J. Math. Soc. Japan
Vol. 38, No. 4, 1986

Quasi-arithmetic means of continuous functions

By Takashi ITO and Chiê NARA*

(Received April 25, 1985)

Introduction.

Let I be an interval, containing more than one point, of real numbers. A quasi-arithmetic mean with weight of two numbers a, b in I is defined as

$$\phi^{-1}\{t\phi(a)+(1-t)\phi(b)\}, \tag{1}$$

where ϕ is fixed as a strictly increasing or decreasing, real valued continuous function defined on I and the weight t is also fixed as a real number with $0<t<1$. This mean will be denoted by $N_{\phi,t}(a, b)$ throughout the paper.

This definition of a mean can be extended naturally to a mean of a continuous function, instead of two numbers, as follows. Let X be a compact Hausdorff space and let $C(X;I)$ be the space of all I-valued continuous functions on X. For a fixed strictly monotoneous continuous function ϕ on I as above and a fixed probability measure μ on X, a mean of a function f in $C(X;I)$ is defined to be

$$\phi^{-1}\left\{\int_X \phi(f)d\mu\right\}. \tag{2}$$

In this paper this mean will be called a quasi-arithmetic mean with weight, simply a *QA-mean* and denoted by $M_{\phi,\mu}(f)$ for f in $C(X;I)$.

It is clear that a QA-mean $M=M_{\phi,\mu}$ is a continuous functional defined on $C(X;I)$ and has the following properties I), II) and (*), here we regard $C(X;I)$ as the space with the topology of uniform convergence and the usual order structure.

I) $\quad M(a1_X)=a \quad$ for all a in I,

where 1_X is the constant 1 function on X.

II) $\quad M(f)\leqq M(g) \quad$ if $f\leqq g$ in $C(X;I)$.

By Fubini's theorem we have the following equation (*), which will be called the *bisymmetry equation*, this terminology is taken from [2],

* The second author was partially supported by Grant-in-Aid for Scientific Research (No. 59740088), Ministry of Education, Science and Culture.

(*) $$M_y[M_x[h(x, y)]] = M_x[M_y[h(x, y)]] \qquad \text{for } h \text{ in } C(X\times X; I),$$

where suffixes x and y of M_x and M_y are understood to apply M to a function of variables x and y respectively.

If the support of μ is identical to X, then $M=M_{\phi,\mu}$ is strictly increasing on $C(X; I)$;

III) $M(f) < M(g)$ for $f \lneqq g$ in $C(X; I)$.

Now we define a *mean* on $C(X; I)$ in general to be a continuous non-linear functional M on $C(X; I)$ satisfying I) and II) above. A mean is called *bisymmetric* if the bisymmetry equation (*) above is satisfied. Note that the both sides of the bisymmetry equation (*) are well defined for any mean in general. The bisymmetry equation is a necessary condition for a mean to be a QA-mean. The first result we will prove is that under the assumption of being strictly increasing the bisymmetry equation (*) is a sufficient condition for a mean to be a QA-mean.

THEOREM 1. *Let X be a compact Hausdorff space and let M be a mean on $C(X; I)$. If M is strictly increasing and satisfies the bisymmetry equation (*), then there are a strictly increasing or decreasing, real valued continuous function ϕ on I and a probability measure μ on X with the support identical to X such that*

$$M(f) = \phi^{-1}\left\{\int_X \phi(f)d\mu\right\} \qquad \text{for } f \text{ in } C(X; I),$$

where the probability measure μ is determined uniquely and the function ϕ is also unique up to an affine equivalence.

Typical QA-means on $C(X)^+$ are the *p-th power root means,* where $C(X)^+$ stands for $C(X; I)$ with $I=(0, +\infty)$. For $-\infty<p<+\infty$, the p-th power root mean is defined to be

(3) $$M_{p,\mu}(f) = \left\{\int_X f^p d\mu\right\}^{1/p} \qquad \text{for } f \text{ in } C(X)^+,$$

for $p=0$, $M_{0,\mu}$ is understood as

(4) $$M_{0,\mu}(f) = \exp\left\{\int_X \log f\, d\mu\right\} \qquad \text{for } f \text{ in } C(X)^+.$$

It is obvious that the p-th power root means are positively homogeneous on $C(X)^+$:

IV) $M(rf) = rM(f)$ for all $r>0$ and f in $C(X)^+$.

Besides these p-th power root means, one can define the following positively homogeneous means on $C(X)^+$. For a given upper semi-continuous function δ

on X with $0\leqq\delta(x)\leqq 1$ and $\sup_{x\in X}\delta(x)=1$, define

$$\text{(5)}\qquad \mathrm{Max}_{\delta}(f)=\max_{x\in X}\{\delta(x)f(x)\}\qquad \text{for } f \text{ in } C(X)^{+}.$$

This mean will be called a *maximum mean with weight* and δ is its *weight*. Similarly for a given lower semi-continuous function γ on X with $1\leqq\gamma(x)\leqq+\infty$ and $\inf_{x\in X}\gamma(x)=1$, define

$$\text{(6)}\qquad \mathrm{Min}_{\gamma}(f)=\min_{x\in X}\{\gamma(x)f(x)\}\qquad \text{for } f \text{ in } C(X)^{+}.$$

This mean is called a *minimum mean with weight* and γ is its *weight*. It is easily checked that maximum and minimum means with weight are bisymmetric means.

The second result we will prove is that any positively homogeneous bisymmetric mean on $C(X)^{+}$ must be one of the means as mentioned above.

THEOREM 2. *Let X be a compact Hausdorff space and let M be a mean on $C(X)^{+}$. If M is positively homogeneous and satisfies the bisymmetry equation* (*), *then M is a p-th power root mean, $M=M_{p,\mu}$ for some $-\infty<p<+\infty$ and some probability measure μ on X or M is a maximum or minimum mean with weight, $M=\mathrm{Max}_{\delta}$ or $M=\mathrm{Min}_{\gamma}$, where the power p, the probability measure μ and the weights δ, γ are determined uniquely except for the case of M being a point evaluation.*

Since the notion of means has its long history, there could be many previously known results which have some connection with our results, particularly with Theorem 1. However we mention only a few closely related results.

In 1930, A. N. Kolmogorov [8] and M. Nagumo [9] independently characterized sequences of quasi-arithmetic means $\phi^{-1}(\{\phi(x_1)+\cdots+\phi(x_n)\}/n)$ $(n=1, 2, \cdots)$. In 1946 and 1948, J. Aczél [1] and [2] discovered significance of the *bisymmetry equation* by using it to characterize finite dimensional quasi-arithmetic, as well as weighted, means. Hence, Theorem 1 can be regarded as an infinite dimensional generalization of the theorem of [2] (see Proposition 1 of this paper). In this paper his theorem will be used fundamentally.

Following after Kolmogorov and Nagumo's work, a few results attempting generalizations from a finite dimensional case to an infinite dimensional case had appeared, for instance, B. Jessen [7] and B. de Finetti [5]. The results in those papers are formulated in an entirely different setting from ours, because the notion of the bisymmetry did not appear in their time yet.

After the authors finished their first draft of this paper, they were informed of the work [3] of J. Aczél, I. Fenyö and J. Horvath and other related works, see the references of [3]. In [3], they work on means defined *a priori* on all *measurable* functions instead of continuous functions, and they give several conditions characterizing their mean to be a QA-mean with respect to the absolute continuous Lebesgue-Stieltjes measures on the unit interval [0, 1]. Although their results do not cover ours, one of their conditions (see condition e) in [**3**]) is actually close to our bisymmetry equation (*).

Finally we add a few sentences about the organization of our paper. Our general procedures of proving Theorem 1 and Theorem 2 are quite similar. However, in Theorem 1 we depend heavily on the property of M being *strictly increasing*, on the other hand, the property of M being *positively homogeneous* will play a crucial role in Theorem 2. In the first section we discuss the preliminaries including the extension of the bisymmetry equation to semi-continuous functions. Theorem 1 and Theorem 2 are proved in the second and the third sections respectively. In the last section we state some generalizations of Theorem 1 and Theorem 2.

§ 1. Preliminaries.

Whenever a mean is given, speaking of technicality, it is more convenient, as it will be evident later, that the domain of the mean is regarded as extended a priori to the broader class of semi-continuous functions rather than the class of continuous functions. We discuss first this extension. Let I^u and I^l be I plus the supremum of I and I plus the infimum of I respectively, where the supremum and the infimum of I are taken in the space of extended real numbers $[-\infty, +\infty]$. Suppose a mean M on $C(X; I)$ is given. Let v (u) be an I^u valued (I^l-valued) lower (upper) semi-continuous function on X. Our extension of M to v or u is defined as

$$(7) \qquad \begin{aligned} M(v) &= \sup\{M(f) \mid v \geqq f,\ f \text{ in } C(X; I)\}, \quad \text{or} \\ M(u) &= \inf\{M(f) \mid u \leqq f,\ f \text{ in } C(X; I)\}. \end{aligned}$$

It should be remarked that the family $\{f \mid v \geqq f,\ f \text{ in } C(X; I)\}$ ($\{f \mid u \leqq f,\ f \text{ in } C(X; I)\}$) is upward (downward) directed and converges pointwise to v (u) on X, and the value of the extended M could be $\pm\infty$.

First of all, we would like to point out here two facts which will cause somewhat delicate argument later. One is that it is not a straightforward matter whether the extended mean is continuous with respect to the uniform convergence. The other one is that the extended mean may not be strictly increasing on the class of semi-continuous functions even if the original one is so on the class of continuous functions.

This extended mean has the order-continuity as follows:

LEMMA 1. *Let $\{v_\alpha\}$ ($\{u_\alpha\}$) be an upward (downward) directed system of I^u (I^l)-valued lower (upper) semi-continuous functions on X and let v (u) be the pointwise limit of $\{v_\alpha\}$ ($\{u_\alpha\}$), that is, $v_\alpha \uparrow_\alpha v$ ($u_\alpha \downarrow_\alpha u$). Then we have*

$$\sup_\alpha M(v_\alpha) = M(v) \qquad (\inf_\alpha M(u_\alpha) = M(u)).$$

Since the proof is quite standard, we omit the details.

We note that if M is a QA-mean, maximum or minimum mean with weight, this extended mean allows the same representation as the given one. Let M be a QA-mean with $M = M_{\phi,\mu}$ on $C(X;I)$. Then it is evident that for any lower (upper) semi-continuous function $v: X \to I^u$ ($u: X \to I^l$) we have

$$M(v) = \phi^{-1}\left\{\int_X \phi(v(x))d\mu(x)\right\} \qquad \text{(the same for } u). \tag{8}$$

Let M be a maximum mean as $M = \mathrm{Max}_\delta$ on $C(X)^+$. Then, for any lower (upper) semi-continuous function $v: X \to (0, +\infty]$ ($u: X \to [0, +\infty)$) we have

$$M(v) = \sup_{x \in X} \{\delta(x)v(x)\} \qquad \text{(the same for } u), \tag{9}$$

where product $0(+\infty)$ is regarded as 0 if $\delta(x)=0$ and $v(x)=+\infty$ on the right side. We omit the proof (the case for v is straightforward but the case for u would require a little bit careful argument). We have a complete analog for a minimum mean with weight. Let M be a minimum mean with $M = \mathrm{Min}_\gamma$ on $C(X)^+$. Then for any upper (lower) semi-continuous function $u: X \to [0, +\infty)$ ($v: X \to (0, +\infty]$) we have

$$M(u) = \inf_{x \in X} \{\gamma(x)u(x)\} \qquad \text{(the same for } v), \tag{10}$$

where the product $(+\infty)0$ is regarded as $+\infty$ if $\gamma(x)=+\infty$ and $u(x)=0$ on the right side.

Here we introduce the notion of commutativity between means, which plays a fundamental role in our paper.

DEFINITION. Let X and Y be compact Hausdorff spaces and let M and L be means on $C(X;I)$ and $C(Y;I)$ respectively. We say that M and L are *commuting* if the following equation holds,

$$L_y[M_x[h(x, y)]] = M_x[L_y[h(x, y)]] \qquad \text{for all } h \text{ in } C(X \times Y; I). \tag{**}$$

According to this definition, one can say that M is a bisymmetric mean if and only if M commutes with M itself.

First it should be remarked that for a commuting pair of means M and L the equation (**) can be extended to semi-continuous functions.

LEMMA 2. *Let M and L be means on $C(X;I)$ and $C(Y;I)$ respectively. If M and L commute with each other, then for any lower (upper) semi-continuous function $v: X\times Y\to I^u$ $(u: X\times Y\to I^l)$ we have*

$$(**) \qquad L_y[M_x[v(x,y)]] = M_x[L_y[v(x,y)]] \qquad (\textit{the same for } u).$$

PROOF. For a given lower semi-continuous function $v: X\times Y\to I^u$, choose an upward directed system $\{h_\alpha\}$ in $C(X\times Y;I)$ such that $h_\alpha\uparrow_\alpha v$ on $X\times Y$. By Lemma 1, for each fixed y in Y we have $M_x[h_\alpha(x,y)]\uparrow_\alpha M_x[v(x,y)]$. Since $M_x[h_\alpha(x,y)]$ is an I-valued continuous function on Y, by using Lemma 1 again $L_y[M_x[h_\alpha(x,y)]]\uparrow_\alpha L_y[M_x[v(x,y)]]$. Similarly we have $M_x[L_y[h_\alpha(x,y)]]\uparrow_\alpha M_x[L_y[v(x,y)]]$. This and the commutativity of M and L, $L_y[M_x[h_\alpha(x,y)]]=M_x[L_y[h_\alpha(x,y)]]$ for each α, yield $L_y[M_x[v(x,y)]]=M_x[L_y[v(x,y)]]$. The argument is completely the same for u. q.e.d.

In our process of proving Theorems 1 and 2 we have to derive new means from a given mean by *keeping commutativity*, by which we mean more precisely that any mean commuting with the original one commutes with the derived new ones. Here we summarize three such methods.

a) *Transition by a continuous map.* Let M be a mean on $C(X;I)$ and let τ be a continuous map from X to a compact Hausdorff space Y. One can define a functional M^τ on $C(Y;I)$ as

$$(11) \qquad M^\tau(f)=M(f\circ\tau) \qquad \text{for } f \text{ in } C(Y;I).$$

It is easy to check that the following holds.

LEMMA 3. *M^τ is a mean on $C(Y;I)$ and we have*
1) *M^τ commutes with L if a mean L commutes with M,*
2) *if M is a bisymmetric mean, then so is M^τ,*
3) *$M^\tau(v)=M(v\circ\tau)$ $(M^\tau(u)=M(u\circ\tau))$ for any lower (upper) semi-continuous function $v: Y\to I^u$ $(u: Y\to I^l)$.*

b) *Derivation of a two dimensional mean.* If a compact Hausdorff space X consists of two points, $C(X;I)$ can be identified with $I\times I=I^2$. We call a mean on I^2 a *two dimensional mean.* Since two dimensional means play a special role in our paper, here we repeat the definition of the bisymmetry equation of a mean for the two dimensional case. A two dimensional mean N is an I-valued continuous function on $I\times I=I^2$ having

I) $N(a,a)=a$ for a in I,

II) $N(a,b)\leqq N(a',b')$ if $(a,b)\leqq(a',b')$ in I^2.

Furthermore, N is called bisymmetric if the following equation is satisfied, for all a, b, c, d in I

$$(*)\qquad N[N(a, b), N(c, d)] = N[N(a, c), N(b, d)] \quad (b \text{ and } c \text{ are exchangeable}).$$

Throughout the paper, we keep using N, instead of M, for denoting two dimensional means.

Later in our proof, the most typical situation where two dimensional means are derived is that a mean defined on $C([0, 1]; I)$ is given and we want to derive a two dimensional mean from that. To explain this situation further, first we introduce the notations $\check{\chi}_{a,b,\xi}$ and $\hat{\chi}_{a,b,\xi}$ for the following semi-continuous functions on the unit interval $[0, 1]$,

$$(12)\qquad \check{\chi}_{a,b,\xi}(x) = \begin{cases} a & \text{if } 0 \leqq x < \xi, \\ b & \text{if } \xi < x \leqq 1, \\ \min\{a, b\} & \text{if } x = \xi, \end{cases} \qquad \hat{\chi}_{a,b,\xi}(x) = \begin{cases} a & \text{if } 0 \leqq x < \xi, \\ b & \text{if } \xi < x \leqq 1, \\ \max\{a, b\} & \text{if } x = \xi. \end{cases}$$

The notations $\check{\chi}_{a,b,\xi}$ and $\hat{\chi}_{a,b,\xi}$ will be kept used in the latter sections.

Let M be a mean on $C([0, 1]; I)$. Suppose there is a ξ in $[0, 1]$ such that

$$(13)\qquad M(\check{\chi}_{a,b,\xi}) = M(\hat{\chi}_{a,b,\xi}) \qquad \text{for all } a, b \text{ in } I,$$

then one can define a function $N^{(M,\xi)}$ on I^2 by setting

$$(14)\qquad N^{(M,\xi)}(a, b) = M(\check{\chi}_{a,b,\xi}) = M(\hat{\chi}_{a,b,\xi}) \qquad \text{for } (a, b) \text{ in } I^2.$$

This function $N^{(M,\xi)}$ has the following property.

LEMMA 4. *Assuming the existence of* ξ *satisfying* (13) *above,* $N^{(M,\xi)}$ *is a two dimensional mean on* I^2 *and we have*

1) $N^{(M,\xi)}$ *commutes with* L *if a mean* L *commutes with* M,
2) *if* M *is a bisymmetric mean, then so is* $N^{(M,\xi)}$,
3) $N^{(M,\xi)}(a, b) = M(\check{\chi}_{a,b,\xi})$ *for all* a, b *in* I^u,
   $N^{(M,\xi)}(c, d) = M(\hat{\chi}_{c,d,\xi})$ *for all* c, d *in* I^l.

PROOF. We denote $N^{(M,\xi)}$ by N simply. It is obvious that this function N is monotoneously increasing on I^2 and $N(a, a) = a$ for all a in I. To see the continuity of N on I^2, it suffices to see the continuity from above and below. By using Lemma 1,

$$\lim_{x\uparrow a,\, y\uparrow b} N(x, y) = \lim_{x\uparrow a,\, y\uparrow b} M(\check{\chi}_{x,y,\xi}) = M(\check{\chi}_{a,b,\xi}) = N(a, b),$$

similarly

$$\lim_{x\downarrow a,\, y\downarrow b} N(x, y) = \lim_{x\downarrow a,\, y\downarrow b} M(\hat{\chi}_{x,y,\xi}) = M(\hat{\chi}_{a,b,\xi}) = N(a, b).$$

Thus N becomes a two dimensional mean on I^2.

To see 1), let L be a mean on $C(X; I)$ commuting with M. Note that the commutativity between N and L means

$$L_x[N[f(x), g(x)]] = N[L(f), L(g)] \tag{15}$$

for all f, g in $C(X; I)$. To show this equality, for given f, g in $C(X; I)$, consider an I-valued lower semi-continuous function $\check{h}$ on $X\times[0, 1]$ and an I-valued upper semi-continuous function $\hat{h}$ on $X\times[0, 1]$ such that $\check{h}(x, y)=\check{\chi}_{f(x), g(x), \xi}(y)$ and $\hat{h}(x, y)=\hat{\chi}_{f(x), g(x), \xi}(y)$ for x in X and y in $[0, 1]$.

From our assumption (13) on ξ, we have

$$M_y[\check{h}(x, y)] = N[f(x), g(x)] = M_y[\hat{h}(x, y)].$$

By applying L to this equation, we have

$$L_x[M_y[\check{h}(x, y)]] = L_x[N[f(x), g(x)]] = L_x[M_y[\hat{h}(x, y)]].$$

Since we have commutativity between M and L, namely

$$L_x[M_y[\check{h}(x, y)]] = M_y[L_x[\check{h}(x, y)]] \quad \text{and}$$
$$L_x[M_y[\hat{h}(x, y)]] = M_y[L_x[\hat{h}(x, y)]],$$

we have

$$L_x[N[f(x), g(x)]] = M_y[L_x[\check{h}(x, y)]] = M_y[L_x[\hat{h}(x, y)]]. \tag{16}$$

On the other hand, we have

$$L_x[\check{h}(x, y)] \leqq \check{\chi}_{L(f), L(g), \xi}(y) \leqq L_x[\hat{h}(x, y)].$$

Thus, by applying M on the both sides of this inequality, we have

$$M_y[L_x[\check{h}(x, y)]] \leqq N[L(f), L(g)] \leqq M_y[L_x[\hat{h}(x, y)]]. \tag{17}$$

Now the equality (15) follows from (16) and (17).

2) follows from 1) (use 1) twice), and 3) is obvious. q. e. d.

c) *Restriction onto a compact subset.* Let M be a positively homogeneous mean on $C(X)^+$ and let K be a compact subset of X with $M(\chi_K)=1$, where χ_K is the characteristic function of K. One can define a functional M^K on $C(K)^+$ as

$$M^K(f) = M(f^K) \qquad \text{for } f \text{ in } C(K)^+, \tag{18}$$

where f^K denotes an upper semi-continuous function on X which is identical to f on K and equal to zero on $X\setminus K$. We have the following.

LEMMA 5. *M^K is a positively homogeneous mean on $C(K)^+$ and we have*
1) *M^K commutes with L if a mean L commutes with M,*
2) *if M is a bisymmetric mean, then so is M^K,*
3) *$M^K(u)=M(u^K)$ for any upper semi-continuous function $u: K\to[0, +\infty)$.*

PROOF. It is obvious that the functional M^K satisfies the following properties; $M^K(a1_K)=a$ for all $a>0$, M^K is increasing and positively homogeneous on $C(K)^+$. It should be noticed that being positively homogeneous together with being monotoneously increasing implies the continuity of the functional. Thus M^K becomes a positively homogeneous mean on $C(K)^+$. To see 1), let L be a mean on $C(Y)^+$ which commutes with M. For each h in $C(K\times Y; I)$, by taking the upper semi-continuous function h^K on $X\times Y$, which is equal to h on $K\times Y$ and to zero on $(X\setminus K)\times Y$, we have $L_y[M_x^K[h(x, y)]]=L_y[M_x[h^K(x, y)]]=M_x[L_y[h^K(x, y)]]=M_x^K[L_y[h(x, y)]]$, where we used the commutativity of M and L and Lemma 2 in the second equality.

2) follows from 1) (use 1) twice), and 3) is obvious. q. e. d.

§ 2. Strictly increasing means.

Our proof of Theorem 1 is divided roughly into three steps. The first step is supposed to prove the two dimensional version of Theorem 1, namely where X consists of two points. As mentioned in our introduction, the finite dimensional version of Theorem 1 is J. Aczél's theorem [**2**], so we state the two dimensional version of his theorem as Proposition 1 without proof. The second step is to prove that every mean commuting with a QA-mean, which is not a point evaluation, is also a QA-mean. This step corresponds to Proposition 2. The last step is to prove that any strictly increasing and bisymmetric mean has a commuting two dimensional QA-mean which is not a point evaluation.

PROPOSITION 1. *Let N be a two dimensional mean on I. If N is strictly increasing and bisymmetric, then there are a strictly increasing or decreasing, continuous function ϕ on I and $0<t<1$ such that*

$$N(a, b)=\phi^{-1}\{t\phi(a)+(1-t)\phi(b)\} \qquad \textit{for all } a, b \textit{ in } I,$$

where t is determined uniquely and ϕ is also unique up to the affine equivalence.

PROOF. See [**2**] or [**4**] pp. 278-287.

A kind of observation in our proof of the following proposition can be seen partially in [**2**] and [**3**] although in different situation.

PROPOSITION 2. *Let X and Y be compact Hausdorff spaces and let M and L be means on $C(X; I)$ and $C(Y; I)$ respectively. If M commutes with L and if L is a QA-mean with $L=M_{\phi,\mu}$ but not a point evaluation, then M is also a QA-mean with $M=M_{\phi,\nu}$ for some probability measure ν on X, where ν is determined uniquely.*

PROOF. The commutativity between M and L, (**) of Lemma 2 and (8) together yield

$$\int_Y \phi(M_x[v(x, y)])d\mu(y) = \phi\Big[M_x\phi^{-1}\Big[\Big(\int_Y \phi[v(x, y)]d\mu(y)\Big)\Big]\Big], \tag{19}$$

where v is any I^u-valued lower semi-continuous function on $X\times Y$.

Setting a functional F on $C(X;\phi(I))$ as

$$F(f) = \phi[M_x[\phi^{-1}(f(x))]] \qquad \text{for } f \text{ in } C(X;\phi(I)),$$

we will first show that there is t with $0<t<1$ such that for all f, g in $C(X;\phi(I))$

$$F(tf+(1-t)g) = tF(f)+(1-t)F(g). \tag{20}$$

To see this, since the support of μ contains more than one point, one can choose a continuous map τ from Y to the interval $[0, 1]$ such that $\tau^{-1}(\{1\})$ and $\tau^{-1}(\{0\})$ both intersect the support of μ. Then one can choose ξ_0 such that $0<\xi_0<1$ and $\mu(\tau^{-1}(\{\xi_0\}))=0$. Thus we have a decomposition of Y as $Y=Y_1\cup Y_2\cup Y_3=\tau^{-1}([0, \xi_0))\cup\tau^{-1}((\xi_0, 1])\cup\tau^{-1}(\{\xi_0\})$. For given f, g in $C(X;\phi(I))$, define an I-valued lower semi-continuous function v on $X\times Y$ as follows,

$$v(x, y) = \begin{cases} \phi^{-1}(f(x)) & \text{if } y \text{ in } Y_1, \\ \phi^{-1}(g(x)) & \text{if } y \text{ in } Y_2, \\ \min\{\phi^{-1}(f(x)), \phi^{-1}(g(x))\} & \text{if } y \text{ in } Y_3. \end{cases}$$

By setting $t=\mu(Y_1)$, we have $\mu(Y_2)=1-t$, $0<t<1$, and furthermore we have

$$\int_Y \phi[M_x[v(x, y)]]d\mu(y) = tF(f)+(1-t)F(g),$$

$$\int_Y \phi[v(x, y)]d\mu(y) = tf(x)+(1-t)g(x).$$

Thus (19) yields (20).

Since $M_{\phi,\mu}=M_{\alpha\phi+\beta,\mu}$ for any $\alpha\neq 0$ and any β, one can assume that $\phi(a)=0$ and $\phi(b)=1$ at two interior points a, b of I. Then $C(X;\phi(I))$ becomes a convex zero-neighborhood containing 1_X in the space $C(X)$ of all continuous functions on X. By (20), we will see that F can be extended uniquely to a continuous linear functional on $C(X)$.

We write V instead of $C(X;\phi(I))$. By setting $f=0$ or $g=0$ in (20) and using $F(0)=0$, we have

$$F(tf) = tF(f), \quad F((1-t)f) = (1-t)F(f) \qquad \text{for all } f \text{ in } V. \tag{21}$$

Let $W=tV\cap(1-t)V\subset V$. Note that $W+W\subset V$ and (20) together with (21) yield

(22) $$F(h+k)=F(h)+F(k) \qquad \text{for all } h,\, k \text{ in } W\,.$$

Using this additivity on W n-times and again m-times, we have

$$F\left(\frac{n}{m}h\right)=\frac{n}{m}F(h)$$

for all h in W and all integers $0 \leqq n < m$. Thus the continuity of F on V yields

(23) $$F(\lambda h)=\lambda F(h) \qquad \text{for all } h \text{ in } W \text{ and all } 0\leqq\lambda\leqq 1\,.$$

By (23), for any f in $C(X)$ and any $0\leqq\lambda_1, \lambda_2\leqq 1$ such that $\lambda_1 f$ and $\lambda_2 f$ belong to W we have

(24) $$\lambda_1 F(\lambda_2 f)=\lambda_2 F(\lambda_1 f)\,.$$

Now, define a functional $\hat{F}$ on $C(X)$ as

$$\hat{F}(f)=\frac{1}{\lambda}F(\lambda f)\,,$$

where f in $C(X)$ and λ with $0<\lambda\leqq 1$ such that λf belongs to W. Because of (24), $\hat{F}$ is well defined. $\hat{F}$ is additive on $C(X)$, because for any f and g in $C(X)$, choose $0<\lambda\leqq 1$ such that λf, λg and $\lambda(f+g)$ belong to W and use (22), it follows $\hat{F}(f+g)=F(\lambda(f+g))/\lambda=[F(\lambda f)+F(\lambda g)]/\lambda=\hat{F}(f)+\hat{F}(g)$. $\hat{F}$ is an extension of F from V to $C(X)$, because note that $t(1-t)f$ belongs to W for any f in V, thus $\hat{F}(f)=F(t(1-t)f)/t(1-t)$ for all f in V. Use (21) twice, we have $F(t(1-t)f)=t(1-t)F(f)$, thus $\hat{F}(f)=F(f)$ for all f in V. Consequently, $\hat{F}$ is additive on $C(X)$ and continuous on the zero-neighborhood V, so $\hat{F}$ is a continuous linear functional on $C(X)$. Applying Riesz' theorem, there is a finite measure ν on X such that

$$F(f)=\int_X f d\nu \qquad \text{for all } f \text{ in } C(X;\phi(I))\,.$$

Furthermore, by observing that $\hat{F}$ is monotoneously increasing and $F(1_X)=1$, ν is a probability measure on X. Finally we have a representation of M with $M(f)=\phi^{-1}\left(\int_X \phi(f)d\nu\right)$ for all f in $C(X;I)$, that is, M is a QA-mean. The uniqueness of ν follows from the argument above. q. e. d.

PROOF OF THEOREM 1. By using Propositions 1 and 2, in order to complete a proof of Theorem 1, it suffices to show that for the mean M on $C(X;I)$ given in Theorem 1 there is a two dimensional, strictly increasing mean on I^2 which commutes with M. We split this proof into two cases depending upon whether X is connected or disconnected.

Suppose X is disconnected. Then there is a continuous map τ from X onto a set consisting of two points. By Lemma 3 it is clear that M^τ is a two dimensional mean which is strictly increasing, bisymmetric and commuting with M.

Suppose X is connected but not a singleton. (There is nothing to prove if X is a singleton.) Choose a continuous map τ from X *onto* the interval $[0, 1]$, then M^{τ} is a strictly increasing mean on $C([0, 1]; I)$. By Lemma 3 M^{τ} is bisymmetric and commutes with M. Thus our task is to construct a two dimensional mean N on I^2 which is strictly increasing, bisymmetric and commuting with any mean which commutes with M^{τ}. This will be done in the following order of steps. The first is to show that there exists $0<\xi_0<1$ such that

$$M^{\tau}(\check{\chi}_{a,b,\xi_0}) = M^{\tau}(\hat{\chi}_{a,b,\xi_0}) \qquad \text{for all } a, b \text{ in } I.$$

Once this is shown, the claimed two dimensional mean N will be given by $N^{(M^{\tau},\xi_0)}$ of Lemma 4 of the previous section. The last little step is to show that this N is actually strictly increasing on I^2. The first long step will be completed by Lemmas 6 and 7. The last step will be shown in Lemma 8.

In the following, for the sake of simplicity of notations M^{τ} is replaced by L. Thus we are working on a strictly increaing bisymmetric mean L defined on $C([0, 1]; I)$. For any fixed ξ with $0<\xi<1$, first introduce a lower semi-continuous function $\check{F}=\check{F}_{\xi}$ and an upper semi-continuous function $\hat{F}=\hat{F}_{\xi}$ on I^2 as follows

$$\check{F}(a, b) = L(\check{\chi}_{a,b,\xi}), \quad \hat{F}(a, b) = L(\hat{\chi}_{a,b,\xi}) \qquad \text{for } a, b \text{ in } I. \tag{25}$$

The following lemma shows that actually $\check{F}$ and $\hat{F}$ are continuous for any choice of ξ with $0<\xi<1$.

LEMMA 6. *$\check{F}_{\xi}$ and $\hat{F}_{\xi}$ are continuous on I^2 for any $0<\xi<1$.*

PROOF. We start with preliminary observations. First, since L is strictly increasing

$$\min\{a, b\} < \check{F}(a, b) < \max\{a, b\} \quad \text{if } a \neq b \text{ in } I \qquad (\text{the same for } \hat{F}) \tag{26}$$

and

$$\lim_{y \downarrow d} \check{F}(c, y) < d \quad \text{if } c<d \text{ in } I \qquad (\lim_{y \uparrow d} \hat{F}(c, y) > d \quad \text{if } c>d). \tag{27}$$

It is obvious to see (26). To see (27), choose a continuous function f on $[0, 1]$ such that $f \lneqq d$ and $\check{\chi}_{c,d,\xi} \leqq f$. Then we have $L(f)<d$ because L is strictly increasing. Using the continuity of L, we can choose $\varepsilon>0$ such that $L(f+\varepsilon) <d$. Hence, for all $d<y\leqq d+\varepsilon$, we have $\check{\chi}_{c,y,\xi} \leqq f+\varepsilon$, thus $\check{F}_{\xi}(c, y)=L(\check{\chi}_{c,y,\xi}) \leqq L(f+\varepsilon)<d$. Similarly we have

$$\lim_{x \downarrow c} \check{F}(x, d) < c \quad \text{if } c>d \qquad (\lim_{x \uparrow c} \hat{F}(x, d) > c \quad \text{if } d>c). \tag{27'}$$

Next, we see that $\check{F}_\xi$ and $\hat{F}_\xi$ satisfy the following equation which is weaker than the bisymmetric equation.

(28) $\check{F}[\check{F}(a, b), \check{F}(a, c)] = \check{F}[a, \check{F}(b, c)]$ if $a \leqq \min\{b, c\}$ or $a \geqq \max\{b, c\}$ (the same for $\hat{F}$).

To observe this, supposing $a \leqq \min\{b, c\}$ (the case for $a \geqq \max\{b, c\}$ can be done similarly), set a lower semi-continuous function v on $[0, 1]\times[0, 1]$ as

$$v(x, y) = \begin{cases} a & \text{if } x \leqq \xi, \\ b & \text{if } x > \xi \text{ and } y < \xi, \\ c & \text{if } x > \xi \text{ and } y > \xi, \\ \min\{b, c\} & \text{if } x > \xi \text{ and } y = \xi, \end{cases}$$

and substitute v into the bisymmetry equation of L, then we have (28). Similarly we have

(28′) $\check{F}[\check{F}(b, a), \check{F}(c, a)] = \check{F}[\check{F}(b, c), a]$ if $a \leqq \min\{b, c\}$ or $a \geqq \max\{b, c\}$ (the same for $\hat{F}$).

Now we are ready to show the continuity of $\check{F}_\xi$ and $\hat{F}_\xi$. We argue only the continuity of $\check{F}=\check{F}_\xi$. Since $\check{F}$ is increasing on I^2, it is sufficient to show that $\check{F}$ is separately continuous with respect to each variable. Fix a point (a, b) in I^2, if $a=b$, the continuity of $\check{F}$ at (a, a) is easy to see. Assume $a>b$ (the proof for $a<b$ will be similar), by using (26), (27) and (28) we will show the continuity of $\check{F}$ at (a, b) with respect to variable y only. (The continuity with respect to the other variable x can be shown similarly by using (26), (27′) and (28′).) To see this, since $\check{F}$ is lower semi-continuous on I^2, it suffices to show $\lim_{y\downarrow b}\check{F}(a, y)=\check{F}(a, b)$. Setting $\alpha=\lim_{y\downarrow b}\check{F}(a, y)$, since $\lim_{y\downarrow b}\check{F}(b, y)=b$, we have

$$\lim_{y\downarrow b}\check{F}[a, \check{F}(b, y)] = \alpha . \tag{29}$$

If there is $y_0>b$ such that $\check{F}(a, y_0)=\alpha$, then (28) implies $\check{F}[\check{F}(a, b), \alpha]=\check{F}[\check{F}(a, b), \check{F}(a, y)]=\check{F}[a, \check{F}(b, y)]$ for all y with $b<y\leqq y_0$. By letting $y\downarrow b$ and using (29) we have $\check{F}[\check{F}(a, b), \alpha]=\alpha$, and from (26) we have $\check{F}(a, b)=\alpha=\lim_{y\downarrow b}\check{F}(a, y)$. If $\check{F}(a, y)>\alpha$ for all $y>b$ in I, then from (28) we have

$$\lim_{x\downarrow \alpha}\check{F}[\check{F}(a, b), x] = \lim_{y\downarrow b}\check{F}[\check{F}(a, b), \check{F}(a, y)] = \lim_{y\downarrow b}\check{F}[a, \check{F}(b, y)] = \alpha .$$

Thus, (27) implies $\check{F}(a, b)=\alpha=\lim_{y\downarrow b}\check{F}(a, y)$. q. e. d.

LEMMA 7. *The following holds except for at most countably many ξ's in* $[0, 1]$,

$$L(\check{\chi}_{a,b,\xi}) = L(\hat{\chi}_{a,b,\xi}) \qquad \textit{for all } a, b \textit{ in } I.$$

PROOF. For fixed a and b in I, the function $\check{F}_\xi(a, b)=L(\check{\chi}_{a,b,\xi})$ of variable ξ is monotoneously increasing or decreasing on $[0, 1]$. Hence, except for at most countably many ξ's in $[0, 1]$, ξ is a continuous point of this function. Note that if this function is continuous at $\xi=\xi_0$, then we have $\check{F}_{\xi_0}(a, b)=\hat{F}_{\xi_0}(a, b)$, that is, $L(\check{\chi}_{a,b,\xi_0})=L(\hat{\chi}_{a,b,\xi_0})$. Now one can choose ξ in $[0, 1]$, with at most countably many exceptions, such that the following holds

$$\check{F}_\xi(a, b) = \hat{F}_\xi(a, b) \qquad \text{for all rational numbers } a, b \text{ in } I.$$

For such choice of ξ, by using Lemma 6 we have actually $\check{F}_\xi(a, b)=\hat{F}_\xi(a, b)$ for *all* a, b in I, namely $L(\check{\chi}_{a,b,\xi})=L(\hat{\chi}_{a,b,\xi})$ for all a, b in I. q. e. d.

Suppose that ξ_0 with $0<\xi_0<1$ satisfies the conditions of Lemma 7, then by Lemma 4, $N=N^{(L,\xi_0)}$ becomes a two dimensional bisymmetric mean on I^2. Furthermore, we have

LEMMA 8. *$N=N^{(L,\xi_0)}$ is strictly increasing on I^2.*

PROOF. Suppose $N(a, b)=N(a', b)$ for some $a<a'$ and b (the argument will be the same for supposing $N(b, a)=N(b, a')$). We will show

$$N(x, b) = N(a, b) \qquad \text{for all } x \text{ in } I.$$

This would imply $N(a, b)=N(a', b)=N(b, b)=b$, which contradicts (26). Set $c=\inf\{x \mid N(x, b)=N(a, b)\}$ and $d=\sup\{x \mid N(x, b)=N(a, b)\}$, then we have $-\infty\leqq c\leqq a<a'\leqq d\leqq+\infty$.

Suppose $d<\sup I$. We have $a<N(a, d)=L(\hat{\chi}_{a,d,\xi_0})<d$. Thus one can choose f in $C([0, 1]; I)$ such that $f(\xi_0)>d$, $f(x)\geqq a$ for all x in $[0, 1]$ and $a<L(f)<d$. By using the commutativity between N and L, it follows $L_x[N[f(x), b]]=N[L(f), b]=N(a, b)$. This yields $N[f(x), b]=N(a, b)$ for all x in $[0, 1]$, because $N[f(x), b]\geqq N(a, b)$ for all x in $[0, 1]$ and L is strictly increasing. In particular we have $N[f(\xi_0), b]=N(a, b)$, which contradicts our choice of f with $f(\xi_0)>d$. Thus we proved $d=\sup I$. Similarly one can show $c=\inf I$. By combining these two, we have $N(x, b)=N(a, b)$ for all x in I. q. e. d.

Finally, $N=N^{(L,\xi_0)}$ becomes a two dimensional, strictly increasing and bisymmetric mean, furthermore it commutes with any mean which commutes with L. By applying Propositions 1 and 2 this completes to prove that M given in Theorem 1 is a QA-mean, $M=M_{\phi,\mu}$. The uniqueness of ϕ and μ follows from the uniqueness in Propositions 1 and 2. q. e. d.

§3. Positively homogeneous means.

Our general procedure of proving Theorem 2 is the same as the one shown in the proof of Theorem 1. It has also three steps which are the same as those mentioned at the beginning of §2. Let us start with two dimensional positively homogeneous means. Recall that concrete examples of two dimensional positively homogeneous means are p-th power root means $N_{p,t}$ for $-\infty<p<+\infty$, $0\leqq t\leqq 1$ and maximum or minimum means with weight as follows,

$$N_{p,t}(a, b)=\{ta^{p}+(1-t)b^{p}\}^{1/p} \quad \text{for } a, b>0, \tag{30}$$

if $p=0$,

$$N_{0,t}(a, b)=a^{t}b^{1-t} \quad \text{for } a, b>0, \tag{31}$$

$$\mathrm{Max}_{\delta}(a, b)=\max\{\delta_{1}a, \delta_{2}b\} \quad \text{for } a, b>0, \tag{32}$$

where $\delta=(\delta_{1}, \delta_{2})$ with $0\leqq\delta_{1}, \delta_{2}\leqq 1$ and $\max\{\delta_{1}, \delta_{2}\}=1$, and

$$\mathrm{Min}_{\gamma}(a, b)=\min\{\gamma_{1}a, \gamma_{2}b\} \quad \text{for } a, b>0, \tag{33}$$

where $\gamma=(\gamma_{1}, \gamma_{2})$ with $1\leqq\gamma_{1}, \gamma_{2}\leqq+\infty$ and $\min\{\gamma_{1}, \gamma_{2}\}=1$.

These types of two dimensional positively homogeneous means are mutually exclusive except for point evaluations. We call N a *right point evaluation* if $N(a, b)=b$ for all $a, b>0$ and a *left point evaluation* if $N(a, b)=a$ for all $a, b>0$.

The following proposition is the two dimensional version of Theorem 2, namely where X consists of two points.

PROPOSITION 3. *If a two dimensional positively homogeneous mean N satisfies the bisymmetry equation, then we have $N=N_{p,t}$ for some $-\infty<p<+\infty$ and some $0\leqq t\leqq 1$, $N=\mathrm{Max}_{\delta}$ or $N=\mathrm{Min}_{\gamma}$ for some weight δ or γ.*

We start with some preliminaries. The proof of this proposition will be completed after Lemma 12.

Let c, d, $\bar{c}$ and $\bar{d}$ be as follows,

$$c=\inf\{y>0 \mid N(1, y)=1\}, \qquad d=\sup\{y>0 \mid N(1, y)=1\},$$
$$\bar{c}=\inf\{x>0 \mid N(x, 1)=1\}, \qquad \bar{d}=\sup\{x>0 \mid N(x, 1)=1\}.$$

Clearly we see

$$0\leqq c,\ \bar{c}\leqq 1 \quad \text{and} \quad 1\leqq d,\ \bar{d}\leqq+\infty.$$

About these four numbers we have

LEMMA 9. *One of the following cases occurs for c and d.*

$$\text{i)}\quad c=0,\qquad \text{ii)}\quad d=+\infty\qquad \textit{or}\qquad \text{iii)}\quad c=1=d\,.$$

The same occurs for $\bar{c}$ *and* $\bar{d}$.

PROOF. We give a proof for only c and d. The case for $\bar{c}$ and $\bar{d}$ can be done similarly.

Assuming $0<c\leqq 1\leqq d<+\infty$, we want to conclude that $c=1$ and $d=1$. First we show $c=1$. Suppose $0<c<1$. Since $N(c, cd)=c$ and $d=\sup\{y\,|\,N(1, y)=1\}$, by the continuity of N one can choose y' sufficiently close to cd and $y'>cd$ such that

$$c<N(c, y')<1\,.$$

Then, one can choose c' sufficiently close to c and $c'<c$ such that

$$c<N(c', y')<1\,.$$

Now, the bisymmetry equation yields a contradiction;

$$\begin{aligned} 1=N[1, N(c', y')] &= N[N(1, c), N(c', y')]=N[N(1, c'), N(c, y')] \\ &\leqq \max\{N(1, c'), N(c, y')\}<1\,. \end{aligned}$$

To show $d=1$, consider the following positively homogeneous bisymmetric mean N^{-1} derived from N,

$$N^{-1}(a, b)=N(a^{-1}, b^{-1})^{-1}\qquad \text{for all } a, b>0\,.$$

The above argument applied to N^{-1} implies $d=1$. q.e.d.

LEMMA 10. 1) *If* $c=0$ *and* $d=1$, *then we have* $N(a, b)=\max\{a, b\}$ *for all* $a, b>0$.

2) *If* $c=1$ *and* $d=+\infty$, *then we have* $N(a, b)=\min\{a, b\}$ *for all* $a, b>0$.

PROOF. Assuming $c=0$ and $d=1$, we will show

$$N(x, 1)=1\qquad \text{for all } 0<x\leqq 1\,. \tag{34}$$

Suppose there is x_0 with $0<x_0<1$ such that $N(x_0, 1)<1$. Take $y_0>1$, then $N(1, y_0)>1$ because $d=1$. By the continuity of N one can choose a point (x', y') on the segment joining two points $(x_0, 1)$ and $(1, y_0)$ such that $x_0<x'<1$, $1<y'<y_0$ and

$$N(x', y')=1\,.$$

Thus, the bisymmetry equation yields again a contradiction;

$$1=N[1, N(x', y')]=N[N(1, x'), N(1, y')]=N[1, N(1, y')]>1\,.$$

On the other hand, since $c=0$, we have

$$N(1, y)=1\qquad \text{for all } 0<y\leqq 1\,. \tag{35}$$

Combining (34) and (35), for all $a, b>0$

$$N(a, b)=\left\{\begin{array}{ll} aN(1, b/a)=a & \text{if } a>b \\ bN(a/b, 1)=b & \text{if } a<b \end{array}\right\}=\max\{a, b\}.$$

2) To see this, consider N^{-1} and apply the previous case 1) to N^{-1}. q.e.d.

LEMMA 11. *One can classify N, depending upon the four numbers c, d, $\bar{c}$ and $\bar{d}$,*

The values of $c, \bar{c}, d, \bar{d}$	$N(a, b)$ for all $a, b>0$
$c=0,\ 1\leqq d<+\infty$	$N(a, b)=\max\{a, b/d\}$
$c=0,\ d=+\infty$	$N(a, b)=a$
$0<c\leqq 1,\ d=+\infty$	$N(a, b)=\min\{a, b/c\}$
$\bar{c}=0,\ 1\leqq\bar{d}<+\infty$	$N(a, b)=\max\{a/\bar{d}, b\}$
$\bar{c}=0,\ \bar{d}=+\infty$	$N(a, b)=b$
$0<\bar{c}\leqq 1,\ \bar{d}=+\infty$	$N(a, b)=\min\{a/\bar{c}, b\}$

PROOF. First, if $c=0$ and $1\leqq d<+\infty$, setting

$$\hat{N}(a, b)=N(a, db) \qquad \text{for } a, b>0$$

$\hat{N}$ becomes a positively homogeneous and bisymmetric mean and it has

$$\inf\{y>0 \mid \hat{N}(1, y)=1\}=0, \qquad \sup\{y>0 \mid \hat{N}(1, y)=1\}=1.$$

Thus, by Lemma 10, $\hat{N}(a, b)=\max\{a, b\}$, so

$$N(a, b)=\hat{N}(a, b/d)=\max\{a, b/d\}.$$

Secondly, if $c=0$ and $d=+\infty$, then $N(a, b)=aN(1, b/a)=a$ for all $a, b>0$. Thirdly, if $0<c\leqq 1$ and $d=+\infty$, the proof is completely similar to the first case; $c=0$ and $1\leqq d<+\infty$. Thus we have $N(a, b)=\min\{a, b/c\}$ for all $a, b>0$.

The classification of N depending upon $\bar{c}$ and $\bar{d}$ can be given by considering a mean $\tilde{N}$ derived from N as

$$\tilde{N}(a, b)=N(b, a) \qquad \text{for all } a, b>0. \qquad \text{q. e. d.}$$

LEMMA 12. *If $c=1=d$ and $\bar{c}=1=\bar{d}$, then N is strictly increasing and consequently N is a p-th power mean $N=N_{p,t}$ for some $-\infty<p<+\infty$ and some $0<t<1$.*

PROOF. Suppose N were not strictly increasing, namely there were a, b with $0<a<b$ such that

$$N(1, a) = N(1, b) \quad \text{or} \quad N(a, 1) = N(b, 1).$$

Assume $N(1, a)=N(1, b)$ (the argument for $N(a, 1)=N(b, 1)$ goes similarly). Let α and β be as follows

$$\alpha = \inf\{x>0 \mid N(1, x)=N(1, b)\} \quad \text{and} \quad \beta = \sup\{x>0 \mid N(1, x)=N(1, b)\}.$$

Clearly we see $0\leqq\alpha\leqq a<b\leqq\beta\leqq+\infty$. Since $c=1=d$, we have $1<\alpha<\beta\leqq+\infty$ or $0\leqq\alpha<\beta<1$. Assume $1<\alpha<\beta\leqq+\infty$ (the other case goes similarly). Take $1<x_0<\alpha$ and $\alpha<y_0<\beta$, then we have $N(x_0, \alpha)<\alpha$ because $\bar{c}=1$ and $N(\alpha, y_0)>\alpha$ because $d=1$. By using the continuity of N, one can choose a point (x', y') on the segment joining two points (x_0, α) and (α, y_0) such that $1<x_0<x'<\alpha$, $\alpha<y'<y_0<\beta$ and $N(x', y')=\alpha$. Note that $N(1, y')=N(1, \alpha)=N(1, b)$ because $\alpha<y'<\beta$ and $N(1, x')<N(1, \alpha)=N(1, b)$ because $x'<\alpha$. Apply the bisymmetry equation as

$$\begin{aligned} N(1, \alpha) &= N[1, N(x', y')] = N[N(1, x'), N(1, y')] \\ &= N[N(1, x'), N(1, \alpha)], \end{aligned}$$

hence $1=N[N(1, x')/N(1, \alpha), 1]$ and $N(1, x')/N(1, \alpha)<1$, which contradicts $\bar{c}=1$.

Thus we have proved that N is strictly increasing. Now one can apply Proposition 1 to N, therefore N is a QA-mean with $N=N_{\phi,t}$ for some ϕ and some $0<t<1$. By the result of de Finetti-Jessen-Nagumo (see pp. 68-69 of [**6**]), the property of N being positively homogeneous yields that this function ϕ is a p-th power function; $\phi(x)=x^p$ $(x>0)$ for some $-\infty<p\neq 0<+\infty$ or $\phi(x)=\log x$ $(x>0)$. Thus we conclude $N=N_{p,t}$. q. e. d.

Proof of Proposition 3. By combining Lemmas 11 and 12 we have a complete proof of Proposition 3.

Proposition 4. *Let X and Y be compact Hausdorff spaces and let M and L be positively homogeneous means on $C(X;I)$ and $C(Y;I)$ respectively. If M commutes with L and if L is a maximum (minimum) mean with weight, but not a point evaluation, then M is also a maximum (minimum) mean with weight, $M=\mathrm{Max}_\delta$ $(M=\mathrm{Min}_\gamma)$, where the weight δ (γ) is determined uniquely.*

Proof. Suppose L is a maximum mean with weight ρ on $C(Y)^+$. The commutativity between M and L says

$$(36) \qquad M_x[\max_{y\in Y}\{\rho(y)h(x, y)\}] = \max_{y\in Y}\{\rho(y)M_x[h(x, y)]\} \quad \text{for all } h \text{ in } C(X\times Y)^+.$$

It should be noticed that (36) holds for any lower (upper) semi-continuous function $v: X\times Y\to(0, +\infty]$ $(u: X\times Y\to[0, +\infty))$ replacing h (see Lemma 2 and (9)).

First, we observe that

$$M_x[\max\{f(x), g(x)\}] = \max\{M(f), M(g)\} \qquad \text{for all } f, g \text{ in } C(X)^+. \tag{37}$$

Since L is not a point evaluation, one can choose two point $y_1 \neq y_2$ in Y such that $\rho(y_1)>0$ and $\rho(y_2)>0$. Choose open neighborhoods U_1 and U_2, with the disjoint closures, of y_1 and y_2 respectively. Consider a lower semi-continuous function $v: X\times Y \to (0, +\infty]$ such that

$$v(x, y) = \begin{cases} \dfrac{1}{\rho(y)} f(x) & \text{for } y \text{ in } U_1, \\ \dfrac{1}{\rho(y)} g(x) & \text{for } y \text{ in } U_2, \\ \dfrac{1}{\rho(y)} \min\{f(x), g(x)\} & \text{for } y \text{ in } Y\setminus\{U_1 \cup U_2\}. \end{cases}$$

Replacing h in (36) by this v, we have (37).

Secondly, we show that (37) implies that M is a maximum mean with weight. Set

$$\delta(x) = M(\chi_{\{x\}}) \qquad \text{for } x \text{ in } X. \tag{38}$$

It is not hard to see that δ is an upper semi-continuous function on X and $0 \leqq \delta(x) \leqq 1$ for all x in X. We see

$$\delta(x)f(x) = f(x)M(\chi_{\{x\}}) = M_y[f(x)\chi_{\{x\}}(y)] = M_y[f(y)\chi_{\{x\}}(y)] \leqq M_y[f(y)] = M(f).$$

Thus we have $\max_{x\in X}\{\delta(x)f(x)\} \leqq M(f)$. Conversely, because of $\delta(x)f(x) = M(f\chi_{\{x\}})$, for any $\varepsilon>0$ and for each x in X, one can choose g_x in $C(X)^+$ such that

$$f(x) < g_x(x) \qquad \text{and} \qquad M(g_x) < \delta(x)f(x)+\varepsilon.$$

By using the compact topology of X, one can choose finitely many $g_{x_1}, \cdots, g_{x_n}$ such that $f \leqq g_{x_1} \vee \cdots \vee g_{x_n}$, where the function $g_{x_1} \vee \cdots \vee g_{x_n}$ is the pointwise maximum function of $g_{x_1}, \cdots, g_{x_n}$. Setting $g = g_{x_1} \vee \cdots \vee g_{x_n}$ (37) yields $M(g) = \max\{M(g_{x_1}), \cdots, M(g_{x_n})\}$, hence

$$\begin{aligned} M(f) \leqq M(g) &= \max\{M(g_{x_1}), \cdots, M(g_{x_n})\} \\ &\leqq \max\{\delta(x_1)f(x_1)+\varepsilon, \cdots, \delta(x_n)f(x_n)+\varepsilon\} \leqq \max_{x\in X}\{\delta(x)f(x)\}+\varepsilon. \end{aligned}$$

Thus we have $M(f) \leqq \max_{x\in X}\{\delta(x)f(x)\}$. Consequently M is Max_δ. Uniqueness of the weight follows from (38).

Suppose L is a minimum mean with weight. Consider $M^{-1}(f) = \{M(f^{-1})\}^{-1}$ for f in $C(X)^+$ and consider $L^{-1}(g) = \{L(g^{-1})\}^{-1}$ for g in $C(Y)^+$. Then L^{-1} becomes a maximum mean with weight and M^{-1} commutes with L^{-1}. Thus the previous argument shows that M^{-1} is a maximum mean with weight δ, so M is a minimum mean with weight $1/\delta$. q. e. d.

We observe two lemmas before we actually go into our proof of Theorem 2.

LEMMA 13. *Let L be a positively homogeneous mean on $C([0, 1])^{+}$. The following holds except for at most countably many ξ's in* $[0, 1]$,

$$L(\check{\chi}_{a,b,\xi}) = L(\hat{\chi}_{a,b,\xi}) \qquad \textit{for all } a, b>0.$$

(Recall (12) for the notations of $\check{\chi}_{a,b,\xi}$ and $\hat{\chi}_{a,b,\xi}$).

PROOF. For ξ in $[0, 1]$, set, as in § 1,

$$\check{F}_{\xi}(a, b) = L(\check{\chi}_{a,b,\xi}), \qquad \hat{F}_{\xi}(a, b) = L(\hat{\chi}_{a,b,\xi}) \qquad \text{for all } a, b>0.$$

Then $\check{F}_{\xi}$ and $\hat{F}_{\xi}$ are positively homogeneous and monotonic increasing functions on $(0, +\infty)\times(0, +\infty)$ and furthermore $\check{F}_{\xi}(a, a)=\hat{F}_{\xi}(a, a)=a$ for all $a>0$. It should be noticed that being positively homogeneous together with being monotonic increasing implies the continuity of the function. Therefore $\check{F}_{\xi}$ and $\hat{F}_{\xi}$ turn out automatically continuous on $(0, +\infty)\times(0, +\infty)$. As in the proof of Lemma 7 for Theorem 1, one can choose ξ from $[0, 1]$, with at most countably many exceptions, such that $\check{F}_{\xi}$ coincides with $\hat{F}_{\xi}$ at every rational point of $(0, +\infty)\times(0, +\infty)$. For such choice of ξ, the continuity of $\check{F}_{\xi}$ and $\hat{F}_{\xi}$ yields $\check{F}_{\xi}=\hat{F}_{\xi}$ on $(0, +\infty)\times(0, +\infty)$. q. e. d.

By combining Lemma 13 and Lemma 4 in § 1, one can conclude that from any positively homogeneous and bisymmetric mean L on $C([0, 1])^{+}$ we can derive two dimensional positively homogeneous and bisymmetric means $N^{(L,\xi)}$ for almost all ξ in $[0, 1]$ (more precisely, with at most countably many exceptions of ξ) such that $N^{(L,\xi)}$ commutes with any mean which commutes with L.

Now, let L be a positively homogeneous and bisymmetric mean on $C(Y)^{+}$ and let τ be a continuous map from Y into $[0, 1]$. As mentioned in the previous paragraph, one can have two dimensional positively homogeneous and bisymmetric means $N^{(L^{\tau},\xi)}$ derived from L^{τ}. About these two dimensional means we have the following observation.

LEMMA 14. *If $N=N^{(L^{\tau},\xi)}$ is a right (left) point evaluation, then by setting $K=\tau^{-1}([\xi, 1])$ ($K=\tau^{-1}([0, \xi])$), we have*

$$L(\chi_K)=1 \qquad \textit{and}$$

$$\inf_{y\in K} f(y) \leqq L(f) \leqq \sup_{y\in K} f(y) \qquad \textit{for all } f \textit{ in } C(Y)^{+}.$$

PROOF. First, from 3) of Lemma 3 and 3) of Lemma 4, we see $L(\chi_K)= L^{\tau}(\hat{\chi}_{0,1,\xi})=N^{(L^{\tau},\xi)}(0, 1)=1$. Next, for any given f in $C(Y)^{+}$, setting

$$a=\sup_{y\in Y} f(y), \qquad b=\sup_{y\in K} f(y), \qquad c=\inf_{y\in Y} f(y), \qquad d=\inf_{y\in K} f(y),$$

one can define a lower semi-continuous function v and an upper semi-continuous function u on Y as

$$v(y)=\begin{cases} b & \text{for } y \text{ in } K, \\ a & \text{for } y \text{ in } Y\setminus K, \end{cases} \qquad u(y)=\begin{cases} d & \text{for } y \text{ in } K, \\ c & \text{for } y \text{ in } Y\setminus K. \end{cases}$$

It is clear that $f\leqq v\leqq\hat{\chi}_{a,b,\xi}\circ\tau$ and $f\geqq u\geqq\check{\chi}_{c,d,\xi}\circ\tau$. Hence, from 3) of Lemma 3, we have $L(f)\leqq L(v)\leqq L(\hat{\chi}_{a,b,\xi})=N^{(L^{\tau},\xi)}(a, b)=b=\sup_{y\in K}f(y)$. Similarly we have $L(f)\geqq d$. q. e. d.

PROOF OF THEOREM 2. Suppose M is a positively homogeneous and bisymmetric mean on $C(X)^{+}$ and suppose M is not a point evaluation. By using Propositions 1, 2, 3 and 4, it suffices to show that M commutes with a two dimensional positively homogeneous and bisymmetric mean N which is not a point evaluation. In order to have such N, first observe that for any compact subset K of X with $M(\chi_{K})=1$ the restriction M^{K} of M onto $C(K)^{+}$ is a positively homogeneous and bisymmetric mean on $C(K)^{+}$ and M commutes with M^{K}. Secondly, for any continuous map τ from K into $[0, 1]$ the transition $(M^{K})^{\tau}=M^{K,\tau}$ of M^{K} by τ is a positively homogeneous and bisymmetric mean on $C([0,1])^{+}$ and M commutes with $M^{K,\tau}$. Thirdly, two dimensional means $N=N^{(M^{K,\tau},\xi)}$ derived from $M^{K,\tau}$ are two dimensional positively homogeneous and bisymmetric means and M commutes with those N. Finally, therefore our task is to choose proper K, τ, ξ such that $N^{(M^{K,\tau},\xi)}$ is not a point evaluation.

Let $\mathcal{K}=\{K\subset X \mid K \text{ is compact and } M(\chi_{K})=1\}$. By Zorn's lemma, one can see that $\mathcal{K}$ has a minimal element K_{0}, because for any downward directed system $\{K_{\alpha}\}$ from $\mathcal{K}$ we see $M(\chi_{\cap_{\alpha}K_{\alpha}})=\inf_{\alpha}M(\chi_{K_{\alpha}})=1$ from Lemma 1, so $\bigcap_{\alpha}K_{\alpha}$ belongs to $\mathcal{K}$.

Case 1. Suppose K_{0} contains two elements $x_{1}\neq x_{2}$. Let τ be a continuous map from K_{0} into $[0, 1]$ with $\tau(x_{1})=0$ and $\tau(x_{2})=1$. Then one can see that $N=N^{(M^{K_{0}},\tau,\xi)}$ with $0<\xi<1$ is not a point evaluation. If N were a right point evaluation, then $K_{1}=\tau^{-1}([\xi, 1])\subset K_{0}$ and x_{1} were in $K_{0}\setminus K_{1}$, so $K_{1}\subsetneqq K_{0}$, however from Lemma 14 we would have $M^{K_{0}}(\chi_{K_{1}})=1$, and from 3) of Lemma 5 $M(\chi_{K_{1}})=M^{K_{0}}(\chi_{K_{1}})=1$. Thus K_{1} would belong to $\mathcal{K}$, which contradicts the minimality of K_{0}. Supposing that N is a left point evaluation yields a contradiction in a completely similar manner.

Case 2. Suppose K_{0} consists of a single point x_{0}. Since M is not a point evaluation, there is f_{0} in $C(X)^{+}$ such that $M(f_{0})\neq f_{0}(x_{0})$. Assume $M(f_{0})>f_{0}(x_{0})$, (the other case goes similarly), then choose an open neighborhood U of x_{0} such that $M(f_{0})>\sup_{x\in U}f(x)$. Take a continuous map τ from X into $[0, 1]$ such that $\tau(x_{0})=1$ and $\tau(x)=0$ for all x in $X\setminus U$. Now one can see that the two dimensional mean $N=N^{(M^{\tau},\xi)}$ with $0<\xi<1$ is not a point evaluation. Suppose it were a point evaluation, then N must be a right point evaluation, because $1\geqq N(0, 1)$

$=M^{\tau}(\hat{\chi}_{0,1,\xi})=M(\hat{\chi}_{0,1,\xi}\circ\tau)\geqq M(\chi_{\{x_0\}})=1$, so $N(0, 1)=1$. By Lemma 14 we have $M(f_0)\leqq\sup\{f_0(x) \mid x \text{ in } \tau^{-1}([\xi, 1])\}\leqq\sup\{f_0(x)\mid x \text{ in } U\}<M(f_0)$, which is a contradiction. q. e. d.

We add one remark here. We have been studying strictly bisymmetric means which are strictly increasing or positively homogeneous. However there are many bisymmetric means which are neither strictly increasing nor positively homogeneous. For instance, the following is such a two dimensional example.

EXAMPLE. Let $p>0$, $0<t<1$ and let $N:(0, +\infty)\times(0, +\infty)\to(0, +\infty)$ as follows,

$$N(a, b)=\begin{cases}\max\{a, b\} & \text{if } \max\{a, b\}\leqq 1,\\ 1 & \text{if } \min\{a, b\}\leqq 1\leqq\max\{a, b\},\\ \{t(a-1)^p+(1-t)(b-1)^p\}^{1/p}+1 & \text{if } \min\{a, b\}>1.\end{cases}$$

§ 4. Complements.

We state generalizations of Theorems 1 and 2 without proof.

1. One can generalize Theorem 1 slightly, that is, without assuming "$M(a1_X)=a$ for all a in I", one has also a representation of M similar to Theorem 1. Such generalization can be proved by reducing M to the situation where Theorem 1 is applicable. It should be mentioned that the idea of such reduction appeared in [**3**] and the two dimensional version of this generalization can be seen on pp. 287-290 of J. Aczél's book [**4**].

THEOREM 3. *Let X be a compact Hausdorff space and let F be an I-valued continuous functional on $C(X;I)$. If F is strictly increasing and satisfies the bisymmetry equation*

$$F_y[F_x[h(x, y)]]=F_x[F_y[h(x, y)]] \quad \textit{for all } h \textit{ in } C(X\times X;I),$$

there are a strictly increasing or decreasing, real valued continuous function ϕ defined on I, a finite positive measure μ on X with support identical to X and a constant c such that

$$F(f)=\phi^{-1}\left\{\int_X\phi(f)d\mu+c\right\} \quad \textit{for all } f \textit{ in } C(X;I),$$

where ϕ is determined uniquely up to an affine equivalence, μ is also determined uniquely and ϕ, μ and c are related as $c=\phi[F(a1_X)]-\phi(a)\mu(X)$ for all a in I.

2. Theorem 2 can be generalized to a locally compact Hausdorff space. Let $C_c^+(X)$ be the space of all *non-negative* continuous functions with compact support on X.

Let M be a functional on $C_c^+(X)$, we call M a *positively homogeneous mean* on $C_c^+(X)$ if M has

i) $M(f_\alpha)\downarrow_\alpha M(f)$ if $f_\alpha \downarrow_\alpha f$ pointwise on X in $C_c^+(X)$,

ii) $M(f)\leqq M(g)$ if $f\leqq g$ in $C_c^+(X)$,

iii) $M(rf)=rM(f)$ for all $r\geqq 0$ and f in $C_c^+(X)$.

Note that we *do not* assume the continuity of M with respect to the topology of uniform convergence on $C_c^+(X)$. However, if X is compact, then i), ii) and iii) together imply the continuity of M with respect to the uniform convergence and M turns out a constant multiple of the positively homogeneous mean studied in the previous section.

In order to define the bisymmetry equation of M, we need to discuss the extension of M to upper semi-continuous non-negative functions u with compact supports on X. Denote by $\mathcal{U}_c^+(X)$ the space of all such functions u. For a given u in $\mathcal{U}_c^+(X)$, $\{f \mid u\leqq f,\ f \text{ in } C_c^+(X)\}$ is a downward directed system and converges to u pointwise on X. Define

$$M(u)=\inf\{M(f) \mid u\leqq f,\ f \text{ in } C_c^+(X)\}\,.$$

It is not hard to see that this extended functional M satisfies the properties i), ii), iii) mentioned above in the space $\mathcal{U}_c^+(X)$ instead of $C_c^+(X)$. One can prove the following.

LEMMA 15. *For any u in $\mathcal{U}_c^+(X\times X)$, regarding $M_x[u(x, y)]$ as a function of y, $M_x[u(x, y)]$ belongs to $\mathcal{U}_c^+(X)$.*

For any u in $\mathcal{U}_c(X\times X)^+$, using this lemma, $M_y[M_x[u(x, y)]]$ and $M_x[M_y[u(x, y)]]$ are well defined. Thus one can define the bisymmetry equation for a positively homogeneous mean M on $C_c(X)^+$ as

$$(*)\qquad M_y[M_x[u(x, y)]]=M_x[M_y[u(x, y)]] \qquad \text{for all } u \text{ in } \mathcal{U}_c(X\times X)^+\,.$$

Now we are ready to state a generalization of Theorem 2 to a locally compact Hausdorff space.

THEOREM 4. *Let X be a locally compact Hausdorff space and let M be a positively homogeneous mean on $C_c^+(X)$. If M satisfies the bisymmetry equation $(*)$, then M can be represented as one of the followings*

a) *there are $-\infty<p<+\infty$ and a non-negative Radon measure μ, not necessarily finite, such that*

$$M(f)=\left(\int_X f^p d\mu\right)^{1/p} \qquad \text{for all } f \text{ in } C_c^+(X)\,,$$

where μ has a compact support if $p\leqq 0$,

b) *there is an upper semi-continuous function δ on X with $0\leqq\delta(x)<+\infty$ such that*

$$M(f)=\sup_{x\in X}\{\delta(x)f(x)\} \qquad \textit{for all } f \textit{ in } C_c^+(X),$$

c) *there is a lower semi-continuous function* γ *on* X *with* $0<\gamma(x)\leqq+\infty$ *such that the closure of* $\{x \mid \gamma(x)<+\infty\}$ *is compact and*

$$M(f)=\inf_{x\in X}\{\gamma(x)f(x)\} \qquad \textit{for all } f \textit{ in } C_c^+(X),$$

where $\gamma(x)f(x)$ *is regarded as* $+\infty$ *if* $\gamma(x)=+\infty$ *and* $f(x)=0$.

These representations of M *are determined uniquely if neither* M *is a point evaluation nor identically zero.*

ACKNOWLEDGEMENT. The authors wish to express their thanks for many valuable suggestions to Professors T. Ando, I. Sawashima, and the participants of the functional analysis seminars held at Ochanomizu University, and special thanks to Professor F. Niiro who stimulated them to study the bisymmetry equation.

References

[1] J. Aczél, The notion of mean values, Norke Vid. Selsk. Forh., **19** (1946), 83-86.
[2] J. Aczél, On mean values, Bull. Amer. Math. Soc., **54** (1948), 392-400.
[3] J. Aczél, I. Fenyö and J. Horváth, Sur certaines classes de fonctionnelles, Portugal. Math., **8** (1949), 1-11.
[4] J. Aczél, Lectures on functional equations and their applications, Academic Press, New York and London, 1966.
[5] B. de Finetti, Sur concetto di media, Gionale dell'Instituto Italiano degli Attuarii, **2** (1931), 369-396.
[6] G. H. Hardy, J. E. Littlewood and G. Pólya, Inequalities, Cambridge Univ. Press, 1934.
[7] B. Jessen, Bemaerkninger om konvekse Funktioner og Uligheder imellem Middelvaerdier. I., Mat. Tidsskrift B, **2** (1931), 17-28.
[8] A. N. Kolmogorov, Sur la notion de la moyenne, Atti Accad. Naz. Lincei Rend., (6) **12** (1930), 388-391.
[9] N. Nagumo, Über eine Klasse von Mittelwerten, Japan. J. Math., **7** (1930), 71-79.

Takashi ITO
Department of Mathematics
Musashi Institute of Technology
Tamazutsumi, Setagaya
Tokyo 158
Japan

Chiê NARA
Department of Mathematics
Musashi Institute of Technology
Tamazutsumi, Setagaya
Tokyo 158
Japan