J. Math. Soc. Japan
Vol. 38, No. 4, 1986

Behavior of modes of a class of processes with independent increments

By Ken-iti SATO

(Received April 16, 1985)

1. Introduction.

We consider a stochastic process X_t, $0\leqq t<\infty$, with homogeneous independent increments of class L, starting at the origin. That is, X_t is a process with homogeneous independent increments with characteristic function

$$E\exp(izX_t)=\exp\left[t\left(i\gamma z-2^{-1}\sigma^2z^2+\int_{-\infty}^{\infty}g(z,\ x)x^{-1}k(x)dx\right)\right], \tag{1.1}$$

where

$$g(z,\ x)=e^{izx}-1-izx(1+x^2)^{-1}, \tag{1.2}$$

γ real, $\sigma^2\geqq 0$, $k(x)$ is non-negative, non-increasing on $(0, \infty)$ and non-positive, non-increasing on $(-\infty, 0)$, and

$$\int_{|x|<1}xk(x)dx+\int_{|x|>1}x^{-1}k(x)dx<\infty. \tag{1.3}$$

Yamazato [9] proves that the distribution of X_t is unimodal for each t. If a distribution is unimodal, then either its mode is unique or the set of its modes is a closed interval. When X_t has a unique mode, we denote it by $a(t)$. The purpose of the present paper is to study behavior of $a(t)$ as a function of t. In Section 2 we treat the case that X_t is an increasing process (subordinator). Results and techniques of the paper [6] are employed. Asymptotic behavior of $a(t)$ as $t\to\infty$ is found when $k(x)$ is slowly varying at infinity. When X_t is not an increasing process, behavior of $a(t)$ is hard to obtain except some asymptotic results. In Section 3 processes attracted to stable distributions are discussed. Other miscellaneous results are gathered in Section 4.

2. Increasing processes.

Assume that the process X_t is increasing. This is equivalent to assuming

$$E\exp(izX_t)=\exp\left[t\left(i\gamma_0 z+\int_0^{\infty}(e^{izx}-1)x^{-1}k(x)dx\right)\right], \tag{2.1}$$

where $\gamma_0 \geqq 0$, $k(x)$ is non-negative, non-increasing, and

$$\int_0^1 k(x)dx + \int_1^\infty x^{-1}k(x)dx < \infty . \tag{2.2}$$

For simplicity we assume that $\gamma_0=0$ and that $k(x)$ does not identically vanish. Let $m=EX_1 \leqq \infty$. We have $m=\int_0^\infty k(x)dx$, including the case of infinity. Let $\lambda=k(0+)$. The behavior of the mode of X_t is different according to the following three cases:

Case I: $\lambda<\infty$ and $k(x)<\lambda$ for all $x>0$.

Case II: $\lambda<\infty$ and there is a $\beta>0$ such that $k(x)=\lambda$ for $0<x<\beta$ and $k(x)<\lambda$ for $x>\beta$.

Case III: $\lambda=\infty$.

THEOREM 2.1. *The distribution of X_t has a unique mode $a(t)$ for each t except one epoch in Case II. The only exception is that the set of modes is the interval $[0, \beta]$ at the epoch $t=\lambda^{-1}$ in Case II. In Case I, $a(t)=0$ for $t\in[0, \lambda^{-1}]$, and $a(t)$ is continuous, strictly increasing for $t\in[\lambda^{-1}, \infty)$. In Case II, $a(t)=0$ for $t\in[0, \lambda^{-1})$, $a(t)$ is continuous, strictly increasing for $t\in(\lambda^{-1}, \infty)$, and $a(t)\downarrow\beta$ as $t\downarrow\lambda^{-1}$. In Case III, $a(0)=0$, and $a(t)$ is continuous, strictly increasing for $t\in[0, \infty)$. We have*

$$t^{-1}a(t) < m \tag{2.3}$$

unless t is the exceptional epoch of Case II. In all cases,

$$t^{-1}a(t) \to m \qquad \text{as } t\to\infty . \tag{2.4}$$

If we define a function $b(t)$ on (λ^{-1}, ∞) by

$$b(t) = \sup\{x>0 : k(x)\geqq t^{-1}\}, \tag{2.5}$$

then

$$a(t) > b(t) \qquad \text{for } t>\lambda^{-1} . \tag{2.6}$$

PROOF. The assertions on the set of modes of X_t (whether a singleton or the interval $[0, \beta]$) and on the position of $a(t)$ (whether zero or positive) are consequences of Theorem 1.3 of [**6**]. We have $a(t)\geqq\beta$ for $t>\lambda^{-1}$ in Case II by using Lemma 6.1 of [**6**]. Consider Case III. The distribution of X_t has density $f_t(x)$, which is of class C^∞ in $x\in(-\infty, \infty)$ for each $t>0$. It follows from

$$f_{t+s}(x) = \int_0^x f_t(x-y)f_s(y)dy$$

that

$$f'_{t+s}(x)=\int_0^x f'_t(x-y)f_s(y)dy \tag{2.7}$$

for $t>0$, $s>0$. Since $f'_t>0$ on $(0, a(t))$ by Theorem 1.3 of [**6**] and $f_s>0$ on $(0, \infty)$, we obtain $f'_{t+s}>0$ on $(0, a(t)]$. Hence $a(t+s)>a(t)$. Next, let us consider Cases I and II. For each $t>\lambda^{-1}$, the density $f_t(x)$ of the distribution of X_t is continuous on $(-\infty, \infty)$ and of class C^1 on $(0, \infty)$. We claim that (2.7) holds for $t>\lambda^{-1}$, $s>0$, $x\in(0, a(t)]$. In fact, since $f'_t\geqq 0$ on $(0, a(t)]$, we have

$$\int_0^x dy\int_0^y f'_t(y-z)f_s(z)dz=\int_0^x dz\int_z^x f'_t(y-z)f_s(z)dy$$
$$=\int_0^x f_t(x-z)f_s(z)dz=f_{t+s}(x)=\int_0^x f'_{t+s}(y)dy\,.$$

Thus we get (2.7) almost everywhere on $(0, a(t)]$. Writing the right-hand side of (2.7) as $\int_0^x f'_t(y)f_s(x-y)dy$, we can prove that it is continuous in $x>0$. Hence (2.7) holds everywhere on $(0, a(t)]$. Now it follows that $f'_{t+s}(a(t))>0$ and hence $a(t+s)>a(t)$. Continuity of $a(t)$ is a consequence of Lemma 2.1 below. By Theorem 6.1 (i) of [**6**], $a(t)<EX_t=tm$, that is (2.3). The limit behavior (2.4) can be shown more generally (1 in Section 4 and [**5**] Theorem 3.3), but we give here a proof that uses [**6**] Theorem 6.1 (iii) or [**8**]. Truncate $k(x)$ at ξ, define $X_t^{(\xi)}$ by

$$E\exp(izX_t^{(\xi)})=\exp\left[t\int_0^\xi(e^{izx}-1)x^{-1}k(x)dx\right], \tag{2.8}$$

and use the theorem to obtain

$$t\int_0^\xi k(x)dx-\xi<a^{(\xi)}(t)$$

for the mode $a^{(\xi)}(t)$ of $X_t^{(\xi)}$ for $t>\lambda^{-1}$. Note that $a^{(\xi)}(t)\leqq a(t)$ by Lemma 6.1 of [**6**]. Hence

$$\liminf_{t\to\infty} t^{-1}a(t)\geqq\int_0^\xi k(x)dx\,.$$

Thus (2.4) follows. Unlike in [**6**], we are not assuming right-continuity of $k(x)$. But the function $b(t)$ remains the same if we use in (2.5) the right-continuous modification of $k(x)$. Hence (2.6) is a direct consequence of [**6**] Theorem 6.1 (vi). The proof is complete.

LEMMA 2.1. *Let $\{\mu_n\}$ be a sequence of unimodal probability measures. Let a_n be a mode of μ_n. If μ_n weakly converges to a probability measure μ, then the set $\{a_n\}$ is bounded, μ is unimodal, and any limit point of $\{a_n\}$ is a mode of μ.*

This lemma is evident from the proof of [3] § 32, Theorem 4.

When m is infinity, we are interested in the problem how fast the mode $a(t)$ grows as $t\to\infty$. In the following theorem we consider the case where $k(x)$ is slowly varying at infinity. If $k(x)$ is regularly varying at infinity with exponent $0<\alpha<1$ (that is, $k(x)=x^{-\alpha}L(x)$ with a slowly varying $L(x)$), then

$$\int_x^\infty y^{-1}k(y)dy \sim \alpha^{-1}x^{-\alpha}L(x)\,, \qquad x\to\infty\,,$$

by [1] VIII, 9, Theorem 1 (and conversely by Lemma 3.1 in the next section). Asymptotic behavior of $a(t)$ in this case will be described in Section 3.

THEOREM 2.2. (i) *If $k(x)$ is slowly varying at infinity, then*

$$\log a(t) = o(t)\,, \qquad t\to\infty\,. \tag{2.9}$$

(ii) *If $k(e^x)$ is regularly varying with exponent $-\beta$ ($\beta\geqq 1$) as $x\to\infty$, then $\log a(t)$ is regularly varying with exponent β^{-1} as $t\to\infty$, and*

$$\log a(t) \sim \log b(t)\,, \qquad t\to\infty\,, \tag{2.10}$$

where $b(t)$ is given by (2.5).

(iii) *Suppose that $k(e^x)=x^{-1}L(x)$ where $L(x)$ is monotone for large x and slowly varying at infinity. Then*

$$\log a(t) = o(t/\log t)\,, \qquad t\to\infty\,. \tag{2.11}$$

PROOF. (i) We have

$$t\int_0^{a(t)} k(y)dy > a(t) \qquad \text{for } t>\lambda^{-1} \tag{2.12}$$

by Theorem 6.1 (ii) of [6]. Since

$$\int_0^x k(y)dy \sim xk(x)\,, \qquad x\to\infty\,,$$

by slow variation of $k(x)$ ([1] VIII, 9, Theorem 1), it follows that

$$(1+o(1))\,t\,k(a(t)) > 1\,. \tag{2.13}$$

Since we have

$$k(x)\log x \to 0\,, \qquad x\to\infty\,, \tag{2.14}$$

from (2.2) (see Lemma 2.1 of [6]), it follows that

$$\log a(t) < (1+o(1))\,t\,k(a(t))\log a(t) = o(t)\,,$$

that is (2.9).

(ii) Let $l(x)=k(e^x)^{-1}$ and $c(t)=\log b(t)$. We have

$$c(t)=\sup\{x>-\infty : l(x)\leqq t\}$$

from (2.5). Since $l(x)$ is regularly varying with exponent β, the function $c(t)$ is the asymptotic inverse function of $l(x)$ (that is, $c(t)\to\infty$ and $l(c(t))\sim t$ as $t\to\infty$) and $c(t)$ is regularly varying with exponent β^{-1} ([7] p. 24). As $k(e^x)$ is of the form $x^{-\beta}L(x)$ with a slowly varying function $L(x)$, we have

$$k(x)=(\log x)^{-\beta}L(\log x)\,. \tag{2.15}$$

The function $L(\log x)$ is slowly varying ([7] p. 19) and so is $k(x)$. Therefore we have (2.13) again and the definition (2.5) of b implies

$$b(t(1+o(1)))\geqq a(t)\,.$$

For any $\varepsilon>0$ we have

$$c(t(1+o(1)))\leqq c(t(1+\varepsilon))\sim(1+\varepsilon)^{1/\beta}c(t)\,,\qquad t\to\infty\,.$$

Therefore

$$\log a(t)\leqq(1+\varepsilon)^{1/\beta}(1+\varepsilon)c(t)$$

for all large t. This, combined with (2.6), shows (2.10).

(iii) We have (2.15) with $\beta=1$. For every $\varepsilon>0$ the function $\log a(t)$ is bigger than $t^{1-\varepsilon}$ for large t, because (ii) says that $\log a(t)$ is regularly varying with exponent 1. The function $L(x)$ decreases to infinity as $x\to\infty$ by (2.14) and by its monotonicity. Therefore

$$(\log t)k(a(t))\log a(t)=(\log t)L(\log a(t))$$
$$\leqq(\log t)L(t^{1-\varepsilon})=(1-\varepsilon)^{-1}(\log\log x)L(\log x)$$

where $x=\exp(t^{1-\varepsilon})$. Now we claim that

$$(\log\log x)L(\log x)\to 0\,,\qquad x\to\infty\,. \tag{2.16}$$

This will prove

$$(\log t)k(a(t))\log a(t)\to 0\,,\qquad t\to\infty\,,$$

which shows (2.11) just like the proof of (i). We have, for large $x_1<x_2$,

$$\int_{x_1}^{x_2}x^{-1}k(x)dx=\int_{x_1}^{x_2}(\log\log x)'L(\log x)dx \tag{2.17}$$
$$\geqq[(\log\log x)L(\log x)]_{x_1}^{x_2}$$

using integration by parts, since $L(\log x)$ is monotone decreasing. Let θ_1 and θ_2 be the lower and upper limits of $(\log\log x)L(\log x)$, respectively. If $\theta_1<\theta_1+\delta<\theta_2$ for some δ, then we would have contradiction, because by (2.2) we can find x_1 and x_2 such that the extreme left member of (2.17) is smaller than δ while the extreme right member of (2.17) is larger than δ. Hence $\theta_1=\theta_2$. If

$\theta_1=\theta_2>0$, then, for large $x_1<x_2$,

$$\int_{x_1}^{x_2} x^{-1}k(x)dx > 2^{-1}\theta_1\int_{x_1}^{x_2} x^{-1}(\log x)^{-1}(\log\log x)^{-1}dx$$
$$= 2^{-1}\theta_1[\log\log\log x]_{x_1}^{x_2},$$

which contradicts (2.2). Therefore we have (2.16), completing the proof.

Note that we did not use the assumption $\beta\geqq 1$ in the proof of (ii). But, if $\beta<1$, then we have $\int_1^\infty x^{-1}k(x)dx=\infty$, which violates (2.2).

EXAMPLE 2.1. If $k(x)$ is of class C^1 in a neighborhood of infinity and satisfies

$$k'(x)/k(x) \sim -\beta/(x\log x), \qquad x\to\infty,$$

for some $\beta\neq 0$, then $k(e^x)$ is regularly varying with exponent $-\beta$. In fact we have, for $K(x)=k(e^x)$,

$$K'(x)/K(x) \sim -\beta/x, \qquad x\to\infty,$$

and

$$\log(K(hx)/K(x)) = \int_x^{hx}(K'(y)/K(y))dy \to -\beta\log h$$

for every $h>1$.

EXAMPLE 2.2. Define iterated logarithmic functions $L_n(x)$, $n\geqq 1$, by $L_1(x)=\log x$ and $L_n(x)=\log L_{n-1}(x)$. Let $\varepsilon>0$. Let, for large x, $k_1(x)=L_1(x)^{-1-\varepsilon}$ and

$$k_n(x) = \left(\prod_{j=1}^{n-1} L_j(x)\right)^{-1} L_n(x)^{-1-\varepsilon} \qquad \text{for } n\geqq 2.$$

Then the functions $k_n(e^x)$ are regularly varying with exponents $-1-\varepsilon$ (for $n=1$) and -1 (for $n\geqq 2$). Let $b_n(t)$ be the function defined by (2.5) for $k_n(x)$. This is the inverse function of $k_n(x)^{-1}$ for large x. We have

$$\log b_1(t) = t^{1/(1+\varepsilon)} \qquad \text{for large } t$$

and, by the method of asymptotic expansion,

$$\log b_2(t) = t(\log t)^{-1-\varepsilon}(1+o(1)),$$
$$\log b_n(t) = t\left(\prod_{j=1}^{n-2} L_j(t)\right)^{-1} L_{n-1}(t)^{-1-\varepsilon}(1+o(1)) \qquad \text{for } n\geqq 3$$

as $t\to\infty$.

We add a result on stable subordinators.

THEOREM 2.3. *Let* $0<\alpha<1$ *and let*

$$(2.18)\qquad E\exp(izX_t)=\exp\left[t\int_0^\infty(e^{izx}-1)x^{-1-\alpha}dx\right].$$

Denote the mode of X_t *by* $a_\alpha(t)$. *Then*

$$(2.19)\qquad a_\alpha(t)=t^{1/\alpha}a_\alpha(1),$$

$$(2.20)\qquad (1-\alpha)^{-1/\alpha}>a_\alpha(1)>\max\{1,\ \alpha(1-\alpha)^{-1}\}.$$

PROOF. (2.19) is obvious because X_t and $t^{1/\alpha}X_1$ have a common distribution. The bound $(1-\alpha)^{-1/\alpha}>a_\alpha(1)>1$ is given in [6] p. 307. To see $a_\alpha(1)>\alpha(1-\alpha)^{-1}$, consider the process $X_t^{(\xi)}$ defined by (2.8). Then we have

$$a_\alpha(1)>\int_0^\xi x^{-\alpha}dx-\xi.$$

Now let $\xi=1$.

COROLLARY. *As* $\alpha\downarrow 0$, *we have*

$$a_\alpha(t)\to 0\qquad for\ 0<t<1,$$
$$a_\alpha(t)\to\infty\qquad for\ 1<t<\infty.$$

3. Processes attracted to stable distributions.

Let X_t be a process given in Section 1. Let ν be its Lévy measure, that is,

$$\nu(B)=\int_B x^{-1}k(x)dx\qquad \text{for } B \text{ Borel}.$$

We assume that, for some α and p satisfying $0<\alpha<2$ and $0\leqq p\leqq 1$ and for some slowly varying function $L(x)$,

$$(3.1)\qquad \nu[x,\infty)+\nu(-\infty,-x]\sim\alpha^{-1}x^{-\alpha}L(x),$$

$$(3.2)\qquad \frac{\nu[x,\infty)}{\nu[x,\infty)+\nu(-\infty,-x]}\to p$$

as $x\to\infty$. Note that if $0<\alpha<1$ then $\int_{|x|>1}|x|\nu(dx)=\infty$ and hence $E|X_t|=\infty$. If $1<\alpha<2$, then $\int_{|x|>1}|x|\nu(dx)<\infty$ and $E|X_t|<\infty$. Choose $b_t>0$ in such a way that $b_t\to\infty$ and

$$(3.3)\qquad b_t^{-\alpha}L(b_t)\sim t^{-1},\qquad t\to\infty.$$

Such choice is possible and b_t is regularly varying with exponent $1/\alpha$. We may take the function $b(t)$ of (2.5) as b_t ([7] pp. 21-24). We denote the indicator function of a set B by $\chi_B(x)$ and use a notation

(3.4) $$\chi_{a,b}(x)=a\chi_{(0,\infty)}(x)+b\chi_{(-\infty,0)}(x)\,.$$

THEOREM 3.1. *If* $0<\alpha<1$, *then*

(3.5) $$b_t^{-1}a(t)\to a\,,\qquad t\to\infty\,,$$

where a *is the unique mode of the stable distribution* μ *with index* α *with characteristic function*

(3.6) $$\hat{\mu}(z)=\exp\int_{-\infty}^{\infty}(e^{izx}-1)\chi_{p,1-p}(x)|x|^{-1-\alpha}dx\,.$$

PROOF. The conditions (3.1), (3.2) imply similar conditions on tails of the distribution of X_1 by Zolotarev's result [**10**] if $p\neq 0$, 1 (and also in case $p=0$ or 1 after some consideration). So the distribution of $Y_t=b_t^{-1}X_t$ tends to μ by a well-known theorem ([**1**] XVII, 5, Theorem 3), and (3.5) follows from Lemma 2.1. But our condition on tails of the Lévy measure allows us to give a direct proof, which we present here. Define ν_t by

(3.7) $$\nu_t(B)=t\int\chi_B(b_t^{-1}x)\nu(dx)\,.$$

We have

$$E\exp(izY_t)=\exp(itb_t^{-1}\gamma_t z-2^{-1}tb_t^{-2}\sigma^2z^2+S_1+S_2)\,,$$

$$S_1=\int_{|x|\leq b_t^{-1}}g(z,x)\nu_t(dx),\qquad S_2=\int_{|x|>b_t^{-1}}(e^{izx}-1)\nu_t(dx)\,,$$

$$\gamma_t=\gamma-\int_{|x|>1}x(1+x^2)^{-1}\nu(dx)+\int_{|x|\leq 1}x((1+b_t^{-2}x^2)^{-1}-(1+x^2)^{-1})\nu(dx)\,.$$

Clearly γ_t is bounded as $t\to\infty$. We have $tb_t^{-1}\to 0$ from (3.3). Conditions (3.1) and (3.2) imply

(3.8) $$\nu[x,\infty)/(\alpha^{-1}x^{-\alpha}L(x))\to p\quad\text{and}\quad\nu(-\infty,-x]/(\alpha^{-1}x^{-\alpha}L(x))\to 1-p$$

as $x\to\infty$. Hence it follows from (3.3) that

(3.9) $$\nu_t[x,\infty)\to p\alpha^{-1}x^{-\alpha}\quad\text{and}\quad\nu_t(-\infty,-x]\to(1-p)\alpha^{-1}x^{-\alpha}$$

for any $x>0$ as $t\to\infty$. For any $\delta>0$,

$$\int_{|x|\leq\delta}x^2\nu_t(dx)=tb_t^{-2}\int_{|x|\leq b_t\delta}x^2\nu(dx)$$
$$\sim(2-\alpha)^{-1}tb_t^{-\alpha}L(b_t\delta)\delta^{2-\alpha}\sim(2-\alpha)^{-1}\delta^{2-\alpha}\,,\qquad t\to\infty\,,$$

by [**1**] VIII, 9, Theorem 2 (although ν is not a probability measure, the result is true). Therefore

(3.10) $$\lim_{\delta\to 0}\limsup_{t\to\infty}\int_{|x|\leq\delta} x^2\nu_t(dx)=0\,.$$

Similarly

(3.11) $$\lim_{\delta\to 0}\limsup_{t\to\infty}\int_{b_t^{-1}<|x|\leq\delta}|x|\nu_t(dx)=0\,,$$

since

$$\int_{b_t^{-1}<|x|\leq\delta}|x|\nu_t(dx)=tb_t^{-1}\int_{1<|x|\leq b_t\delta}|x|\nu(dx)$$
$$\sim(1-\alpha)^{-1}tb_t^{-\alpha}L(b_t\delta)\delta^{1-\alpha}\sim(1-\alpha)^{-1}\delta^{1-\alpha}\,,\qquad t\to\infty\,.$$

It follows that, as $t\to\infty$,

$$S_1\to 0\qquad\text{and}\qquad S_2\to\int_{-\infty}^{\infty}(e^{izx}-1)\chi_{p,1-p}(x)|x|^{-1-\alpha}dx\,.$$

Hence the distribution of Y_t tends to μ, which completes the proof.

THEOREM 3.2. *If* $1<\alpha<2$, *then*

(3.12) $$b_t^{-1}(a(t)-mt)\to a\,,\qquad t\to\infty\,,$$

where

(3.13) $$m=EX_1=\gamma+\int_{-\infty}^{\infty}x^3(1+x^2)^{-1}\nu(dx)$$

and a *is the unique mode of the stable distribution* μ *with index* α *characterized by*

(3.14) $$\hat{\mu}(z)=\exp\int_{-\infty}^{\infty}(e^{izx}-1-izx)\chi_{p,1-p}(x)|x|^{-1-\alpha}dx\,.$$

PROOF. Let $Y_t=b_t^{-1}(X_t-mt)$. It suffices to prove convergence of the distribution of Y_t to μ. We have

$$E\exp(izX_t)=\exp\left[t\left(imz-2^{-1}\sigma^2z^2+\int(e^{izx}-1-izx)\nu(dx)\right)\right].$$

Hence

$$E\exp(izY_t)=\exp\left[-2^{-1}tb_t^{-2}\sigma^2z^2+\int(e^{izx}-1-izx)\nu_t(dx)\right],$$

where ν_t is given by (3.7). We get (3.8), (3.9), (3.10), and $tb_t^{-2}\to 0$. Also we have

$$\lim_{c\to\infty}\limsup_{t\to\infty}\int_{|x|>c}|x|\nu_t(dx)=0\,,$$

noting that $\int_{|y|\leq x}y^2\nu(dy)$ is regularly varying with exponent $2-\alpha$ and thus observing

$$\int_{|x|>c}|x|\nu_t(dx)=tb_t^{-1}\int_{|x|>b_tc}|x|\nu(dx)$$
$$\sim(\alpha-1)^{-1}tb_t^{-\alpha}L(b_tc)c^{1-\alpha}\sim(\alpha-1)^{-1}c^{1-\alpha}\,,\qquad t\to\infty\,,$$

in application of [1] VIII, 9, Theorem 2, to the measure ν. These together imply $E\exp(izY_t)\to\hat{\mu}(z)$. The proof is complete.

In the following we use functions $L_*(x)$ and $L^*(x)$ defined by

$$L_*(x)=\int_1^x y^{-1}L(y)dy\,,\qquad L^*(x)=\int_x^\infty y^{-1}L(y)dy\,.$$

By [1] VIII, 9, Theorem 1, the function $L_*(x)$ is slowly varying at infinity.

THEOREM 3.3. *Suppose that* $\alpha=1$. *Define*

$$c_t^{(1)}=\int_0^\infty x((1+b_t^{-2}x^2)^{-1}-(1+x^2)^{-1})\nu(dx)\,,$$
$$c_t^{(2)}=\int_{-\infty}^0 |x|((1+b_t^{-2}x^2)^{-1}-(1+x^2)^{-1})\nu(dx)\,,$$
$$c_t=\gamma+c_t^{(1)}-c_t^{(2)}\,.$$

Let μ *be the stable distribution with index* 1 *characterized by*

$$\hat{\mu}(z)=\exp\int_{-\infty}^\infty g(z,\,x)\chi_{p,1-p}(x)x^{-2}dx\,. \tag{3.15}$$

Let a *be the unique mode of* μ. *Then,*

(i) $$b_t^{-1}(a(t)-c_tt)\to a\,,\qquad t\to\infty\,.$$

(ii) *Suppose that* $L_*(\infty)=\infty$. *Then* $E|X_1|=\infty$, *and*

$$c_t^{(1)}/L_*(b_t)\to p\,,\qquad c_t^{(2)}/L_*(b_t)\to 1-p\qquad \text{as } t\to\infty\,. \tag{3.16}$$

Moreover,

$$c_t\sim(2p-1)L_*(b_t)\qquad \text{if } p\neq 2^{-1}\,. \tag{3.17}$$

(iii) *Suppose that* $L_*(\infty)<\infty$. *Then,* $E|X_1|$ *is finite,* (3.13) *holds,* $L^*(x)$ *is slowly varying at infinity and*

$$(c_t^{(1)}-m^{(1)})/L^*(b_t)\to -p,\quad (c_t^{(2)}-m^{(2)})/L^*(b_t)\to -(1-p)\quad \text{as } t\to\infty\,, \tag{3.18}$$

where

$$m^{(1)}=\int_0^\infty x^3(1+x^2)^{-1}\nu(dx)\,,\qquad m^{(2)}=\int_{-\infty}^0|x|^3(1+x^2)^{-1}\nu(dx)\,.$$

Moreover,

$$c_t-m\sim-(2p-1)L^*(b_t)\qquad \text{if } p\neq 2^{-1}\,. \tag{3.19}$$

LEMMA 3.1. *Let $u(x)$ be positive, non-increasing, and let*

$$U(x)=\int_x^\infty u(y)dy<\infty .$$

If

$$U(x)\sim x^{-\rho}L(x), \qquad x\to\infty,$$

where $\rho>0$ and $L(x)$ is slowly varying, then

$$u(x)\sim \rho x^{-1}U(x), \qquad x\to\infty .$$

Proof is similar to that of [1] XIII, 5, Lemma.

PROOF OF THEOREM 3.3. Let $Y_t=b_t^{-1}(X_t-c_t t)$. Then

$$E\exp(izY_t)=\exp\left[-2^{-1}\sigma^2 t b_t^{-2}z^2+\int g(z, x)\nu_t(dx)\right]$$

with ν_t defined by (3.7). We get (3.8), (3.9), (3.10), and $tb_t^{-2}\to 0$ again. Hence $E\exp(izY_t)$ tends to $\hat{\mu}(z)$. Thus we have (i) by Lemma 2.1.

We have $L_*(\infty)=\infty$ if and only if $\int_{|x|>1}|x|\nu(dx)=\infty$. Hence, $L_*(\infty)=\infty$ is equivalent to $E|X_1|=\infty$. Now it is clear that (3.13) holds if $L_*(\infty)<\infty$. Suppose that $p\neq 0$. Let $l(x)=xk(x)$. Using Lemma 3.1, we get

$$x^{-1}k(x)\sim px^{-2}L(x), \qquad x\to\infty ,$$

from the first relation in (3.8). Hence $l(x)$ is slowly varying, $l(x)\sim pL(x)$, as $x\to\infty$. Let

$$h(s)=\int_0^\infty((1+x^2)^{-1}-(s+x^2)^{-1})xl(x)dx , \qquad s>0 .$$

We have $c_t^{(1)}=h(b_t^2)$ and

$$h'(s)=\int_0^\infty(s+x^2)^{-2}xl(x)dx=s^{-1}\int_0^\infty y(1+y^2)^{-2}l(s^{1/2}y)dy .$$

We claim that

$$h'(s)\sim 2^{-1}ps^{-1}L(s^{1/2}), \qquad s\to\infty . \tag{3.20}$$

By [7] Theorem 2.6,

$$\int_1^\infty y(1+y^2)^{-2}l(s^{1/2}y)dy\sim l(s^{1/2})\int_1^\infty y(1+y^2)^{-2}dy\sim 4^{-1}pL(s^{1/2}) .$$

Let $\tilde{l}(x)=l(x)$ for $x\geqq 1$ and $\tilde{l}(x)=0$ for $0<x<1$. We have

$$\int_{s^{-1/2}}^1 y(1+y^2)^{-2}l(s^{1/2}y)dy=\int_0^1 y(1+y^2)^{-2}\tilde{l}(s^{1/2}y)dy$$

$$\sim\tilde{l}(s^{1/2})\int_0^1 y(1+y^2)^{-2}dy\sim 4^{-1}pL(s^{1/2})$$

by [7] Theorem 2.7. Moreover

$$\int_0^{s^{-1/2}} y(1+y^2)^{-2}l(s^{1/2}y)dy \leqq \int_0^{s^{-1/2}} yl(s^{1/2}y)dy = s^{-1}\int_0^1 x^2 k(x)dx\,.$$

Hence (3.20) obtains.

Now the proof of (ii) is as follows. We have

$$\int_1^\infty s^{-1}L(s^{1/2})ds = 2\int_1^\infty x^{-1}L(x)dx = \infty\,.$$

Therefore, if $p\neq 0$, then (3.20) implies

$$\int_1^s h'(r)dr \sim 2^{-1}p\int_1^s r^{-1}L(r^{1/2})dr = pL_*(s^{1/2})\,, \qquad s\to\infty\,,$$

$$c_t^{(1)} = h(b_t^2) = \int_1^{b_t^2} h'(r)dr \sim pL_*(b_t)\,, \qquad t\to\infty\,.$$

Analogously, if $p\neq 1$, then

$$c_t^{(2)} \sim (1-p)L_*(b_t)\,.$$

If $p=0$, then, letting

$$k_\varepsilon(x) = k(x)+\varepsilon|k(-x)|\,,$$

$$c_t^{(1,\varepsilon)} = \int_0^\infty((1+b_t^{-2}x^2)^{-1}-(1+x^2)^{-1})k_\varepsilon(x)dx$$

for a fixed $\varepsilon>0$, we have, as $x\to\infty$,

$$\int_x^\infty y^{-1}k_\varepsilon(y)dy\Big/(x^{-1}L(x)) \to \varepsilon\,,$$

$$x^{-1}k_\varepsilon(x) \sim \varepsilon x^{-2}L(x)\,,$$

and hence

$$c_t^{(1,\varepsilon)} \sim \varepsilon L_*(b_t)\,,$$

which, combined with $c_t^{(2)}\sim L_*(b_t)$ and $c_t^{(1)}=c_t^{(1,\varepsilon)}-\varepsilon c_t^{(2)}$, shows that $c_t^{(1)}/L_*(b_t)\to 0$. The proof of the second relation in (3.16) for $p=1$ is analogous. (3.17) is an obvious consequence of (3.16).

In order to prove (iii), we note that, if $p\neq 0$, then it follows from (3.20) that

$$\int_s^\infty h'(r)dr \sim 2^{-1}p\int_s^\infty r^{-1}L(r^{1/2})dr = pL^*(s^{1/2})\,, \qquad s\to\infty\,,$$

under the condition that

$$\int_1^\infty s^{-1}L(s^{1/2})ds = 2\int_1^\infty x^{-1}L(x)dx < \infty\,.$$

Hence

$$c_t^{(1)}-m^{(1)} = h(b_t^2)-\int_0^\infty x^2(1+x^2)^{-1}k(x)dx = h(b_t^2)-h(\infty) \sim pL^*(b_t)$$

if $p\neq 0$. In an analogous manner, we have

$$c_t^{(2)}-m^{(2)}\sim-(1-p)L^*(b_t)$$

if $p\neq 1$. Extension of the first relation of (3.18) to $p=0$ and the second relation to $p=1$ is similar to (ii). (3.19) follows directly from (3.18). The proof of Theorem 3.3 is complete.

4. Miscellaneous notes.

We gather behaviors of modes other than given in the preceding sections and the paper [5]. Let X_t, $0\leqq t<\infty$, be a process with homogeneous independent increments and $X_0=0$. Let

$$E\exp(izX_t)=\exp\Big[t\Big(i\gamma z-2^{-1}\sigma^2z^2+\int_{-\infty}^{\infty}g(z,\ x)\nu(dx)\Big)\Big], \tag{4.1}$$

where γ real, $\sigma^2\geqq 0$, $\nu(\{0\})=0$, $\int_{|x|<1}x^2\nu(dx)+\int_{|x|\geqq 1}\nu(dx)<\infty$ and $g(z,\ x)$ is given by (1.2). Assume that, for every large t, the distribution of X_t is unimodal with a mode $a(t)$.

1. If

$$\int_{|x|>t}\nu(dx)=o(t^{-1}),\qquad t\rightarrow\infty, \tag{4.2}$$

then

$$a(t)=t\Big(\gamma+\int_{|x|\leqq t}x^3(1+x^2)^{-1}\nu(dx)\Big)+o(t),\qquad t\rightarrow\infty. \tag{4.3}$$

This is a consequence of an analogue of the generalized weak law of large numbers for sums of i.i.d. random variables ([1] VII, 7). The case of finite mean ($a(t)\sim mt$ if $EX_1=m$) is a special case (cf. [5]). To see (4.3), let

$$b_t=t\Big(\gamma+\int_{|x|\leqq t}x^3(1+x^2)^{-1}\nu(dx)\Big)\qquad\text{and}\qquad Y_t=t^{-1}(X_t-b_t).$$

Then

$$E\exp(izY_t)=\exp(-2^{-1}t^{-1}\sigma^2z^2+S_1+S_2),$$

$$S_1=t\int_{|x|\leqq t}(e^{it^{-1}zx}-1-it^{-1}zx)\nu(dx),$$

$$S_2=t\int_{|x|>t}(e^{it^{-1}zx}-1-it^{-1}zx(1+x^2)^{-1})\nu(dx).$$

We have $S_2=o(1)$ by (4.2), and

$$|S_1|\leqq 2^{-1}t^{-1}z^2\int_{|x|\leqq t}x^2\nu(dx)=O(t^{-1})+O\Big(t^{-1}\int_{1<|x|\leqq t}x^2\nu(dx)\Big).$$

Let $N(t)=\int_{|x|>t}\nu(dx)$. Then

$$t^{-1}\int_{1<|x|\leq t}x^2\nu(dx) = -t^{-1}\int_1^t x^2 dN(x)$$
$$\leqq t^{-1}N(1)+2t^{-1}\int_1^t xN(x)dx \to 0$$

by (4.2). Thus $E\exp(izY_t)\to 1$, that is, Y_t converges to 0 in distribution. Hence we have (4.3) by Lemma 2.1.

2. Assume that $E|X_t|=\infty$. Let $b(t)$ be a right-continuous function such that $t^{-1}b(t)$ is non-decreasing and goes to infinity with t. Let $g(x)=\sup\{t: b(t)\leqq x\}$, the inverse function of $b(t)$. If

$$\int_{|x|>x_0} g(|x|)\nu(dx) < \infty \tag{4.4}$$

for some $x_0>0$, then

$$a(t) = o(b(t)), \qquad t\to\infty. \tag{4.5}$$

The case $g(|x|)=|x|^p$, $0<p<1$, is treated in [5] Theorem 3.3.

In fact, define a measure $\tilde{\nu}$ on $(0, \infty)$ by $\tilde{\nu}(x_1, x_2]=\nu(x_1, x_2]+\nu[-x_2, -x_1)$. Since $b(t)$ is strictly increasing, $g(x)$ is continuous and

$$\int_{g(x_0)}^{\infty}\tilde{\nu}[b(t), \infty)dt = \int_{x_0}^{\infty}\tilde{\nu}[x, \infty)dg(x).$$

The right-hand side is finite if and only if (4.4) holds. Hence (4.4) implies $b(t)^{-1}X_t\to 0$ a.s. as $t\to\infty$ if X_t is an increasing process, by Fristedt's result [**2**]. By splitting the Lévy measure ν into three parts, we can represent X_t as the sum of three independent processes $X_t^{(j)}$, $j=1, 2, 3$, each with homogeneous independent increments, such that their Lévy measures are supported on $[1, \infty)$, $(-\infty, -1]$, and $[-1, 1]$, respectively, and $X_t^{(1)}$ and $X_t^{(2)}$ are without Gaussian components. Since $b(t)^{-1}X_t^{(1)}$ and $b(t)^{-1}X_t^{(2)}$ tend to 0 a.s. and $t^{-1}X_t^{(3)}$ tends to a finite constant, we get $b(t)^{-1}X_t\to 0$ a.s. Thus (4.5) follows.

3. Suppose that $EX_1=0$, $EX_1^2=1$, and that there are $\alpha\in[2, 3]$, $p\in[0, 1/2)\cup(1/2, 1]$, and a slowly varying function $L(x)$ such that

$$\nu[x, \infty)/(x^{-\alpha}L(x))\to p, \quad \nu(-\infty, -x]/(x^{-\alpha}L(x))\to 1-p \quad \text{as } x\to\infty. \tag{4.6}$$

Let ν_{ac} be the absolutely continuous part of ν. Suppose that there are $c>0$ and $\delta>0$ such that

$$-\frac{d}{dx}(\nu_{\mathrm{ac}}[x, \infty)+\nu_{\mathrm{ac}}(-\infty, -x])\geqq\frac{c}{x} \quad \text{a.e. on } (0, \delta). \tag{4.7}$$

If $2\leqq\alpha<3$, then we have

(4.8) $$a(t)\sim -C_{\alpha}(2p-1)t^{(3-\alpha)/2}L(t^{1/2})\,,\qquad t\to\infty\,,$$

where

$$C_{\alpha}=-2^{(\alpha+1)/2}\pi^{-1/2}\Gamma(1-\alpha)\Gamma(1+2^{-1}\alpha)\sin(2^{-1}\alpha\pi)>0\qquad\text{for } 2<\alpha<3\,,$$

$$C_2=(2\pi)^{1/2}\,.$$

If $\alpha=3$ and $\int_1^{\infty}x^{-1}L(x)dx=\infty$, then

(4.9) $$a(t)\sim -\frac{3}{2}(2p-1)\int_1^{t^{1/2}}x^{-1}L(x)dx\,,\qquad t\to\infty\,.$$

This is a continuous parameter analogue of Corollary 1 of Hall [**4**].

Proof is as follows. Let μ be the distribution of X_1. The condition (4.7) implies that the symmetrization of μ has a component of class L characterized by $\exp\left[c\int_{-\delta}^{\delta}(e^{izx}-1)x^{-1}dx\right]$. Hence, for every $\varepsilon>0$, $|\hat{\mu}(z)|^2=o(|z|^{-2c+\varepsilon})$ as $|z|\to\infty$ by [**6**] Lemma 2.4. In proving an analogue of [**4**] Corollary 1, this property replaces the condition that, for every large t, the distribution of X_t has a density $f_t(x)$ such that $f_t'(x)$ exists and is integrable. Suppose, for some while, that $p\neq 0, 1$. By [**10**], (4.6) implies that

(4.10) $$\mu[x,\infty)/(x^{-\alpha}L(x))\to p\,,\qquad \mu(-\infty,-x]/(x^{-\alpha}L(x))\to 1-p$$

as $x\to\infty$. Now we can follow the proof in [**4**] to get

$$a(t)\sim C_{\alpha}(p^{-1}-2)t^{3/2}\mu[t^{1/2},\infty)\qquad\text{if } 2\leqq\alpha<3\,,$$

$$a(t)\sim\frac{3}{2}(p^{-1}-2)\int_0^{t^{1/2}}x^2\mu[x,\infty)dx\qquad\text{if } \alpha=3 \text{ and } E|X_1|^3=\infty\,.$$

Hence (4.8) and (4.9) are obtained. Note that $\int_1^{\infty}x^{-1}L(x)dx=\infty$ and $E|X_1|^3=\infty$ are equivalent. If $p=1$, then the first relation of (4.10) holds by [**10**], and the second relation is also true. In fact, if $p=1$, we define μ_{ε} by

$$\hat{\mu}_{\varepsilon}(z)=\hat{\mu}(z)\exp\left[\varepsilon\int_1^{\infty}(e^{-ixz}-1)\nu(dx)\right]$$

for $\varepsilon>0$, and see that the Lévy measure ν_{ε} of μ_{ε} satisfies $\nu_{\varepsilon}(-\infty,-x]/(x^{-\alpha}L(x))\to\varepsilon$ as $x\to\infty$ and hence $\mu_{\varepsilon}(-\infty,-x]/(x^{-\alpha}L(x))\to\varepsilon$ by [**10**], from which follows $\mu(-\infty,-x]/(x^{-\alpha}L(x))\to 0$ since $\mu(-\infty,-x]\leqq\mu_{\varepsilon}(-\infty,-x]$. Thus, in the same way, we get (4.8) and (4.9) in case $p=1$, too. The case $p=0$ is transformed to the case $p=1$ by reflection.

4. If $E|X_1|^n<\infty$, then we can see from (4.1) that the j-th order cumulant

κ_j of X_1 is as follows:

$$\kappa_1 = \gamma + \int x^3(1+x^2)^{-1}\nu(dx) = EX_1 ,$$

$$\kappa_2 = \sigma^2 + \int x^2\nu(dx) = \mathrm{Var}\, X_1 ,$$

$$\kappa_j = \int x^j\nu(dx) \qquad \text{for } 3 \leqq j \leqq n .$$

If $EX_1=0$, $EX_1^2=1$, and $E|X_1|^3<\infty$, then

$$a(t) \to -2^{-1}\kappa_3 = -2^{-1}EX_1^3 , \qquad t\to\infty .$$

If, moreover, $E|X_1|^{3+\delta}<\infty$ for some $\delta\in(0, 2]$, then

$$a(t) = -2^{-1}\kappa_3+O(t^{-\delta/2}) , \qquad t\to\infty .$$

In case $\delta=1$, the term $O(t^{-\delta/2})$ can be replaced by $o(t^{-1/2})$. In case $\delta=2$, the following more precise asymptotic obtains:

$$a(t) = -\frac{1}{2}\kappa_3+\frac{1}{t}\Big(\frac{1}{8}\kappa_5-\frac{5}{12}\kappa_3\kappa_4+\frac{1}{4}\kappa_3^3\Big)+o\Big(\frac{1}{t}\Big) .$$

These are analogues of Corollary 2 and Theorem 3 of Hall [4].

5. For stable processes we get the whole behavior of the mode $a(t)$, $0\leqq t<\infty$. Let X_t be stable with index $0<\alpha<2$. That is, $\sigma=0$ and $\nu(dx)=\chi_{c_1,c_2}(x)|x|^{-1-\alpha}dx$, $c_1\geqq 0$, $c_2\geqq 0$, $c_1+c_2>0$, where $\chi_{c_1,c_2}(x)$ is given in (3.4). Then

$$E\exp(izX_t) = \exp[t\lambda(i\delta z-|z|^\alpha+i\beta(\tan 2^{-1}\alpha\pi)z|z|^{\alpha-1})] \qquad \text{for } \alpha\neq 1 ,$$

$$E\exp(izX_t) = \exp[t\lambda(i\delta z-|z|-i\beta 2\pi^{-1}z\log|z|)] \qquad \text{for } \alpha=1 ,$$

where $\lambda=-\Gamma(-\alpha)(c_1+c_2)\cos 2^{-1}\pi\alpha>0$ (for $\alpha\neq 1$) or $\lambda=2^{-1}\pi(c_1+c_2)>0$ (for $\alpha=1$), δ real, and $\beta=(c_1-c_2)/(c_1+c_2)$. By Yamazato [9], the distribution is unimodal for each t. By [6], its mode is unique for each t and denoted by $a(t)$. Assume that $\lambda=1$. Let us denote by $a_0(t)$ the mode of the process obtained by letting $\delta=0$. Then it is easy to see that

$$a(t) = t^{1/\alpha}a_0(1)+t\delta \qquad \text{for } \alpha\neq 1 , \tag{4.11}$$

$$a(t) = ta_0(1)+2\pi^{-1}\beta t\log t+t\delta \qquad \text{for } \alpha=1 . \tag{4.12}$$

Zolotarev [11] shows that

$$\operatorname{sgn} a_0(1) = \begin{cases} \operatorname{sgn}\beta & \text{for } 0<\alpha<1 \\ -\operatorname{sgn}\beta & \text{for } 1<\alpha<2 . \end{cases}$$

The expressions (4.11) and (4.12) show that $a(t)$ is either convex for $0\leqq t<\infty$ or concave for $0\leqq t<\infty$. But it should be emphasized that $a(t)$ is not always monotone as a function of t. In case $\alpha \neq 1$ and $\beta \neq 0$, the non-monotonicity of $a(t)$ comes out by the influence of the drift term δ; it happens when and only when $\operatorname{sgn} a_0(t) = -\operatorname{sgn} \delta$. In case $\alpha = 1$, however, non-monotonicity of $a(t)$ is of intrinsic character; it always occurs so long as $\beta \neq 0$. Some results in case $\alpha = 1$ are given in [**5**].

ACKNOWLEDGMENT. The assertion (ii) of Theorem 2.2 was originally restricted to the case of Example 2.1. The author owes to M. Yamazato its improvements to the present form. For that the author is grateful to him.

References

[1] W. Feller, An introduction to probability theory and its applications, Vol. II, Second edition, Wiley, New York, 1971.

[2] B. E. Fristedt, Sample function behavior of increasing processes with stationary, independent increments, Pacific J. Math., **21** (1967), 21-33.

[3] B. V. Gnedenko and A. N. Kolmogorov, Limit distributions for sums of independent random variables (English translation), Second edition, Addison-Wesley, Reading, 1968.

[4] P. Hall, On the limiting behaviour of the mode and median of a sum of independent random variables, Ann. Probab., **8** (1980), 419-430.

[5] K. Sato, Bounds of modes and unimodal processes with independent increments, Nagoya Math. J., **104** (1986), 29-42.

[6] K. Sato and M. Yamazato, On distribution functions of class L, Z. Wahrsch. Verw. Gebiete, **43** (1978), 273-308.

[7] E. Seneta, Regularly varying functions, Lecture Notes in Math., **508**, Springer, 1976.

[8] S. J. Wolfe, Inequalities for modes of L functions, Ann. Math. Statist., **42** (1971), 2126-2130.

[9] M. Yamazato, Unimodality of infinitely divisible distribution functions of class L, Ann. Probab., **6** (1978), 523-531.

[10] V. M. Zolotarev, On the asymptotic behaviour of a class of infinitely divisible laws, Theory Probab. Appl., **6** (1961), 304-307.

[11] V. M. Zolotarev, One-dimensional stable distributions (in Russian), Nauka, Moscow, 1983.

Ken-iti SATO
Department of Mathematics
College of General Education
Nagoya University
Nagoya 464
Japan