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Cyclical coincidences of multivalued maps

Dedicated to Professor A. Granas

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0. Introduction.

If $T:X\to 2^{Y}$ and $U:Y\to 2^{X}$, $(x, y)\in X\times Y$ is a *coincidence* of T and U if $Tx\ni y$ and $Uy\ni x$. In a recent paper, Browder, [4], has proved the existence of coincidences in a variety of situations. In this paper we shall extend Browder's results to the case of m ($\geqq 2$) spaces. The basic tool that we use is Brouwer's fixed-point theorem for a simplex (though we could equally well use the KKM theorem). We prove in Corollary 3.2 that if, for each $i=0, \cdots, m-1$, X_i is a nonempty convex subset of a topological vector space and $T_i: X_i\to 2^{X_{i+1}}$ has nonempty convex values (with $(m-1)+1$ interpreted as 0) then there exists $(x_0, \cdots, x_{m-1})\in X_0\times\cdots\times X_{m-1}$ such that

$$\text{for all}\quad i=0, \cdots, m-1, \qquad T_i x_i\ni x_{i+1}$$

provided that each T_i is either "Browder-Fan" (Definition 1.2) or of "Kakutani type" (Definition 2.3). These definitions will require that some (but possibly not all) of the sets X_i are compact and some (but possibly not all) of the underlying topological vector spaces are locally convex. It is curious that the two kinds of map can be mixed in any order. Corollary 3.2 is a consequence of the main existence theorem, Theorem 3.1, in which we allow some of the maps to satisfy a weaker condition than "Browder-Fan". This weaker condition does not require the vector spaces to be topologized, since it is stated in terms of the "polytopology", which is an intrinsic topology defined on any nonempty convex subset of a vector space (Definition 1.1). The proof of Theorem 3.1 goes by way of two special cases, Theorem 1.4 and Theorem 2.5.

If X is a nonempty compact convex subset of a topological vector space and for all $f\in E'$,

$$Sf=\{x : x\in X,\ f(x)=\max f(X)\}$$

then $S: E'\to 2^{X}$ has nonempty closed convex values. In Theorem 2.2, we prove a coincidence theorem that involves the map S. This result (which was proved in a different way in [11]) seems to be entirely independent of Theorem 3.1 and can best be thought of as a generalization of Kakutani's fixed-point theorem to the non-locally-convex case. In fact, it has as immediate consequences generalizations of Kakutani's fixed-point theorem due to Browder, [4], Theorems 8 and 9, Fan, [5], Theorems 5 and 6 and Takahashi, [12], Theorem 8 and [13], Theorem 11. These applications are discussed fully in [11], Remark 4.6.

We now introduce some abbreviations and notation. *Vs* stands for "real vector space", *tvs* for "real Hausdorff topological vector space", *lcs* for "real locally convex Hausdorff topological vector space", *lsc* for "lowersemicontinuous" and *usc* for "uppersemicontinuous". If E is a tvs we write E' for its topological dual. If $m\geqq 1$ we write σ_m for the subset

$$\{(\lambda_1, \cdots, \lambda_m) : \lambda_1, \cdots, \lambda_m \geqq 0,\ \lambda_1+\cdots+\lambda_m=1\}$$

of $\boldsymbol{R}^m$ and $\boldsymbol{Z}_m=\{0, 1, \cdots, m-1\}$ with addition modulo m. Finally, if $f: X\times Y \to \boldsymbol{R}$ we say that f has property (P) *in its first variable* if, for all $y\in Y$, $f(\cdot, y)$ has property (P); we define *in its second variable* analogously.

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1. Browder-Fan maps.

DEFINITION 1.1. Let X be a nonempty convex subset of a vs E. If $m\geqq 1$ and $x_1, \cdots, x_m\in X$ we write $q[x_1, \cdots, x_m]$ for the map of σ_m into X defined by

$$q[x_1, \cdots, x_m](\lambda)=\lambda_1x_1+\cdots+\lambda_mx_m \qquad (\lambda=(\lambda_1, \cdots, \lambda_m)\in\sigma_m).$$

The *polytopology* of X is the finest topology on X with respect to which all the maps $q[x_1, \cdots, x_m]$ are continuous. (A subset Y of X is polyopen $\Leftrightarrow$ for all $m\geqq 1$ and $x_1, \cdots, x_m\in X$, $q[x_1, \cdots, x_m]^{-1}(Y)$ is open in σ_m.)

We now give some motivation for the above definition. In order to establish our results under the weakest possible hypotheses we shall assume that our vs's are tvs's only when it seems strictly necessary. The polytopology of X is very useful in this connection since it is derived purely from the convexity of X. (The word "polytopology" is an amalgam of "polytope" and "topology"). Any real affine function on X is poly-continuous. If E is a tvs then the polytopology of X is at least as fine as the relative topology.

DEFINITION 1.2. We say that $T: X\to 2^Y$ is *Browder-Fan* (*B-F*) if

X is a nonempty compact convex subset of a tvs,

Y is a nonempty convex subset of a vs,

for all $x\in X$, Tx is a nonempty convex subset of Y

and

$$\text{for all } y \in Y, \quad T^{-1}y \text{ is open in } X.$$

We say that $T: X \to 2^Y$ is *poly-Browder-Fan* (*p-B-F*) if

X and Y are nonempty convex subsets of vs's,

for all $x \in X$, Tx is a nonempty convex subset of Y

and

for all $y \in Y$, $T^{-1}y$ is poly-open in X.

We note that if T is B-F then it is automatically p-B-F. See Remark 1.5 for the reason for the terminology "Browder-Fan."

The following results on selections are suggested by the proof of Browder, [3], Theorem 1, p. 285.

LEMMA 1.3. (a) *Let $T: X \to 2^Y$ be B-F. Then there exist $y_1, \cdots, y_n \in Y$ and a continuous map $f: X \to \sigma_n$ such that*

$$\textit{for all } x \in X, \qquad Tx \ni q[y_1, \cdots, y_n]f(x).$$

(b) *Let $T: X \to 2^Y$ be p-B-F and $x_1, \cdots, x_m \in X$. Then there exist $y_1, \cdots, y_n \in Y$ and a continuous map $f: \sigma_m \to \sigma_n$ such that*

$$\textit{for all } \lambda \in \sigma_m, \qquad Tq[x_1, \cdots, x_m](\lambda) \ni q[y_1, \cdots, y_n]f(\lambda).$$

(c) *Let $T: X \to 2^Y$ be B-F. Let Z be a nonempty convex subset of a vs and $U: X \times Y \to 2^Z$. For all $y_1, \cdots, y_n \in Y$ define $U[y_1, \cdots, y_n]: X \times \sigma_n \to 2^Z$ by*

$$U[y_1, \cdots, y_n](x, \lambda) = U(x, q[y_1, \cdots, y_n](\lambda)) \qquad ((x, \lambda) \in X \times \sigma_n).$$

If

for all $x \in X$ and $y \in Tx$, $U(x, y)$ is a nonempty convex subset of Z

and

for all $z \in Z$ and $y_1, \cdots, y_n \in Y$, $U[y_1, \cdots, y_n]^{-1}(z)$ is open in $X \times \sigma_n$

then there exists a map $g: X \to Y$ such that the map of X into 2^Z defined by

$$x \longrightarrow U(x, g(x))$$

is B-F.

PROOFS. (a) For all $x \in X$, $Tx \neq \emptyset$ hence

$$X = \bigcup_{y \in Y} T^{-1}y.$$

Since X is compact and the sets $T^{-1}y$ are open, there exist $y_1, \cdots, y_n \in Y$ such that

$$X = \bigcup_{j=1}^{n} T^{-1}y_j.$$

Let $p_1, \cdots, p_n$ be a continuous partition of unity on X subordinate to this open covering. Define $f: X \to \sigma_n$ by

$$f(x) = (p_1(x), \cdots, p_n(x)) \qquad (x \in X).$$

Clearly f is continuous. If $x \in X$ let $J = \{j : p_j(x) > 0\}$. Then

$$q[y_1, \cdots, y_n]f(x) = \sum_{j=1}^{n} p_j(x)y_j = \sum_{j \in J} p_j(x)y_j.$$

Now

$$j \in J \implies x \in T^{-1}y_j \implies Tx \ni y_j.$$

Hence, since Tx is convex,

$$Tx \ni q[y_1, \cdots, y_n]f(x)$$

as required.

(b) This follows from (a) since the map of σ_m into 2^Y defined by

$$\lambda \longrightarrow Tq[x_1, \cdots, x_m](\lambda)$$

is B-F.

(c) Let $y_1, \cdots, y_n \in Y$ and $f: X \to \sigma_n$ be as in (a). The result follows with $g = q[y_1, \cdots, y_n]f$. (We note that if $z \in Z$ then $U(x, g(x)) \ni z \Leftrightarrow (x, f(x)) \in U[y_1, \cdots, y_n]^{-1}(z)$.)

We now come to our first cyclic coincidence theorem. This result will eventually be incorporated into Theorem 3.1.

THEOREM 1.4. *Let $m \geqq 1$ and, for each $i \in \boldsymbol{Z}_m$, let $T_i: X_i \to 2^{X_{i+1}}$ be p-B-F. Suppose that there exists $i_0 \in \boldsymbol{Z}_m$ such that T_{i_0} is B-F. Then there exists $(x_0, \cdots, x_{m-1}) \in X_0 \times \cdots \times X_{m-1}$ such that,*

$$\textit{for all } i \in \boldsymbol{Z}_m, \qquad T_i x_i \ni x_{i+1}.$$

PROOF. *Case* 1 ($m=1$). This is similar to, but simpler than, case 2 discussed below and so we leave the details to the reader.

Case 2 ($m \geqq 2$). Without loss of generality we suppose that T_0 is B-F. From Lemma 1.3 (a) there exist $y_1^1, \cdots, y_{n(1)}^1 \in X_1$ and a continuous map $f_0: X_0 \to \sigma_{n(1)}$ such that

$$\text{for all } x \in X_0, \qquad T_0 x \ni q_1 f_0(x), \tag{1.4.1}$$

where q_1 is written for $q[y_1^1, \cdots, y_{n(1)}^1]: \sigma_{n(1)} \to X_1$. We now apply Lemma 1.3 (b) $m-1$ times and, for each $r \in \boldsymbol{Z}_m \setminus \{0\}$, find $y_1^{r+1}, \cdots, y_{n(r+1)}^{r+1} \in X_{r+1}$ and a continuous map $f_r: \sigma_{n(r)} \to \sigma_{n(r+1)}$ such that

$$\text{for all } \lambda \in \sigma_{n(r)}, \qquad T_r q_r(\lambda) \ni q_{r+1} f_r(\lambda), \tag{1.4.2}$$

where q_{r+1} is written for $q[y_1^{r+1}, \cdots, y_{n(r+1)}^{r+1}]: \sigma_{n(r+1)} \to X_{r+1}$. Since $f_{m-1} f_{m-2} \cdots$

$\cdots f_1 f_0 q_0$ is a continuous map of the simplex $\sigma_{n(0)}$ into itself (recall that, in $\boldsymbol{Z}_m$, if $r=m-1$ then $r+1=0$), from Brouwer's theorem there exists $\lambda_0 \in \sigma_{n(0)}$ such that

$$f_{m-1} \cdots f_1 f_0 q_0(\lambda_0) = \lambda_0 . \tag{1.4.3}$$

The result follows from (1.4.1), (1.4.2) and (1.4.3) with

$$x_0 = q_0(\lambda_0)$$

and, for all $r \in \boldsymbol{Z}_m \setminus \{0\}$,

$$x_r = q_r f_{r-1} \cdots f_0 q_0(\lambda_0) .$$

REMARK 1.5. Our next result goes back to Browder, [3], Theorem 1, p. 285 and is equivalent to Fan's minimax inequality, [6], Theorem 1, p. 103. These results are our motivation for the terminology "Browder-Fan".

COROLLARY 1.6. *If $T: X \to 2^X$ is B-F then there exists $x \in X$ such that $Tx \ni x$.*

COROLLARY 1.7. *Let $T: X \to 2^Y$ be p-B-F and $U: Y \to 2^X$ be B-F. Then there exists $(x, y) \in X \times Y$ such that $Tx \ni y$ and $Uy \ni x$.*

Using Corollary 1.7 we can prove the following generalization of a two-function inequality given in Simons [10], Theorem 1 (c), p. 380 and [11], Theorem 1.4. See also Liu [9], Theorem 1, p. 517.

THEOREM 1.8. *Let*

X be a nonempty convex subset of a vs,

Y be a nonempty compact convex subset of a tvs,

$f: X \times Y \to \boldsymbol{R}$ be quasiconcave in its first variable and lsc in its second variable,

$g: X \times Y \to \boldsymbol{R}$ be poly-usc in its first variable and quasiconvex in its second variable

and

$$f \leqq g \quad \text{on } X \times Y.$$

Then

$$\min_Y \sup_X f \leqq \sup_X \inf_Y g .$$

PROOF. If the result were false, we could choose $r \in \boldsymbol{R}$ so that

$$\min_Y \sup_X f > r > \sup_X \inf_Y g .$$

We could then contradict Corollary 1.7 with

$$Tx = \{y : y \in Y,\ g(x, y) < r\} \qquad (x \in X)$$

and

$$Uy = \{x : x \in X,\ f(x, y) > r\} \qquad (y \in Y).$$

REMARK 1.9. Following Ben-El-Mechaiekh, Deguire and Granas [2], Définition 2, p. 381 we say that $T: X \to 2^Y$ is ϕ^* if

X is a nonempty compact convex subset of a tvs,

Y is a nonempty convex subset of a vs,

for all $x \in X$, Tx is a convex subset of Y

and there exists $\tilde{T}: X \to 2^Y$ such that

for all $x \in X$, $\emptyset \neq \tilde{T}x \subset Tx$

and

for all $y \in Y$, $\tilde{T}^{-1}y$ is open in X.

We say that $T: X \to 2^Y$ is *poly-*ϕ^* if

X and Y are nonempty convex subsets of vs's

and

for all $x_1, \cdots, x_m \in X$, $T \circ q[x_1, \cdots, x_m]: \sigma_m \to 2^Y$ is ϕ^*.

Then Lemma 1.3, Theorem 1.4, Corollary 1.6 and Corollary 1.7 remain true with "B-F" replaced by "ϕ^*" and "p-B-F" replaced by "poly-ϕ^*" throughout. The proofs are essentially identical with those given in the text. The modified Lemma 1.3 (a) is then a slightly more precise version of [2], Théorème 2.2, p. 381, the modified Corollary 1.6 is [2], Théorème 3.1, p. 382 and the modified Corollary 1.7 is a generalization of [2], Corollaire 3.4, p. 382. Theorem 1.8 can also be modified to give a generalization of [2], Corollaire 5.5, p. 384.

2. The support map and generalizations of Kakutani's fixed-point theorem.

DEFINITION 2.1. Let X be a nonempty compact convex subset of a tvs E. We define the support map $S: E' \to 2^X$ by

$$Sy = \{x : x \in X,\ \langle x, y \rangle = \max\langle X, y \rangle\} \qquad (y \in E').$$

Theorem 2.2 was proved in a different way in [**11**], Theorem 4.5. It was also shown in [**11**], Remark 4.6 that this result unifies a number of fixed-point theorems for multivalued maps in tvs, lcs and normed space situations due to Browder, Fan and Takahashi. Theorem 2.2 can be deduced from a recent result in a similar vein of Fan [**7**], Theorem 8, p. 526. It can also be deduced from a recent result of a totally different character of Granas and Liu, [**8**], Théorème 2.1, p. 329.

THEOREM 2.2. *Let X be a nonempty compact convex subset of a tvs E and $T: X\to 2^{E'}$ be B-F. Then* (*with S as above*) *there exists $(x, y)\in X\times E'$ such that $Tx\ni y$ and $Sy\ni x$.*

PROOF. Define $U: X\times E'\to 2^X$ by

$$U(x, y)=\{z : z\in X, \langle z, y\rangle>\langle x, y\rangle\} \qquad ((x, y)\in X\times E').$$

Clearly, for all $(x, y)\in X\times E'$, $U(x, y)$ is a (possibly empty) convex subset of X and, for all $z\in X$ and $y_1, \cdots, y_n\in E'$,

$$U[y_1, \cdots, y_n]^{-1}(z)=\{(x, \lambda) : x\in X,\ \lambda\in\sigma_n,\ \sum_{j=1}^{n}\lambda_j\langle z, y_j\rangle>\sum_{j=1}^{n}\lambda_j\langle x, y_j\rangle\}$$

(see Lemma 1.3 (c)) which is open in $X\times\sigma_n$. So if

$$\text{for all } x\in X \text{ and } y\in Tx, \qquad U(x, y)\neq\emptyset \tag{2.2.1}$$

then we could apply Lemma 1.3 (c) and obtain a map $g: X\to E'$ such that the map of X into 2^X defined by $x\to U(x, g(x))$ would be B-F. It would then follow from Corollary 1.6 that there would exist $x\in X$ such that $U(x, g(x))\ni x$. This is impossible from the definition of U. Thus (2.2.1) is false and so

$$\text{there exist } x\in X \text{ and } y\in Tx \text{ such that}$$
$$\text{for all } z\in X, \qquad \langle z, y\rangle\leqq\langle x, y\rangle.$$

This clearly implies that $x\in Sy$ and completes the proof of the theorem.

Here we shall consider just one application of Theorem 2.2, namely we shall show how to deduce Kakutani's original fixed point theorem from it. First we make a definition.

DEFINITION 2.3. We say that $T: X\to 2^Y$ is *of Kakutani type* (*Kt*) if

X is a nonempty convex subset of a tvs,

Y is a nonempty compact convex subset of a lcs F,

for all $x\in X$, Tx is a nonempty closed convex subset of Y

and

T is usc.

LEMMA 2.4 (Kakutani). *Let X be a nonempty compact convex subset of a lcs E and $V: X\to 2^X$ be Kt. Then there exists $x\in X$ such that $Vx\ni x$.*

PROOF. Let $y\in E'$ and $x\in Sy$. Since $\emptyset\neq Vx\subset X$

$$\min\langle Vx, y\rangle\leqq\max\langle X, y\rangle=\langle x, y\rangle. \tag{2.4.1}$$

Define $T: X\to 2^{E'}$ by

$$Tx=\{y:y\in E',\ \langle x, y\rangle<\min\langle Vx, y\rangle\} \qquad (x\in X).$$

It clearly follows from (2.4.1) that

$$x\in Sy \implies y\notin Tx.$$

From Theorem 2.2, T is not B-F. Since V is usc,

$$\text{for all } y\in E',\quad T^{-1}y \text{ is open in } X.$$

Further, by definition,

$$\text{for all } x\in X,\quad Tx \text{ is convex.}$$

It follows that

$$\text{there exists } x\in X \text{ such that } Tx=\emptyset.$$

Since Vx is closed and convex and E is locally convex,

$$\begin{aligned} Tx=\emptyset &\iff \text{for all } y\in E',\quad \min\langle Vx, y\rangle\leqq\langle x, y\rangle \\ &\implies Vx\ni x. \end{aligned}$$

This gives the required result.

We now come to our second cyclic coincidence theorem. As was the case with Theorem 1.4, this result will eventually be incorporated into Theorem 3.1 The case with $k=2$ is Browder [**4**], Theorem 1, p. 70.

THEOREM 2.5. *Let $k\geqq1$ and, for each $h\in \boldsymbol{Z}_k$, let Y_h be a nonempty compact convex subset of a lcs F_h and $V_h: Y_h\to 2^{Y_{h+1}}$ be Kt. Then there exists $(y_0, \cdots, y_{k-1})\in Y_0\times\cdots\times Y_{k-1}$ such that,*

$$\textit{for all } h\in\boldsymbol{Z}_k,\qquad V_h y_h\ni y_{h+1}.$$

PROOF. We suppose that $k\geqq2$ since the case when $k=1$ is essentially Lemma 2.4. Let $X=Y_0\times\cdots\times Y_{k-1}$ and $E=F_0\times\cdots\times F_{k-1}$ and define $V: X\to 2^X$ by

$$V(y_0, \cdots, y_{k-1})=V_{k-1}y_{k-1}\times V_0y_0\times\cdots\times V_{k-2}y_{k-2} \qquad ((y_0, \cdots, y_{k-1})\in Y_0\times\cdots\times Y_{k-1}).$$

Then V is Kt. From Lemma 2.4, there exists

$$x=(y_0, \cdots, y_{k-1})\in X \quad \text{such that}\quad Vx\ni x.$$

This gives the required result.

REMARK 2.6. Theorem 2.2 remains true with "B-F" replaced by "ϕ^*" (see Remark 1.9) throughout.

3. A general cyclic coincidence theorem.

THEOREM 3.1. *Let $m \geqq 1$ and, for each $i \in \boldsymbol{Z}_m$ let $T_i: X_i \to 2^{X_{i+1}}$ be p-B-F or Kt subject to the following restrictions.*

(a) *If T_i is Kt then T_{i+1} is B-F or Kt.*

(b) *There exists $i_0 \in \boldsymbol{Z}_m$ such that T_{i_0} is B-F or Kt.*

Then there exists $(x_0, \cdots, x_{m-1}) \in X_0 \times \cdots \times X_{m-1}$ such that

$$\text{for all } i \in \boldsymbol{Z}_m, \qquad T_i x_i \ni x_{i+1}. \tag{3.1.1}$$

PROOF. In view of Corollary 1.6 and Lemma 2.4, we can suppose that $m \geqq 2$ and, in view of Theorem 1.4, we can suppose that there exists $s \in \boldsymbol{Z}_m$ such that T_s is Kt. Let $s(0) < \cdots < s(k-1)$ be exactly those values of $s \in \boldsymbol{Z}_m$ for which T_s is Kt. For each $h \in \boldsymbol{Z}_k$ let $t(h) = s(h)+1 \in \boldsymbol{Z}_m$ and $Y_h = X_{t(h)}$. For each $h \in \boldsymbol{Z}_k$ we are going to define $V_h: Y_h \to 2^{Y_{h+1}}$.
Case 1 ($t(h+1) = t(h)+1 \in \boldsymbol{Z}_m$). Then $Y_{h+1} = X_{t(h)+1}$. We define $V_h = T_{t(h)}$.
Case 2 ($t(h+1) \neq t(h)+1 \in \boldsymbol{Z}_m$). Here $T_{t(h)}$ is B-F, for all $r = t(h)+1, \cdots, s(h+1)-1$, T_r is p-B-F and $T_{s(h+1)}$ is Kt. Arguing as in Theorem 1.4, there exist continuous maps

$$f_{t(h)}: X_{t(h)} \longrightarrow \sigma_{n(t(h)+1)},$$

$$f_r: \sigma_{n(r)} \longrightarrow \sigma_{n(r+1)} \qquad (r = t(h)+1, \cdots, s(h+1)-1)$$

and

$$q_r: \sigma_{n(r)} \longrightarrow X_r \qquad (r = t(h)+1, \cdots, s(h+1))$$

such that,

$$\text{for all } x \in X_{t(h)}, \qquad T_{t(h)} x \ni q_{t(h)+1} f_{t(h)}(x)$$

and,

$$\text{for all } r = t(h)+1, \cdots, s(h+1)-1 \text{ and } \lambda \in \sigma_{n(r)}, \qquad T_r q_r(\lambda) \ni q_{r+1} f_r(\lambda).$$

We define $V_h: Y_h \to 2^{Y_{h+1}}$ (i.e. $X_{t(h)} \to 2^{X_{s(h+1)+1}}$) by

$$V_h = T_{s(h+1)} q_{s(h+1)} f_{s(h+1)-1} \cdots f_{t(h)}.$$

For all $h \in \boldsymbol{Z}_k$, V_h is Kt. In case 1 this is immediate from the definitions and in case 2 this follows since $T_{s(h+1)}$ is Kt and $q_{s(h+1)}, f_{s(h+1)-1}, \cdots, f_{t(h)}$ are continuous. Thus, from Theorem 2.5, there exists $(y_0, \cdots, y_{k-1}) \in Y_0 \times \cdots \times Y_{k-1}$ such that

$$\text{for all } h \in \boldsymbol{Z}_k, \qquad V_h y_h \ni y_{h+1}.$$

For all $h \in \boldsymbol{Z}_k$ we write

$$x_{t(h)} = y_h \in Y_h = X_{t(h)}.$$

If $r \in \boldsymbol{Z}_m \setminus \{t(h): h \in \boldsymbol{Z}_k\}$ then there exists $h \in \boldsymbol{Z}_k$ such that

$$r \in \{t(h)+1, \cdots, s(h+1)\}.$$

We then write

$$x_r = q_r f_{r-1} \cdots f_{t(h)}(x_{t(h)}) \in X_r .$$

We leave it to the reader to show that (3.1.1) is satisfied.

If we are prepared to assume that the X_i's are all subsets of tvs's then we obtain the following version of Theorem 3.1 that has a somewhat simpler statement. The case with $m=2$, T_0 B-F and T_1 Kt generalizes Browder [**3**], Theorem 7, p. 290, and [**4**], Theorem 3, p. 71, in that the topological conditions are slightly weaker.

COROLLARY 3.2. *Let $m \geqq 1$ and, for each $i \in \boldsymbol{Z}_m$, $T_i : X_i \to 2^{X_{i+1}}$ be B-F or Kt. Then there exists $(x_0, \cdots, x_{m-1}) \in X_0 \times \cdots \times X_{m-1}$ such that*

$$\text{for all } i \in \boldsymbol{Z}_m, \qquad T_i x_i \ni x_{i+1} .$$

Our final result is a version of Corollary 3.2 in terms of sets.

COROLLARY 3.3. *Let $m \geqq 1$ and, for each $i \in \boldsymbol{Z}_m$, X_i be a nonempty convex subset of a tvs E_i, $A_i \subset X_i \times X_{i+1}$ and*

$$\text{for all } x \in X_i, \quad \{y : y \in X_{i+1}, (x, y) \in A_i\} \text{ be nonempty and convex.}$$

Suppose that, for all $i \in \boldsymbol{Z}_m$, either

$$X_i \text{ is compact and, for all } y \in X_{i+1}, \ \{x : x \in X_i, (x, y) \in A_i\} \text{ is open in } X_i$$

or

$$E_{i+1} \text{ is a lcs, } X_{i+1} \text{ is compact in } E_{i+1} \text{ and } A_i \text{ is closed in } X_i \times X_{i+1} .$$

Then there exists $(x_0, \cdots, x_{m-1}) \in X_0 \times \cdots \times X_{m-1}$ such that

$$\text{for all } i \in \boldsymbol{Z}_m , \qquad (x_i, x_{i+1}) \in A_i .$$

PROOF. Immediate from Corollary 3.2 with

$$y \in T_i x \iff (x, y) \in A_i .$$

REMARK 3.4. Theorem 3.1 and Corollary 3.2 remain true with "B-F" replaced by "ϕ^*" and "p-B-F" replaced by "poly-ϕ^*" (see Remark 1.9) throughout.

The results of Browder and Fan discussed in this paper have been extended in a totally different direction in Ben-El-Mechaiekh, Deguire and Granas [**1**], Théorème 2, p. 339.

References

[1] H. Ben-El-Mechaiekh, P. Deguire and A. Granas, Points fixes et coïncidences pour les applications multivoques (applications de Ky Fan), C.R. Acad. Sci. Paris, **295** (1982), 337-340.

[2] H. Ben-El-Mechaiekh, P. Deguire and A. Granas, Points fixes et coïncidences pour les fonctions multivoques II. (Applications de type ϕ et ϕ^*), C.R. Acad. Sci. Paris, **295** (1982), 381-384.

[3] F.E. Browder, The fixed-point theory of multivalued mappings in topological vector spaces, Math. Ann., **177** (1968), 283-301.

[4] F.E. Browder, Coincidence theorems, minimax theorems and variational inequalities, Contemporary Math., **26** (1984), 67-80.

[5] K. Fan, Extensions of two fixed point theorems of F.E. Browder, Math. Z., **112** (1969), 234-240.

[6] K. Fan, A minimax inequality and applications, Inequalities III, Academic Press, 1972, pp. 103-113.

[7] K. Fan, Some properties of convex sets related to fixed point theorem, Math. Ann., **266** (1984), 519-537.

[8] A. Granas and F-C Liu, Théorèmes du minimax, C.R. Acad. Sci. Paris, **298** (1984), 329-332.

[9] F-C Liu, A note on the von Neumann - Sion minimax principle, Bull. Inst. Math. Acad. Sinica, **6** (1978), 517-524.

[10] S. Simons, Minimax and variational inequalities, are they of fixed-point or Hahn-Banach type ?, Game Theory and Mathematical Economics, North-Holland, Amsterdam, 1981, pp. 379-387.

[11] S. Simons, Two-function minimax theorems and variational inequalities for functions on compact and noncompact sets, with some comments on fixed-point theorems, to appear in the proceedings of the 1983 AMS Summer Institute on Nonlinear Functional Analysis and Applications.

[12] W. Takahashi, Nonlinear variational inequalities and fixed point theorems, J. Math. Soc. Japan, **28** (1976), 168-181.

[13] W. Takahashi, Recent results in fixed point theory, Southeast Asian Bull. Math., **4** (1980), 59-85.

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