

Two intrinsic pseudo-metrics with pseudoconvex indicatrices and starlike circular domains

Dedicated to Professor Tadashi Kuroda on his 60th birthday

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Introduction.

It is well-known ([7]) that at each point p of a complex manifold M the indicatrix of the *Carathéodory pseudo-metric* C^M of M is always a convex circular domain in the holomorphic tangent space $T_p(M)$ at p . By using a family of bounded plurisubharmonic functions on a complex manifold M , Sibony [19] also defined a biholomorphically invariant pseudo-metric on M whose indicatrices are always convex. On the other hand, Suzuki [21] and Barth [4] showed, independently, that if M is a pseudoconvex starlike circular domain with center at the origin 0 in \mathbf{C}^m , an m -dimensional complex Euclidian space, then the indicatrix of the *Kobayashi pseudo-metric* K^M of M at 0 coincides with M , the tangent space $T_0(M)$ being identified with \mathbf{C}^m in the natural manner. In particular, the indicatrix of K^M at 0 is pseudoconvex. It seems that indicatrices of the Kobayashi pseudo-metric K^M for a complex manifold M are, in general, not necessarily pseudoconvex, for the proofs of the result of Suzuki and Barth mentioned above essentially depend on the pseudoconvexity of the original domain. The main purpose of this paper is to construct, intrinsically, two biholomorphically invariant pseudo-metrics, denoted by B^M and P^M , on a complex manifold M , for each of which the indicatrix at each point is always pseudoconvex in the tangent space.

The paper is organized as follows. In §1, we recall some results concerning starlike circular domains and prove that if M is a pseudoconvex bounded starlike circular domain in \mathbf{C}^m with a continuous boundary, then the Bergman metric of M is complete (Theorem 1.11). Since the boundary of a pseudoconvex Reinhardt domain is continuous, Theorem 1.11 is a generalization of the following theorem of Skwarczyński [20; Theorem 3.16]: The Bergman metric of a pseudoconvex bounded Reinhardt domain in \mathbf{C}^m is complete. Recently, the theorem of Skwarczyński was also improved by Pflug [16]. He proved that the Carathéodory metric of a pseudoconvex bounded Reinhardt domain in \mathbf{C}^m is

complete.

The pseudo-metrics B^M and P^M mentioned above are constructed in §§ 2 and 3, respectively. The pseudo-metric B^M is defined on an m -dimensional complex manifold M by using a family of square-integrable holomorphic m -forms on M . The pseudo-metric B^M does not coincide with the square-root of the Bergman pseudo-metric g^M . However, it has the following two properties similar to $(g^M)^{1/2}$: $C^M \leq B^M$ on the tangent bundle $T(M)$ of M and, if M' is a domain in M , then $B^M \leq B^{M'}$ on $T(M')$ (Proposition 2.4). The pseudo-metric P^M is defined on a complex manifold M by using a family of negative plurisubharmonic functions on M . The construction is similar to that of the pseudo-metric defined by Sibony mentioned above. The pseudo-metric P^M possesses the distance-decreasing property for holomorphic mappings and it holds that $C^M \leq P^M \leq K^M$ on $T(M)$ for every complex manifold M (Proposition 3.9).

In the last section, we investigate the structures of B^M and P^M at the center 0 of a starlike circular domain M in \mathbb{C}^m . For a pseudo-metric F on M , let $F_0 = F|_{T_0(M)}$, where the tangent space $T_0(M)$ at 0 is identified with \mathbb{C}^m in the natural manner. We first note that Schwarz' lemma formulated by Sadullaev [18] can be improved as follows: If Φ is a holomorphic mapping of the unit disk U in \mathbb{C} into M and if $\Phi(0) = 0$, then $P_0^M(\Phi(\lambda)) \leq |\lambda|$ for $\lambda \in U$ and $P_0^M(\Phi'(0)) \leq 1$ (Proposition 4.2). It is well-known ([7]) that the indicatrix of C_0^M coincides with the convex hull of M . In connection with this, we show that the indicatrix of P_0^M is exactly the holomorphic hull of M (Theorem 4.3). On the other hand, the indicatrix of B_0^M is characterized as the domain of convergence of the Bergman kernel of M (Theorem 4.6). As a result, we have $C_0^M \leq B_0^M \leq P_0^M$ (Theorem 4.8). Finally, we show that $B_0^M = P_0^M$ for a bounded complete Reinhardt domain M (Corollary 4.11).

NOTATIONS. Throughout this paper, we regard the function identically equal to $-\infty$ as a plurisubharmonic function.

A pseudo-metric on a complex manifold M is by definition a non-negative real-valued function F on the holomorphic tangent bundle $T(M)$ of M satisfying $F(\lambda X) = |\lambda|F(X)$ for $X \in T(M)$ and $\lambda \in \mathbb{C}$. If, in particular, $F(X) = 0$ implies $X = 0$, then F is called a metric. Let $M \rightarrow F^M$ be an assignment from a complex manifold M to a pseudo-metric F^M on M . The assignment F^M is said to be biholomorphically invariant or invariant, if $\Phi^*F^{M'} = F^M$ on $T(M)$ for any biholomorphic mapping $\Phi \in \text{Bihol}(M, M')$ from a complex manifold M onto another M' . Here, $\Phi^*F^{M'}(X) = F^{M'}(\Phi_*X)$ for $X \in T(M)$. We say the assignment F^M to be distance-decreasing for holomorphic mappings or to possess the decreasing property, if $\Phi^*F^{M'} \leq F^M$ on $T(M)$ for any holomorphic mapping $\Phi \in \text{Hol}(M, M')$ from a complex manifold M into another M' . If F^M possesses the decreasing property, then F^M is invariant. Typical examples of pseudo-metrics with the decreasing

property are the *Carathéodory pseudo-metric* C^M on M defined by

$$(0.1) \quad C^M(X) = \sup \{ \rho(f_*X) ; f \in \text{Hol}(M, U) \},$$

and the *Kobayashi pseudo-metric* K^M on M defined by

$$(0.2) \quad K^M(X) = \inf \{ \rho(Y) ; Y \in T(U), f \in \text{Hol}(U, M) \text{ with } f_*Y = X \}.$$

Here, $X \in T(M)$ and ρ is the *Poincaré metric* on the unit disk $U = \{ \lambda \in \mathbf{C} ; |\lambda| < 1 \}$ in \mathbf{C} , that is,

$$(0.3) \quad \rho\left(\xi \frac{d}{d\lambda}\right) = \frac{|\xi|}{1 - |\lambda|^2}, \quad (\lambda, \xi) \in U \times \mathbf{C}.$$

Schwarz' lemma implies that $C^M \leq K^M$ on $T(M)$ for any M and that $C^U = K^U = \rho$. Furthermore, the following fact is immediate from the definition (cf. [11]):

(0.4) If an assignment F^M of pseudo-metrics possesses the decreasing property with $F^U = \rho$, then $C^M \leq F^M \leq K^M$ on $T(M)$ for any complex manifold M .

When we refer to the complex Euclidian space \mathbf{C}^m , $\|u\|$ always means the Euclidian norm $(\sum_{i=1}^m |u^i|^2)^{1/2}$ of $u = (u^1, \dots, u^m) \in \mathbf{C}^m$.

§ 1. Starlike circular domains.

The interior of the indicatrix of a pseudo-metric on a complex manifold is always a starlike circular domain in the holomorphic tangent space. Following Barth [4] and [1] and using the terminology in [4], we recall some results concerning such domains.

Let V be a complex vector space of finite dimension endowed with the structure of a metric space defined by a norm. A domain M , a non-empty connected open subset, in V is called *starlike circular*, if $\lambda M \subset M$ for any $\lambda \in \mathbf{C}$ with $|\lambda| = 1$. We denote by $\mathcal{C}(V)$ or \mathcal{C} the totality of starlike circular domains in V . A non-negative real-valued function N on V is called a *semi-gauge*, if N is upper semi-continuous and satisfies $N(\lambda v) = |\lambda| N(v)$ for $\lambda \in \mathbf{C}$ and $v \in V$. We denote by $\mathcal{G}(V)$ or \mathcal{G} the totality of semi-gauges on V . For every $M \in \mathcal{C}$, the function defined by

$$N^M(v) = \inf \{ \lambda > 0 ; v \in \lambda M \} \quad (v \in V)$$

is a semi-gauge on V and is called the *semi-gauge defining* M . On the other hand, for every $N \in \mathcal{G}$, the *indicatrix* IN of N is defined by

$$IN = \{ v \in V ; N(v) < 1 \};$$

clearly $IN \in \mathcal{C}$. Two mappings $\mathcal{C} \ni M \mapsto N^M \in \mathcal{G}$ and $\mathcal{G} \ni N \mapsto IN \in \mathcal{C}$ are mutually

inverse and order-reversing, i. e., $M_1 \subset M_2$ if and only if $N^{M_1} \geq N^{M_2}$. We also note that $N^{\lambda M} = \lambda^{-1} N^M$ for $\lambda > 0$. Every $N \in \mathcal{G}$ is continuous at the origin 0. In fact, there exists a norm $\|\cdot\|$ on V such that $N \leq \|\cdot\|$. If there exists another norm $\|\cdot\|'$ on V such that $N \geq \|\cdot\|'$, then we call N a *gauge*. When $V = \mathbf{C}^m$, $M \in \mathcal{C}$ is called a *Reinhardt domain* in \mathbf{C}^m if $(e^{i\theta^1}u^1, \dots, e^{i\theta^m}u^m) \in M$ for any $(u^1, \dots, u^m) \in M$ and $(\theta^1, \dots, \theta^m) \in \mathbf{R}^m$; furthermore, M is called a *complete Reinhardt domain* in \mathbf{C}^m if $(u^1, \dots, u^m) \in M$ and $|v^a| \leq |u^a|$ ($a=1, \dots, m$) imply $(v^1, \dots, v^m) \in M$.

The following is easily shown by definition.

LEMMA 1.1. *Let $M \in \mathcal{C}(V)$ with $N = N^M$.*

- (i) *M is bounded if and only if N is a gauge.*
- (ii) *M is convex if and only if N is a seminorm.*
- (iii) *When $V = \mathbf{C}^m$, M is a Reinhardt domain if and only if $|u^a| = |v^a|$ ($a=1, \dots, m$) imply $N(u^1, \dots, u^m) = N(v^1, \dots, v^m)$.*
- (iv) *When $V = \mathbf{C}^m$, M is a complete Reinhardt domain if and only if $|u^a| \leq |v^a|$ ($a=1, \dots, m$) imply $N(u^1, \dots, u^m) \leq N(v^1, \dots, v^m)$.*

In the following proposition, the assertion (a) \Leftrightarrow (b₁) is proved in Barth [4; Theorem 1, (c)] and (a) \Leftrightarrow (b₂) in [1; Theorem 5.4].

PROPOSITION 1.2. *For $M \in \mathcal{C}(V)$ with $N = N^M$, the following three statements are mutually equivalent:*

- (a) *M is pseudoconvex.*
- (b₁) *N is plurisubharmonic.*
- (b₂) *$\log N$ is plurisubharmonic.*

Here, $\log N$ may be $-\infty$ identically.

Proposition 1.2 yields a byproduct which has its own interest.

COROLLARY 1.3. *Assume that N is a non-negative real-valued function on V and satisfies $N(\lambda v) = |\lambda| N(v)$ for $\lambda \in \mathbf{C}$ and $v \in V$. Then, N is plurisubharmonic if and only if $\log N$ is plurisubharmonic.*

Corollary 1.3 is also proved directly (without use of the pseudoconvexity of the corresponding domain) as follows: We may assume $V = \mathbf{C}^m$. "If" part of the assertion follows from Jensen's inequality. To prove the converse, suppose that N is plurisubharmonic on \mathbf{C}^m . Set

$$f(v) = N(e^{v^1+v^m}, \dots, e^{v^{m-1}+v^m}, e^{-v^1-\dots-v^{m-1}+v^m})$$

for $v = (v^1, \dots, v^m) \in \mathbf{C}^m$, where $e(\xi) = \exp \xi$. Then, by the assumption on N we have

$$(1.1) \quad f(v) = e(\operatorname{Re} v^m) g(v^1, \dots, v^{m-1}),$$

where $g(v^1, \dots, v^{m-1}) = N(e(v^1), \dots, e(v^{m-1}), e(-v^1 - \dots - v^{m-1}))$. Since N is plurisubharmonic on \mathbb{C}^m , so is f . We want to show that $\log f$ is plurisubharmonic. To prove this, we may assume that f is of class C^2 , because f is approximated from above by smooth plurisubharmonic functions (cf., e.g., [22; §10.9]). In this case, we can easily show the plurisubharmonicity of $\log f$ by using the relation (1.1) and by representing the complex Hessians of f and $\log f$ in terms of g . If these are done, it follows from the definition of f that $\log N$ is plurisubharmonic on $(\mathbb{C} - \{0\})^m$. Since a plurisubharmonic function bounded from above is uniquely extended beyond a principal analytic set ([9; Satz 3]), $\log N$ is plurisubharmonic on the whole \mathbb{C}^m , as desired.

Proposition 1.2 also provides an alternative proof of the following well-known fact which we use later.

COROLLARY 1.4 ([8; Théorèmes 37 et 39]). *The holomorphic hull of a starlike circular domain in V is schlicht and starlike circular.*

PROOF. Let $M \in \mathcal{C}(V)$ and let $f \in \text{Hol}(M)$ be a holomorphic function on M with the homogeneous expansion $\sum_{j=0}^{\infty} f_j$ around 0, where f_j is a homogeneous polynomial of degree j . Set

$$M_f = \text{Int} \left\{ v \in V ; \sum_{j=0}^{\infty} |f_j(v)| < +\infty \right\},$$

$$T_f(v) = \limsup_{u \rightarrow v} \limsup_{j \rightarrow \infty} |f_j(u)|^{1/j}$$

for $v \in V$, where $\text{Int} A$ means the interior of a subset A of V . Since the family $|f_j|^{1/j}$ of plurisubharmonic functions on V is locally uniformly bounded, T_f is also plurisubharmonic ([22; p. 74]). It is easily seen that $T_f \in \mathcal{Q}(V)$ and $T_f = N^{M_f}$. We consider the domain

$$\tilde{M} = \text{Int} \cap \{M_f ; f \in \text{Hol}(M)\}$$

which belongs to $\mathcal{C}(V)$. Every $f \in \text{Hol}(M)$ is then extended to a function in $\text{Hol}(\tilde{M})$ via the homogeneous expansion of f around 0, because $\tilde{M} \subset M_f$. On the other hand, it can be seen that

$$N^{\tilde{M}}(v) = \limsup_{u \rightarrow v} \sup_{f \in \text{Hol}(M)} T_f(u)$$

for $v \in V$ so that $N^{\tilde{M}}$ is plurisubharmonic. By Proposition 1.2, \tilde{M} is pseudoconvex; therefore, \tilde{M} is the holomorphic hull of M .

Now, a domain $M \in \mathcal{C}(V)$ is called *strictly* starlike circular if $\lambda M \supset \text{Cl} M$ for every $\lambda > 1$, where $\text{Cl} M$ means the closure of M in the ambient space V . We shall characterize such a domain M in terms of the semi-gauge N^M defining M . For $M \in \mathcal{C}(V)$ with $N = N^M$, we first note the following facts which, in general, hold and are easily checked:

$$(1.2) \quad M = \text{Int} \{v \in V; N(v) \leq 1\}$$

$$(1.3) \quad \bigcap_{\lambda > 1} \lambda M = \{v \in V; N(v) \leq 1\}$$

$$(1.4) \quad \text{Cl}M \supset \{v \in V; N(v) \leq 1\}.$$

PROPOSITION 1.5 ([1; Proposition 4.1 and its Corollary]). *For $M \in \mathcal{C}(V)$ with $N = N^M$, the following statements are mutually equivalent:*

- (a) *N is continuous.*
- (ab) *The opposite inclusion of (1.4) holds.*
- (b) *M is strictly starlike circular.*

PROOF. Since the condition (b) is equivalent to $\text{Cl}M \subset \bigcap_{\lambda > 1} \lambda M$, the equality (1.3) induces the equivalence of (ab) and (b). The implication (a) \Rightarrow (ab) is trivial. To prove the opposite implication, suppose that (ab) holds, and put

$$\tilde{N}(v) = \liminf_{u \rightarrow v, u \neq v} N(u)$$

for $v \in V$. Fix $v \in V$ and take an arbitrary real number η with $\tilde{N}(v) < \eta$. Considering a sequence (v_n) in V such that $v_n \neq v$, $v_n \rightarrow v$, and $\lim N(v_n) = \tilde{N}(v)$, we see that $v/\eta \in \text{Cl}M$. It follows from (ab) that $N(v) \leq \eta$. Thus we have $\tilde{N}(v) \geq N(v)$ so that the upper semi-continuity of N implies the continuity of N or the assertion (a).

COROLLARY 1.6. *If a domain $M \in \mathcal{C}(V)$ is strictly starlike circular, then M is fat, that is, $\text{Int Cl}M = M$.*

This is easily obtained by applying Proposition 1.5 to the formula (1.2).

COROLLARY 1.7 ([1; Proposition 4.2]). *If $M \in \mathcal{C}(V)$ is convex, then M is strictly starlike circular.*

PROOF. By (ii) in Lemma 1.1, N^M is a seminorm so that it is continuous. It follows from Proposition 1.5 that M is strictly starlike circular.

COROLLARY 1.8 ([1; Proposition 4.3]). *If $M \in \mathcal{C}(\mathbf{C}^m)$ is a complete Reinhardt domain, then M is strictly starlike circular.*

PROOF. Suppose $N(v) > 1$ for a vector $v = (v^1, \dots, v^m)$ and put

$$W = \{(u^1, \dots, u^m) \in \mathbf{C}^m; |u^a| > |v^a|/N(v) \ (a=1, \dots, m)\}.$$

Then W is a neighborhood of v , and $M \cap W = \emptyset$ by Lemma 1.1, (iv). Thus, v belongs to the exterior of M . Hence we see that the condition (ab) in Proposition 1.5 holds.

In the remainder of this section, we give an application of Corollary 1.6 to a problem concerning completeness of the Bergman metric.

Let $hL^2(M)$ be the Hilbert space of all square-integrable holomorphic functions on a bounded domain M in \mathbf{C}^m , and let $k(u, \bar{v})$ be the Bergman kernel of M . The kernel hull $\mathcal{K}(M)$ of M is defined to be the largest, not necessarily schlicht, domain containing M on which the Bergman function $k(v, \bar{v})$ is real-analytically extensible. We recall the following sufficient condition of Kobayashi [10; (A.4), p. 284] for completeness of the Bergman metric:

(1.5) For every infinite sequence S in M which has no adherent point in M and for each $f \in hL^2(M)$ there exists a subsequence (v_n) of S such that $|f(v_n)|^2/k(v_n, v_n)$ converges to 0.

We need the following two lemmas.

LEMMA 1.9 (Pflug [15; Folgerung 4]). *If M is a bounded domain in \mathbf{C}^m and has a schlicht holomorphic hull $\mathcal{H}(M)$, then $\mathcal{H}(M) \subset \mathcal{K}(M) \subset \text{Int Cl } \mathcal{H}(M)$.*

LEMMA 1.10 (Skwarczyński [20; Theorem 3.15]). *Let M be a bounded domain in \mathbf{C}^m . Suppose that for every boundary point v of M , $\lim_{u \rightarrow v, u \in M} k(u, \bar{u}) = +\infty$ and the set of all holomorphic functions bounded in a neighborhood of v is dense in $hL^2(M)$. Then Kobayashi's condition (1.5) holds for M .*

We can now prove the following generalization of a theorem of Skwarczyński [20; Theorem 3.16] announced in Introduction of this paper.

THEOREM 1.11. *Let M be a bounded, strictly starlike circular domain in \mathbf{C}^m . Then the following four statements are mutually equivalent:*

- (i) $\mathcal{K}(M) = M$.
- (ii) Kobayashi's condition (1.5) holds for M .
- (iii) The Bergman metric on M is complete.
- (iv) M is pseudoconvex.

REMARK 1.12. Under the assumption of Theorem 1.11, it is known [4] that M is pseudoconvex if and only if M is a taut manifold in the sense of Wu [23; p. 199].

PROOF OF THEOREM 1.11. Since M is starlike circular and bounded, the space of all holomorphic polynomials is dense in $hL^2(M)$ so that (i) implies (ii) by Lemma 1.10. Implications (ii) \Rightarrow (iii) and (iii) \Rightarrow (iv) are well-known (cf. [5], [10]). Suppose that (iv) holds. Then $\mathcal{H}(M) = M$. By Corollary 1.6 we have $M = \text{Int Cl } M$ so that Lemma 1.9 implies $\mathcal{K}(M) = M$, that is, (i) holds.

§ 2. Square-integrable holomorphic m -forms.

Throughout this section, a complex manifold under consideration is always assumed to be connected and paracompact.

Denote by $HL^2(M)$ the Hilbert space consisting of all *square-integrable holomorphic m -forms* on a complex manifold M of dimension m , and by $(\cdot, \cdot)_M$ the inner product on $HL^2(M)$. If we denote by \bar{M} the conjugate manifold of M , and by $A_{\bar{p}}^{(m,0)}(\bar{M})$ the space of all $(m, 0)$ -forms at $\bar{p} \in \bar{M}$, then the *Bergman form* ${}^M K$ on M is the unique $(2m, 0)$ -form on the product manifold $M \times \bar{M}$ such that for every $p \in M$, the form ${}^M K(\cdot, \bar{p})$ belongs to $HL^2(M) \wedge A_{\bar{p}}^{(m,0)}(\bar{M})$ and satisfies

$$\alpha(p) = (\alpha, {}^M K(\cdot, \bar{p}))_M$$

for any $\alpha \in HL^2(M)$. The mapping $M \ni p \rightarrow {}^M K(p, \bar{p})$ is also considered as an (m, m) -form on M .

Given a multi-index $I = (i_1, \dots, i_m)$ of non-negative integers, and a local holomorphic coordinate $z = (z^1, \dots, z^m)$ of M , we put

$$(\partial^z)^I = \frac{\partial^{|I|}}{(\partial z^1)^{i_1} \dots (\partial z^m)^{i_m}},$$

where $|I| = i_1 + \dots + i_m$. For a non-negative integer n and a point $p \in M$, we consider the subspace

$$H_n^M(p) = \{\alpha \in HL^2(M) ; (\partial^z)^I \alpha(p) = 0 \ (|I| < n)\},$$

where z is a coordinate around p . The space $H_n^M(p)$ does not depend on the choice of z . For a holomorphic tangent vector $X \in T_p(M)$ at p , we also consider an (m, m) -form

$$\mu_n^M(X) = \max \{(X^n \alpha \wedge \overline{X^n \alpha})(p) ; \alpha \in H_n^M(p), \|\alpha\|_M = 1\}$$

at p , where the maximum is taken under the natural order in the space of (m, m) -forms at p , and $\|\cdot\|_M$ is the norm corresponding to $(\cdot, \cdot)_M$. We note that

$$(2.1) \quad \mu_0^M(X) = {}^M K(p, \bar{p})$$

$$(2.2) \quad \mu_n^M(\lambda X) = |\lambda|^{2n} \mu_n^M(X)$$

for $X \in T_p(M)$ and $\lambda \in \mathbb{C}$ (see [3] for details).

LEMMA 2.1. *If M' is a domain in M , then $\mu_n^M \leq \mu_n^{M'}$ on $T(M') \subset T(M)$ for any n .*

PROOF. If ι is the inclusion mapping of M' into M and $p \in M'$, then $\iota^* H_n^M(p) \subset H_n^{M'}(p)$ and $\|\iota^* \alpha\|_{M'} \leq \|\alpha\|_M$ for any $\alpha \in H_n^M(p)$. Therefore, for every $X \in T_p(M')$,

$$\mu_n^M(X) \leq \max \{(X^n \alpha \wedge \overline{X^n \alpha})(p) ; \alpha \in H_n^M(p), \|\iota^* \alpha\|_{M'} = 1\} \leq \mu_n^{M'}(X)$$

as desired.

For a holomorphic coordinate z around a point $p \in M$, we consider the func-

tion $\mu_n^{M(p,z)} = \mu_n^M / (dz \wedge d\bar{z})_p$ on $T_p(M)$, where $dz = dz^1 \wedge \dots \wedge dz^m$ for $z = (z^1, \dots, z^m)$. If w is another coordinate around p , then

$$(2.3) \quad \mu_n^{M(p,w)} = \mu_n^{M(p,z)} |J_z^w(p)|^2$$

on $T_p(M)$. Here, J_z^f means the Jacobian of a system $f = (f^1, \dots, f^m)$ of holomorphic functions in a neighborhood of p with respect to the coordinate $z = (z^1, \dots, z^m)$. Similarly, if $\Phi \in \text{Bihol}(M, M')$ and if w is a coordinate around $q = \Phi(p)$, then

$$(2.4) \quad \Phi^* \mu_n^{M'(q,w)} = \mu_n^{M(p,z)} |J_z^{w \circ \Phi}(p)|^2$$

on $T_p(M)$.

If $M \in \mathcal{C}(\mathbb{C}^m)$ and S_n is a complete orthonormal system of the space of square-integrable homogeneous polynomials of degree n with

$$(2.5) \quad k_n(u, \bar{v}) = \sum_{f \in S_n} f(u) \overline{f(v)}$$

($n=0, 1, \dots$), and if z is the natural coordinate of M , i.e., $z(v) = v$, $v \in M$, then the Bergman form on M is given by ${}^M K = k(z, \bar{z}) dz \wedge d\bar{z}$ with $k(u, \bar{v}) = \sum_n k_n(u, \bar{v})$. Furthermore, it is known ([2]) that

$$(2.6) \quad \frac{\mu_n^{M(0,z)}((\partial_v^z)_0)}{(n!)^2} = k_n(v, \bar{v})$$

for $v \in \mathbb{C}^m$. Here, we use the following convention:

$$(2.7) \quad \partial_v^z = \sum_{a=1}^m v^a \frac{\partial}{\partial z^a}$$

for $v = (v^1, \dots, v^m) \in \mathbb{C}^m$.

LEMMA 2.2. *Let z be a holomorphic coordinate around a point p of a complex manifold M .*

- (i) *The function $\log \mu_n^{M(p,z)}$ is plurisubharmonic on $T_p(M)$ for any n .*
- (ii) *The family $(2n)^{-1} \log(\mu_n^{M(p,z)} / (n!)^2)$ ($n=1, 2, \dots$) is locally uniformly bounded from above.*

PROOF. Referring to (2.7), we put

$$h_n(v) = (2n)^{-1} \log \left(\frac{\mu_n^{M(p,z)}((\partial_v^z)_p)}{(n!)^2} \right)$$

for $v \in \mathbb{C}^m$.

(i) It is known ([1]) that there exists a positive semi-definite Hermitian matrix $(g_{I\bar{J}}^{(n)})_{|I|=|J|=n}$ of order $N = \binom{m+n-1}{n}$ such that

$$h_n(v) = (2n)^{-1} \log \sum_{|I|=|J|=n} v_I^{[n]} g_{I\bar{J}}^{(n)} \overline{v_J^{[n]}}$$

for any $v \in \mathbf{C}^m$, where

$$v_I^{[n]} = \frac{n!}{i_1! \cdots i_m!} (v^1)^{i_1} \cdots (v^m)^{i_m}$$

for $I=(i_1, \dots, i_m)$. Since the mapping $\mathbf{C}^m \ni v \mapsto (v_I^{[n]}) \in \mathbf{C}^N$ is holomorphic, for the proof of (i) it is sufficient to show that the function

$$\mathbf{C}^N \ni (v_I)_{|I|=n} \mapsto \log \sum_{I,J} v_I g_I^{(n)} \bar{v}_J \in [-\infty, +\infty)$$

is plurisubharmonic. By a linear change of variables it is sufficient to show that the function

$$\mathbf{C}^N \ni (v^1, \dots, v^N) \mapsto \log \sum_{\alpha=1}^r |v^\alpha|^2 \in [-\infty, +\infty)$$

is plurisubharmonic for any $r=0, \dots, N$. The last assertion actually holds so that the part (i) is proved.

(ii) Take a ball B with center $z(p)$ in the image $z(U_z) \subset \mathbf{C}^m$ of the coordinate neighborhood U_z of z and put $M'=z^{-1}(B)$. Consider a new coordinate w given by $w(q)=(z(q)-z(p))/R$, $q \in U_z$, where R is the radius of B . Then, $M'=\{q \in U_z; \|w(q)\| < 1\}$ and the Bergman form $k(q, \bar{q}') dw \wedge d\bar{w}$ of M' is explicitly calculated and satisfies

$$k(q, \bar{q}) = V_m^{-1} (1 - \|w(q)\|^2)^{-m-1}, \quad q \in M',$$

where V_m is the volume of the unit ball in \mathbf{C}^m so that by (2.6) we have

$$\frac{\mu_n^{M'(p,w)}((\partial_w^z)_p)}{(n!)^2} = \frac{(n+m)(n+m-1) \cdots (n+1)}{m! V_m} \|v\|^{2n}$$

for $v \in \mathbf{C}^m$. Since $(\partial_w^z)_p = R(\partial_v^z)_p$ and $|J_z^w(p)|^2 = R^{-2m}$, (2.2) and (2.3) imply

$$\frac{\mu_n^{M'(p,z)}((\partial_v^z)_p)}{(n!)^2} = \frac{(n+m)(n+m-1) \cdots (n+1)}{m! V_m} \|v\|^{2n}.$$

An application of Lemma 2.1 gives

$$h_n(v) \leq (2n)^{-1} \log \left(\frac{(n+m) \cdots (n+1)}{m! V_m} \right) + \log \|v\|$$

for any $v \in \mathbf{C}^m$ and n ; therefore, h_n are uniformly bounded from above on $\|v\| < r$ for every positive number r , which proves the part (ii).

Let C^M be the Carathéodory pseudo-metric on a complex manifold M (see (0.1)). We prove the following lemma, which implies a result announced in [3; Remark 4.3].

LEMMA 2.3. *For every non-negative integer n , $(n+1)^2(C^M)^2 \mu_n^M \leq \mu_{n+1}^M$ on $T(M)$.*

PROOF. The proof is done by the same argument as in the proof of [6; Theorem 1]. Let $X \in T_p(M)$. We may assume $C^M(X) > 0$. Assume that $f \in \text{Hol}(M, U)$ is non-constant and $f(p) = 0$. Let $\alpha \in H_n^M(p)$ with $\|\alpha\|_M = 1$. Then $f\alpha \in H_{n+1}^M(p)$, $0 < \|f\alpha\|_M \leq 1$, and $X^{n+1}(f\alpha)(p) = (n+1)Xf(p)X^n\alpha(p)$. Thus,

$$\begin{aligned} (n+1)^2 \rho(f_*X)^2 (X^n\alpha \wedge \overline{X^n\alpha})(p) &= (X^{n+1}(f\alpha) \wedge \overline{X^{n+1}(f\alpha)})(p) \\ &\leq \|f\alpha\|_M^2 \mu_{n+1}^M(X) \leq \mu_{n+1}^M(X). \end{aligned}$$

From this we get $(n+1)^2 C^M(X)^2 \mu_n^M(X) \leq \mu_{n+1}^M(X)$, which is the desired.

For a tangent vector $X \in T_p(M)$ of a complex manifold M , we put

$$B^M(X) = \limsup_{Y \rightarrow X, Y \in T_p(M)} \limsup_{n \rightarrow \infty} \left(\frac{\mu_n^{M(p,z)}(Y)}{(n!)^2} \right)^{1/2n},$$

where z is a coordinate around p . Since $\lim_{n \rightarrow \infty} \lambda^{1/2n} = 1$ for any $\lambda > 0$, (2.3) shows that $B^M(X)$ does not depend on the choice of z .

PROPOSITION 2.4. (i) For every connected paracompact complex manifold M , B^M is a pseudo-metric on M .

(ii) The assignment $M \rightarrow B^M$ is biholomorphically invariant and distance-decreasing for open submanifolds, that is, if M' is a domain in M , then $B^M \leq B^{M'}$ on $T(M')$.

(iii) Assume that for every $p \in M$ there exists a form $\alpha \in H(M)$ which does not vanish at p , i. e., M satisfies Kobayashi's condition (A.1) in [10]. Then $C^M \leq B^M$ on $T(M)$.

(iv) The indicatrix of B^M at every point p of M is a pseudoconvex starlike circular domain in $T_p(M)$.

PROOF. A property ([22; p. 74]) on plurisubharmonic functions and Lemma 2.2 imply that

$$(2.8) \quad \log B^M \text{ is plurisubharmonic on every tangent space.}$$

In particular, B^M is real-valued. This and an application of (2.2) imply the assertion (i). The first assertion of (ii) follows from the relation (2.4), and the second from Lemma 2.1. Let z be a coordinate around a point p of M . The assumption of (iii) implies that the constant function $\mu_0^{M(p,z)}$ on $T_p(M)$ is positive (see (2.1)). Therefore, by Lemma 2.3 we have

$$C^M \leq \left(\frac{\mu_n^{M(p,z)}}{\mu_0^{M(p,z)}(n!)^2} \right)^{1/2n}$$

on $T_p(M)$ for any positive integer n . Thus the desired inequality of (iii) follows from the definition of B^M . The assertion (iv) follows from (2.8) and Prop-

osition 1.2. The proof is complete.

In the last section of this paper, we show that if M is a symmetric bounded domain in \mathbf{C}^m , then B^M coincides with C^M , contrary to the usual Bergman metric.

§3. Negative plurisubharmonic functions.

For a point p of a connected complex manifold M , we denote by $\text{LHC}(p)$ the totality of *local holomorphic curves* passing through p , that is,

$$\text{LHC}(p) = \bigcup_{\varepsilon > 0} \{ \varphi \in \text{Hol}(\varepsilon U, M); \varphi(0) = p \},$$

where $\varepsilon U = \{ \lambda \in \mathbf{C}; |\lambda| < \varepsilon \}$. Thus we see

$$T_p(M) = \{ \varphi_*(d/d\lambda)_0; \varphi \in \text{LHC}(p) \}.$$

We denote by $\text{PS}^M(p)$ the family of all *negative plurisubharmonic functions* f on M satisfying the following *singularity condition* $(S)_p$ and *coordinate condition* $(C)_p$:

- (S)_p $\limsup_{u \rightarrow 0, u \neq 0} (\exp f \circ z^{-1})(u) / \|u\| < +\infty$ for some holomorphic coordinate z with $z(p) = 0$.
- (C)_p If $\varphi_i \in \text{LHC}(p)$ ($i=1, 2$) and $\varphi_{1*}(d/d\lambda)_0 = \varphi_{2*}(d/d\lambda)_0 \neq 0$, then $L_f[\varphi_1] = L_f[\varphi_2]$.

Here

$$L_f[\varphi] = \limsup_{\lambda \rightarrow 0, \lambda \neq 0} \frac{(\exp f \circ \varphi)(\lambda)}{|\lambda|}$$

for $\varphi \in \text{LHC}(p)$. The inequality in the condition $(S)_p$ is equivalent to the existence of positive numbers η and δ such that $(\exp f \circ z^{-1})(u) / \|u\| \leq \eta$ for any $u \in \mathbf{C}^m$ with $0 < \|u\| < \delta$ so that every $f \in \text{PS}^M(p)$ takes the value $-\infty$ at the point p , because $\limsup_{q \rightarrow p, q \neq p} f(q) = f(p)$. The condition $(S)_p$ does not depend on the choice of the coordinate z with $z(p) = 0$ (cf. the proof of Lemma 3.4 below). The family $\text{PS}^M(p)$ always contains the constant function $-\infty$, and it may consist only of one element $-\infty$. (This case occurs when M is compact or $M = \mathbf{C}^m$.) If $f \in \text{PS}^M(p)$ and $\varphi \in \text{LHC}(p)$ and if z is a holomorphic coordinate with $z(p) = 0$, then

$$(3.1) \quad \frac{(\exp f \circ \varphi)(\lambda)}{|\lambda|} = \frac{(\exp f \circ z^{-1})(z \circ \varphi(\lambda))}{\|z \circ \varphi(\lambda)\|} \left\| \frac{z \circ \varphi(\lambda)}{\lambda} \right\|$$

for all sufficiently small $\lambda \in \mathbf{C}$ with $\varphi(\lambda) \neq p$.

REMARK 3.1. When M is one-dimensional, one can drop the condition $(C)_p$ in the definition of $\text{PS}^M(p)$. Indeed, assume that f is a negative subharmonic

function on M satisfying $(S)_p$ and that $\varphi_i \in \text{LHC}(p)$ ($i=1, 2$) satisfy $\varphi_{i*}(d/d\lambda)_0 = X \in T_p(M) - \{0\}$. If we regard φ_2^{-1} as a coordinate around p , the functions $g_i(\lambda) = (\exp f \circ \varphi_i)(\lambda) / |\lambda|$, $\lambda \in \varepsilon U$ satisfy

$$g_1(\lambda) = g_2(\varphi_2^{-1} \circ \varphi_1(\lambda)) \left| \frac{\varphi_2^{-1} \circ \varphi_1(\lambda)}{\lambda} \right|$$

for all sufficiently small $\lambda \neq 0$ by (3.1). It follows from $\lim_{\lambda \rightarrow 0, \lambda \neq 0} \varphi_2^{-1} \circ \varphi_1(\lambda) / \lambda = 1$ that $L_f[\varphi_1] = L_f[\varphi_2]$; therefore, the condition $(C)_p$ is automatically satisfied.

LEMMA 3.2. For $f \in \text{PS}^M(p)$ and $\varphi \in \text{LHC}(p)$ with $\varphi_*(d/d\lambda)_0 = 0$, it holds that $L_f[\varphi] = 0$.

PROOF. If z is a coordinate appeared in the assumption $(S)_p$ on f , then by (3.1) we can find a positive number η such that

$$\frac{(\exp f \circ \varphi)(\lambda)}{|\lambda|} \leq \eta \left\| \frac{z \circ \varphi(\lambda)}{\lambda} \right\|$$

for all sufficiently small $\lambda \neq 0$; therefore, the desired assertion follows from $\varphi_*(d/d\lambda)_0 = 0$ or $\lim_{\lambda \rightarrow 0, \lambda \neq 0} z \circ \varphi(\lambda) / \lambda = 0$.

For $f \in \text{PS}^M(p)$ and $X \in T_p(M)$, we set $L_f(X) = L_f[\varphi]$ for some $\varphi \in \text{LHC}(p)$ with $\varphi_*(d/d\lambda)_0 = X$. By $(S)_p$ and (3.1) we have

$$(3.2) \quad L_f(X) \in [0, +\infty).$$

By $(C)_p$ and Lemma 3.2, we see that $L_f(X)$ does not depend on the representation of X in terms of φ . We also see that

$$(3.3) \quad L_f(\lambda X) = |\lambda| L_f(X)$$

for any $X \in T_p(M)$ and $\lambda \in \mathbb{C}$. Indeed, the assertion for $\lambda \neq 0$ follows immediately from definition and for $\lambda = 0$ from Lemma 3.2 and (3.2).

LEMMA 3.3. For $f \in \text{PS}^M(p)$ and $\varphi \in \text{LHC}(p)$,

$$\log L_f(X) = \lim_{r \rightarrow 0^+} \left((2\pi)^{-1} \int_0^{2\pi} f \circ \varphi(re^{i\theta}) d\theta - \log r \right),$$

where $X = \varphi_*(d/d\lambda)_0$.

PROOF. The function $g = f \circ \varphi - \log |\cdot|$ is subharmonic on $\varepsilon U - \{0\}$ and is bounded from above in a deleted neighborhood of 0 so that it can be extended uniquely to a subharmonic function \tilde{g} on εU by the requirement $\tilde{g}(0) = \limsup_{\lambda \rightarrow 0, \lambda \neq 0} g(\lambda) = \log L_f(X)$. Thus, our assertion is equivalent to

$$\tilde{g}(0) = \lim_{r \rightarrow 0^+} (2\pi)^{-1} \int_0^{2\pi} \tilde{g}(re^{i\theta}) d\theta,$$

which follows from the properties $\tilde{g}(0) \leq (2\pi)^{-1} \int_0^{2\pi} \tilde{g}(re^{i\theta}) d\theta$ and

$$\limsup_{r \rightarrow 0^+} \tilde{g}(re^{i\theta}) \leq \limsup_{\lambda \rightarrow 0, \lambda \neq 0} \tilde{g}(\lambda) = \tilde{g}(0)$$

and from Fatou's lemma.

LEMMA 3.4. *Let $\Phi \in \text{Hol}(M, M')$ and $p \in M$. If $f \in \text{PS}^{M'}(\Phi(p))$ and if $X \in T_p(M)$, then $f \circ \Phi \in \text{PS}^M(p)$ and $L_{f \circ \Phi}(X) = L_f(\Phi_*X)$.*

PROOF. Let z and w be coordinates in M and M' with $z(p) = 0$ and $w(q) = 0$, respectively, where $q = \Phi(p)$. Then, by (3.1) we have

$$\begin{aligned} \limsup_{u \rightarrow 0, u \neq 0} \frac{(\exp f \circ \Phi \circ z^{-1})(u)}{\|u\|} &= \limsup_{u \rightarrow 0, \Phi \circ z^{-1}(u) \neq q} \frac{(\exp f \circ \Phi \circ z^{-1})(u)}{\|u\|} \\ &\leq \|(\Phi \circ z^{-1})'(0)\| \limsup_{v \rightarrow 0, v \neq 0} \frac{(\exp f \circ w^{-1})(v)}{\|v\|}, \end{aligned}$$

where $\Psi'(0)$ indicates the linear part of a mapping Ψ at 0 and $\|\Psi'(0)\| = \sup_{\|u\|=1} \|\Psi'(0)u\|$ its operator norm. From this inequality we see that $f \circ \Phi$ satisfies the condition (S)_p. Furthermore, if $\varphi \in \text{LHC}(p)$ and if $\varphi_*(d/d\lambda)_0 = X$, then $\Phi \circ \varphi \in \text{LHC}(q)$ and $(\Phi \circ \varphi)_*(d/d\lambda)_0 = \Phi_*X$. It follows that $f \circ \Phi$ satisfies also the condition (C)_p and that

$$L_{f \circ \Phi}(X) = \limsup_{\lambda \rightarrow 0, \lambda \neq 0} \frac{(\exp f \circ \Phi \circ \varphi)(\lambda)}{|\lambda|} = L_f(\Phi_*X),$$

which proves the lemma.

LEMMA 3.5. *Let $M \in \mathcal{C}(\mathbb{C}^m)$ with $N = N^M$, and let z be the natural coordinate on M . Then $L_f((\partial_z^2)_0) \leq N(u)$ for any $f \in \text{PS}^M(0)$ and $u \in \mathbb{C}^m$ (see (2.7)).*

PROOF. Take any $\eta > N(u)$. Since $N(\eta^{-1}u) < 1$, the function $\varphi(\lambda) = \lambda\eta^{-1}u$, $\lambda \in U$ belongs to $\text{Hol}(\text{Cl}U, M) \cap \text{LHC}(0)$ with $\varphi_*(d/d\lambda)_0 = \eta^{-1}X$, where $X = (\partial_z^2)_0$. Furthermore, the function $f \circ \varphi - \log|\cdot|$ is extended to a subharmonic function on $\text{Cl}U$ and is negative on the boundary of U so that the maximum principle implies

$$\log L_f(\eta^{-1}X) = \limsup_{\lambda \rightarrow 0, \lambda \neq 0} (f \circ \varphi(\lambda) - \log|\lambda|) \leq 0.$$

Hence $\eta \geq L_f(X)$ by (3.3). This shows $L_f(X) \leq N(u)$.

For a holomorphic tangent vector $X \in T_p(M)$ of a connected complex manifold M , we consider the extremal quantity

$$Q^M(X) = \sup \{L_f(X); f \in \text{PS}^M(p)\}.$$

LEMMA 3.6. (i) *For every $\Phi \in \text{Hol}(M, M')$, it holds that $\Phi_*Q^{M'} \leq Q^M$.*

(ii) For every $p \in M$, there exists a semi-gauge N_p on $T_p(M)$ such that $Q^M|_{T_p(M)} \leq N_p$.

PROOF. The assertion (i) immediately follows from Lemma 3.4. Let (w, U_w) be a holomorphic chart of M around p with $w(p)=0$ and let M' be a starlike circular domain included in $w(U_w) \subset \mathbb{C}^m$ with a semi-gauge N' . By (i) and by Lemma 3.5, we get

$$Q^M((\partial_u^w)_p) \leq Q^{M'}((\partial_u^z)_p) \leq N'(u)$$

for $u \in \mathbb{C}^m$, where z is the natural coordinate of M' ; therefore the semi-gauge N_p defined by $N_p((\partial_u^w)_p) = N'(u)$ has the desired property of (ii).

LEMMA 3.7. (i) Q^M is a pseudo-metric on M .

(ii) $\limsup_{X \rightarrow 0, X \in T_p(M)} Q^M(X) = 0$.

(iii) The assignment $M \rightarrow Q^M$ possesses the decreasing property.

(iv) For the unit disk U , Q^U coincides with the Poincaré metric ρ on U (see (0.3)).

PROOF. The statement (ii) in Lemma 3.6 implies that Q^M is real-valued and that (ii) holds. By (3.3) we have $Q^M(\lambda X) = |\lambda| Q^M(X)$ which proves (i). The assertion (iii) then follows from Lemma 3.6, (i). The assertion (iv) is proved as follows: We first get $Q^U((d/d\lambda)_0) \leq N^U(1) = 1$ by Lemma 3.5. The function $f(\lambda) = \log |\lambda|$, $\lambda \in U$, satisfies $(S)_0$ so that $f \in \text{PS}^U(0)$ by Remark 3.1; while $L_f((d/d\lambda)_0) = 1$. Therefore, $Q^U((d/d\lambda)_0) = 1$. Combining this with (i) we have $Q^U = \rho$ on $T_0(U)$. It follows from the homogeneity of U that $Q^U = \rho$ on the whole $T(U)$.

For $X \in T_p(M)$, we set

$$P^M(X) = \limsup_{Y \rightarrow X, Y \in T_p(M)} Q^M(Y).$$

The function P^M is thus upper semi-continuous on every tangent space $T_p(M)$.

LEMMA 3.8. For every $p \in M$, the function $l = \log P^M|_{T_p(M)}$ satisfies

$$l(X) \leq (2\pi)^{-1} \int_0^{2\pi} l(X + e^{i\xi} Y) d\xi$$

for any $X, Y \in T_p(M)$.

PROOF. Take a coordinate z with $z(p)=0$ and set $X = (\partial_u^z)_p$, $Y = (\partial_v^z)_p$. For any $f \in \text{PS}^M(p)$, by Lemma 3.3 we have

$$\begin{aligned} l(X + e^{i\xi} Y) &\geq \log Q^M(X + e^{i\xi} Y) \\ &\geq \lim_{r \rightarrow 0+} \left((2\pi)^{-1} \int_0^{2\pi} f \circ z^{-1}(r e^{i\theta}(u + e^{i\xi} v)) d\theta - \log r \right). \end{aligned}$$

Fatou's lemma and Fubini's theorem show

$$\int_0^{2\pi} l(X + e^{i\xi} Y) d\xi \geq \limsup_{r \rightarrow 0^+} \int_0^{2\pi} d\theta \left((2\pi)^{-1} \int_0^{2\pi} f \circ z^{-1}(r e^{i\theta} u + r e^{i(\theta+\xi)} v) d\xi - \log r \right).$$

Since $f \circ z^{-1}$ is plurisubharmonic in a neighborhood of 0 in \mathbf{C}^m , we have

$$(2\pi)^{-1} \int_0^{2\pi} f \circ z^{-1}(r e^{i\theta} u + r e^{i(\theta+\xi)} v) d\xi \geq f \circ z^{-1}(r e^{i\theta} u)$$

for all sufficiently small r , so that by Lemma 3.3 we get

$$(2\pi)^{-1} \int_0^{2\pi} l(X + e^{i\xi} Y) d\xi \geq \log L_r(X).$$

From this we see

$$(3.4) \quad (2\pi)^{-1} \int_0^{2\pi} l(X + e^{i\xi} Y) d\xi \geq \log Q^M(X).$$

Take a sequence $X_j \in T_p(M)$ converging to X with $\lim_{j \rightarrow \infty} Q^M(X_j) = P^M(X)$. Since l is upper semi-continuous it follows from Fatou's lemma and (3.4) that

$$\begin{aligned} (2\pi)^{-1} \int_0^{2\pi} l(X + e^{i\xi} Y) d\xi &\geq (2\pi)^{-1} \int_0^{2\pi} \limsup_{j \rightarrow \infty} l(X_j + e^{i\xi} Y) d\xi \\ &\geq \limsup_{j \rightarrow \infty} (2\pi)^{-1} \int_0^{2\pi} l(X_j + e^{i\xi} Y) d\xi \\ &\geq \limsup_{j \rightarrow \infty} \log Q^M(X_j) = l(X). \end{aligned}$$

We have obtained the desired formula.

PROPOSITION 3.9. (i) *For every connected complex manifold M , P^M is a pseudo-metric on M .*

(ii) *The assignment $M \rightarrow P^M$ possesses the decreasing property.*

(iii) *$C^M \leq P^M \leq K^M$ on $T(M)$ for any M .*

(iv) *For every $p \in M$, the indicatrix of P^M at p is a pseudoconvex starlike circular domain in $T_p(M)$.*

PROOF. Lemma 3.6, (ii) and Lemma 3.7, (i) imply that P^M is real-valued and that $P^M(\lambda X) = |\lambda| P^M(X)$ for $\lambda \in \mathbf{C} - \{0\}$ and $X \in T(M)$; while the last formula for $\lambda = 0$, which is equivalent to $P^M(0) = 0$, follows from Lemma 3.7, (ii). This proves the assertion (i). The statements (iii) and (iv) in Lemma 3.7 imply (ii) and the assertion $P^U = \rho$, respectively. Observing (0.4) and using (i), (ii) and the fact $P^U = \rho$, we get (iii). Lemma 3.8 shows that $\log P^M$ is plurisubharmonic on $T_p(M)$ so that Proposition 1.2 yields (iv).

§ 4. Invariant pseudo-metrics on a starlike circular domain.

In this section we assume that M is a domain belonging to $\mathcal{C}(\mathbb{C}^m)$ with semi-gauge N^M and that z is the natural coordinate of M . Put $F_0(u) = F((\partial_u^z)_0)$ for $u \in \mathbb{C}^m$, where $F = C^M, K^M, B^M$, or P^M (see (0.1), (0.2) as well as (2.7)). Proposition 3.9 implies the first two inequalities of the following:

$$(4.1) \quad C_0^M \leq P_0^M \leq K_0^M \leq N^M$$

on \mathbb{C}^m ; while the last inequality is proved as follows ([4]): Fix $u \in \mathbb{C}^m$ and take $\eta > N^M(u)$. The function $f(\lambda) = \lambda \eta^{-1} u, \lambda \in U$, then belongs to $\text{Hol}(U, M)$ and satisfies $f_* \eta(d/d\lambda)_0 = (\partial_u^z)_0$ with $\rho(\eta(d/d\lambda)_0) = \eta$ (see (0.3)). By the definition (0.2) we see that $K_0^M(u) \leq \eta$. This gives $K_0^M(u) \leq N^M(u)$.

REMARK 4.1. The last inequality of (4.1) provides a simple proof of the following result of Kodama [13; Theorem 2]: M is bounded if and only if M is hyperbolic in the sense of Kobayashi [11], [12]. In fact, suppose that M is hyperbolic in the sense of Kobayashi. By a result of Royden [17; Theorem 2] there then exists a positive constant η such that $K_0^M \geq \eta \|\cdot\|$. Combining this with (4.1) we see that N^M is a gauge or M is bounded (Lemma 1.1, (i)). The converse is well-known ([11], [12]).

We get also the following improvement of Schwarz' lemma due to Sadullaev [18; Lemma 1].

PROPOSITION 4.2. For $\Phi \in \text{Hol}(U, M)$ with $M \in \mathcal{C}(\mathbb{C}^m)$ and $\Phi(0) = 0$, the following inequalities hold:

- (i) $P_0^M(\Phi(\lambda)) \leq |\lambda|$ for $\lambda \in U$;
- (ii) $P_0^M(\Phi'(0)) \leq 1$.

Furthermore, either if the equality in (i) holds at some $\lambda \neq 0$ or if the equality in (ii) holds, then the equality in (i) holds for all λ .

PROOF. By Proposition 3.9 we see that the function

$$h(\lambda) = \begin{cases} \log P_0^M(\Phi(\lambda)) - \log |\lambda|, & \lambda \in U - \{0\} \\ \log P_0^M(\Phi'(0)), & \lambda = 0 \end{cases}$$

is subharmonic on U . By (4.1) we see $h \leq \log N^M \circ \Phi - \log |\cdot| \leq -\log |\cdot|$ on $U - \{0\}$. The maximum principle then gives the inequality $h \leq 0$ and all the statements desired.

We next prove assertions about P^M similar to the following well-known facts ([7; Sätze 5 und 7]) about C^M :

$$(4.2) \quad M \text{ is convex if and only if } C_0^M = N^M.$$

(4.3) The indicatrix IC_0^M of C_0^M coincides with the convex hull of M .

THEOREM 4.3. Let $M \in \mathcal{C}(\mathbf{C}^m)$ with semi-gauge N^M .

(i) M is pseudoconvex if and only if $P_0^M = N^M$.

(ii) The indicatrix IP_0^M of P_0^M coincides with the holomorphic hull of M (see Corollary 1.4).

PROOF. “If” part in (i) follows from Proposition 3.9, (iv) and Proposition 1.2. Conversely, suppose that M is pseudoconvex. We consider the function $f = \log N^M$, which is negative and plurisubharmonic on M by Proposition 1.2. There exists a unique $(0, +\infty]$ -valued function R on the complex projective space $\mathbf{P} = \mathbf{P}^{m-1}$ such that $N^M = \|\cdot\|/R \circ \pi$ on $\mathbf{C}^m - \{0\}$, where $\pi: \mathbf{C}^m - \{0\} \rightarrow \mathbf{P}$ is the canonical projection defining \mathbf{P} . Since N^M is upper semi-continuous, the function R is lower semi-continuous on \mathbf{P} (cf. [1; § 3]) so that

$$\limsup_{u \rightarrow 0, u \neq 0} (f \circ z^{-1}(u) - \log \|u\|) \leq -\min \{ \log R(\xi) ; \xi \in \mathbf{P} \} < +\infty,$$

where z is the natural coordinate of M ; therefore, f satisfies the condition (S)₀. Let $\varphi \in \text{LHC}(0) \cap \text{Hol}(\varepsilon U, M)$ with $\varphi_*(d/d\lambda)_0 = (\partial_u^z)_0$, $u \in \mathbf{C}^m$. Then, $f \circ \varphi(\lambda) - \log |\lambda| = f(\varphi(\lambda)/\lambda)$ for $\lambda \in \varepsilon U - \{0\}$. Since the function ψ on εU defined by

$$\psi(\lambda) = \begin{cases} \varphi(\lambda)/\lambda, & \lambda \in \varepsilon U - \{0\} \\ u, & \lambda = 0 \end{cases}$$

is holomorphic, $f \circ \psi$ is subharmonic on εU so that we have $\log L_f[\varphi] = f \circ \psi(0) = f(u) = \log N^M(u)$; this means that f satisfies (C)₀ (hence $f \in PS^M(0)$) and that $P_0^M(u) \geq N^M(u)$. Thus, “only if” part in the assertion (i) is proved.

The holomorphic hull of M is schlicht and starlike circular (Corollary 1.4). Hence, to prove (ii), it is sufficient to show that IP_0^M is the smallest pseudoconvex starlike circular domain including M . Suppose that $M' \in \mathcal{C}(\mathbf{C}^m)$ is pseudoconvex and includes M . Since M' is pseudoconvex, (i) shows $N^{M'} = P_0^{M'}$. On the other hand, since $M \subset M'$, (ii) of Proposition 3.9 implies $P_0^{M'} \leq P_0^M$. Thus, $N^{M'} \leq P_0^M$ or $IP_0^M \subset M'$ as desired. This completes the proof.

COROLLARY 4.4 (Suzuki [21; Theorem 1], Barth [4; Theorem 2]). If M is pseudoconvex, then $K_0^M = N^M$.

PROOF. By the use of (i) of Theorem 4.3 and (4.1), we have this corollary.

COROLLARY 4.5 (Sadullaev [18; Theorem 1]). Let $\Phi \in \text{Hol}(M, M')$ with $M \in \mathcal{C}(\mathbf{C}^m)$, $M' \in \mathcal{C}(\mathbf{C}^{m'})$, and $\Phi(0) = 0$. Assume that M' is pseudoconvex. Then the linear part $\Phi'(0)$ of Φ at 0 maps M into M' .

PROOF. By the decreasing property of P^M , we see that $P_0^{M'} \circ \Phi'(0) \leq P_0^M$ on \mathbf{C}^m . Since M' is pseudoconvex, (i) of Theorem 4.3 gives $N^{M'} = P_0^{M'}$. Using

(4.1), we see that $N^{M'} \cdot \Phi'(0) \leq N^M$ on C^m , which proves the desired result.

THEOREM 4.6. *For every $M \in \mathcal{C}(C^m)$, the indicatrix IB_0^M of B_0^M coincides with the kernel hull $\mathcal{K}(M)$ of M (see § 1).*

PROOF. Let $k(u, \bar{v})dz \wedge d\bar{z}$ be the Bergman form on M and let $k(u, \bar{v}) = \sum_{n=0}^{\infty} k_n(u, \bar{v})$, where k_n is the function given by (2.5). Since k is holomorphic on $M \times \bar{M}$, the function $(\lambda, u, \bar{v}) \rightarrow \sum_n \lambda^{2n} k_n(u, \bar{v}) = k(\lambda u, \lambda \bar{v})$ is holomorphic in a neighborhood of 0 in C^{1+m+m} . If we define a function r on C^m by

$$r(u) = \sup \{ \lambda > 0; \sum_n \lambda^{2n} k_n(u, \bar{v}) \text{ converges in a neighborhood of } u \},$$

then it follows from Cauchy-Hadamard formula that

$$(4.4) \quad r^{-1} = B_0^M \quad \text{on } C^m.$$

Since, in this case, $\mathcal{K}(M)$ coincides with the domain $\text{Int} \{ u \in C^m; \sum_n k_n(u, \bar{u}) < +\infty \}$ which belongs to $\mathcal{C}(C^m)$ (cf. [14]), the semi-gauge $N^{\mathcal{K}(M)}$ of $\mathcal{K}(M)$ is given by

$$\begin{aligned} N^{\mathcal{K}(M)}(u) &= \inf \{ \lambda > 0; u \in \lambda \mathcal{K}(M) \} \\ &= (\sup \{ \lambda > 0; \lambda u \in \mathcal{K}(M) \})^{-1} \\ &= r(u)^{-1} \end{aligned}$$

for $u \in C^m$. Combining this with (4.4), we get $N^{\mathcal{K}(M)} = B_0^M$, which proves the theorem.

LEMMA 4.7. *For every $M \in \mathcal{C}(C^m)$, the holomorphic hull $\mathcal{H}(M)$ is included in the kernel hull $\mathcal{K}(M)$ of M .*

PROOF. The proof of the same assertion for a bounded M was given by Mehring and Sommer [14; Satz 2]. Their argument is also valid for an unbounded M .

THEOREM 4.8. *Assume that M is a domain belonging to $\mathcal{C}(C^m)$ and that, for every $p \in M$, there exists a form $\alpha \in HL^2(M)$ such that $\alpha(p) \neq 0$. Then the following inequalities hold:*

$$C_0^M \leq B_0^M \leq P_0^M \leq K_0^M \leq N^M$$

on C^m ; therefore the following inclusions also hold:

$$IC_0^M \supset IB_0^M \supset IP_0^M \supset IK_0^M \supset M.$$

PROOF. The last two inequalities were shown in (4.1). The second one follows from Lemma 4.7, since $IP_0^M = \mathcal{H}(M)$ and $IB_0^M = \mathcal{K}(M)$ (Theorems 4.3 and 4.6). The first one is a consequence of Proposition 2.4, (iii).

COROLLARY 4.9. *If M is a symmetric bounded domain in \mathbf{C}^m , then $C^M = B^M = P^M = K^M$ on the tangent bundle $T(M)$.*

PROOF. Since M is assumed to be convex and starlike circular, Theorem 4.8 as well as (4.2) gives $C_0^M = B_0^M = P_0^M = K_0^M$. Since M is homogeneous, the biholomorphic invariance of these metrics implies the desired formulas.

Finally, we give sufficient conditions for B_0^M to coincide with P_0^M .

PROPOSITION 4.10. *If $M \in \mathcal{C}(\mathbf{C}^m)$ is bounded and if the holomorphic hull $\mathcal{H}(M)$ of M is strictly starlike circular, then $B_0^M = P_0^M$ on \mathbf{C}^m .*

PROOF. By Corollary 1.6, $\mathcal{H}(M)$ is fat. Therefore, by Lemma 1.9 we have $\mathcal{H}(M) = \mathcal{H}(M)$ so that Theorems 4.3 and 4.6 imply the assertion.

COROLLARY 4.11. *If M is a bounded, complete Reinhardt domain in \mathbf{C}^m , then $B_0^M = P_0^M$ on \mathbf{C}^m .*

PROOF. It is well-known that, in this case, $\mathcal{H}(M)$ is also a bounded, complete Reinhardt domain. Hence, $\mathcal{H}(M)$ is strictly starlike circular (Corollary 1.8). Applying Proposition 4.10, we have our corollary.

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References

- [1] K. Azukawa, Hyperbolicity of circular domains, Tôhoku Math. J., **35** (1983), 403-413.
- [2] K. Azukawa, Square-integrable holomorphic functions on a circular domain in \mathbf{C}^n , Tôhoku Math. J., **37** (1985), 15-26.
- [3] K. Azukawa and J. Burbea, Hessian quartic forms and the Bergman metric, Kodai Math. J., **7** (1984), 133-152.
- [4] T. J. Barth, The Kobayashi indicatrix at the center of a circular domain, Proc. Amer. Math. Soc., **88** (1983), 527-530.
- [5] H. J. Bremerman, Holomorphic Continuation of the Kernel Function and the Bergman Metric in Several Complex Variables, Lectures on Functions of a Complex Variable, Univ. Michigan Press, 1955.
- [6] J. Burbea, Inequalities between intrinsic metrics, Proc. Amer. Math. Soc., **67** (1977), 50-54.
- [7] C. Carathéodory, Über die Geometrie der analytischen Abbildungen, die durch analytische Funktionen von zwei Veränderlichen vermittelt werden, Abh. Math. Sem. Univ. Hamburg, **6** (1928), 96-145.
- [8] H. Cartan, Les fonctions de deux variables complexes et le problème de la représentation analytique, J. Math. Pures Appl. (9), **10** (1931), 1-114.
- [9] H. Grauert und R. Remmert, Plurisubharmonische Funktionen in komplexen Räume, Math. Z., **65** (1956), 175-194.

- [10] S. Kobayashi, Geometry of bounded domains, *Trans. Amer. Math. Soc.*, **92** (1959), 267-290.
- [11] S. Kobayashi, Intrinsic distances, measures and geometric function theory, *Bull. Amer. Math. Soc.*, **82** (1976), 357-416.
- [12] S. Kobayashi, *Hyperbolic Manifolds and Holomorphic Mappings*, Marcel Dekker, New York, 1977.
- [13] A. Kodama, On boundedness of circular domains, *Proc. Japan Acad.*, **58** (1982), 227-230.
- [14] J. Mehring und F. Sommer, Kernfunktion und Hüllenbildung in der Funktionentheorie mehrerer Veränderlichen, *Math. Ann.*, **131** (1956), 1-16.
- [15] P. Pflug, Quadrantegrable holomorphe Funktionen und die Serre-Vermutung, *Math. Ann.*, **216** (1975), 285-288.
- [16] P. Pflug, About the Carathéodory completeness of all Reinhardt domains, *Functional Analysis, Holomorphy and Approximation Theory II*, G. I. Zapata (ed.), North-Holland, Amsterdam, 1984, pp. 331-337.
- [17] H. L. Royden, Remarks on the Kobayashi metric, *Lecture Notes in Math.*, **185**, Springer, 1971, pp. 125-137.
- [18] A. Sadullaev, Schwarz lemma for circular domains and its applications, *Math. Notes Aca. Sci. USSR*, **27** (1980), 120-125.
- [19] N. Sibony, A class of hyperbolic manifolds, *Ann. of Math. Studies*, **100**, Princeton Univ. Press, 1981, pp. 357-372.
- [20] M. Skwarczyński, Biholomorphic invariants related to the Bergman functions, *Dissertationes Math.*, **173** (1980), 1-59.
- [21] M. Suzuki, The intrinsic metrics on the circular domains in C^n , *Pacific J. Math.*, **112** (1984), 249-256.
- [22] V. S. Vladimirov, *Methods of the Theory of Functions of Many Complex Variables*, M. I. T. Press, Massachusetts, 1966.
- [23] H. Wu, Normal families of holomorphic mappings, *Acta Math.*, **119** (1967), 193-233.

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