Smooth perturbations in ordered Banach spaces and similarity for the linear transport operators

By Tomio UMEDA

1. Introduction.

The aim of this paper is to develop a new method for establishing the similarity for a pair of linear operators in an ordered Banach space X over C. Consider a pair of (in general unbounded) linear operators B_1 and $B_2=B_1+A$ with A a bounded operator, and assume that both $-B_1$ and $-B_2$ generate the bounded C_0-groups on X. Assume, in addition, that e^{-tB_1} and $-A$ are positivity preserving, and that A is $(-iB_1, 1)$-smooth. Then we show that B_2 is similar to B_1 (see Theorem 1 in section 2). The similarity of B_2 to B_1 is established by constructing both the wave operator $W_+(B_2, B_1)=\text{s-}\lim_{t \to +\infty} e^{tB_2}e^{-tB_1}$ and the inverse wave operator $W_+(B_1, B_2)=\text{s-}\lim_{t \to +\infty} e^{tB_1}e^{-tB_2}$. To do this, we simply use Cook's method. Our technique depends heavily upon both e^{-tB_1} and $-A$ being positivity preserving.

There is some literature on the theory of smooth perturbations. Kato [7] dealt with a perturbed operator of the form $T(\kappa)=T+\kappa V$ (κ being a small complex-parameter) in a Hilbert space and established the similarity of $T(\kappa)$ to T. Kato's result has been extended to a reflexive Banach space setting by Lin [9, 10], and to a not necessarily reflexive Banach space setting by Evans [3]. These authors need to factorize the perturbation V into the form D^*C, where C is T-smooth and D is T^*-smooth. We do not use such factorization, however.

Our theory of smooth perturbations is applicable to the linear transport operator (Boltzmann operator) in multiple scattering problem. We consider the linear transport operator

$$(-Bu)(x, \xi) = -\xi \cdot \nabla_x u(x, \xi) - \sigma(x, \xi)u(x, \xi)$$

$$+\int_{R^d} k(x, \xi', \xi)u(x, \xi')d\xi' \quad (x \in R^d, \xi \in R^d)$$

as a perturbation of the collisionless transport operator $-B_0=-\xi \cdot \nabla_x$. Here σ

and k denote the collision frequency and the scattering kernel respectively. (For references to the literature on the transport operator in multiple scattering problem, see Kaper-Lekkerkerker-Hejtmanek [6], Reed-Simon [12].) Our basic assumptions are that the pair (k, σ) is regular and has finite mean free path (see section 3). We work in the Banach space $L^1(R_{x,\xi}^{2d})$, since $L^1(R_{x,\xi}^{2d})$ is a natural space for the linear transport operator.

While scattering theory for the transport operator has received much attention in recent years (see [2, 4, 5, 13, 14, 15]), there are no known results concerning the similarity for the transport operators. We are unable to use the results of [7, 9, 10] to establish the similarity of the transport operator $-B$ to the collisionless transport operator $-B_0$, since our work is carried out in the non-reflexive Banach space L^1. Even in the setting of [3], it is difficult to treat the pair B, B_0.

In section 2 we discuss a theorem concerning smooth perturbations in ordered Banach spaces. As an application we deal with the transport operator and establish the similarity of B to B_0 in section 3.

This research was inspired by Reed-Simon [12, section 12].

It is a pleasure to thank Professor T. Ikebe and Doctor H. Isozaki for helpful discussions.

2. Smooth perturbations.

The task of this section is to establish the similarity for generators of some bounded C_0-groups on ordered Banach spaces over C. Before stating the main theorem of this section we give some preliminaries.

Let $(E, E_+, \|\cdot\|)$ be an ordered Banach space. Throughout this section we assume that the positive cone E_+ is generating, i. e.,

(2.1) $E = E_+ - E_+ ,$

and that the norm is α-monotone, i. e.,

(2.2) $0 \leq a \leq b$ implies $\|a\| \leq \alpha \|b\| .$

For terminology on ordered Banach spaces, we refer to Batty-Robinson [1].

We define the complexification of $(E, E_+, \|\cdot\|)$ to be the complex Banach space $X = E + iE$ equipped with the norm

$$\|u\| = \frac{1}{4} \int_0^{2\pi} \|(\cos\theta)a + (\sin\theta)b\| d\theta$$

for $u = a + ib \in X$, with $a, b \in E$. Then X is called an *ordered Banach space over C*.

A bounded linear operator T on an ordered Banach space X over C is defined to be *positivity preserving* $(T \geq 0)$ if $TE_+ \subseteq E_+$. We write $S \geq T$ if $S - T \geq 0$.

The following lemma is easy to prove.

LEMMA 2.1. *Let* S, T *and* R *be bounded linear operators on an ordered Banach space over* C.

   (a) *If* $S \geq T$ *and* $R \geq 0$, *then* $SR \geq TR$ *and* $RS \geq RT$.

   (b) *If* $S \geq T \geq 0$, *then* $S^n \geq T^n \geq 0$ *for all integers* $n \geq 1$.

We now consider a pair B_1, B_2 of linear operators in an ordered Banach space X over C, where $B_2 = B_1 + A$ and A is a bounded linear operator on X. We make the following assumptions on B_1, B_2 and A.

   (I) The operator $-B_1$ generates a C_0-group on X such that

$$\| e^{-tB_1} \| \leq M_1$$

for all $t \in R$, for some constant $M_1 > 0$.

   (II) $-A \geq 0$ and $e^{-tB_1} \geq 0$ for all $t \in R$.

   (III) There exists a constant $M_2 > 0$ such that

$$\| e^{-tB_2} \| \leq M_2$$

for all $t \in R$.

REMARK. Since A is a bounded operator on X, $-B_2$ generates a C_0-group on X; see Kato [8, Theorem 2.1, p. 497] or Umeda [14, Proposition in Appendix].

We can now state the main theorem in this section which establishes the similarity of B_2 to B_1.

THEOREM 1. *Suppose* (I), (II) *and* (III) *hold. Suppose also*

$$(2.3) \qquad \int_{-\infty}^{\infty} \| A e^{-tB_1} u \| dt < +\infty$$

for all $u \in E_+$. *Then the wave operators* $W_+(B_2, B_1)$, $W_+(B_1, B_2)$ *exist, and possess the following properties:*

   (i) $W_+(B_2, B_1) W_+(B_1, B_2) = W_+(B_1, B_2) W_+(B_2, B_1) = I$.

   (ii) $B_2 = W_+(B_2, B_1) B_1 W_+(B_2, B_1)^{-1}$.

To prove this theorem, we need a few lemmas. The following lemma plays a crucial role in our analysis.

LEMMA 2.2. *Suppose* (I) *and* (II) *hold. Then* $e^{-tB_2} \geq e^{-tB_1}$ *for all* $t \geq 0$. *In particular,* $e^{-tB_2} \geq 0$ *for all* $t \geq 0$.

PROOF. Recall first that the positive cone E_+ is a closed subset of X.

Now the statement is immediate from Lemma 2.1 and the Dyson-Phillips expansion of e^{-tB_2} in terms of e^{-tB_1}; cf. Kato [8, Theorem 2.1, p. 497]. Q.E.D.

Lemma 2.3.   *Suppose* (I), (II) *and* (III) *hold.   Suppose also*

$$(2.4) \qquad \int_{-\infty}^{0} \|Ae^{-tB_1}u\|dt < +\infty$$

for all $u \in E_+$.   *Then*

$$\int_{0}^{\infty} \|Ae^{-tB_2}u\|dt < +\infty$$

for all $u \in E_+$.

REMARK.   In terms of smooth perturbations (see Lin [9, Definition 2.4]), the condition (2.4) means that A is $(-iB_1, 1, -)$-smooth.   So the lemma above may be stated as follows: Suppose (I), (II) and (III) hold, and that A is $(-iB_1, 1, -)$-smooth.   Then A is $(-iB_2, 1, +)$-smooth.

PROOF OF LEMMA 2.3.   First, note that (2.4) holds for all $u \in X$, since E_+ is generating (see (2.1)), that is, since every u belonging to X can be written in the form $u = u_1 - u_2 + i(u_3 - u_4)$ with $u_j \in E_+$ where $j = 1, \cdots, 4$.   Then, applying the principle of uniform boundedness (see Kato [8, Theorem 1.29, p. 136]), we find a constant $C > 0$ such that

$$(2.5) \qquad \int_{-\infty}^{0} \|Ae^{-tB_1}u\|dt \leqq C\|u\|$$

for all $u \in X$.

Next, let $r > 0$ be given, and suppose $0 \leqq t \leqq r$.   By Lemma 2.1 (a) and Lemma 2.2, we see that

$$e^{-rB_2} \geqq e^{-(r-t)B_1}e^{-tB_2} \geqq 0 .$$

Using Assumption (II) and Lemma 2.1 (a), we get

$$-Ae^{(r-t)B_1}e^{-rB_2}u \geqq -Ae^{-tB_2}u \geqq 0$$

for all $u \in E_+$.   Hence we have by (2.2)

$$\|Ae^{-tB_2}u\| \leqq \alpha\|Ae^{(r-t)B_1}e^{-rB_2}u\|$$

for all $u \in E_+$.   Then, integration with respect to the variable t and a change of variable give

$$\int_{0}^{r} \|Ae^{-tB_2}u\|dt \leqq \alpha \int_{-r}^{0} \|Ae^{-tB_1}e^{-rB_2}u\|dt .$$

Due to (2.5) and Assumption (III), the right-hand side of this inequality is bounded by $\alpha C M_2\|u\|$.   Since r was arbitrary, the conclusion follows.   Q. E. D.

We are in a position to prove the main theorem in this section.

PROOF OF THEOREM 1. Since A is a bounded operator on X, a standard argument shows that

(2.6)
$$e^{tB_2}e^{-tB_1}u = u + \int_0^t e^{sB_2}Ae^{-sB_1}u\,ds,$$

(2.7)
$$e^{tB_1}e^{-tB_2}u = u - \int_0^t e^{sB_1}Ae^{-sB_2}u\,ds$$

for all $u \in X$. By (2.3) and Assumption (III), the strong limit for $t \to \infty$ of the left-hand side of (2.6) exists for all $u \in E_+$. Then the existence of the wave operator $W_+(B_2, B_1)$ follows from the fact that E_+ is generating. Similarly, the formula (2.7), together with (2.3), Assumption (I) and Lemma 2.3, implies the existence of the wave operator $W_+(B_1, B_2)$.

It remains to show the properties (i) and (ii). Notice that

$$e^{tB_2}e^{-tB_1} \cdot e^{tB_1}e^{-tB_2} = e^{tB_1}e^{-tB_2} \cdot e^{tB_2}e^{-tB_1} = I.$$

By letting $t \to \infty$, we obtain (i). To show (ii), it suffices to note the intertwining property

$$e^{-tB_2}W_+(B_2, B_1) = W_+(B_2, B_1)e^{-tB_1}.$$

Indeed, by taking the Laplace transform of both sides, we can easily deduce (ii).
Q. E. D.

3. An application to the transport operators.

In this section we shall apply Theorem 1 obtained in the preceding section in order to establish the similarity for the transport operator $-B$ to the collisionless transport operator $-B_0$. As mentioned in the introduction, we work in the complex Banach space L^1. It should be noted that L^1 equipped with the usual norm $\|\cdot\|_1$ coincides with the complexification of the ordered Banach space $(L_R^1, L_+^1, \|\cdot\|_1)$; here L_R^1 and L_+^1 respectively denote the space of real functions in L^1 and the cone of positive functions in L^1. It should also be noted that the positive cone L_+^1 is generating, and that the norm $\|\cdot\|_1$ is 1-monotone.

We first introduce a definition. The pair (k, σ) of the scattering kernel and the collision frequency is defined to be *admissible* if the following conditions hold:

(i) $k(x, \xi', \xi)$ is a nonnegative measurable function on R^{3d} and $\sigma(x, \xi)$ is a nonnegative measurable function on R^{2d};

(ii) For each (x, ξ'), $k(x, \xi', \cdot)$ is in $L^1(R_\xi^d)$;

(iii) $\sigma(x, \xi)$ and $\sigma_s(x, \xi) = \int k(x, \xi, \xi')d\xi'$ are bounded functions on R^{2d}.

We make the following assumptions on the admissible pair.

(A) $F(\sigma) = \mathrm{ess}\sup_{x,\xi} \int_{-\infty}^{\infty} \sigma(x-\tau\xi,\ \xi)d\tau < +\infty$.

(B) $F(\sigma_s) < +\infty$.

We now define the collisionless transport operator $-B_0$ to be the closure of the operator defined on $C_0^\infty(\mathbf{R}_{x,\xi}^{2d})$ by

$$(-B_0 u)(x,\ \xi) = -\xi\cdot\nabla_x u(x,\ \xi) .$$

It is well-known [12, p. 244] that $-B_0$ is the infinitesimal generator of a C_0-group of positivity preserving isometries on $L^1(\mathbf{R}^{2d})$ and that

(3.1) $(e^{-tB_0}u)(x,\ \xi) = u(x-t\xi,\ \xi) .$

The penetration operator $-B_1$ is defined to be the perturbation of the collision-less transport operator $-B_0$ by the bounded operator

$$(-A_1 u)(x,\ \xi) = -\sigma(x,\ \xi)u(x,\ \xi) , \qquad u \in L^1(\mathbf{R}^{2d}) ,$$

i. e., $-B_1 = -B_0 - A_1$. As mentioned in the remark following Assumption (III) in the preceding section, $-B_1$ generates the C_0-group on $L^1(\mathbf{R}^{2d})$. Then it is easily seen that e^{-tB_1} is given by the expression

(3.2) $(e^{-tB_1}u)(x,\ \xi) = u(x-t\xi,\ \xi)\exp\left(-\int_0^t \sigma(x-\tau\xi,\ \xi)d\tau\right)$

for all $t \in \mathbf{R}$. Finally, we define the transport operator $-B$ to be the perturbation of the penetration operator $-B_1$ by the bounded operator

(3.3) $-(A_2 u)(x,\ \xi) = \int k(x,\ \xi',\ \xi)u(x,\ \xi')d\xi' , \qquad u \in L^1(\mathbf{R}^{2d}) ,$

i. e., $-B = -B_1 - A_2$. Then, $-B$ also generates the C_0-group on $L^1(\mathbf{R}^{2d})$.

We now state the main theorem in this section which establishes the similarity for the transport operator $-B$ to the collisionless transport operator $-B_0$.

THEOREM 2. *Let $(k,\ \sigma)$ be admissible. Suppose* (A) *and* (B) *hold. Suppose also*

(3.4) $F(\sigma_s)\exp(F(\sigma)) < 1$.

Then the wave operators $W_+(B,\ B_0)$, $W_+(B_0,\ B)$ exist, and possess the following properties:

   (i) $W_+(B,\ B_0)W_+(B_0,\ B) = W_+(B_0,\ B)W_+(B,\ B_0) = I$.
   (ii) $B = W_+(B,\ B_0)B_0 W_+(B,\ B_0)^{-1}$.

Physically speaking, Theorem 2 has an interesting corollary. Before stating it, we introduce two definitions. The admissible pair $(k,\ \sigma)$ is said to be *regular*

if there is a compact set D in R_x^d such that $k(x, \xi', \xi)$ and $\sigma(x, \xi)$ vanish whenever $x \notin D$. The admissible pair is said to have *finite mean free path* if

$$M(\sigma) = \sup_{x, \xi} \frac{1}{|\xi|} \sigma(x, \xi) < +\infty .$$

We also need $M(\sigma_s)$ defined in the same manner as above.

COROLLARY. *Let (k, σ) be regular and have finite mean free path. Suppose*

$$M(\sigma_s)(\operatorname{diam} D) \exp(M(\sigma) \operatorname{diam} D) < 1 .$$

Then the conclusions of Theorem 2 hold.

PROOF. By a change of variable we have

$$\int_{-\infty}^{\infty} \sigma(x - \tau\xi, \xi) d\tau \leqq M(\sigma) \operatorname{diam} D ,$$

hence

$$F(\sigma) \leqq M(\sigma) \operatorname{diam} D .$$

Similarly, we have

$$F(\sigma_s) \leqq M(\sigma_s) \operatorname{diam} D .$$

Thus we can apply Theorem 2.   Q. E. D.

Theorem 2 is an immediate consequence of the following two theorems. The first establishes the similarity for the penetration operator $-B_1$ to the collisionless transport operator $-B_0$, whereas the second establishes the similarity for the transport operator $-B$ to the penetration operator $-B_1$.

THEOREM 3. *Let (k, σ) be admissible. Suppose* (A) *holds. Then all the wave operators $W_{\pm}(B_1, B_0)$, $W_{\pm}(B_0, B_1)$ exist, and possess the following properties:*
 (i) $W_{\pm}(B_1, B_0)W_{\pm}(B_0, B_1) = W_{\pm}(B_0, B_1)W_{\pm}(B_1, B_0) = I.$
 (ii) $B_1 = W_{\pm}(B_1, B_0)B_0W_{\pm}(B_1, B_0)^{-1}.$

THEOREM 4. *Let (k, σ) be admissible. Suppose* (A) *and* (B) *hold. Suppose also* (3.4) *holds. Then the wave operators $W_+(B, B_1)$, $W_+(B_1, B)$ exist, and possess the following properties:*
 (i) $W_+(B, B_1)W_+(B_1, B) = W_+(B_1, B)W_+(B, B_1) = I.$
 (ii) $B = W_+(B, B_1)B_1W_+(B, B_1)^{-1}.$

We first prove Theorem 3 which is rather trivial. The expressions (3.1) and (3.2) enable us to show, without using Cook's method, the existence of the wave operators.

PROOF OF THEOREM 3. It suffices to show the existence of the wave operators $W_{\pm}(B_1, B_0)$ and $W_{\pm}(B_0, B_1)$. For $u \in L^1(R^{2d})$ and $t \in R$, we have

$$e^{tB_1}e^{-tB_0}u = u \exp\left(-\int_0^{-t} \sigma(x-\tau\xi,\ \xi)d\tau\right),$$

$$e^{tB_0}e^{-tB_1}u = u \exp\left(-\int_{-t}^0 \sigma(x-\tau\xi,\ \xi)d\tau\right),$$

by (3.1) and (3.2). Then, the Lebesgue dominated convergence theorem, together with (A), shows the existence of $W_\pm(B_1, B_0)$ and $W_\pm(B_0, B_1)$.   Q. E. D.

LEMMA 3.1. *For* $u \in L^1(\mathbf{R}^{2d})$

$$\int_{-\infty}^\infty \|A_2 e^{-tB_1}u\| dt \leqq F(\sigma_s)\exp(F(\sigma))\|u\| .$$

PROOF. By the definitions of A_2, σ_s, and $F(\sigma)$, we have

(3.5) $\|A_2 e^{-tB_1}u\| \leqq \exp(F(\sigma))\int_{R^{2d}} |u(x,\ \xi)|\, \sigma_s(x+t\xi,\ \xi)dxd\xi .$

Here we made a change of variables. Integrating both sides of (3.5) with respect to the variable t, and using the definition of $F(\sigma_s)$, we obtain the desired inequality.   Q. E. D.

Finally, we prove Theorem 4.

PROOF OF THEOREM 4. We apply Theorem 1 in the preceding section. We take $-B_2$, $-B_1$ and $-A$ in Theorem 1 respectively to be the transport operator $-B$, the penetration operator $-B_1$, and the operator $-A_2$ defined by (3.3). Then, using (3.2) and (A), we can easily check that Assumption (I) holds. It is clear from (3.2) and (3.3) that Assumption (II) holds. Furthermore, Lemma 3.1 implies that (2.3) in Theorem 1 holds for all $u \in L^1$. Finally, we have to verify Assumption (III). To this end, we exploit Duhamel's formula:

$$e^{-tB} = e^{-tB_1} - \int_0^t e^{-(t-s)B}A_2 e^{-sB_1}ds .$$

Now let $T > 0$ be given, and suppose $0 \leqq t \leqq T$. Then, we have

$$\|e^{-tB}u\| \leqq \|u\| + \left(\sup_{0 \leqq s \leqq T}\|e^{-sB}\|\right)\int_0^t \|A_2 e^{-sB_1}u\|ds$$

for all $t \in [0, T]$ and all $u \in L^1$. By Lemma 3.1, the integral in the right-hand side of the above inequality is bounded by $F(\sigma_s)\exp(F(\sigma))\|u\|$. Hence, we get

$$\sup_{0 \leqq t \leqq T}\|e^{-tB}\| \leqq 1 + \left(\sup_{0 \leqq t \leqq T}\|e^{-tB}\|\right)M$$

(M denoting the constant $F(\sigma_s)\exp(F(\sigma))$) which by (3.4) implies that

$$\sup_{0 \leqq t \leqq T}\|e^{-tB}\| \leqq \frac{1}{1-M} < +\infty .$$

Similarly, we have

$$\sup_{-T \leq t \leq 0} \|e^{-tB}\| \leq \frac{\exp(F(\sigma))}{1-M} < +\infty.$$

Since T was arbitrary, Assumption (Ⅲ) is verified. Thus all assumptions of Theorem 1 are satisfied, and the conclusions of Theorem 4 follow. Q. E. D.

References

[1] C. J. K. Batty and D. W. Robinson, Positive one-parameter semigroups on ordered Banach spaces, Acta Appl. Math., **2** (1984), 221–296.

[2] H. Emamirad, On the Lax-Phillips scattering theory for transport equation, J. Funct. Anal., **62** (1985), 276–303.

[3] D. E. Evans, Smooth perturbations in non-reflexive Banach spaces, Math. Ann., **221** (1976), 183–194.

[4] J. Hejtmanek, Scattering theory of the linear Boltzmann operator, Comm. Math. Phys., **43** (1975), 109–120.

[5] J. Hejtmanek, The problem of reconstructing objects from projections as an inverse problem in scattering theory of the linear transport operator, G. T. Herman and F. Natterer ed., Mathematical Aspects of Computerized Tomography, Lecture Notes in Medical Informatics, **8**, Springer, 1981, pp. 29–35.

[6] H. G. Kaper, C. G. Lekkerkerker and J. Hejtmanek, Spectral Methods in Linear Transport Theory, Birkhäuser, Basel, 1982.

[7] T. Kato, Wave operators and similarity for some non-selfadjoint operators, Math. Ann., **162** (1966), 258–279.

[8] T. Kato, Perturbation Theory for Linear Operators, 2nd ed., Springer, Berlin - Heidelberg - New York, 1976.

[9] S.-C. Lin, Wave operators and similarity for generators of semigroups in Banach spaces, Trans. Amer. Math. Soc., **139** (1969), 469–494.

[10] S.-C. Lin, A stationary approach to perturbation of operators in Banach spaces, J. Math. Anal. Appl., **32** (1970), 352–369.

[11] M. Reed and B. Simon, Methods of Modern Mathematical Physics, II. Fourier Analysis, Self-Adjointness, Academic Press, New York, 1975.

[12] M. Reed and B. Simon, Methods of Modern Mathematical Physics, III. Scattering Theory, Academic Press, New York, 1979.

[13] B. Simon, Existence of the scattering matrix for the linearized Boltzmann equation, Comm. Math. Phys., **41** (1975), 99–108.

[14] T. Umeda, Scattering and spectral theory for the linear Boltzmann operator, J. Math. Kyoto Univ., **24** (1984), 205–218.

[15] J. Voigt, On the existence of the scattering operator for the linear Boltzmann equation, J. Math. Anal. Appl., **58** (1977), 541–558.

Tomio UMEDA
Department of Mathematics
Osaka University
Toyonaka, Osaka 560
Japan