completely positive, and hence so is ϕ. Moreover we have $\|\phi(a)\|\leqq\|\rho\|$, $\|a\|<1$.

(i)⇒(ii). By Theorems 1.1 and 1.6, there exist $\rho_{mn}\in CP(\mathcal{A}^m\otimes\bar{\mathcal{A}}^n, \mathcal{M})$, $m, n\geqq 0$, such that

$$\phi(a)=\sum_{m,n=0}^{\infty}\rho_{mn}(a^{\otimes m}\otimes\bar{a}^{\otimes n}),\ \text{ norm convergent},\qquad \|a\|<1.$$

Regarding $\mathcal{A}^m\otimes\bar{\mathcal{A}}^n\subset e^{\mathcal{A}}\otimes e^{\bar{\mathcal{A}}}$ naturally, we define $\rho_k\in CP(e^{\mathcal{A}}\otimes e^{\bar{\mathcal{A}}}, \mathcal{M})$ by $\rho_k=\sum_{m,n=0}^{k}\rho_{mn}$, $k\geqq 1$. Then $\{\rho_k\}$ is increasing. Let $\{e_\lambda\}$ be an approximate identity of $\mathcal{A}$ contained in ball $\mathcal{A}$. Then it is easily seen that $\{\Gamma(e_\lambda)\otimes\Gamma(\bar{e}_\lambda)\}$ is an approximate identity of $e^{\mathcal{A}}\otimes e^{\bar{\mathcal{A}}}$. Therefore

$$\begin{aligned}\|\rho_k\|&=\sup_\lambda\|\rho_k(\Gamma(e_\lambda)\otimes\Gamma(\bar{e}_\lambda))\|\\&=\sup_\lambda\left\|\sum_{m,n=0}^{k}\rho_{mn}(e_\lambda^{\otimes m}\otimes\bar{e}_\lambda^{\otimes n})\right\|\\&\leqq\sup_\lambda\|\phi(e_\lambda)\|<\infty.\end{aligned}$$

For every positive $x\in e^{\mathcal{A}}\otimes e^{\bar{\mathcal{A}}}$, $\{\rho_k(x)\}$ is increasing and bounded, so it converges strongly in $\mathcal{M}$. Hence $\{\rho_k(x)\}$ converges strongly for every $x\in e^{\mathcal{A}}\otimes e^{\bar{\mathcal{A}}}$. Now define $\rho\in CP(e^{\mathcal{A}}\otimes e^{\bar{\mathcal{A}}}, \mathcal{M})$ by

$$\rho(x)=s\text{-}\lim_{k\to\infty}\rho_k(x), \qquad x\in e^{\mathcal{A}}\otimes e^{\bar{\mathcal{A}}}.$$

We then have

$$\phi(a)=\lim_{k\to\infty}\rho_k(\Gamma(a)\otimes\Gamma(\bar{a}))=\rho(\Gamma(a)\otimes\Gamma(\bar{a})), \qquad \|a\|<1,$$

and

$$\|\rho\|\leqq\sup_\lambda\|\phi(e_\lambda)\|\leqq\sup_{\|a\|<1}\|\phi(a)\|.$$

The uniqueness of ρ is seen from the uniqueness properties in Theorems 1.1 and 1.6. □

When the range algebra is a general C^*-algebra, we obtain

THEOREM 1.8. *If $\phi:\{a\in\mathcal{A}:\|a\|<r\}\to\mathcal{B}$ is a bounded completely positive map with $r>1$, then there exists a unique $\rho\in CP(e^{\mathcal{A}}\otimes e^{\bar{\mathcal{A}}}, \mathcal{B})$ such that*

$$\phi(a)=\rho(\Gamma(a)\otimes\Gamma(\bar{a})), \qquad \|a\|<1.$$

PROOF. By Theorems 1.1 and 1.6, there exist $\rho_{mn}\in CP(\mathcal{A}^m\otimes\bar{\mathcal{A}}^n, \mathcal{B})$, $m, n\geqq 0$, such that

$$\phi(a)=\sum_{m,n=0}^{\infty}\rho_{mn}(a^{\otimes m}\otimes\bar{a}^{\otimes n}), \quad \text{absolutely norm convergent}, \qquad \|a\|<r.$$

Taking an approximate identity $\{e_\lambda\}$ of $\mathcal{A}$, we have

$$\begin{aligned}\|\rho_{mn}\|&=\sup_\lambda\|\rho_{mn}(e_\lambda^{\otimes m}\otimes\bar{e}_\lambda^{\otimes n})\|\\&=\sup_\lambda s^{-(m+n)}\|\rho_{mn}((se_\lambda)^{\otimes m}\otimes(s\bar{e}_\lambda)^{\otimes n})\|\\&\leqq\sup_\lambda s^{-(m+n)}\|\phi(se_\lambda)\|\\&\leqq s^{-(m+n)}\sup_{\|a\|\leqq s}\|\phi(a)\|\end{aligned}$$

where $1<s<r$, so that $\sum_{m,n=0}^{\infty}\|\rho_{mn}\|<\infty$. Hence a desired $\rho\in CP(e^{\mathcal{A}}\otimes e^{\bar{\mathcal{A}}}, \mathcal{B})$ can be given by $\rho=\sum_{m,n=0}^{\infty}\rho_{mn}$. The uniqueness follows from Theorems 1.1 and 1.6 again. □

2. Continuity of completely positive maps.

It was proved in [3] that completely positive complex-valued functions on a unital C^*-algebra are continuous. The purpose of this section is to establish the local uniform continuity of completely positive maps between arbitrary C^*-algebras.

We first introduce the notion of local boundedness for completely positive maps. For a completely positive map $\phi:\mathcal{A}\to\mathcal{B}$ (resp. $\phi:\text{ball}\,\mathcal{A}\to\mathcal{B}$), define

$$r(\phi)=\sup\{r\in[0,\,R)\,:\,\sup_{\|a\|\leq r}\|\phi(a)\|<\infty\}$$

where $R=\infty$ (resp. $R=1$). We call ϕ to be *locally bounded* if $r(\phi)=R$. When $\mathcal{A}$ is unital, it follows that any completely positive map $\phi:\mathcal{A}$ (resp. $\text{ball}\,\mathcal{A})\to\mathcal{B}$ is locally bounded. Indeed we have (cf. [**3**, Lemma 4.2])

$$\sup_{\|a\|\leq r}\|\phi(a)\|=\|\phi(r1)\|,\qquad 0\leq r<R.$$

The following simple example shows that this is not the case when $\mathcal{A}$ is not unital.

EXAMPLE 2.1. Let c_0 be the C^*-algebra consisting of all convergent sequences with limit 0. Take any $r>0$ and define $\phi:c_0\to\boldsymbol{C}$ by

$$\phi(a)=\sum_{n=1}^{\infty}(z_n/r)^n,\qquad a=(z_1,\,z_2,\,\cdots)\in c_0.$$

Then ϕ is a completely positive function on c_0, but we have $r(\phi)=r$.

Now let $\phi:\mathcal{A}$ (or $\text{ball}\,\mathcal{A})\to\mathcal{B}$ be a completely positive map. By Theorems 1.1 and 1.6, ϕ is represented as

$$\phi(a)=\sum_{m,n=0}^{\infty}\phi_{mn}(a)=\sum_{m,n=0}^{\infty}\rho_{mn}(a^{\otimes m}\otimes\bar{a}^{\otimes n})$$

where $\phi_{mn}:\mathcal{A}\to\mathcal{B}$ is a completely positive $(m,\,n)$-mixed homogeneous map and $\rho_{mn}\in CP(\mathcal{A}^{\otimes m}\otimes\bar{\mathcal{A}}^{\otimes n},\,\mathcal{B})$ with $\|\phi_{mn}\|=\|\rho_{mn}\|$.

LEMMA 2.2. *For every completely positive map* $\phi:\mathcal{A}$ (*resp.* $\text{ball}\,\mathcal{A})\to\mathcal{B}$,

$$\begin{aligned}r(\phi)&=\sup\Big\{r\in[0,\,R)\,:\,\sum_{m,n=0}^{\infty}r^{m+n}\|\phi_{mn}\|<\infty\Big\}\\&=\min\Big\{R,\,1/\varlimsup_{k\to\infty}\Big(\max_{m+n=k}\|\phi_{mn}\|^{1/k}\Big)\Big\}\end{aligned}$$

where $R=\infty$ (*resp.* $R=1$).

PROOF. Let $0\leq r<s<r(\phi)$. If $\|a_1\|\leq 1$, then

$$\begin{aligned}\|\phi_{mn}(a_1)\|&=s^{-(m+n)}\|\phi_{mn}(sa_1)\|\\&\leq s^{-(m+n)}\sup_{\|a\|\leq s}\|\phi(a)\|.\end{aligned}$$

We hence have

$$\sum_{m,n=0}^{\infty} r^{m+n}\|\phi_{mn}\| \leqq \left(\frac{s}{s-r}\right)^2 \sup_{\|a\|\leqq s}\|\phi(a)\| < \infty .$$

Conversely if $\sum_{m,n=0}^{\infty} r^{m+n}\|\phi_{mn}\|<\infty$ for $r\in[0, R)$, then $r\leqq r(\phi)$ obviously. Therefore we get the first equality. Since

$$\overline{\lim_{k\to\infty}}\Big(\sum_{m+n=k}\|\phi_{mn}\|\Big)^{1/k} = \overline{\lim_{k\to\infty}}\Big(\max_{m+n=k}\|\phi_{mn}\|^{1/k}\Big)$$

by $\lim_{k\to\infty} k^{1/k}=1$, the second equality is obtained. □

THEOREM 2.3. *If $\phi:\mathcal{A}$ (or $\mathrm{ball}\,\mathcal{A}$)$\to\mathcal{B}$ is a completely positive map, then $r(\phi)>0$ and ϕ is Lipschitz continuous on $\{a\in\mathcal{A}:\|a\|\leqq r\}$ for every $r\in[0, r(\phi))$.*

PROOF. Suppose $r(\phi)=0$, then by Lemma 2.2 there is a sequence of (m_k, n_k), $k=1, 2, \cdots$, such that $\|\phi_{m_k n_k}\|>(4^k)^{m_k+n_k}$. For each $k\geqq 1$, we can choose a positive element a_k in $\mathrm{ball}\,\mathcal{A}$ with $\|\phi_{m_k n_k}(a_k)\|>(4^k)^{m_k+n_k}$. Letting $a=\sum_{k=1}^{\infty}2^{-k}a_k$, we get $a\in\mathrm{ball}\,\mathcal{A}$ and

$$\|\phi(a)\| \geqq \|\phi_{m_k n_k}(a)\| \geqq \|\phi_{m_k n_k}(2^{-k}a_k)\| > (2^k)^{m_k+n_k}, \qquad k\geqq 1,$$

a contradiction. Hence $r(\phi)>0$.

Next, if $a, b\in\mathcal{A}$ and $\|a\|, \|b\|\leqq r<r(\phi)$, then we have

$$\|\rho_{mn}(a^{\otimes m}\otimes\bar{a}^{\otimes n}-b^{\otimes m}\otimes\bar{b}^{\otimes n})\| \leqq \|\rho_{mn}\|(m+n)r^{m+n-1}\|a-b\|$$

and so

$$\|\phi(a)-\phi(b)\| \leqq \sum_{m,n=0}^{\infty}(m+n)r^{m+n-1}\|\phi_{mn}\|\cdot\|a-b\| .$$

Hence Lemma 2.2 implies the theorem. □

The following corollary generalizes [**3**, Theorem 4.1].

COROLLARY 2.4. *If $\phi:\mathrm{ball}\,\mathcal{A}\to\mathcal{B}$ is completely positive, then ϕ is continuous on $\mathrm{ball}\,\mathcal{A}$.*

PROOF. For every $a_0\in\mathrm{ball}\,\mathcal{A}$, define a map $\psi:\{a\in\mathcal{A}:\|a\|<1-\|a_0\|\}\to \boldsymbol{M}_2(\mathcal{B})$ by

$$\psi(a)=\begin{pmatrix}\phi(a+|a_0|) & \phi(a+a_0^*)\\ \phi(a+a_0) & \phi(a+|a_0^*|)\end{pmatrix}, \qquad \|a\|<1-\|a_0\| .$$

Since $\begin{pmatrix}|a_0| & a_0^*\\ a_0 & |a_0^*|\end{pmatrix}$ is positive (cf. [**1**]), it is easily checked that ψ is completely positive. Hence, applying Theorem 2.3 to a scaling transform $\psi(ra)$ with $r=1-\|a_0\|$, we see that ψ is uniformly continuous in a neighborhood of 0 This implies that ϕ is continuous at a_0. □

3. Extensions of completely positive maps.

The fundamental extension theorem for completely positive linear maps is stated as follows (cf. [2, Theorem 1.2.3], [9, Theorem 4.2]): If $\mathcal{A}\subset\mathcal{B}$ are C^*-algebras and $\phi\in CP(\mathcal{A}, B(\mathcal{H}))$, then there is a $\psi\in CP(\mathcal{B}, B(\mathcal{H}))$ such that $\phi=\psi\restriction\mathcal{A}$ and $\|\psi\|=\|\phi\|$. It turns out in this section that this type of extension theorem does not always hold for nonlinear completely positive maps.

Before discussing the general case of $\mathcal{A}\subset\mathcal{B}$, we give some completely positive extension theorems in special cases. For a completely positive map ϕ on $\mathcal{A}$ (or ball $\mathcal{A}$), let ϕ_{mn} and ρ_{mn} be given as in Section 2 by Theorems 1.1 and 1.6. Let $\mathcal{A}_{\mathrm{I}}$ denote the C^*-algebra obtained by adjoining an identity to $\mathcal{A}$.

THEOREM 3.1. *For each completely positive map $\phi:\mathcal{A}$ (resp.* ball $\mathcal{A}$*$)\to\mathcal{M}$ where $\mathcal{M}$ is a von Neumann algebra, the following conditions are equivalent:*

(i) *ϕ is locally bounded,*

(ii) *there exists a completely positive map $\psi:\mathcal{A}_{\mathrm{I}}$ (resp.* ball $\mathcal{A}_{\mathrm{I}}$*$)\to\mathcal{M}$ such that $\phi=\psi\restriction\mathcal{A}$ (resp. $\phi=\psi\restriction$* ball $\mathcal{A}$*).*

In this case, ψ can be chosen so that

$$\sup_{\substack{a\in\mathcal{A}\\ \|a\|\leq r}}\|\phi(a)\| = \sup_{\substack{a\in\mathcal{A}_{\mathrm{I}}\\ \|a\|\leq r}}\|\psi(a)\|, \qquad r\geqq 0 \quad (\textit{resp. } 0\leqq r<1).$$

PROOF. Since any completely positive map is locally bounded in the unital case, (ii)$\Rightarrow$(i) is immediate. We now show (i)$\Rightarrow$(ii). For each m, n fixed, assuming $\mathcal{M}\subset B(\mathcal{H})$, we can get a triple $\{\mathcal{K}, (\pi_1, \cdots, \pi_{m+n}), V\}$ as in Theorem 1.2 with $\mathcal{A}_1=\cdots=\mathcal{A}_m=\mathcal{A}$ and $\mathcal{A}_{m+1}=\cdots=\mathcal{A}_{m+n}=\bar{\mathcal{A}}$ such that

$$\rho_{mn}(a_1\otimes\cdots\otimes a_{m+n}) = V^*\cdot\sum_{k=1}^{m+n}\pi_k(a_k)\cdot V, \qquad a_k\in\mathcal{A}_k$$

Take representations $\tilde{\pi}_k:(\mathcal{A}_k)_{\mathrm{I}}\to B(\mathcal{K})$ by

$$\tilde{\pi}_k(a+z1) = \pi_k(a)+z1_{\mathcal{K}}, \qquad a\in\mathcal{A}_k, \quad z\in\boldsymbol{C}.$$

Noting $(\bar{\mathcal{A}})_{\mathrm{I}}=\overline{\mathcal{A}_{\mathrm{I}}}$, we define

$$\psi_{mn}(a) = V^*\cdot\prod_{k=1}^{m}\tilde{\pi}_k(a)\prod_{k=m+1}^{m+n}\tilde{\pi}_k(\bar{a})\cdot V, \qquad a\in\mathcal{A}_{\mathrm{I}}.$$

Since $\tilde{\pi}_k((\mathcal{A}_k)_{\mathrm{I}})$, $k=1, \cdots, m+n$, commute mutually and

$$V^*\cdot\prod_{k=1}^{m+n}\tilde{\pi}_k((\mathcal{A}_k)_{\mathrm{I}})\cdot V\subset\mathcal{M},$$

it follows that ψ_{mn} is a completely positive (m, n)-mixed homogeneous map from $\mathcal{A}_{\mathrm{I}}$ to $\mathcal{M}$ with $\phi_{mn}=\psi_{mn}\restriction\mathcal{A}$ and $\|\psi_{mn}\|=\|V\|^2=\|\phi_{mn}\|$. Hence Lemma 2.2

implies that a desired completely positive map ψ is given by

$$\psi(a)=\sum_{m,n=0}^{\infty}\psi_{mn}(a), \qquad a\in\mathcal{A}_{\mathrm{I}} \quad (\text{resp. } a\in \operatorname{ball}\mathcal{A}_{\mathrm{I}}).$$

Moreover, for each $r\geqq 0$ (resp. $0\leqq r<1$), since

$$\sum_{m,n=0}^{k}\psi_{mn}(r1)=s\text{-}\lim_{\lambda}\sum_{m,n=0}^{k}\phi_{mn}(re_{\lambda})$$

and

$$\Big\|\sum_{m,n=0}^{k}\phi_{mn}(re_{\lambda})\Big\|\leqq\|\phi(re_{\lambda})\|, \qquad k\geqq 1,$$

we have

$$\sup_{\substack{a\in\mathcal{A}_{\mathrm{I}}\\ \|a\|\leq r}}\|\psi(a)\|=\|\psi(r1)\|\leqq\sup_{\lambda}\|\phi(re_{\lambda})\|$$

where $\{e_{\lambda}\}$ is an approximate identity of $\mathcal{A}$. □

It is said that $\mathcal{A}$ is *seminuclear* (or equivalently $\mathcal{A}$ has the WEP) if $\mathcal{A}\otimes\mathcal{C}\subset\mathcal{B}\otimes\mathcal{C}$ (the projective C^*-tensor products) for any C^*-algebras $\mathcal{B}$ and $\mathcal{C}$ with $\mathcal{A}\subset\mathcal{B}$ (see [**10**]). This notion is weaker than that of nuclear C^*-algebras.

Theorem 3.2. *Let $\mathcal{A}$ be seminuclear, $\mathcal{A}\subset\mathcal{B}$, and $\mathcal{M}$ a von Neumann algebra. If $\phi:\mathcal{A}$ (resp.* ball $\mathcal{A}$*)$\to\mathcal{M}$ is a locally bounded completely positive map, then there exists a locally bounded completely positive map $\psi:\mathcal{B}$ (resp.* ball $\mathcal{B}$*)$\to\mathcal{M}$ such that $\phi=\psi\restriction\mathcal{A}$ (resp. $\phi=\psi\restriction$* ball $\mathcal{A}$*) and*

$$\sup_{\|a\|\leq r}\|\phi(a)\|=\sup_{\|b\|\leq r}\|\psi(b)\|, \qquad r\geqq 0 \quad (\textit{resp. } 0\leqq r<1).$$

Proof. Since $\mathcal{A}_{\mathrm{I}}$ is seminuclear if so is $\mathcal{A}$ (cf. [**7, 9**]), we may assume by Theorem 3.1 that $\mathcal{B}$ is unital and $\mathcal{A}$ contains the identity of $\mathcal{B}$. For each m, n fixed, we have a triple $\{\mathcal{K},(\pi_1,\cdots,\pi_{m+n}),V\}$ representing $\rho_{mn}(a_1\otimes\cdots\otimes a_{m+n})$, $a_k\in\mathcal{A}_k$, where $\mathcal{A}_k=\mathcal{A}$ or $\bar{\mathcal{A}}$ according as $k\leqq m$ or $k>m$. Since $\mathcal{A}$ (and hence $\bar{\mathcal{A}}$) is seminuclear, by [7, Theorem 6.3] there exist $\eta_k\in CP(\mathcal{B}_k,\pi_k(\mathcal{A}_k)'')$, $k=1,\cdots,m+n$, such that $\pi_k=\eta_k\restriction\mathcal{A}_k$ where $\mathcal{B}_k=\mathcal{B}$ or $\bar{\mathcal{B}}$ according as $k\leqq m$ or $k>m$. Define

$$\psi_{mn}(b)=V^*\cdot\prod_{k=1}^{m}\eta_k(b)\prod_{k=m+1}^{m+n}\eta_k(\bar{b})\cdot V, \qquad b\in\mathcal{B}.$$

By an argument as in the proof of Theorem 3.1 with $\pi_k(\mathcal{A}_k)''$ instead of $\tilde{\pi}_k((\mathcal{A}_k)_{\mathrm{I}})$, ψ_{mn} is a completely positive (m,n)-mixed homogeneous map from $\mathcal{B}$ to $\mathcal{M}$, and a completely positive map $\psi:\mathcal{B}$ (resp. ball $\mathcal{B}$)$\to\mathcal{M}$ can be given by $\psi(b)=\sum_{m,n=0}^{\infty}\psi_{mn}(b)$. Then $\phi=\psi\restriction\mathcal{A}$ (resp. $\phi=\psi\restriction$ ball $\mathcal{A}$). The last assertion of equality follows from the assumption that $\mathcal{A}$ contains $1\in\mathcal{B}$. □

In a similar way, we obtain

THEOREM 3.3. *Let* $\mathcal{A}_k \subset \mathcal{B}_k$, $k=1, \cdots, n$, *be* C^*-*algebras and* $\mathcal{M}$ *a von Neumann algebra. If each* $\mathcal{A}_k$ *is seminuclear and* $\Phi: \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \to \mathcal{M}$ *is completely positive multilinear map, then there exists a completely positive multilinear map* $\Psi: \mathcal{B}_1 \times \cdots \times \mathcal{B}_n \to \mathcal{M}$ *such that* $\Phi = \Psi \upharpoonright \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ *and* $\|\Phi\| = \|\Psi\|$.

We now present necessary and sufficient conditions for nonlinear completely positive extensibility. In the sequel, let $\mathcal{A} \subset \mathcal{B}$ be general C^*-algebras and $\mathcal{H}$ $(\neq \{0\})$ an arbitrary Hilbert space.

THEOREM 3.4. *For each* $m, n \geqq 0$, *the following conditions are equivalent:*

(i) *if* $\phi: \mathcal{A} \to B(\mathcal{H})$ *is a completely positive* (m, n)-*mixed homogeneous map, then there exists a completely positive* (m, n)-*mixed homogeneous map* $\psi: \mathcal{B} \to B(\mathcal{H})$ *such that* $\phi = \psi \upharpoonright \mathcal{A}$,

(ii) $\mathcal{A}^{\otimes m} \otimes \bar{\mathcal{A}}^{\otimes n} \subset \mathcal{B}^{\otimes m} \otimes \bar{\mathcal{B}}^{\otimes n}$,

(iii) $\mathcal{A}^m \otimes \bar{\mathcal{A}}^n \subset \mathcal{B}^m \otimes \bar{\mathcal{B}}^n$.

PROOF. (i)⇒(ii). Let $\theta: \mathcal{A}^{\otimes m} \otimes \bar{\mathcal{A}}^{\otimes n} \to \mathcal{B}^{\otimes m} \otimes \bar{\mathcal{B}}^{\otimes n}$ be the natural homomorphism. It is obvious that

$$\theta \circ \alpha_{\sigma,\tau} = \beta_{\sigma,\tau} \circ \theta, \qquad (\sigma, \tau) \in S_m \times S_n,$$

where α and β are the natural actions of $S_m \times S_n$ on $\mathcal{A}^{\otimes m} \otimes \bar{\mathcal{A}}^{\otimes n}$ and $\mathcal{B}^{\otimes m} \otimes \bar{\mathcal{B}}^{\otimes n}$, respectively. If (ii) does not hold, then there exists a nonzero positive element w in $\mathcal{A}^{\otimes m} \otimes \bar{\mathcal{A}}^{\otimes n}$ with $\theta(w)=0$. A positive linear functional f of $\mathcal{A}^{\otimes m} \otimes \bar{\mathcal{A}}^{\otimes n}$ can be chosen so that $f(w)>0$. Let $f_0 = \sum_{\sigma,\tau} f \circ \alpha_{\sigma,\tau}$. Then f_0 is $S_m \times S_n$-invariant and $f_0(w) \geqq f(w) > 0$. It follows from (i) that there exists a completely positive (m, n)-mixed homogeneous map $\psi: \mathcal{B} \to B(\mathcal{H})$ such that

$$f_0(a^{\otimes m} \otimes \bar{a}^{\otimes n}) 1_{\mathcal{H}} = \psi(a), \qquad a \in \mathcal{A}.$$

By Theorem 1.6, there is an $S_m \times S_n$-invariant $\rho \in CP(\mathcal{B}^{\otimes m} \otimes \bar{\mathcal{B}}^{\otimes n}, B(\mathcal{H}))$ such that

$$\psi(b) = \rho(b^{\otimes m} \otimes \bar{b}^{\otimes n}), \qquad b \in \mathcal{B}.$$

Then $\rho \circ \theta$ is $S_m \times S_n$-invariant and

$$f_0(a^{\otimes m} \otimes \bar{a}^{\otimes n}) 1_{\mathcal{H}} = \rho \circ \theta(a^{\otimes m} \otimes \bar{a}^{\otimes n}), \qquad a \in \mathcal{A}.$$

The uniqueness property in Theorem 1.6 implies $f_0(\cdot) 1_{\mathcal{H}} = \rho \circ \theta$, contradicting $f_0(w) > 0$.

(ii)⇒(iii) follows from Lemma 1.5, and (iii)⇒(i) follows from Theorem 1.6 and the extension theorem for completely positive linear maps. □

Similarly we have

THEOREM 3.5. *Let* $\mathcal{A}_k \subset \mathcal{B}_k$, $k=1, \cdots, n$, *be* C^**-algebras. Then the following conditions are equivalent:*

(i) *if* $\Phi: \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \to B(\mathcal{H})$ *is a completely positive multilinear map, then there exists a completely positive multilinear map* $\Psi: \mathcal{B}_1 \times \cdots \times \mathcal{B}_n \to B(\mathcal{H})$ *such that* $\Phi = \Psi \restriction \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$,

(ii) $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \subset \mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_n$.

The following theorem is our main characterization of completely positive extensibility.

THEOREM 3.6. *The following conditions are equivalent:*

(i) *if* $\phi: \mathcal{A} \to B(\mathcal{H})$ *is a locally bounded completely positive map, then there exists a locally bounded completely positive map* $\psi: \mathcal{B} \to B(\mathcal{H})$ *such that* $\phi = \psi \restriction \mathcal{A}$,

(ii) *if* $\phi: \mathrm{ball}\, \mathcal{A} \to B(\mathcal{H})$ *is a bounded (resp. locally bounded) completely positive map, then there exists a bounded (resp. locally bounded) completely positive map* $\psi: \mathrm{ball}\, \mathcal{B} \to B(\mathcal{H})$ *such that* $\phi = \psi \restriction \mathrm{ball}\, \mathcal{A}$,

(iii) $\mathcal{A}^{\otimes m} \otimes \bar{\mathcal{A}}^{\otimes n} \subset \mathcal{B}^{\otimes m} \otimes \bar{\mathcal{B}}^{\otimes n}$ *for all* $m, n \geqq 0$,

(iv) $\mathcal{A}^m \otimes \bar{\mathcal{A}}^n \subset \mathcal{B}^m \otimes \bar{\mathcal{B}}^n$ *for all* $m, n \geqq 0$, *i.e.*, $e^{\mathcal{A}} \otimes e^{\bar{\mathcal{A}}} \subset e^{\mathcal{B}} \otimes e^{\bar{\mathcal{B}}}$.

PROOF. Theorem 3.4 shows (iii)⇔(iv).

(ii)⇒(iv). For each $m, n \geqq 0$, let $\phi: \mathcal{A} \to B(\mathcal{H})$ be a completely positive (m, n)-mixed homogeneous map and $\psi: \mathrm{ball}\, \mathcal{B} \to B(\mathcal{H})$ a completely positive extension of $\phi \restriction \mathrm{ball}\, \mathcal{A}$. Let $\psi = \sum_{k,l} \psi_{kl}$ be the decomposition as in Theorem 1.1. For every $a \in \mathrm{ball}\, \mathcal{A}$ and $z \in \boldsymbol{C}$, $|z|<1$, we then have

$$z^m \bar{z}^n \phi(a) = \sum_{k,l} z^k \bar{z}^l \psi_{kl}(a),$$

and so $\phi(a) = \psi_{mn}(a)$. Therefore $\phi = \psi_{mn} \restriction \mathcal{A}$. Hence (iv) holds by Theorem 3.4.

(iv)⇒(ii). Theorem 1.7 shows that (iv) implies the bounded case of (ii). Also the locally bounded case of (ii) is derived from Theorems 1.1, 1.6 and Lemma 2.2.

The proof of (i)⇔(iv) is analogous. □

Theorem 3.2 asserts that if $\mathcal{A}$ is seminuclear, then $\mathcal{A}$ satisfies the conditions in Theorem 3.6 for any $\mathcal{B}$ with $\mathcal{A} \subset \mathcal{B}$. In fact, condition (iii) is directly checked by definition of seminuclearity. Note that Theorems 3.4–3.6 hold also when $B(\mathcal{H})$ is replaced by an injective C^*-algebra (see [5]).

We conclude this section with a counter-example for nonlinear completely positive extension.

EXAMPLE 3.7. As a non-nuclear C^*-subalgebra of a nuclear C^*-algebra, Choi [4] constructed $\mathcal{A} \subset \mathcal{B}$ such that $\mathcal{A}$ is isomorphic to the regular group C^*-algebra $C_r^*(G)$ of a non-amenable discrete group G and $\mathcal{B}$ is isomorphic to

Cuntz's algebra $\mathcal{O}_2$. If $\mathcal{A}\otimes\bar{\mathcal{A}}\subset\mathcal{B}\otimes\bar{\mathcal{B}}$, then we have $\mathcal{A}\otimes\bar{\mathcal{A}}=\mathcal{A}\otimes_{\min}\bar{\mathcal{A}}$, the injective C^*-tensor product, since $\mathcal{B}$ is nuclear. As stated in [6], this condition implies that $\mathcal{A}$ is nuclear, when $\mathcal{A}=C_r^*(G)$ with discrete G. Hence $\mathcal{A}\otimes\bar{\mathcal{A}}\subset\mathcal{B}\otimes\bar{\mathcal{B}}$ does not hold. So, by Theorem 3.4, there is a completely positive (1, 1)-mixed homogeneous function $\phi:\mathcal{A}\to\boldsymbol{C}$ having no completely positive extension on $\mathcal{B}$.

4. Normal extensions of completely positive maps.

Let $\mathcal{N}$ and $\mathcal{M}$ be von Neumann algebras. As for positive linear maps, we call a completely positive map $\phi:\mathcal{N}$ (resp. ball $\mathcal{N}$)$\to\mathcal{M}$ to be *normal* if $\phi(a_\lambda)\nearrow\phi(a)$ for every increasing net $\{a_\lambda\}$ and a in $\mathcal{N}$ (resp. ball $\mathcal{N}$) with $0\leqq a_\lambda\nearrow a$. Let $\mathcal{N}_k$, $k=1,\cdots,n$, be von Neumann algebras. A completely positive map $\Phi:\mathcal{N}_1\times\cdots\times\mathcal{N}_n\to\mathcal{M}$ is said to be normal if $\Phi(a_{1\lambda},\cdots,a_{n\lambda})\nearrow\Phi(a_1,\cdots,a_n)$ whenever $0\leqq a_{k\lambda}\nearrow a_k$ in $\mathcal{N}_k$, $k=1,\cdots,n$.

Let $\mathcal{A}^{**}$ be the enveloping von Neumann algebra of a C^*-algebra $\mathcal{A}$. We first give the normal extension of a completely positive multilinear map.

THEOREM 4.1. *Let $\mathcal{A}_k$, $k=1,\cdots,n$, be C^*-algebras. If $\Phi:\mathcal{A}_1\times\cdots\times\mathcal{A}_n\to\mathcal{M}$ is a completely positive multilinear map, then there exists a unique normal completely positive multilinear map $\tilde{\Phi}:\mathcal{A}_1^{**}\times\cdots\times\mathcal{A}_n^{**}\to\mathcal{M}$ such that $\Phi=\tilde{\Phi}\restriction\mathcal{A}_1\times\cdots\times\mathcal{A}_n$. Further $\|\Phi\|=\|\tilde{\Phi}\|$ holds.*

PROOF. Let $\{\mathcal{K},(\pi_1,\cdots,\pi_n),V\}$ be the Stinespring representation of Φ, and $\tilde{\pi}_k:\mathcal{A}_k^{**}\to\pi_k(\mathcal{A}_k)''$, $k=1,\cdots,n$, be the normal extension of π_k (cf. [14, p. 121]). If we define $\tilde{\Phi}:\mathcal{A}_1^{**}\times\cdots\times\mathcal{A}_n^{**}\to\mathcal{M}$ by

$$\tilde{\Phi}(a_1,\cdots,a_n)=V^*\cdot\prod_{k=1}^n\tilde{\pi}_k(a_k)\cdot V,\qquad a_k\in\mathcal{A}_k^{**},$$

then $\tilde{\Phi}$ is a normal completely positive multilinear map satisfying $\Phi=\tilde{\Phi}\restriction\mathcal{A}_1\times\cdots\times\mathcal{A}_n$ and $\|\Phi\|=\|\tilde{\Phi}\|$. To show the uniqueness, let $\Psi:\mathcal{A}_1^{**}\times\cdots\times\mathcal{A}_n^{**}\to\mathcal{M}$ be a normal completely positive multilinear map with $\Phi=\Psi\restriction\mathcal{A}_1\times\cdots\times\mathcal{A}_n$, and $\{\mathcal{K}',(\pi_1',\cdots,\pi_n'),V'\}$ be the Stinespring representation of Ψ. Since Ψ is normal, it follows as in the case of completely positive linear maps that π_k', $k=1,\cdots,n$, are normal. Hence $\pi_k'(\mathcal{A}_k^{**})=\pi_k^0(\mathcal{A}_k)''$ where $\pi_k^0=\pi_k'\restriction\mathcal{A}_k$, and so $\{\mathcal{K}',(\pi_1^0,\cdots,\pi_n^0),V'\}$ is the Stinespring representation of Φ. By Theorem 1.2, there is a unitary operator $U:\mathcal{K}\to\mathcal{K}'$ such that $\pi_k^0(a_k)=U\pi_k(a_k)U^*$, $a_k\in\mathcal{A}_k$, $k=1,\cdots,n$, and $V'=UV$. By the normality of $\tilde{\pi}_k$ and π_k', we have

$$\pi_k'(a_k)=U\tilde{\pi}_k(a_k)U^*,\qquad a_k\in\mathcal{A}_k^{**},$$

so that $\Psi=\tilde{\Phi}$. □

The main result in this section is

THEOREM 4.2. *If* $\phi:\mathcal{A}$ (*resp.* ball $\mathcal{A}$)$\to\mathcal{M}$ *is a locally bounded completely positive map, then there exists a unique normal completely positive map* $\tilde{\phi}:\mathcal{A}^{**}$ (*resp.* ball $\mathcal{A}^{**}$)$\to\mathcal{M}$ *such that* $\phi=\tilde{\phi}\upharpoonright\mathcal{A}$ (*resp.* $\phi=\tilde{\phi}\upharpoonright$ ball $\mathcal{A}$).

To prove the theorem, we need some lemmas.

LEMMA 4.3. *If* $\phi:\mathcal{N}\to\mathcal{M}$ *is a normal completely positive map, then so is* $\phi_n:\boldsymbol{M}_n(\mathcal{N})\to\boldsymbol{M}_n(\mathcal{M})$ *for every* $n\geqq 1$.

PROOF. Let $x_\lambda=\begin{pmatrix}a_\lambda & b_\lambda^*\\ b_\lambda & c_\lambda\end{pmatrix}$ and $x=\begin{pmatrix}a & b^*\\ b & c\end{pmatrix}$ be in $\boldsymbol{M}_2(\mathcal{N})$ with $0\leqq x_\lambda\nearrow x$. Then $0\leqq a_\lambda\nearrow a$ and $0\leqq c_\lambda\nearrow c$, so that $\phi(a_\lambda)\nearrow\phi(a)$ and $\phi(c_\lambda)\nearrow\phi(c)$. Since $\{\phi_2(x_\lambda)\}$ is increasing with $\phi_2(x_\lambda)\leqq\phi_2(x)$, $\{\phi_2(x_\lambda)\}$ converges weakly to some element in $\boldsymbol{M}_2(\mathcal{M})$. Hence it follows that $\{\phi(b_\lambda)\}$ converges weakly to some $y\in\mathcal{M}$. We now get

$$\begin{pmatrix}0 & (\phi(b)-y)^*\\ \phi(b)-y & 0\end{pmatrix}\geqq 0.$$

This implies that $y=\phi(b)$, and so $\phi_2(x_\lambda)\nearrow\phi_2(x)$. Therefore ϕ_2 is normal. Repeating this argument, we deduce that ϕ_n is normal for $n=2^k$, $k\geqq 1$, showing the desired conclusion. □

LEMMA 4.4. *If* $\phi:\mathcal{A}\to\mathcal{M}$ *is a completely positive* (m, n)*-mixed homogeneous map, then there exists a unique normal completely positive* (m, n)*-mixed homogeneous map* $\tilde{\phi}:\mathcal{A}^{**}\to\mathcal{M}$ *such that* $\phi=\tilde{\phi}\upharpoonright\mathcal{A}$. *Further* $\|\phi\|=\|\tilde{\phi}\|$ *holds.*

PROOF. Let $\Phi:\mathcal{A}^{(m)}\times\bar{\mathcal{A}}^{(n)}\to\mathcal{M}$ be a completely positive multilinear map taken for ϕ by Theorem 1.4. Theorem 4.1 gives the normal completely positive multilinear map $\tilde{\Phi}:\mathcal{A}^{**(m)}\times\bar{\mathcal{A}}^{**(n)}\to\mathcal{M}$ such that $\Phi=\tilde{\Phi}\upharpoonright\mathcal{A}^{(m)}\times\bar{\mathcal{A}}^{(n)}$ and $\|\Phi\|=\|\tilde{\Phi}\|$. Identifying $\bar{\mathcal{A}}^{**}$ with $\overline{\mathcal{A}^{**}}$, we can define a desired $\tilde{\phi}:\mathcal{A}^{**}\to\mathcal{M}$ by

$$\tilde{\phi}(a)=\tilde{\Phi}(a^{(m)}, \bar{a}^{(n)}),\qquad a\in\mathcal{A}^{**}.$$

Since $\|\phi\|=\|\Phi\|$ and $\|\tilde{\phi}\|=\|\tilde{\Phi}\|$ (see the proof of Theorem 1.6), we get $\|\phi\|=\|\tilde{\phi}\|$. To show the uniqueness, let $\psi:\mathcal{A}^{**}\to\mathcal{M}$ be a normal completely positive (m, n)-mixed homogeneous map with $\phi=\psi\upharpoonright\mathcal{A}$, and $\Psi:\mathcal{A}^{**(m)}\times\bar{\mathcal{A}}^{**(n)}\to\mathcal{M}$ be taken for ψ by Theorem 1.4. Indeed Ψ is given by the formula $\Psi=\Theta\circ\psi_l\circ\Lambda$ where $l=4^{m+n}$ as stated just after Theorem 1.4. It is immediate that Λ and Θ are normal. Also ψ_l is normal by Lemma 4.3. Hence Ψ is normal. Moreover, since $\phi=\psi\upharpoonright\mathcal{A}$, we have $\Phi=\Psi\upharpoonright\mathcal{A}^{(m)}\times\bar{\mathcal{A}}^{(n)}$. Therefore $\Psi=\tilde{\Phi}$ follows from Theorem 4.1, and so $\psi=\tilde{\phi}$. □

PROOF OF THEOREM 4.2. Let $\phi=\sum_{m,n}\phi_{mn}$ be the decomposition by Theorem 1.1. For each $m, n\geqq 0$, Lemma 4.4 gives the normal completely positive (m, n)-mixed homogeneous map $\tilde{\phi}_{mn}:\mathcal{A}^{**}\to\mathcal{M}$ such that $\phi_{mn}=\tilde{\phi}_{mn}\upharpoonright\mathcal{A}$ and

$\|\phi_{mn}\|=\|\tilde{\phi}_{mn}\|$. Since ϕ is locally bounded, Lemma 2.2 shows that a desired $\tilde{\phi}:\mathcal{A}^{**}$ (resp. ball $\mathcal{A}^{**})\to\mathcal{M}$ can be defined by $\tilde{\phi}=\sum_{m,n}\tilde{\phi}_{mn}$. For the uniqueness, let $\psi:\mathcal{A}^{**}$ (resp. ball $\mathcal{A}^{**})\to\mathcal{M}$ be a normal completely positive map with $\phi=\psi\upharpoonright\mathcal{A}$, and $\psi=\sum_{m,n}\psi_{mn}$ be the decomposition by Theorem 1.1. Then it is readily seen that ψ_{mn} is normal and $\phi_{mn}=\psi_{mn}\upharpoonright\mathcal{A}$ for every $m, n\geqq 0$. Hence $\psi=\tilde{\phi}$ follows from Lemma 4.4. □

Lastly we note one aspect of the normality of nonlinear completely positive maps. If a completely positive map between von Neumann algebras is continuous with respect to the σ-weak topologies, then it is normal clearly. However the converse is false in the nonlinear case. Indeed we have

EXAMPLE 4.5. We consider the map $f\in L^{\infty}(\boldsymbol{R})\mapsto|f|^2=f\bar{f}\in L^{\infty}(\boldsymbol{R})$, which is a normal completely positive (1, 1)-mixed homogeneous map. Let $f_s(t)=e^{ist}$, $s, t\in\boldsymbol{R}$. Then $f_s\to 0$ in $\sigma(L^{\infty}, L^1)$ as $s\to\infty$, a consequence of Riemann-Lebesgue lemma. But $|f_s|^2=1$.

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Fumio HIAI
Division of Applied Mathematics
Research Institute of
Applied Electricity
Hokkaido University
Sapporo 060
Japan

Yoshihiro NAKAMURA
Division of Applied Mathematics
Research Institute of
Applied Electricity
Hokkaido University
Sapporo 060
Japan