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A certain class of infinite dimensional diffusion processes arising in population genetics

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1. Introduction.

Let S be a countable set, and let

$$X=\{x=(x_i)_{i\in S} : x_i\geqq 0\ (i\in S),\ \sum_{i\in S} x_i=1\}$$

be the totality of probability vectors on S, which is equipped with the weak topology. Suppose that we are given a second order differential operator L of the following type:

$$L=\frac{1}{2}\sum_{i\in S}\sum_{j\in S}(x_i\beta_i\delta_{ij}+x_ix_j(\sum_{k\in S}x_k\beta_k-\beta_i-\beta_j))\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i\in S}b_i(x)\frac{\partial}{\partial x_i}, \tag{1.1}$$

where $(\beta_i)_{i\in S}$ are non-negative constants satisfying that $\sup_{i\in S}\beta_i<+\infty$, δ_{ij} stands for the Kronecker symbol, and the domain $\mathcal{D}(L)$ of L is the set of all C^2-functions defined on X depending on only finitely many coordinates.

Let $W=C([0, \infty)\to X)$ be the space of all continuous functions $w:[0, \infty)\ni t\to w(t)\in X$ with the topology of uniform convergence on bounded intervals, and let $\mathcal{F}$ ($\mathcal{F}_t$) be the σ-field on W generated by cylinder sets (up to time t).

By an (X, L)-diffusion we mean a system $\{P_x, x\in X\}$ of probability distributions on $(W, \mathcal{F})$ that is strongly Markovian and satisfies the following two conditions:

$$P_x\{w: w(0)=x\}=1 \quad \text{for every } x\in X, \tag{1.2}$$

$$f(w(t))-f(w(0))-\int_0^t Lf(w(s))ds \text{ is a } (P_x, \mathcal{F}_t)\text{-martingale for every } f\in\mathcal{D}(L). \tag{1.3}$$

In order to construct an (X, L)-diffusion we need boundary conditions and a regularity condition on the drift coefficients $(b_i(x))_{i\in S}$.

ASSUMPTION [B]. $(b_i(x))_{i\in S}$ are real functions defined on X which satisfy the following three conditions:

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(B.1) $b_i(x) \geqq 0$ if $x_i=0$ $(i \in S)$,

(B.2) $\sum_{i \in S} b_i(x) = 0$ uniformly in $x \in X$,

(B.3) there exists a matrix $(q_{ij})_{i,j \in S}$ such that $q_{ij} \geqq 0$ for every i and j of S, $\sup_{j \in S} \sum_{i \in S} q_{ij} < +\infty$, and

$$|b_i(x)-b_i(x')| \leqq \sum_{j \in S} q_{ij} |x_j - x'_j| \quad \text{for every } x \text{ and } x' \text{ of } X, \ (i \in S).$$

Our main result is the following.

THEOREM 1.1. *Suppose that the operator L of* (1.1) *satisfies the Assumption* [B]. *Then there exists a unique (X, L)-diffusion.*

The diffusion operator L of (1.1) was first introduced by Gillespie [**2**] in case that S consists of two points. Then L is a one-dimensional diffusion operator. In case that S is an arbitrary finite set Sato [**5**] derived the operator L by a diffusion approximation from Markov chain models.

As to the well-posedness problem of the (X, L)-diffusions Okada [**4**] solved it in case that S consists of three points, and Shiga [**7**] also gave a partial result in case that S is an arbitrary finite set.

In particular, if $\beta_i = \beta$ for every $i \in S$ in (1.1), L reduces to an infinite allelic diffusion model of the Wright-Fisher type, which was discussed by Ethier [**1**] and Shiga [**8**]. In this case the diffusion coefficients are polynomials of order 2, so that the diffusion part of L transforms every polynomial of order n into a polynomial of the same order n, which makes an analytical treatment extremely tractable (cf. [**1**]).

On the other hand we notice that the diffusion coefficients of (1.1) are polynomials of order 3, so that the method of [**1**] is not applicable. Our method is due to stochastic differential equations. A key point is an observation that the (X, L)-diffusion of (1.1) is derived from a simpler diffusion corresponding to the stochastic differential equation (2.5) by making use of normalization and a time change transformation. Furthermore, for the equation (2.5) we can apply a method to be adopted in a one-dimensional solvable case (cf. [**9**]).

2. Proof of Theorem 1.1.

We here discuss the existence and uniqueness of (X, L)-diffusions by the method of stochastic differential equations.

We will formulate a stochastic differential equation which describes an (X, L)-diffusion process. We first choose $\alpha(x) = (\alpha_{ij}(x))_{i,j \in S}$ as follows:

(2.1) $$\alpha_{ij}(x) = (\delta_{ij} - x_i)\sqrt{\beta_j x_j}, \quad (i, j \in S).$$

Then it satisfies

(2.2) $$\sum_{k\in S}\alpha_{ik}(x)\alpha_{jk}(x)=x_i\beta_i\delta_{ij}+x_ix_j(\sum_{k\in S}x_k\beta_k-\beta_i-\beta_j).$$

Consider the following stochastic differential equation on X:

(2.3) $$dx_i(t)=\sum_{k\in S}\alpha_{ik}(x(t))dB_k(t)+b_i(x(t))dt\,,\qquad i\in S\,.$$

Following Ikeda-Watanabe [**3**], by a solution of equation (2.3) we mean a system of stochastic processes

$$\mathcal{X}=\{x(t)=(x_i(t))_{i\in S},\ B(t)=(B_i(t))_{i\in S}\}$$

defined on a probability space $(\Omega, \mathcal{B}, P)$ with a reference family $(\mathcal{B}_t)_{t\geq 0}$ such that

(i) $x(t)$ is a continuous $(\mathcal{B}_t)$-adapted process taking values in X,

(ii) $B(t)$ is an independent system of one-dimensional $(\mathcal{B}_t)$-adapted Brownian motions, and

(iii) with probability one,

$$x_i(t)=x_i(0)+\sum_{k\in S}\int_0^t\alpha_{ik}(x(s))dB_k(s)+\int_0^t b_i(x(s))ds\,,\qquad i\in S\,.$$

We say that the law uniqueness of solutions for (2.3) holds if whenever $\mathcal{X}$ and $\mathcal{X}'$ are any two solutions of (2.3) whose initial law coincides, then the probability laws of $\{x(t)\}$ and $\{x'(t)\}$ on $(W, \mathcal{F})$ coincide.

It is known that the (X, L)-diffusion $\{P_x,\ x\in X\}$ exists uniquely if and only if for every probability μ on $(X, \mathcal{B}_X)$ there exists a solution of (2.3) such that the law of $x(0)$ coincides with μ and the law uniqueness of solutions holds for (2.3).

Accordingly for the proof of Theorem 1.1 it is sufficient to show the existence and the law uniqueness of solutions for the equation (2.3).

THEOREM 2.1. *Suppose that* $(b_i(x))_{i\in S}$ *satisfies the Assumption* [B]. *Then the existence and the law uniqueness of solutions hold for the equation* (2.3).

In order to prove the theorem let us introduce another stochastic differential equation. Let

$$Y=\{y=(y_i)_{i\in S}\ :\ y_i\geqq 0\,(i\in S),\ 0<\sum_{i\in S}y_i<+\infty\}.$$

Define a mapping $\pi: Y\to X$ by

$$\pi y=(\pi_i y)_{i\in S},\quad \pi_i y=y_i\Big/\sum_{k\in S}y_k\qquad \text{for } y=(y_i)_{i\in S}\in Y\,.$$

Let $c>0$ be a fixed constant satisfying $c>(1/2)\sup_{i\in S}\beta_i$ and set

$$(2.4)\qquad \tilde{b}_i(y) = b_i(\pi y)+c\pi_i y+\pi_i y(\beta_i-\sum_{k\in S}\pi_k y\beta_k), \qquad i\in S.$$

Consider the following stochastic differential equation on Y:

$$(2.5)\qquad dy_i(t) = \sqrt{\beta_i y_i(t)}dB_i(t)+\tilde{b}_i(y(t))dt, \qquad i\in S.$$

The notion of solutions for the equation (2.5) is defined analogously to the equation (2.3) replacing X by Y, that is, a system of stochastic processes

$$\mathcal{Y} = \{y(t)=(y_i(t))_{i\in S},\ B(t)=(B_i(t))_{i\in S}\}$$

defined on a probability space $(\Omega, \mathcal{B}, P)$ with a reference family $(\mathcal{B}_t)_{t\geq 0}$ is a solution of the equation (2.5) if

(i)′ $y(t)$ is a continuous $(\mathcal{B}_t)$-adapted process taking values in Y,

(ii)′ $B(t)$ is an independent system of one-dimensional $(\mathcal{B}_t)$-adapted Brownian motions, and

(iii)′ with probability one,

$$y_i(t) = y_i(0)+\int_0^t\sqrt{\beta_i y_i(s)}dB_i(s)+\int_0^t\tilde{b}_i(y(s))ds, \qquad i\in S.$$

We say that the pathwise uniqueness of solutions for (2.5) holds if whenever $\mathcal{Y}$ and $\mathcal{Y}'$ are any two solutions defined on the same probability space $(\Omega, \mathcal{B}, P)$ with the same reference family $(\mathcal{B}_t)$ and the same independent system of $(\mathcal{B}_t)$-adapted one-dimensional Brownian motions $B(t)$ such that $y(0)=y'(0)$ a. s., then $y(t)=y'(t)$ for all $t\geqq 0$ a. s.

The law uniqueness of solutions for (2.5) is also defined analogously to (2.3).

It is known that the pathwise uniqueness of solutions implies the law uniqueness of solutions, (see [**3**], p. 152).

THEOREM 2.2. *Under the same assumption as in Theorem* 2.1 *the existence and the pathwise uniqueness of solutions hold for the equation* (2.5). *In particular, the law uniqueness of solutions for* (2.5) *holds.*

PROOF. For any $\varepsilon>0$ let

$$Y_\varepsilon = \{y=(y_i)_{i\in S}\in Y : \sum_{i\in S} y_i\geqq\varepsilon\}.$$

Note that $(\tilde{b}_i(y))_{i\in S}$ are continuous on Y_ε, and satisfy boundary conditions:

$$\tilde{b}_i(y)\geqq 0 \quad \text{if } y\in Y_\varepsilon \quad \text{and} \quad y_i=0, \quad i\in S.$$

Then it is easy to see that if $y(0)\in Y_\varepsilon$ there exists a Y_ε-valued solution $\mathcal{Y}=(y(t), B(t))$ of (2.5) up to $\tau_\varepsilon=\inf\{t>0: \sum_{i\in S}y_i(t)=\varepsilon\}$, i. e. with probability one,

$$y(t)\in Y_\varepsilon \quad \text{for } 0\leqq t\leqq\tau_\varepsilon, \qquad \text{and}$$

$$y_i(t\wedge\tau_\varepsilon)=y_i(0)+\int_0^{t\wedge\tau_\varepsilon}\sqrt{\beta_i y_i(s)}\,dB_i(s)+\int_0^{t\wedge\tau_\varepsilon}\bar{b}_i(y(s))ds\,.$$

(See [**6**] Theorem 2.1, and [**8**] Theorem 2.1 for the details). Consequently this implies that if $y(0)\in Y$ there exists a Y-valued solution $\mathcal{Y}=(y(t), B(t))$ of (2.5) up to $\tau=\lim_{\varepsilon\downarrow 0}\tau_\varepsilon$. Also, it is easily checked that $r(t)=\sum_{i\in S}y_i(t)$ is continuous in $t\in[0, \tau)$ a. s.

Let $\sigma_M=\inf\{t>0: r(t)\geqq M\}$ for $M>0$, and set $\zeta=\zeta_{\varepsilon,M}=\tau_\varepsilon\wedge\sigma_M$. Since $\sum_{i\in S}\bar{b}_i(y)=c$, using Ito's formula we see

$$\log r(t\wedge\zeta)-\log r(0)-\int_0^{t\wedge\zeta}\frac{1}{r(s)}\Big(c-\frac{1}{2}\sum_{i\in S}\beta_i y_i(s)\Big/r(s)\Big)ds$$

is a martingale. Recalling that $c>(1/2)\sup_{i\in S}\beta_i$ we have

$$E\{\log r(t\wedge\zeta)\}\geqq E\{\log r(0)\}>-\infty$$

for any $\varepsilon>0$, any $M>0$, and any $t>0$, which implies that

$$P\{\tau=+\infty\}=1\,,$$

since $\lim_{\varepsilon\downarrow 0}\tau_\varepsilon=\tau$ and $\lim_{M\to\infty}\sigma_M=\infty$. Thus we have shown that there exists a solution of the equation (2.5) taking values in Y through a full time interval $[0, \infty)$.

Now we proceed to prove the pathwise uniqueness of solutions. Suppose that $\{y(t)\}$ and $\{y'(t)\}$ be two solutions of (2.5) taking values in Y with $y(0)=y'(0)$ for the same independent system of one-dimensional Brownian motions $B(t)=(B_i(t))_{i\in S}$ on a probability space $(\Omega, \mathcal{B}, P; (\mathcal{B}_t))$. Let $r(t)=\sum_{i\in S}y_i(t)$ and $r'(t)=\sum_{i\in S}y'_i(t)$, and for any $\varepsilon>0$ and $M>0$ define

$$\zeta=\inf\{t>0: r(t)\notin(\varepsilon, M) \text{ or } r'(t)\notin(\varepsilon, M)\}.$$

Then

$$(2.6)\qquad y_i(t\wedge\zeta)-y'_i(t\wedge\zeta)=\int_0^{t\wedge\zeta}(\sqrt{\beta_i y_i(s)}-\sqrt{\beta_i y'_i(s)})dB_i(s)+\int_0^{t\wedge\zeta}(\bar{b}_i(y(s))-\bar{b}_i(y'(s)))ds\,.$$

We choose a sequence of smooth functions $(\phi_n(u))_{n\geq 1}$ defined on $\boldsymbol{R}^1$ such that

(i) $|\phi_n(u)|\leqq|u|$ and $\lim_{n\to\infty}\phi_n(u)=|u|$,

(ii) $\lim_{n\to\infty}\phi'_n(u)=\begin{cases}1 & (u>0)\\ 0 & (u=0)\\ -1 & (u<0)\end{cases}$ boundedly,

(iii) $\lim_{n\to\infty}u\phi''_n(u)=0$ boundedly.

Applying Ito's formula for (2.7) with $\phi_n(u)$ and letting $n\to\infty$ we have

$$E\{|y_i(t\wedge\zeta)-y_i'(t\wedge\zeta)|\} \leqq \int_0^t E\{|\bar{b}_i(y(s\wedge\zeta))-\bar{b}_i(y'(s\wedge\zeta))|\}ds.$$

From the Assumption [B] and (2.4) it follows that there exists a constant $C_\varepsilon>0$ such that for every y and y' of Y_ε

$$\sum_{i\in S}|\bar{b}_i(y)-\bar{b}_i(y')| \leqq C_\varepsilon\sum_{i\in S}|y_i-y_i'|.$$

Thus we obtain

$$\sum_{i\in S}E\{|y_i(t\wedge\zeta)-y_i'(t\wedge\zeta)|\} \leqq C_\varepsilon\int_0^t\sum_{i\in S}E\{|y_i(s\wedge\zeta)-y_i'(s\wedge\zeta)|\}ds.$$

Hence, by Gronwall's inequality we have

$$P\{y(t)=y'(t) \text{ for } 0\leqq t\leqq\zeta\} = 1,$$

which implies the pathwise uniqueness of solutions for (2.5) because of $\lim_{M\to\infty}\lim_{\varepsilon\downarrow 0}\zeta=+\infty$ a. s. Therefore the proof of Theorem 2.2 is complete.

PROOF OF THEOREM 2.1. Let $\mathcal{Y}=(y(t), B(t))$ be a solution of the equation (2.5) defined on a probability space $(\Omega, \mathcal{B}, P; (\mathcal{B}_t))$. Let

$$r(t) = \sum_{i\in S}y_i(t), \qquad \text{and set} \quad A_t = \int_0^t\frac{ds}{r(s)}.$$

Then it holds that A_t is strictly increasing and $\lim_{t\to\infty}A_t=\infty$ a. s. because $\lim_{t\to\infty}r(t)/t=c$ a. s. follows from the equality $\sum_{i\in S}\bar{b}_i(y)=c$ in (2.4).

Denoting by A_t^{-1} the inverse function of A_t we define a new process $x(t)$ by

$$x(t) = y(A_t^{-1})/r(A_t^{-1}).$$

Then we will find an independent system of one-dimensional Brownian motions $\hat{B}(t)=(\hat{B}_i(t))_{i\in S}$ such that $(x(t), \hat{B}(t))$ is a solution of the equation (2.3).

Let $\tilde{x}(t)=y(t)/r(t)$. By Ito's formula we have

$$(2.7)\qquad d\tilde{x}_i(t) = \frac{1}{r(t)}dy_i(t)-\frac{y_i(t)}{r^2(t)}dr(t)+\frac{y_i(t)}{r^3(t)}d\langle r\rangle(t)-\frac{1}{r^2(t)}d\langle y_i, r\rangle(t),$$

$$(2.8)\qquad dr(t) = \sum_{j\in S}\sqrt{\beta_j y_j(t)}dB_j(t)+cdt,$$

$$(2.9)\qquad d\langle r\rangle(t) = \sum_{i\in S}\beta_i y_i(t)dt,$$

$$(2.10)\qquad d\langle y_i, r\rangle(t) = \beta_i y_i(t)dt.$$

Substituting (2.8)-(2.10) into (2.7) we see

$$(2.11)\qquad d\tilde{x}_i(t) = \frac{1}{\sqrt{r(t)}}\sum_{j\in S}(\delta_{ij}-\tilde{x}_i(t))\sqrt{\beta_j\tilde{x}_j(t)}dB_j(t)+\frac{1}{r(t)}b_i(\tilde{x}(t))dt.$$

Let

$$\hat{B}_i(t) = \int_0^t \frac{1}{\sqrt{r(A_s^{-1})}} dB_i(A_s^{-1}) \qquad \text{for every } i \in S .$$

Then, as is easily seen, $\hat{B}(t)=(\hat{B}_i(t))$ is an independent system of one-dimensional Brownian motions, and (2.11) turns into

$$x_i(t) = x_i(0) + \sum_{j\in S}\int_0^t (\delta_{ij} - x_i(s))\sqrt{\beta_j x_j(s)}\, d\hat{B}_j(s) + \int_0^t b_i(x(s))ds . \tag{2.12}$$

Thus $(x(t), \hat{B}(t))$ is a solution of the equation (2.3). Consequently we have shown the existence of solutions for the equation (2.5).

Now, suppose that $\mathscr{X}=(x(t), B(t))$ be a solution of the equation (2.3) with an arbitrarily given initial law μ on $(X, \mathscr{B}_X)$. For the solution $\mathscr{X}=(x(t), B(t))$ let us consider the following one-dimensional stochastic integral equation:

$$z(t) = 1 + \int_0^t z(s) \sum_{i\in S} \sqrt{\beta_i x_i(s)}\, dB_i(s) + \int_0^t cz(s)ds . \tag{2.13}$$

Since the coefficients of the equation (2.13) are Lipschitz continuous in z, there exists a unique solution, which is obtained by a standard successive iteration procedure.

Moreover it holds that the solution $z(t)$ satisfies

$$P\{z(t)>0 \text{ for any } t\geqq 0, \text{ and } \lim_{t\to\infty} z(t)=\infty\} = 1 . \tag{2.14}$$

Because, applying Ito's formula for (2.13) with $\log x$ we see

$$\log z(t\wedge\sigma_\varepsilon) = \int_0^{t\wedge\sigma_\varepsilon} \sum_{i\in S}\sqrt{\beta_i x_i(s)}\, dB_i(s) + \int_0^{t\wedge\sigma_\varepsilon}\Big(c - \frac{1}{2}\sum_{i\in S}\beta_i x_i(s)\Big)ds \tag{2.15}$$

where $\sigma_\varepsilon=\inf\{t>0: z(t)\leqq\varepsilon\}$. Accordingly, by $c>(1/2)\sum_{i\in S}\beta_i x_i$ we have

$$E\{\log z(t\wedge\sigma_\varepsilon)\} \geqq 0 ,$$

from which it follows that $P\{\lim_{\varepsilon\downarrow 0}\sigma_\varepsilon>t\}=1$ for any $t>0$, so that $P\{z(t)>0$ for any $t\geqq 0\}=1$ holds.

Furthermore, the equation (2.15) turns into

$$\log z(t) = \int_0^t \sum_{i\in S}\sqrt{\beta_i x_i(s)}\, dB_i(s) + \int_0^t\Big(c - \frac{1}{2}\sum_{i\in S}\beta_i x_i(s)\Big)ds . \tag{2.16}$$

Hence from

$$c - \frac{1}{2}\sum_{i\in S}\beta_i x_i \geqq c - \frac{1}{2}\sup_{i\in S}\beta_i > 0$$

it follows that

$$\lim_{t\to\infty} z(t) = +\infty \qquad \text{a. s.},$$

which yields (2.14).

Next, we set $\tilde{y}_i(t)=z(t)x_i(t)$ for each $i\in S$. Then by Ito's formula

$$d\tilde{y}_i(t) = z(t)dx_i(t)+x_i(t)dz(t)+d\langle z,\ x_i\rangle(t)\,.$$

Into this substituting (2.3), (2.13) and

$$d\langle z,\ x_i\rangle(t) = z(t)\sum_{j\in S}(\delta_{ij}-x_i(t))\beta_j x_j(t)dt\,,$$

we obtain

$$d\tilde{y}_i(t) = \sqrt{z(t)}\sqrt{\beta_i\tilde{y}_i(t)}\,dB_i(t)+z(t)\tilde{b}_i(\tilde{y}(t))dt\,,\qquad i\in S\,. \tag{2.17}$$

Define a time change function C_t by

$$C_t = \int_0^t z(s)ds\,.$$

By (2.14) C_t is strictly increasing, continuous, and $\lim_{t\to\infty}C_t=\infty$ a.s. Denoting by C_t^{-1} the inverse function of C_t we define

$$y(t) = \tilde{y}(C_t^{-1})\,,\qquad \tilde{\mathcal{B}}_t = \mathcal{B}_{C_t^{-1}}\,,\qquad \text{and}\qquad \tilde{B}_i(t) = \int_0^t \sqrt{z(C_s^{-1})}dB_i(C_s^{-1})\,.$$

Then $\tilde{B}(t)=(\tilde{B}_i(t))_{i\in S}$ is an independent system of one-dimensional $(\tilde{\mathcal{B}}_t)$-adapted Brownian motions, and $(y(t),\ \tilde{B}(t))$ is a solution of the equation (2.5). Hence, by virtue of Theorem 2.2 the probability law of $\{y(t)\}$ is uniquely determined from μ (the law of $y(0)=x(0)$).

Let

$$r(t) = \sum_{i\in S} y_i(t),\qquad \text{and}\qquad A_t = \int_0^t \frac{1}{r(s)}ds\,.$$

Then, noticing that $C_t^{-1}=A_t$ holds we obtain

$$x(t) = y(C_t)/z(t) = y(A_t^{-1})/r(A_t^{-1})\,.$$

Therefore the probability law of $\{x(t)\}$ is uniquely determined by the initial law μ. Thus we have completed the proof of Theorem 2.1, which is equivalent to Theorem 1.1.

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