J. Math. Soc. Japan
Vol. 38, No. 2, 1986

# The $L^p$-boundedness of pseudodifferential operators with estimates of parabolic type and product type

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(Received May 18, 1984)
(Revised Oct. 12, 1984)

## § 0. Introduction.

In this paper we consider symbols $P(x, \xi)$ on $\boldsymbol{R}^n$ whose derivatives do not necessarily converge to 0 as $|\xi|\to\infty$, and we give some sufficient conditions for the $L^p$-boundedness of the associated pseudodifferential operators $P(x, D)$. Some modifications of the Fourier multiplier theorem of Mikhlin type and Stein type are also obtained, together with those of the Littleweed-Paley decomposition of the space $L^p(\boldsymbol{R}^n)$. Part of the results of this paper has been announced in Yamazaki [**15**].

The $L^p$-boundedness of pseudodifferential operators on $\boldsymbol{R}^n$ with non-smooth symbols has been studied by many authors. See Mossaheb-Okada [**8**], Nagase [**10**], Coifman-Meyer [**4**], Muramatu-Nagase [**9**] and Bourdaud [**2**]. They considered symbols $P(x, \xi)$ on $\boldsymbol{R}^n$ satisfying the estimate $|\partial_\xi^\alpha P(x, \xi)|\leqq C_\alpha(1+|\xi|)^{-|\alpha|}$ for every multi-index $\alpha$ satisfying $|\alpha|\leqq n+1$ (or $|\alpha|\leqq n+2$), and obtained the $L^p$-boundedness of the associated pseudodifferential operators $P(x, D)$ defined by the formula

$$P(x, D)u(x)=\int e^{ix\cdot\xi}P(x, \xi)\hat{u}(\xi)\bar{d}\xi$$

under some assumptions on the regularity of the symbol $P(x, \xi)$ with respect to $x$. Here $\bar{d}\xi$ denotes $(2\pi)^{-n}d\xi$, and $\hat{u}(\xi)$ denotes the Fourier transform of $u(x)$. Here and hereafter we assume $1<p<\infty$ and denote $L^p=L^p(\boldsymbol{R}^n)$, and the integrals are done over $\boldsymbol{R}^n$ unless otherwise specified.

On the other hand, Stein [**11**] proved the $L^p$-boundedness of the Fourier multiplier $m(\xi)$ satisfying the estimates $|\xi^\alpha\partial_\xi^\alpha m(\xi)|\leqq C$ for all $\alpha\in\boldsymbol{N}^n$ such that $\alpha_l=0$ or 1 for every $l=1, 2, \cdots, n$. Here the space $\boldsymbol{R}^n$ is regarded as the direct product of $n$ copies of $\boldsymbol{R}$.

Fefferman [**6**] and Fefferman-Stein [**7**] regarded $\boldsymbol{R}^n$ as $\boldsymbol{R}^{n-l}\times\boldsymbol{R}^l$, and obtained several boundedness properties of the singular integrals with kernels $K(y, z)$ ($y\in\boldsymbol{R}^{n-l}$, $z\in\boldsymbol{R}^l$) satisfying the estimate $|K(y, z)|\leqq C|y|^{-n+l}|z|^{-l}$ under some hypotheses.

The purpose of this paper is to obtain the $L^p$-boundedness of the pseudodifferential operators whose symbols satisfy the estimates corresponding to one of the identifications $\boldsymbol{R}^n=\boldsymbol{R}^{n(1)}\times\cdots\times\boldsymbol{R}^{n(N)}$, where $n(1),\cdots,n(N)$ are positive integers satisfying $n(1)+\cdots+n(N)=n$.

For example, we can prove the following result corresponding to the identification $\boldsymbol{R}^n=\boldsymbol{R}\times\cdots\times\boldsymbol{R}$:

THEOREM 1. *Let $\omega$ be a continuous, monotone-increasing, concave function on $\boldsymbol{R}^+=\{t; t\geqq 0\}$ into itself satisfying the condition*

$$\int_0^1 t^{-1}(-\log t)^{n-1}\omega(t)^2dt<\infty.$$

*Suppose that a symbol $P(x,\xi)$ satisfies the estimates*

$$|\partial_{\xi_l}^k P(x,\xi)|\leqq C(1+\xi_l^2)^{-k/2}$$

*and*

$$|\partial_{\xi_l}^k P(x,\xi)-\partial_{\xi_l}^k P(y,\xi)|\leqq C\omega(|x-y|)(1+\xi_l^2)^{-k/2}$$

*for every $x, y, \xi\in\boldsymbol{R}^n$, $l=1, 2,\cdots, n$ and $k=0, 1,\cdots, n+1$. Then the associated pseudodifferential operator $P(x, D)$ is bounded on $L^p$.*

This theorem is an immediate consequence of our main theorem (Theorem 2). The latter can also be applied to the symbols satisfying estimates of parabolic type with respect to the weight function introduced by Fabes-Rivière [**5**].

The outline of this paper is as follows. In Section 1 we state our main theorem and derive Theorem 1 from it. For this purpose we generalize the notion of the modulus of continuity introduced in Coifman-Meyer [**4**], and we introduce several notations.

In Section 2 we establish a generalization of the multiplier theorem of Mikhlin-Hörmander type for the functions satisfying anisotropic estimates with shift. For references, see Triebel [**12**] and the papers cited there. Our proof is a modification of the method used in [**12**].

In Section 3 we obtain some versions of the Littlewood-Paley decomposition theorem of $L^p$ of parabolic and product type. The results in this section will also be used in the forthcoming papers [**14**].

In Section 4 we prove the necessity of the condition (1.1) in Theorem 2 by constructing a symbol which is not bounded on $L^p$ for any $1<p<\infty$.

The sufficiency of (1.1) will be proved in Section 5. Our methods employed in these two sections are modifications of those of Coifman-Meyer [**4**].

Finally, in Section 6, we give two generalizations of Theorem 2.

## §1. Notations and statement of the main theorem.

First we put $\Lambda(\nu)=\{l\in\boldsymbol{N};\ n(1)+\cdots+n(\nu-1)<l\leqq n(1)+\cdots+n(\nu)\}$ for each

$\nu=1, \cdots, N$ and denote $x\in \boldsymbol{R}^n$ as $(x^{(1)}, \cdots, x^{(N)})$, where $x^{(\nu)}=(x_l)_{l\in\Lambda(\nu)}\in \boldsymbol{R}^{n(\nu)}$. We give a weight $M=(M^{(1)}, \cdots, M^{(N)})$ to $\boldsymbol{R}^n$, where each $M^{(\nu)}=(m_l)_{l\in\Lambda(\nu)}$ satisfies $\min_{l\in\Lambda(\nu)}m_l=1$, and we put $m=\max_{l=1,\dots,n}m_l$ and $|M^{(\nu)}|=\sum_{l\in\Lambda(\nu)}m_l$ for each $\nu=1, \cdots, N$.

Next, as in Fabes-Rivière [5] and Calderón-Torchinsky [3], we define the action of $t\in \boldsymbol{R}^+$ to $y=(y_l)_{l\in\Lambda(\nu)}\in \boldsymbol{R}^{n(\nu)}$ by $t^{M^{(\nu)}}y=(t^{m_l}y_l)_{l\in\Lambda(\nu)}$, and denote by $[y]_\nu$ the only positive number $t$ satisfying $t^{-M^{(\nu)}}y=(t^{-1})^{M^{(\nu)}}y\in\{y\in \boldsymbol{R}^{n(\nu)}; |y|=1\}$. For $y=0$ we set $[0]=0$.

If $f(x)$ is a function on $\boldsymbol{R}^n$, then we denote by $\Delta_y^\nu$ the difference of first order with respect to the $\nu$-th part of the coordinate variables; that is, we write

$$\Delta_y^\nu f(x)=f(x^{(1)}, \cdots, x^{(\nu)}-y, \cdots, x^{(N)})-f(x)$$

for $\nu=1, 2, \cdots, N$ and $y\in \boldsymbol{R}^{n(\nu)}$.

Next we generalize the notion of the modulus of continuity.

DEFINITION. We call a set of functions $\{\omega_1(t_1), \omega_2(t_1, t_2), \cdots, \omega_N(t_1, t_2, \cdots, t_N)\}$ a *modulus of continuity* if it satisfies the following three conditions:

1) For each $\nu=1, \cdots, N$ the function $\omega_\nu(t_1, \cdots, t_\nu)$ is continuous of $(\boldsymbol{R}^+)^\nu$ into $\boldsymbol{R}^+$, and is concave, monotone-increasing for each $t_l$.

2) $\omega_\nu(t_1, \cdots, t_\nu)$ is invariant under any permutation on the variables $t_1, \cdots, t_\nu$.

3) For each $1\leqq\mu<\nu\leqq N$ we have

$$\omega_\nu(t_1, \cdots, t_\nu)\leqq 2^{\nu-\mu}\omega_\mu(t_1, \cdots, t_\mu).$$

Using the above definition, we consider the conditions $(*\mu)$ $(\mu=0, 1, \cdots, N)$ on symbols $P(x, \xi)$ as follows:

$(*0)$ For every $\nu=1, 2, \cdots N$, $l\in\Lambda(\nu)$ and $k=0, 1, \cdots, n+1$ we have

$$|\partial_{\xi_l}^k P(x, \xi)|\leqq C(1+[\xi^{(\nu)}]_\nu)^{-m_l k}.$$

$(*\mu)$ $(\mu=1, 2, \cdots, N)$ For every $\nu=1, 2, \cdots, N$, $l\in\Lambda(\nu)$, $k=0, 1, \cdots, n+1$, $1\leqq\nu(1)<\cdots<\nu(\mu)\leqq N$ and $y(1)\in \boldsymbol{R}^{n(\nu(1))}, \cdots, y(\mu)\in \boldsymbol{R}^{n(\nu(\mu))}$ we have

$$|\Delta_{y(1)}^{\nu(1)}\cdots\Delta_{y(\mu)}^{\nu(\mu)}\partial_{\xi_l}^k P(x, \xi)|\leqq C\omega_\mu(|y(1)|, \cdots, |y(\mu)|)(1+[\xi^{(\nu)}]_\nu)^{-m_l k}.$$

REMARK. Since the inequality

$$\sup_{x\in \boldsymbol{R}^n}|\Delta_{y(1)}^{\nu(1)}\cdots\Delta_{y(\mu)}^{\nu(\mu)}\partial_{\xi_l}^k P(x, \xi)|\leqq 2^{\mu-\lambda}\sup_{x\in \boldsymbol{R}^n}|\Delta_{y(\kappa(1))}^{\nu(\kappa(1))}\cdots\Delta_{y(\kappa(\lambda))}^{\nu(\kappa(\lambda))}\partial_{\xi_l}^k P(x, \xi)|$$

holds for every $1\leqq\lambda<\mu$ and $1\leqq\kappa(1)<\cdots<\kappa(\lambda)\leqq\mu$, the condition 3) in the definition causes no loss of generality.

Now we can state our main theorem.

THEOREM 2. *The following three conditions concerning moduli of continuity are equivalent:*

1) *For every* $\nu=1, 2, \cdots, N$ *we have*

$$\int_0^1\cdots\int_0^1 \frac{\omega_\nu(t_1, \cdots, t_\nu)^2}{t_1\cdots t_\nu} dt_1\cdots dt_\nu<\infty. \tag{1.1}$$

2) *If a symbol* $P(x, \xi)$ *satisfies the condition* $(*\mu)$ *for all* $\mu=0, 1, \cdots, N$, *then the associated operator* $P(x, D)$ *is bounded on* $L^p$ *for every* $1<p<\infty$.
3) *For every symbol* $P(x, \xi)$ *satisfying the conditions* $(*\mu)$ *for all* $\mu=0, 1, \cdots, N$ *there exists* $1<p<\infty$ *such that the operator* $P(x, D)$ *is bounded on* $L^p$.

In order to find a condition on $\omega_1(t)$ which implies the hypothesis 1), suppose that $\{\omega_1(t_1), \cdots, \omega_N(t_1, \cdots, t_N)\}$ is a modulus of continuity. Then we have

$$\begin{aligned}
&\int_0^1\cdots\int_0^1 \frac{\omega_\nu(t_1, \cdots, t_\nu)^2}{t_1\cdots t_\nu} dt_1\cdots dt_\nu \\
&\leqq\int_0^1\cdots\int_0^1 \frac{\{2^{\nu-1}\omega_1(\min\{t_j\})\}^2}{t_1\cdots t_\nu} dt_1\cdots dt_\nu \\
&=\nu\int_0^1\int_{t_1}^1\cdots\int_{t_1}^1 \frac{dt_2\cdots dt_\nu}{t_2\cdots t_\nu}\cdot 4^{\nu-1}\frac{\omega_1(t_1)^2}{t_1} dt_1 \\
&=\nu 4^{\nu-1}\int_0^1(-\log t)^{\nu-1}\cdot\frac{\omega_1(t)^2}{t} dt.
\end{aligned} \tag{1.2}$$

Hence the hypothesis 1) is satisfied if

$$\int_0^1(-\log t)^{N-1}\frac{\omega_1(t)^2}{t} dt<\infty. \tag{1.3}$$

Putting $N=n$ and $n(1)=n(2)=\cdots=n(n)=1$, we have Theorem 1 immediately.

Conversely, let $\omega_1(t)$ be a continuous, monotone-increasing, concave function which does not satisfy (1.3). Then, by putting $\omega_\nu(t_1, \cdots, t_\nu)=2^{\nu-1}\omega_1(\min\{t_1, \cdots, t_\nu\})$, we can construct a modulus of continuity $\{\omega_1(t_1), \cdots, \omega_N(t_1, \cdots, t_N)\}$ which does not satisfy the condition (1.1), since the equality in (1.2) holds in this case. Hence the condition in Theorem 1 is sharp.

If $\omega_1(t)$ is of the form $(1-\log t)^\delta$, then (1.3) holds if and only if $\delta<-N/2$. If $\omega_1(t)=(1-\log t)^{-N/2}\{1+\log(1-\log t)\}^\delta$, then (1.3) holds if and only if $\delta<-1/2$.

On the other hand, by putting $N=1$, $M^{(1)}=M$ and $[\cdot]_1=[\cdot]$, we have the following result on the symbols satisfying estimates of parabolic type, which is a modification of Theorem 7 in [**14**].

COROLLARY. *Let* $\omega(t)$ *be as above, and assume that*

$$\int_0^1\frac{\omega(t)^2}{t} dt<\infty.$$

*If a symbol* $P(x, \xi)$ *satisfies the estimates*

$$|\partial_{\xi_l}^k P(x, \xi)|\leqq C(1+[\xi])^{-m_l k}$$

*and*

$$|\partial_{\xi_l}^k P(x, \xi)-\partial_{\xi_l}^k P(y, \xi)| \leqq C\omega(|x-y|)(1+[\xi])^{-m_l k}$$

*for every* $l=1, \cdots, n$ *and* $k=0, 1, \cdots, n+1$, *then the associated operator* $P(x, D)$ *is* $L^p$*-bounded.*

Comparing this corollary with Theorem 1, we see easily that the former requires less regularity of $P(x, \xi)$ with respect to $x$. On the other hand, the latter can be applied to the symbols whose derivatives do not necessarily converge to 0 as $|\xi|\to\infty$.

## § 2. Quasi-homogeneous Fourier multipliers.

In this section we consider the case $N=1$, and denote $[\cdot]_1$ simply by $[\cdot]$. For a Lebesgue measurable subset $E$ of $\boldsymbol{R}^n$, let $\mu(E)$ denote the Lebesgue measure of $E$. For a Banach space $X$ and $1\leqq p<\infty$, we denote by $L^p(X)$ the set of strongly measurable $X$-valued functions $f(x)$ on $\boldsymbol{R}^n$ satisfying

$$\|f\|_{L^p(X)}=\left(\int\|f(x)\|_X^p dx\right)^{1/p}<\infty$$

as in Triebel [**12**], and we denote $L^p(\boldsymbol{C})$ simply by $L^p$.

We start with some properties of our weight function $[\cdot]$.

LEMMA 2.1. *For* $\xi, \eta\in\boldsymbol{R}^n$ *and* $0\leqq t<\infty$, *we have the following:*

1) $[\xi+\eta]\leqq[\xi]+[\eta]$.

2) $[t^M\xi]=t[\xi]$.

3) $\min\{|\xi|, |\xi|^{1/m}\}\leqq[\xi]\leqq\max\{|\xi|, |\xi|^{1/m}\}$.

4) $[\xi]$ *is a* $C^\infty$*-function of* $\xi\in\boldsymbol{R}^n\setminus\{0\}$, *and for every real number* $s$ *and for every multi-index* $\alpha$ *there exists a constant* $C_{s,\alpha}$ *such that the estimate* $|\partial_\xi^\alpha([\xi]^s)|\leqq C_{s,\alpha}[\xi]^{s-M\cdot\alpha}$ *holds for every* $\xi\in\boldsymbol{R}^n$.

PROOF. The assertion 1) is exactly the same as Remark 1 of Fabes-Rivière [**5**]. To prove the assertions 2) and 3), put $s=[\xi]$. Then we have $[(st)^{-M}t^M\xi]=[s^{-M}\xi]=1$, which implies the assertion 2). Also we have $|s^{-m}\xi|\leqq|s^{-M}\xi|\leqq|s^{-1}\xi|$ or $|s^{-m}\xi|\geqq|s^{-M}\xi|\geqq|s^{-1}\xi|$ according as $s\geqq1$ or $s\leqq1$, which implies the assertion 3). The smoothness of the function $[\cdot]$ follows from the implicit function theorem. Finally, we can derive the estimate of the derivatives from the quasi-homogeneity (the assertion 2)) of the derivative $\partial_\xi^\alpha[\xi]$.

Next we shall show a general statement on the boundedness of convolution operators. For this purpose we need the following

LEMMA 2.2. *Let* $f(x)$ *be a function in* $L^1$ *such that* $f(x)\geqq0$ *a.e., and* $\tau$ *be a positive number. Then there exist a sequence of positive numbers* $\{e_k\}$ *and a*

*sequence of rectangles* $\{I_k\}$ *satisfying the following conditions:*
1) *The edges of each rectangle are parallel to the coordinate axes, and the length of the edges of* $I_k$ *parallel to the* $x_l$*-axis is equal to* $e_k^{m_l}$.

2) $I_j^\circ \cap I_k^\circ = \emptyset$ *if* $j \neq k$.

3) $f(x) \leqq \tau$ *for almost all* $x \notin I$, *where* $I = \bigcup_{k=1}^{\infty} I_k$.

4) $\tau \leqq \mu(I_k)^{-1} \int_{I_k} f(x)dx \leqq C_0\tau$, *where* $C_0$ *is a constant independent of* $\tau > 0$.

This lemma is the same as the sublemma to Lemma 2 of Fabes-Rivière [**5**]. By virtue of this lemma we can prove the following

PROPOSITION 2.3. *Let* $X$, $Y$ *be Banach spaces and* $K(x)$ *be a locally strongly integrable mapping of* $\boldsymbol{R}^n$ *into* $L(X, Y)$. *For an* $X$*-valued simple function* $f(y)$, *we define a* $Y$*-valued measurable function* $K^*f(x)$ *by* $K^*f(x) = \int K(x-y)f(y)dy$. *Suppose* $A > 0$, $1 < p \leqq r < \infty$, $1/p - 1/r = 1 - 1/q$ *and that the following two conditions hold:*
1) *There exists a constant* $B > 2$ *such that, for any* $t > 0$ *and* $y \in \boldsymbol{R}^n$ *satisfying* $[y] \leqq tB^{-1}$, *we have*

$$\int_{[x] \geqq tB} \|K(x-y) - K(x)\|_{L(X,Y)}^q dx \leqq A^q.$$

2) *For any* $X$*-valued simple function* $f(y)$ *we have*

$$\|K^*f\|_{L^r(Y)} \leqq A\|f\|_{L^p(X)}. \tag{2.1}$$

*Then, for every constants* $s$ *and* $\sigma$ *satisfying* $1 < s \leqq \sigma < \infty$ *and* $1/s - 1/\sigma = 1 - 1/q$, *there exists a positive constant* $C$ *depending only on* $n$, $M$, $B$, $p$, $r$, $s$ *such that* $\|K^*f\|_{L^\sigma(Y)} \leqq CA\|f\|_{L^s(X)}$ *holds for every* $X$*-valued simple function* $f(x)$.

REMARK. From the conclusion of the proposition and the fact that the set of $X$-valued simple functions is dense in $L^s(X)$ $(1 < s < \infty)$, it follows immediately that the operator $K^*$ can be extended to $L^s(X)$ and that the same inequality holds for all $f \in L^s(X)$.

PROOF OF THE PROPOSITION. By multiplying $K$ by a constant, we may assume $A = 1$. First we consider the case $1 < s < p$.

Let $I$ and $I'$ be rectangles defined by

$$I = \{x \in \boldsymbol{R}^n;\ |x_l - z_l| \leqq L^{m_l}\} \quad \text{and} \quad I' = \{x \in \boldsymbol{R}^n;\ |x_l - z_l| \leqq \sqrt{n} L^{m_l} B^2\},$$

where $z \in \boldsymbol{R}^n$. If $y \in I$, then it follows that

$$L^{-1}[y-z] = [L^{-M}(y-z)] \leqq [(1, 1, \cdots, 1)] \leqq \sqrt{n}.$$

Hence, putting $t=\sqrt{n}LB$, we have $[y-z]\leqq tB^{-1}$ if $y\in I$. On the other hand, if $[x-z]\leqq tB=\sqrt{n}LB^2$, then $x\in I'$.

Next, let $w(y)$ be an $X$-valued simple function satisfying $w(y)=0$ $(y\notin I)$ and $\int_I w(y)dy=0$. Then we have

$$K*w(x)=\int_{y\in I}K(x-y)w(y)dy=\int_{[y-z]\leqq tB^{-1}}(K(x-y)-K(x))w(y)dy.$$

Hence, by the generalized Minkowski inequality, we obtain

$$\Big(\int_{[x-z]\geqq tB}\|K*w(x)\|_Y^q dx\Big)^{1/q}$$
$$\leqq\Big(\int_{[x-z]\geqq tB}\Big(\int_{[y-z]\leqq tB^{-1}}\|K(x-y)-K(x)\|_{L(X,Y)}\|w(y)\|_X dy\Big)^q dx\Big)^{1/q}$$
$$\leqq\int\|w(y)\|_X dy,$$

that is,

$$\Big(\int_{x\in I'}\|K*w(x)\|_Y^q dx\Big)^{1/q}\leqq\|w\|_{L^1(X)}.$$

Now suppose that $R>0$ and that $f$ is an $X$-valued simple function. We apply Lemma 2.2 to the function $\|f(x)\|_X\in L^1$ and the number $\tau=R^q\|f(x)\|_{L^1(X)}^{1-q}$, and get the sequences $\{I_k\}$ and $\{e_k\}$.

Next, for every $N\in\boldsymbol{N}$ we put

$$f^{(N)}(x)=\begin{cases}f(x) & \Big(x\notin I \text{ or } x\in\bigcup_{k=1}^{N}I_k\Big)\\ 0 & \Big(x\in\bigcup_{k=N+1}^{\infty}I_k\Big),\end{cases}$$

$$h(x)=\begin{cases}f(x) & (x\notin I)\\ 0 & \Big(x\in I\setminus\bigcup_{k=1}^{N}I_k\Big)\\ \mu(I_k)^{-1}\int_{I_k}f(x)dx & (x\in I_k,\ k\leqq N)\end{cases}$$

and

$$g_k(x)=\begin{cases}0 & (x\notin I_k)\\ f(x)-h(x) & (x\in I_k)\end{cases}$$

for $k=1, 2, \cdots, N$, and set

$$I'_k=\{x\in\boldsymbol{R}^n;\ |x_l-z_l^{(k)}|\leqq\sqrt{n}e_k^{m_l}B^2/2\},$$

where $(z_1^{(k)}, \cdots, z_n^{(k)})$ is the center of the rectangle $I_k$.

Then $h(x)$ and $g_k(x)$ are $X$-valued simple functions, and from the above argument and Lemma 2.2 we obtain

$$\left(\int_{R^n\setminus I'_k}\|K*g_k(x)\|_Y^q dx\right)^{1/q}\leqq\|g_k(x)\|_{L^1(X)}\leqq 2\int_{I_k}\|f(x)\|_X dx.$$

Hence, putting $I'=\bigcup_{k=1}^{\infty}I'_k$, we have

$$\left(\int_{R^n\setminus I'}\left\|K*\sum_{k=1}^{N}g_k(x)\right\|_Y^q dx\right)^{1/q}\leqq\sum_{k=1}^{N}\left(\int_{R^n\setminus I'_k}\|K*g_k\|_Y^q dx\right)^{1/q}$$

$$\leqq\sum_{k=1}^{N}2\int_{I_k}\|f(x)\|_X dx\leqq 2\|f\|_{L^1(X)}.$$

It follows that

(2.2)
$$\mu\left(\left\{x\in \boldsymbol{R}^n;\ \left\|K*\sum_{k=1}^{N}g_k(x)\right\|_Y>R/2\right\}\right)$$

$$\leqq\mu(I')+\mu\left(\left\{x\in\boldsymbol{R}^n\setminus I';\ \left\|K*\sum_{k=1}^{N}g_k(x)\right\|_Y>R/2\right\}\right)$$

$$\leqq\sum_{k=1}^{\infty}(2\sqrt{n})^n\mu(I_k)+(R/2)^{-q}\int_{R^n\setminus I'}\left\|K*\sum_{k=1}^{N}g_k(x)\right\|_Y^q dx$$

$$\leqq(2\sqrt{n})^n\sum_{k=1}^{\infty}\tau^{-1}\int_{I_k}\|f(x)\|_X dx+2^qR^{-q}\cdot 2^q\|f\|_{L^1(X)}^q$$

$$\leqq C(\tau^{-1}\|f\|_{L^1(X)}+R^{-q}\|f\|_{L^1(X)}^q)$$

$$\leqq CR^{-q}\|f\|_{L^1(X)}^q,$$

where $C$ is a constant independent of $N$.

On the other hand, from the fact that $\|h(x)\|_X\leqq C_0\tau$ and

$$\|h(x)\|_{L^1(X)}\leqq\int_{R^n\setminus I}\|f(x)\|_X dx+\sum_{k=1}^{N}\left\|\int_{I_k}f(x)dx\right\|_X\leqq\|f(x)\|_{L^1(X)},$$

we obtain

$$\|h(x)\|_{L^p(X)}\leqq(C_0\tau)^{1-1/p}\|f(x)\|_{L^1(X)}^{1/p},$$

which implies

(2.3)
$$\mu(\{x\in\boldsymbol{R}^n;\ \|K*h(x)\|_Y>R/2\})\leqq(R/2)^{-r}\|K*h(x)\|_{L^r(Y)}^r$$

$$\leqq 2^rR^{-r}\{C_0\tau\|f(x)\|_{L^1(X)}^{-1}\}^{r-r/p}\|f(x)\|_{L^1(X)}^r$$

$$=2^rC_0^{r-r/p}R^{-q}\|f(x)\|_{L^1(X)}^q.$$

Combining (2.2) and (2.3) we obtain

$$\mu(\{x\in\boldsymbol{R}^n;\ \|K*f^{(N)}(x)\|_Y>R\})\leqq CR^{-q}\|f(x)\|_{L^1(X)}^q$$

with a constant $C$ independent of $N$.

It follows that

$$\mu(\{x\in\boldsymbol{R}^n;\ \|K*f(x)\|_Y>R\})\leqq CR^{-q}\|f(x)\|_{L^1(X)}^q$$

for every $X$-valued simple function $f(x)$; that is, the operator $K^*$ is of weak type $(1, q)$. From this and hypothesis 2), we obtain the conclusion in case $1<s<p$ by virtue of the Marcinkiewicz interpolation theorem. We can prove the conclusion in case $p<s\leqq\sigma<\infty$ by the standard duality argument, using the following lemma.

LEMMA 2.4. *Let $X$ be a Banach space and $X'$ be its dual. Suppose $1<p$, $p'<\infty$ and $1/p+1/p'=1$. Then an $X$-valued (resp. $X'$-valued) strongly measurable function $f(x)$ belongs to the space $L^{p'}(X)$ (resp. $L^{p'}(X')$) if and only if the functional $\tilde{f}: g(x)\mapsto\int\langle f(x), g(x)\rangle dx$ belongs to $(L^p(X'))'$ (resp. $(L^p(X))'$). Furthermore, we have $\|f\|_{L^{p'}(X)}=\|\tilde{f}\|_{(L^p(X'))'}$ (resp. $\|f\|_{L^{p'}(X')}=\|\tilde{f}\|_{(L^p(X))'}$).*

This lemma can be verified by approximating $f(x)$ by simple functions, and the proof will be omitted.

Now we can prove the main result of this section. In the following theorem we assume $1<p<\infty$, and that $X$ and $Y$ are Hilbert spaces, and denote by $LH(X, Y)$ the Hilbert space of all Hilbert-Schmidt operators of $X$ into $Y$, equipped with the Hilbert-Schmidt norm. For every $l=1, \cdots, n$, let $k_l$ be the least natural number that satisfies $m_l k_l>|M|/2$, and let $\Psi(t)$ be a real-valued $C^\infty$ function on $\boldsymbol{R}$ satisfying $0\leqq\Psi(t)\leqq1$, $\Psi(t)=1$ if $t\leqq1$, and $\Psi(t)=0$ if $t\geqq4/3$. This function $\Psi(t)$ will be fixed throughout this paper.

THEOREM 2.5. *Let $K(\xi)$ be a continuous function of $\boldsymbol{R}^n$ into $LH(X, Y)$ such that $\partial_{\xi_l}^k K(\xi)$ exists for every $l=1, 2, \cdots, n$ and $k=1, 2, \cdots, k_l$. If there exists a sequence $\{a_j\}_{j\in\boldsymbol{Z}}$ of elements of $\boldsymbol{R}^n$ such that we have the estimate*

$$\int_{2^{j-1}<[\xi]<2^{j+1}}\|\partial_{\xi_l}^k\{\exp(-i2^{-jM}a_j\cdot\xi)K(\xi)\}\|_{LH(X,Y)}^2 d\xi\leqq A^2 2^{j(|M|-2m_l k)}$$

*for every $j\in\boldsymbol{Z}$, $l=1, 2, \cdots, n$ and $k=0, 1, \cdots, k_l$, then we have*

$$\|F^{-1}[K(\xi)\hat{u}(\xi)](x)\|_{L^p(Y)}\leqq CA\log(2+\sup_j|a_j|)\|u\|_{L^p(X)}$$

*for every $X$-valued simple function $u(x)$, where $C$ is a constant independent of the sequence $\{a_j\}$.*

PROOF. We put $\phi_j(\xi)=\Psi(2^{-j}[\xi])-\Psi(2^{1-j}[\xi])$, $K_j(\xi)=\phi_j(\xi)K(\xi)$ and $G_j(x)=F^{-1}[K_j(\xi)](x)$ for $j\in\boldsymbol{Z}$, and $G^{(N)}(x)=\sum_{j=-N}^N G_j(x)$ for $N\in\boldsymbol{N}$. We shall apply Proposition 2.3 to $G^{(N)}$. For $l=1, 2, \cdots, N$ we have

$$\left(\int\|\partial_{\xi_l}^{k_l}\{\exp(-i2^{-jM}a_j\cdot\xi)K_j(\xi)\}\|_{LH(X,Y)}^2 d\xi\right)^{1/2}$$

$$\leqq\sum_{h\leqq k_l}\binom{k_l}{h}\left(\int|\partial_{\xi_l}^{k_l-h}\phi_j(\xi)|^2\cdot\|\partial_{\xi_l}^h\{\exp(-i2^{-jM}a_j\cdot\xi)K(\xi)\}\|_{LH(X,Y)}^2 d\xi\right)^{1/2}$$

$$\leqq \sum_{h\leqq k_l}\binom{k_l}{h}C\cdot 2^{-jm_l(k_l-h)}(A^2 2^{j(|M|-2m_l h)})^{1/2}$$

$$\leqq C'A\cdot 2^{j(|M|/2-m_l k_l)}.$$

Hence, setting $J_l=\{x\in \boldsymbol{R}^n;\ [x]^{-2m_l}x_l^2\geqq 1/n\}$, we obtain

$$\int_{[x]\geqq t,\, x\in J_l}\|G_j(x-2^{-jM}a_j)\|_{LH(X,Y)}dx$$

$$\leqq\left(\int_{[x]\geqq t,\, x\in J_l}x_l^{-2k_l}dx\right)^{1/2}\cdot\left(\int x_l^{2k_l}\|\mathcal{F}^{-1}[\exp(-i2^{-jM}a_j\cdot\xi)K_j(\xi)](x)\|_{LH(X,Y)}^2dx\right)^{1/2}$$

$$\leqq\left(n^{k_l}\int_{[x]\geqq 1}[t^M x]^{-2k_l m_l}\cdot t^{|M|}dx\right)^{1/2}\cdot\left(\int\|\partial_{\xi_l}^{k_l}\{\exp(-i2^{-jM}a_j\cdot\xi)K_j(\xi)\}\|_{LH(X,Y)}^2 \bar{d}\xi\right)^{1/2}$$

$$\leqq(C\cdot t^{-2k_l m_l+|M|})^{1/2}\cdot C'A\cdot 2^{j|M|/2-jk_l m_l}$$

$$=CA(2^j t)^{|M|/2-m_l k_l}$$

by virtue of the Plancherel formula for the Hilbert space $LH(X, Y)$. (See Bergh-Löfström [**1**] and Triebel [**12**].)

Since Lemma 2.1 implies $\boldsymbol{R}^n=\bigcup_{l=1}^n J_l$, we have

$$\int_{[x]\geqq t}\|G_j(x-2^{-jM}a_j)\|_{LH(X,Y)}dx\leqq nCA(2^j t)^{|M|/2-m_l k_l}$$

which implies

$$\int_{[x]\geqq t}\|G_j(x-y)-G_j(x)\|_{LH(X,Y)}dx\leqq 2nCA(2^j t)^{|M|/2-m_l k_l} \tag{2.4}$$

if $[y]\leqq t/2$ and $[2^{-jM}a_j]<t/2$.

Next, in general we have

$$\int\|G_j(x-2^{-jM}a_j)\|_{LH(X,Y)}dx=\int\|G_j(x)\|_{LH(X,Y)}dx \tag{2.5}$$

$$\leqq\left(\int\left(1+\sum_{l=2}^{n}2^{2jk_l m_l}x_l^{2k_l}\right)^{-1}dx\right)^{1/2}$$

$$\cdot\left(\int\Big\{\|\mathcal{F}^{-1}[\exp(-i2^{-jM}a_j\cdot\xi)K_j(\xi)](x)\|_{LH(X,Y)}^2\right.$$

$$\left.+\sum_{l=1}^{n}\|\mathcal{F}^{-1}[2^{jk_l m_l}\partial_{\xi_l}^{k_l}\{\exp(-i2^{-jM}a_j\cdot\xi)K_j(\xi)\}](x)\|_{LH(X,Y)}^2\Big\}\bar{d}\xi\right)^{1/2}$$

$$\leqq\left(2^{-j|M|}\int\left(1+\sum_{l=1}^{n}x_l^{2k_l}\right)^{-1}dx\right)^{1/2}\cdot C'AC2^{j|M|/2}$$

$$=CA.$$

Finally, if $2^j t\leqq 1$ and $[y]\leqq t$ we have

(2.6)
$$\int \|G_j(x-2^{-jM}a_j-y)-G_j(x-2^{-jM}a_j)\|_{LH(X,Y)}dx$$
$$\leqq\left(\int\left(1+\sum_{l=1}^{n}2^{2jk_l m_l}x_l^{2k_l}\right)^{-1}dx\right)^{1/2}$$
$$\cdot\left(\int\left\{\|\mathcal{F}^{-1}[\{\exp(-iy\cdot\xi-i2^{-jM}a_j\cdot\xi)-\exp(-i2^{-jM}a_j\cdot\xi)\}K_j(\xi)](x)\|_{LH(X,Y)}^2\right.\right.$$
$$+\sum_{l=1}^{n}\|\mathcal{F}^{-1}[2^{jk_l m_l}\partial_{\xi_l}^{k_l}(\{\exp(-iy\cdot\xi-i2^{-jM}a_j\cdot\xi)$$
$$\left.\left.-\exp(-i2^{-jM}a_j\cdot\xi)\}K_j(\xi))](x)\|_{LH(X,Y)}^2\right\}dx\right)^{1/2}$$
$$\leqq C\cdot 2^{-j|M|/2}\left(\int\left\{\|\{\exp(-iy\cdot\xi)-1\}\cdot\exp(-i2^{-jM}a_j\cdot\xi)K_j(\xi)\|_{LH(X,Y)}^2\right.\right.$$
$$\left.\left.+\sum_{l=1}^{n}\|2^{jk_l m_l}\partial_{\xi_l}^{k_l}(\{\exp(-iy\cdot\xi)-1\}\cdot\exp(-i2^{-jM}a_j\cdot\xi)K_j(\xi))\|_{LH(X,Y)}^2\right\}\bar{d}\xi\right)^{1/2}.$$

From Lemma 2.1, we obtain the inequalities
$$|e^{-iy\cdot\xi}-1|\leqq|y|\,|\xi|\leqq n\cdot\max_l|y_l|\,|\xi_l|\leqq n\cdot\max_l[y]^{m_l}[\xi]^{m_l}\leqq C2^jt$$
and
$$|\partial_{\xi_l}^k(e^{-iy\cdot\xi}-1)|=|\partial_{\xi_l}^k e^{-iy\cdot\xi}|=|y_l|^k\leqq[y]^{m_l k}\leqq t^{m_l k}$$
$$\leqq(2^jt)^{1-m_l k}\cdot t^{m_l k}=2^jt\cdot 2^{-m_l jk}\qquad (k\geqq 1)$$
for $\xi\in\operatorname{supp}K_j$. It follows from the estimate (2.6) and these inequalities that

(2.7)
$$\int\|G_j(x-2^{-jM}a_j-y)-G_j(x-2^{-jM}a_j)\|_{LH(X,Y)}dx\leqq CA2^jt.$$

Combining (2.4), (2.5) and (2.7), we conclude that

(2.8)
$$\int_{[x]\geqq 2t}\|G_j(x-y)-G_j(x)\|_{LH(X,Y)}dx$$
$$\leqq\begin{cases}CA\cdot 2^jt & (2^jt\leqq 1)\\ CA & (1\leqq 2^jt\leqq 1+2\cdot\sup_k[a_k])\\ CA(2^jt)^{|M|/2-\min_l k_l m_l} & (2^jt\geqq 1+2\cdot\sup_k[a_k])\end{cases}$$
for $y\in\boldsymbol{R}^n$ satisfying $[y]\leqq t/2$.

Let $h_0$ be the greatest integer satisfying $2^{h_0}t\leqq 1$, and $h_1$ be the least integer satisfying $2^{h_1}t\geqq 1+2\cdot\sup_k[a_k]$. Then we have $h_1-h_0\leqq 3+\log_2(1+\sup_k[a_k])$.

We now consider the kernel $G^{(N)}(x)$. First, it follows from (2.8) that
$$\int_{[x]\geqq 2t}\|G^{(N)}(x-y)-G^{(N)}(x)\|_{LH(X,Y)}dx$$
$$\leqq\sum_{j=-N}^{N}\int_{[x]\geqq 2t}\|G_j(x-y)-G_j(x)\|_{LH(X,Y)}dx$$

$$\leqq CA\Big(\sum_{j=-\infty}^{h_0} 2^j t+(h_1-h_0-1)+\sum_{j=h_1}^{\infty}(2^j t)^{|M|/2-\min_l m_l k_l}\Big)$$

$$\leqq CA\{C'+\log_2(1+\sup_k[a_k])\}$$

for $[y]\leqq t/2$. Since $[a_k]\leqq\max\{|a_k|, |a_k|^m\}$, we obtain

$$\int_{[x]\geqq 2t}\|G^{(N)}(x-y)-G^{(N)}(x)\|_{LH(X,Y)}dx\leqq CA\cdot\log(2+\sup_k|a_k|)$$

for $[y]\leqq t/2$ with a constant $C$ independent of $\{a_k\}$ and $N$.

On the other hand, for natural numbers $L$, $N$ $(L<N)$ and every $X$-valued simple function $u(x)$, we have the estimates

$$\Big\|\int G^{(N)}(x-y)u(y)dy\Big\|_{L^2(Y)}^2=\int\Big\|\sum_{j=-N}^{N}K_j(\xi)\hat{u}(\xi)\Big\|_Y^2\bar{d}\xi$$

$$\leqq\int 2\cdot\sup_j\|K_j(\xi)\|_{L(X,Y)}^2\|\hat{u}(\xi)\|_X^2\bar{d}\xi$$

$$\leqq 2\|u(x)\|_{L^2(X)}^2\cdot\Big(\sup_j\int\|G_j(x)\|_{LH(X,Y)}dx\Big)^2$$

$$\leqq 2C^2A^2\|u(x)\|_{L^2(X)}^2$$

by virtue of the estimate (2.5), together with the Riemann-Lebesgue inequality and the Plancherel formula.

In the same way we have

$$\Big\|\int\{G^{(N)}(x-y)-G^{(L)}(x-y)\}u(y)dy\Big\|_{L^2(Y)}^2$$

$$\leqq\int\Big\|\Big(\sum_{j=-N}^{-L-1}+\sum_{j=L+1}^{N}\Big)K_j(\xi)\hat{u}(\xi)\Big\|_Y^2\bar{d}\xi$$

$$\leqq\Big(\int_{[\xi]\leqq 2^{-L-1}}+\int_{[\xi]\geqq 2^L}\Big)2\cdot\sup_j\|K_j(\xi)\|_{L(X,Y)}^2\|\hat{u}(\xi)\|_X^2\bar{d}\xi$$

$$\longrightarrow 0 \quad \text{as } L,\ N\to\infty.$$

Applying Proposition 2.3, we see immediately that

$$\|G^{(N)}*u\|_{L^p(Y)}\leqq CA\cdot\log(2+\sup_k|a_k|)\cdot\|u\|_{L^p(X)}$$

for $1<p<\infty$, where $C$ is a constant independent of $N$, $A$, $\{a_k\}$ and $u(x)$. We also have

$$G^{(N)}*u-G^{(L)}*u\longrightarrow 0 \ \text{ in } L^p(Y) \qquad \text{as } L,\ N\to\infty$$

for any fixed $X$-valued simple function $u(x)$. Hence, the operator $G^{(N)}*$ converges strongly to $G*$ in $L(L^p(X), L^p(Y))$, and the operator norm of $G*$ is dominated by $CA\cdot\log(2+\sup_k|a_k|)$. The proof of Theorem 2.5 is now complete.

## §3. The Littlewood-Paley decomposition of parabolic type and product type.

In this section we prove a generalization of the Littlewood-Paley decomposition theorem. Our theorem is different from the original one on the following points: First, we regard $\boldsymbol{R}^n$ as $\boldsymbol{R}^{n(1)}\times\cdots\times\boldsymbol{R}^{n(N)}$ and consider a decomposition of each $\boldsymbol{R}^{n(j)}$. Secondly, our decomposition of each $\boldsymbol{R}^{n(j)}$ is "parabolic". Thirdly, we estimate the $L^p$-norm not only of $\{\sum|u_K(x)|^2\}^{1/2}$, but of

$$\{\sum|u_K(x^{(1)}+2^{-k(1)M(1)}a^{(1)},\cdots,x^{(N)}+2^{-k(N)M(N)}a^{(N)})|^2\}^{1/2}$$

for general $a\in\boldsymbol{R}^n$.

In the sequel, for $1\leqq j\leqq N$, $k\in\boldsymbol{N}$ and $\eta\in\boldsymbol{R}^{n(j)}$, we put

$$\begin{cases}\Psi_{j,0}(\eta)=\Psi([\eta]_j),\\ \Psi_{j,k}(\eta)=\Psi(2^{-k}[\eta]_j)-\Psi(2^{1-k}[\eta]_j) & \text{for } k\geqq 1.\end{cases}$$

Suppose $1\leqq\nu\leqq N$ and $a^{(j)}\in\boldsymbol{R}^{n(j)}$ for $j=1,2,\cdots,\nu$. For $u\in C_0^\infty(\boldsymbol{R}^n)$, $a=(a^{(1)},\cdots,a^{(\nu)})$ and $K=(k(1),\cdots,k(\nu))\in\boldsymbol{N}^\nu$ we put

$$u_{a,K}^{(\nu)}(x)=\mathcal{F}^{-1}\Big[\exp\Big(i\sum_{j=1}^{\nu}2^{-k(j)M(j)}a^{(j)}\cdot\xi^{(j)}\Big)\prod_{j=1}^{\nu}\Psi_{j,k(j)}(\xi^{(j)})\hat{u}(\xi)\Big](x).$$

Then we have the following

PROPOSITION 3.1. *There exists a constant $C$ independent of $a$ and $u(x)$ such that*

$$\|(\sum_{K\in\boldsymbol{N}^\nu}|u_{a,K}^{(\nu)}(x)|^2)^{1/2}\|_{L^p}\leqq C\prod_{j=1}^{\nu}\log(2+|a^{(j)}|)\cdot\|u\|_{L^p}.$$

PROOF. By induction, we have only to prove

$$\|(\sum_{K\in\boldsymbol{N}^\nu}|u_{a,K}^{(\nu)}(x)|^2)^{1/2}\|_{L^p}\leqq C\log(2+|a^{(\nu)}|)\cdot\|(\sum_{K\in\boldsymbol{N}^{\nu-1}}|u_{a,K}^{(\nu-1)}(x)|^2)^{1/2}\|_{L^p} \tag{3.1}$$

for every $\nu=1,\cdots,N$, where the right-hand side is regarded as $\log(2+|a^{(1)}|)\cdot\|u\|_{L^p}$ if $\nu=1$.

We prove (3.1) by using the Rademacher functions $\{r_k\}_{k\in\boldsymbol{N}}$. The functions $r_k(t)$ ($k\in\boldsymbol{N}$, $t\in[0,1]$) are defined by

$$r_0(t)=\begin{cases}1 & (0\leqq t\leqq 1/2),\\ -1 & (1/2<t\leqq 1)\end{cases}$$

and $r_k(t)=r_0(2^kt-[2^kt])$, where $[2^kt]$ is the greatest integer not greater than $2^kt$. Then we have the following

LEMMA 3.2. *For every $\nu\in\boldsymbol{N}$ and $1<p<\infty$, there exists a constant $C$ such that*

$$C^{-1}\|\sum_{K\in N^\nu} b_K r_{k(1)}(t_1)\cdots r_{k(\nu)}(t_\nu)\|_{L^p([0,1]^\nu)}$$
$$\leqq\|\sum_{K\in N^\nu} b_K r_{k(1)}(t_1)\cdots r_{k(\nu)}(t_\nu)\|_{L^2([0,1]^\nu)}=(\sum_{K\in N^\nu}|b_K|^2)^{1/2}$$
$$\leqq C\|\sum_{K\in N^\nu} b_K r_{k(1)}(t_1)\cdots r_{k(\nu)}(t_\nu)\|_{L^p([0,1]^\nu)}$$

*holds for every family of complex numbers* $\{b_K\}_{K\in N^\nu}$.

This lemma is proved in the Appendix of Stein [**11**].

In view of this lemma, (3.1) is equivalent to

$$\|\sum_{K\in N^\nu} u_{a,K}^{(\nu)}(x) r_{k(1)}(t_1)\cdots r_{k(\nu)}(t_\nu)\|_{L^p(R^n\times[0,1]^\nu)}$$
$$\leqq C\cdot\log(2+|a^{(\nu)}|)\cdot\|\sum_{K\in N^{\nu-1}} u_{a,K}^{(\nu-1)}(x) r_{k(1)}(t_1)\cdots r_{k(\nu-1)}(t_{\nu-1})\|_{L^p(R^n\times[0,1]^{\nu-1})}$$

which will be obtained by integrating

$$\text{(3.2)}\qquad \iint_{R^{n(\nu)}}|\sum_{K\in N^\nu} u_{a,K}^{(\nu)}(x) r_{k(1)}(t_1)\cdots r_{k(\nu)}(t_\nu)|^p dx^{(\nu)}dt_\nu$$
$$\leqq C^p\{\log(2+|a^{(\nu)}|)\}^p\cdot\int_{R^{n(\nu)}}|\sum_{K\in N^{\nu-1}} u_{a,K}^{(\nu-1)}(x) r_{k(1)}(t_1)\cdots r_{k(\nu-1)}(t_{\nu-1})|^p dx^{(\nu)}$$

with respect to $dx^{(1)}\cdots dx^{(\nu-1)}dx^{(\nu+1)}\cdots dx^{(N)}dt_1\cdots dt_{\nu-1}$. Fix $t_1, \cdots, t_{\nu-1}$ and $x^{(j)}$ $(j\neq\nu)$, and put

$$v(y)=\sum_{K\in N^{\nu-1}} u_{a,K}^{(\nu-1)}(x^{(1)},\cdots,x^{(\nu-1)},y,x^{(\nu+1)},\cdots,x^{(N)})\cdot r_{k(1)}(t_1)\cdots r_{k(\nu-1)}(t_{\nu-1}).$$

Then the desired estimate (3.2) can be written as

$$\iint_{R^{n(\nu)}}\left|\sum_{k=0}^\infty r_k(t)\mathcal{F}^{-1}[\exp(ia^{(\nu)}2^{-kM(\nu)}\eta^{(\nu)})\cdot\Psi_{\nu,k}(\eta)\hat{v}(\eta)](y)\right|^p dydt$$
$$\leqq C^p\{\log(2+|a^{(\nu)}|)\}^p\int_{R^{n(\nu)}}|v(y)|^p dy$$

where $y,\ \eta\in \boldsymbol{R}^{n(\nu)}$.

In view of Lemma 3.2, we have only to show

$$\text{(3.3)}\qquad \|\{\mathcal{F}^{-1}[\exp(ia^{(\nu)}2^{-kM(\nu)}\eta^{(\nu)})\Psi_{\nu,k}(\eta)\hat{v}(\eta)](y)\}_{k\in N}\|_{L^p(l^2)}$$
$$\leqq C\cdot\log(2+|a^{(\nu)}|)\|v(y)\|_{L^p}$$

for $v(y)\in L^p(\boldsymbol{R}^{n(\nu)})$.

To verify (3.3), we consider the $LH(\boldsymbol{C}, l^2)$-valued continuous function $K^\lambda(\eta)$ on $\boldsymbol{R}^{n(\nu)}$ for $\lambda=1, 2, 3$ defined by

$$K^\lambda(\eta)=\{\exp(ia^{(\nu)}2^{-kM(\nu)}\cdot\eta)\Psi_{\nu,k}(\eta)\}_{k\in N,\ k\equiv\lambda\,(\mathrm{mod}\,3)}.$$

Then there exists a constant $B$ such that

$$\int_{2^{j-1}\leqq[\eta]_\nu\leqq 2^{j+1}}\|\partial_{\xi_l}^h\{\exp(-i2^{-jM(\nu)}a^{(\nu)}\cdot\eta)K^\lambda(\eta)\}\|_{LH(C,l^2)}^2 d\eta\leqq B^2\cdot 2^{j(|M(\nu)|-2hm_l)}$$

for $h=0, 1, \cdots, n(\nu)+1$.

Applying Theorem 2.5 to each $K^{\lambda}$, we obtain (3.3). This completes the proof.

If $\nu=1$, $M^{(1)}=(1, 1, \cdots, 1)$ and $a^{(1)}=0$, then the above proposition is the Littlewood-Paley decomposition theorem of $L^{p}$. For $\nu=1$ and general $M^{(1)}$ we obtain a decomposition of parabolic type, and for $\nu=n$ and $M^{(1)}=\cdots=M^{(n)}=(1)$ we have a decomposition of product type.

If $a=0$, we have the converse of Proposition 3.1. Fix a constant $B>\sqrt{2}$, and for $j=1, \cdots, N$ set $I_{j,0}=\{\eta\in \boldsymbol{R}^{n(j)};\ [\eta]_{j}\leqq B\}$ and $I_{j,k}=\{\eta\in \boldsymbol{R}^{n(j)};\ 2^{k}B^{-1}\leqq [\eta]_{j}\leqq 2^{k}B\}$ for every positive integer $k$. We also fix $\nu\in\{1, \cdots, N\}$. Then, for $K\in \boldsymbol{N}^{\nu}$ we denote by $I_{K}$ the "parabolic dyadic domain"; that is, we put

$$I_{K}=\{\xi\in \boldsymbol{R}^{n};\ \ \xi^{(j)}\in I_{j,k(j)} \text{ for all } j=1, \cdots, \nu\}.$$

Then we have the following

PROPOSITION 3.3. *Suppose* $1<p<\infty$. *If a family of functions* $\{u_{K}(x)\}_{K\in \boldsymbol{N}^{\nu}}$ *satisfies* $\operatorname{supp}\hat{u}_{K}(\xi)\subset I_{K}$ *and*

$$\|(\sum_{K\in \boldsymbol{N}^{\nu}}|u_{K}(x)|^{2})^{1/2}\|_{L^{p}}\leqq A<\infty,$$

*then the infinite sum* $u(x)=\sum_{K\in \boldsymbol{N}^{\nu}}u_{K}(x)$ *converges in* $L^{p}$, *and it satisfies the estimate* $\|u(x)\|_{L^{p}}\leqq CA$ *for some constant* $C$.

PROOF. We have only to show that there exists a constant $C$ independent of $L\in \boldsymbol{N}$ such that

$$\|u(x)\|_{L^{p}}\leqq C\left\|\left(\sum_{k(1),\cdots,k(\nu)=0}^{L}|u_{K}(x)|^{2}\right)^{1/2}\right\|_{L^{p}} \tag{3.4}$$

holds for every $\{u_{K}(x)\}_{K\in \boldsymbol{N}^{\nu}}$, where $u(x)=\sum_{k(1),\cdots,k(\nu)=0}^{L}u_{K}(x)$. To prove (3.4), we may assume that every $u_{K}(x)$ belongs to $C_{0}^{\infty}(\boldsymbol{R}^{n})$.

For $v(x)\in C_{0}^{\infty}(\boldsymbol{R}^{n})$ and $K'=(k'(1), \cdots, k'(\nu))\in \boldsymbol{Z}^{\nu}$, put

$$v_{K}'(x)=\begin{cases}0 & (k'(j)<0 \text{ for some } j=1, \cdots, \nu),\\ \mathcal{F}^{-1}[\Psi_{1,k'(1)}(\xi^{(1)})\cdots\Psi_{\nu,k'(\nu)}(\xi^{(\nu)})\hat{v}(\xi)](x) & \\ \qquad (k'(j)\geqq 0 \text{ for every } j=1, \cdots, \nu).\end{cases}$$

Then there exists an integer $h$ determined by $B$ such that $|k(j)-k'(j)|\leqq h$ holds for all $j=1, 2, \cdots, \nu$ if

$$\operatorname{supp}\hat{u}_{K}(\xi)\cap\operatorname{supp}\hat{v}_{K'}(\xi)\neq\emptyset.$$

This implies

$$\int u(x)\overline{v(x)}dx=\int\hat{u}(\xi)\overline{\hat{v}(\xi)}\bar{d}\xi$$

$$=\int\sum_{k(1),\cdots,k(\nu)=0}^{L}\hat{u}_{K}(\xi)\cdot\sum_{K'\in \boldsymbol{N}^{\nu}}\overline{\hat{v}_{K'}(\xi)}\bar{d}\xi$$

$$=\int \sum_{k(1),\cdots,k(\nu)=0}^{L} \sum_{|k'(j)-k(j)|\leqq h} \hat{u}_K(\xi)\cdot\overline{\hat{v}_{K'}(\xi)}\bar{d}\xi$$

$$=\int \sum_{\lambda(1),\cdots,\lambda(\nu)=-h}^{h} \sum_{k(1),\cdots,k(\nu)=0}^{L} u_K(x)\cdot\overline{u_{K+\Lambda}(x)}dx.$$

Hence

$$\left|\int u(x)\overline{v(x)}dx\right| \leqq \sum_{\lambda(1),\cdots,\lambda(\nu)=-h}^{h} \int \sum_{k(1),\cdots,k(\nu)=0}^{L} |u_K(x)|\cdot|v_{K+\Lambda}(x)|dx$$

$$\leqq (2h+1)^{\nu}\int(\sum_{K\in N^{\nu}}|u_K(x)|^2)^{1/2}(\sum_{K\in N^{\nu}}|v_K(x)|^2)^{1/2}dx$$

$$\leqq (2h+1)^{\nu}\cdot\left\|\left(\sum_{k(1),\cdots,k(\nu)=0}^{L}|u_K(x)|^2\right)^{1/2}\right\|_{L^p}\cdot\|(\sum_{K\in N^{\nu}}|v_K(x)|^2)^{1/2}\|_{L^{p'}},$$

where $p'=p/(p-1)$.

In view of Proposition 3.1 for $p'$, there exists a constant $C$ such that

$$\left|\int u(x)\overline{v(x)}dx\right| \leqq C\left\|\left(\sum_{k(1),\cdots,k(\nu)=0}^{L}|u_K(x)|^2\right)^{1/2}\right\|_{L^p}\cdot\|v\|_{L^{p'}}.$$

Since $(L^{p'})'=L^p$, we obtain (3.4).

## §4. Proof of Theorem 2: the necessity.

The assertion 2)→3) is trivial. Next we prove the assertion 3)→1) by contradiction; given a modulus of continuity $\{\omega_1(t_1), \cdots, \omega_N(t_1, \cdots, t_N)\}$ where not all of $\omega_\nu$ satisfy the estimate (1.1), we shall construct a symbol $P(x, \xi)$ satisfying the estimates $(*\mu)$ for $\mu=0, 1, \cdots, N$ such that the associated operator $P(x, D)$ is not bounded on $L^p$ for any $1<p<\infty$.

Let $\nu$ be the least integer such that $\omega_\nu$ does not satisfy (1.1). For $K\in N^\nu$ we put $\omega_K=\omega_\nu(2^{-k(1)m}, 2^{-k(2)m}, \cdots, 2^{-k(\nu)m})$, where $m=\max_{l=1,\dots,n}m_l$ as in Section 1. We also put $\Phi_{j,k}(\eta)=\Phi(2^{-k}[\eta]_j)$ for $k\in N$ and $\eta\in R^{n(j)}$, where $\Phi(t)$ is a smooth function on $R^+$ such that $0\leqq\Phi(t)\leqq 1$, $\Phi(t)=0$ $(t\leqq 2/3,\ t\geqq 4/3)$ and $\Phi(t)=1$ $(3/4\leqq t\leqq 5/4)$.

Now put $l(j)=n(1)+\cdots+n(j)$ for $j=1, 2, \cdots, N$ and put

$$P(x, \xi)=\sum_{K\in N^\nu}\omega_K\cdot\exp\left(-i\sum_{j=1}^{\nu}2^{m_{l(j)}k(j)}x_{l(j)}\right)\prod_{j=1}^{\nu}\Phi_{j,k(j)}(\xi^{(j)}).$$

Then, for $j=\nu+1, \nu+2, \cdots, N$, the symbol $P(x, \xi)$ does not depend on $x^{(j)}$ or $\xi^{(j)}$. Hence we have only to show the estimates $(*\mu)$ of the derivatives $\partial_{\xi_l}^k P(x,\xi)$ for $l\in\Lambda(1)\cup\cdots\cup\Lambda(\nu)$.

For every $\mathring{\xi}\in R^n$ there exist a neighborhood $U$ of $\mathring{\xi}$ in $R^n$ and at most one $K\in N^\nu$ such that

$$P(x, \xi)=\omega_K\exp\left(-i\sum_{j=1}^{\nu}2^{m_{l(j)}k(j)}x_{l(j)}\right)\prod_{j=1}^{\nu}\Phi_{j,k(j)}(\xi)$$

holds for every $\xi\in U$.

Then, for every $\mu=1, 2, \cdots\nu$, $h=1, 2, \cdots, N$, $l\in\Lambda(h)$, $k=0, 1, \cdots, n+1$ and $y\in\boldsymbol{R}^n$ such that $|y_l|\leqq 1$ for all $l$, we have

$$\begin{aligned}
&|\Delta^1_{y^{(1)}}(\cdots\{\Delta^{\mu}_{y^{(\mu)}}(\partial^k_{\xi_l}P(x, \mathring{\xi})\}\cdots)|\\
&\leqq\omega_K\prod_{j=1}^{\mu}|\exp\{-i2^{m_{l(j)}k(j)}(x_{l(j)}-y_{l(j)})\}-\exp(-i2^{m_{l(j)}k(j)}x_{l(j)})|\\
&\quad\cdot|\prod_{j\neq h}\Phi_{j,k(j)}(\mathring{\xi})^{(j)})|\cdot|\partial^k_{\xi_l}\Phi_{h,k(h)}(\mathring{\xi}^{(h)})|\\
&\leqq C\cdot\omega_K\cdot\prod_{j=1}^{\mu}\min\{2, 2^{m_{l(j)}k(j)}|y_{l(j)}|\}\cdot 2^{-km_lk(h)}\\
&\leqq 2^{\mu}C\cdot\omega_K\cdot\prod_{j=1}^{\mu}\min\{1, 2^{mk(j)}|y^{(j)}|\}\cdot 2^{-km_lk(h)},
\end{aligned}$$

where $C$ is a constant independent of $y$ and $K$.

Since $\omega_K$ is monotone-increasing, we have

$$\omega_K\leqq\omega_\nu(|y^{(1)}|, 2^{-mk(2)}, \cdots, 2^{-mk(\nu)})$$

if $|y^{(1)}|\geqq 2^{-mk(1)}$. On the other hand, if $|y^{(1)}|\leqq 2^{-mk(1)}$, then it follows from the concavity of $\omega_\nu$ that

$$\begin{aligned}
2^{mk(1)}|y^{(1)}|\omega_K&\leqq 2^{mk(1)}|y^{(1)}|\omega_\nu(2^{-mk(1)}, 2^{-mk(2)}, \cdots, 2^{-mk(\nu)})\\
&\quad+(1-2^{mk(1)}|y^{(1)}|)\omega_\nu(0, 2^{-mk(2)}, \cdots, 2^{-mk(\nu)})\\
&\leqq\omega_\nu(|y^{(1)}|, 2^{-mk(2)}, \cdots, 2^{-mk(\nu)}).
\end{aligned}$$

Repeating this argument $\mu$ times, we obtain

$$\begin{aligned}
&|\Delta^1_{y^{(1)}}(\cdots\{\Delta^{\mu}_{y^{(\mu)}}(\partial^k_{\xi_l}P(x, \mathring{\xi}))\}\cdots)|\\
&\leqq 2^{\mu}C\omega_\nu(|y^{(1)}|, \cdots, |y^{(\mu)}|, 2^{-mk(\mu+1)}, \cdots, 2^{-mk(\nu)})\cdot 2^{-km_lk(h)}\\
&\leqq 2^{\mu}C\cdot 2^{\nu-\mu}\omega_\mu(|y^{(1)}|, \cdots, |y^{(\mu)}|)2^{-km_lk(h)}.
\end{aligned}$$

If $\mathring{\xi}\in\operatorname{supp}\Phi_{h,k(h)}$, then $2^{k(h)}\cdot 2/3\leqq[\mathring{\xi}^{(h)}]_h\leqq 2^{k(h)}\cdot 4/3$. It follows

$$2^{-km_lk(h)}\leqq\left(\frac{3[\mathring{\xi}^{(h)}]_h}{4}\right)^{-km_l}\leqq\left(\frac{1+[\mathring{\xi}^{(h)}]_h}{4}\right)^{-km_l}$$

and hence

$$\begin{aligned}
&|\Delta^1_{y^{(1)}}(\cdots\{\Delta^{\mu}_{y^{(\mu)}}(\partial^k_{\xi_l}P(x, \xi))\}\cdots)|\\
&\leqq C\omega_\mu(|y^{(1)}|, \cdots, |y^{(\mu)}|)\cdot(1+[\mathring{\xi}^{(h)}]_h)^{-km_l}.
\end{aligned}$$

Other differences can be estimated in the same manner, and the proof of $(*\mu)$ is complete. For $\mu=\nu+1, \cdots, N$, the condition $(*\mu)$ is trivial since the correspondent differences vanish identically. This completes the proof of $(*\mu)$.

It remains only to prove that the operator $P(x, D)$ is not bounded on any $L^p$. Put

$$I_K=\{(t_1, \cdots, t_\nu)\in(\boldsymbol{R}^+)^\nu;\ \ 2^{-m(k(j)+1)}\leqq t_j\leqq 2^{-mk(j)} \text{ for every } j=1, \cdots \nu\}.$$

Then, from the inequality

$$\begin{aligned}&\int_0^1\cdots\int_0^1\frac{\omega_\nu(t_1, \cdots, t_\nu)^2}{t_1\cdots t_\nu}dt_1\cdots dt_\nu\\ &\leqq\sum_{K\in\boldsymbol{N}^\nu}\int\cdots\int_{I_K}\frac{\omega_\nu(t_1, \cdots, t_\nu)^2}{t_1\cdots t_\nu}dt_1\cdots dt_\nu\\ &\leqq\sum_{K\in\boldsymbol{N}^\nu}\omega_\nu(2^{-mk(1)}, \cdots, 2^{-mk(\nu)})^2\cdot\log\frac{2^{-mk(j)}}{2^{-m(k(j)+1)}}\\ &=(m\cdot\log 2)\sum_{K\in\boldsymbol{N}^\nu}\omega_K^2,\end{aligned}$$

we have $\sum_{K\in\boldsymbol{N}^\nu}\omega_K^2=\infty$. Hence there exists a sequence of numbers $\{a_K\}_{K\in\boldsymbol{N}^\nu}$ such that $\sum_{K\in\boldsymbol{N}^\nu}a_K^2<\infty$ and $\sum_{K\in\boldsymbol{N}^\nu}a_K\omega_K=\infty$. Let $\phi(\xi)$ be a smooth function on $\boldsymbol{R}^n$ not identically equal to 0 such that $\operatorname{supp}\phi(\xi)\subset\{\xi: |\xi|\leqq 4^{-m}\}$, and

$$u_L(x)=\sum_{|K|\leqq L}a_K\mathcal{F}^{-1}\Big[\phi\Big(\xi-\sum_{j=1}^{\nu}2^{-k(j)m_{l(j)}}e_{l(j)}\Big)\Big](x),$$

where $e_{l(j)}=(0, \cdots, 0, \overset{l(j)}{1}, 0, \cdots, 0)$.

Then, since

$$\begin{aligned}&\operatorname{supp}\phi\Big(\xi-\sum_{j=1}^{\nu}2^{-k(j)m_{l(j)}}e_{l(j)}\Big)\\ &\subset\{\xi;\ |\xi^{(j)}-2^{-k(j)m_{l(j)}}e_{l(j)}|\leqq 4^{-m} \text{ for } j=1, \cdots, \nu,\\ &\qquad\qquad \text{and } |\xi^{(j)}|\leqq 4^{-m} \text{ for } j=\nu+1, \cdots, N\}\\ &\subset\{\xi;\ 2^{k(j)}-1/4\leqq[\xi^{(j)}]_j\leqq 2^{k(j)}+1/4 \text{ for } j=1, \cdots, \nu,\\ &\qquad\qquad \text{and } [\xi^{(j)}]_j\leqq 1/4 \text{ for } j=\nu+1, \cdots, N\},\end{aligned}$$

it follows from Proposition 3.3 that

$$\begin{aligned}\|u_L(x)\|_{L^p}&\leqq C\Big\|\Big(\sum_{|K|\leqq L}|a_K|^2\Big|\mathcal{F}^{-1}\Big[\phi\Big(\xi-\sum_{j=1}^{\nu}2^{-k(j)m_{l(j)}}e_{l(j)}\Big)\Big](x)\Big|^2\Big)^{1/2}\Big\|_{L^p}\\ &=C\sum_{|K|\leqq L}|a_K|^2\|\mathcal{F}^{-1}[\phi](x)\|_{L^p}\\ &\leqq C\sum_{K\in\boldsymbol{N}^\nu}|a_K|^2,\end{aligned}$$

where $C$ is a constant independent of $L\in\boldsymbol{N}^\nu$.

On the other hand, it is easily seen that

$$\begin{aligned}P(x, D)u(x)=&\sum_{K\in\boldsymbol{N}^\nu}\omega_K\exp\Big(-i\sum_{j=1}^{\nu}2^{-k(j)m_{l(j)}}x_{l(j)}\Big)\\ &\times\mathcal{F}^{-1}\Big[\prod_{j=1}^{\nu}\Phi_{j,k(j)}(\xi^{(j)})\sum_{|K'|\leqq L}a_{K'}\phi\Big(\xi-\sum_{j=1}^{\nu}2^{-k'(j)m_{l(j)}}e_{l(j)}\Big)\Big](x)\end{aligned}$$

$$=\sum_{|K|\leq L} a_K\omega_K\mathcal{F}^{-1}[\phi](x).$$

This implies

$$\|P(x, D)u(x)\|_{L^p}=C\cdot\sum_{|K|\leq L} a_K\omega_K \longrightarrow \infty \qquad \text{as } L\to\infty.$$

Thus the operator $P(x, D)$ is not bounded on $L^p$ for any $1<p<\infty$.

## § 5. Proof of Theorem 2: the sufficiency.

In this section we prove the assertion 1)→2) of Theorem 2. Suppose that the symbol $P(x, \xi)$ satisfies the estimates $(*\mu)$ for all $\mu=0, 1, \cdots, N$, where the modulus of continuity $\{\omega_1(t_1), \cdots, \omega_N(t_1, \cdots, t_N)\}$ satisfies the condition (1.1). We shall prove the $L^p$-boundedness of the operator $P(x, D)$. First we introduce functions $\phi_{j,k}(\eta)\in C_0^\infty(\boldsymbol{R}^{n(j)})$ by $\phi_{j,0}(\eta)=\Psi(2^{-1}[\eta]_j)$ and $\phi_{j,k}(\eta)=\Psi(2^{-k-1}[\eta]_j)-\Psi(2^{-k+2}[\eta]_j)$ for $k\geq 1$. Then we have:

$$(5.1)\quad \begin{cases} \phi_{j,k}(\eta)=1 \quad \text{if } \eta\in\operatorname{supp}\Psi_{j,k}, \\ \qquad \text{where } \Psi_{j,k} \text{ has been defined at the beginning of Section 3.} \\ \phi_{j,k}(\eta)=\phi_{j,1}(2^{(1-k)M(j)}\eta) \quad \text{if } k\geq 1. \\ \operatorname{supp}\phi_{j,0}(\eta)\subset\{\eta\,;\,[\eta]_j\leq 8/3\}. \\ \operatorname{supp}\phi_{j,k}(\eta)\subset\{\eta\,;\,2^{k-2}\leq[\eta]_j\leq 2^{k+3}/3\} \quad \text{if } k\geq 1. \end{cases}$$

Next we put

$$Q_K(x, \xi)=P(x, 2^{(k(1)+2)M(1)}\xi^{(1)}, \cdots, 2^{(k(N)+2)M(N)}\xi^{(N)})$$
$$\times\phi_{1,k(1)}(2^{(k(1)+2)M(1)}\xi^{(1)})\cdots\phi_{N,k(N)}(2^{(k(N)+2)M(N)}\xi^{(N)})$$

and decompose $P(x, \xi)$ as follows:

$$(5.2)\quad P(x, \xi)=\sum_{K\in N^N} P(x, \xi)\Psi_{1,k(1)}(\xi^{(1)})\cdots\Psi_{N,k(N)}(\xi^{(N)})$$
$$=\sum_{K\in N^N} P(x, \xi)\phi_{1,k(1)}(\xi^{(1)})\cdots\phi_{N,k(N)}(\xi^{(N)})\cdot\Psi_{1,k(1)}(\xi^{(1)})\cdots\Psi_{N,k(N)}(\xi^{(N)})$$
$$=\sum_{K\in N^N} Q_K(x, 2^{-(k(1)+2)M(1)}\xi^{(1)}, \cdots, 2^{-(k(N)+2)M(N)}\xi^{(N)})$$
$$\cdot\Psi_{1,k(1)}(\xi^{(1)})\cdots\Psi_{N,k(N)}(\xi^{(N)}).$$

Then, since the support of $Q_K$ is contained in the set

$$\{(x, \xi);\ [\xi^{(j)}]_j\leq 2/3 \text{ for all } j=1, \cdots, N\}$$
$$\subset\{(x, \xi);\ |\xi_l|\leq 1 \text{ for all } l=1, \cdots, n\},$$

we can write

$$(5.3)\qquad Q_K(x, \xi)=\sum_{h\in Z^n} a_{K,h}(x)\exp(\pi ih\cdot\xi),$$

where $a_{K,h}(x)$ is determined by

$$a_{K,h}(x)=2^{-n}\cdot\int_{-1}^{1}\cdots\int_{-1}^{1}\exp(-\pi ih\cdot\xi)Q_K(x,\xi)d\xi$$

for $h=(h^{(1)},\cdots,h^{(N)})=(h_1,\cdots,h_n)\in\boldsymbol{Z}^n$. (The estimate (5.6) given later assures the convergence of the right-hand side of (5.3).)

Substituting (5.3) into (5.2) and putting

$$b_{K,h}(\xi)=\exp\Big(\pi i\sum_{j=1}^{N}h^{(j)}\cdot 2^{-(k(j)+2)M(j)}\xi^{(j)}\Big)\prod_{j=1}^{N}\Psi_{j,k(j)}(\xi^{(j)})$$

and $P_h(x,\xi)=\sum_{K\in\boldsymbol{N}^N}a_{K,h}(x)b_{K,h}(\xi)$, we can write

$$\begin{aligned}P(x,\xi)&=\sum_{K\in\boldsymbol{N}^N}\sum_{h\in\boldsymbol{Z}^n}a_{K,h}(x)\exp\Big(\pi i\sum_{j=1}^{N}h^{(j)}\cdot 2^{-(k(j)+2)M(j)}\xi^{(j)}\Big)\cdot\sum_{j=1}^{N}\Psi_{j,k(j)}(\xi)\\&=\sum_{h\in\boldsymbol{Z}^n}P_h(x,\xi).\end{aligned}$$

Here the change of the order of the summations will be justified by the estimate (5.6) later and the fact that each $P_h(x,\xi)$ is a locally finite sum with respect to $K\in\boldsymbol{N}^n$.

Since

$$\sum_{h\in\boldsymbol{Z}^n}\frac{\log(2+|h^{(1)}|)\cdots\log(2+|h^{(N)}|)}{1+|h_1|^{n+1}+\cdots+|h_n|^{n+1}}<\infty,$$

it suffices to prove

$$\|P_h(x,D)u\|_{L^p}<C\frac{\log(2+|h^{(1)}|)\cdots\log(2+|h^{(N)}|)}{1+|h_1|^{n+1}+\cdots+|h_n|^{n+1}}\|u\|_{L^p} \tag{5.4}$$

for all $u(x)\in C_0^\infty(\boldsymbol{R}^n)$, where $C$ is a constant independent of $h$ and $u(x)$.

For $K\in\boldsymbol{N}^N$ and every subset $A$ of $U=\{1,2,\cdots,N\}$ we put

$$\Phi_{K,A}(\xi)=\prod_{j\in A}\Psi(2^{2-k(j)}[\xi^{(j)}]_j)\prod_{j\in U\setminus A}(1-\Psi(2^{2-k(j)}[\xi^{(j)}]_j))$$

and

$$a_{K,h,A}(x)=\mathcal{F}^{-1}[\Phi_{K,A}(\xi)\hat{a}_{K,h}(\xi)](x),$$

where $\hat{a}_{K,h}(\xi)$ denotes the Fourier transform of $a_{K,h}(x)$. We also put $u_{K,h}(x)=\mathcal{F}^{-1}[b_{K,h}(\xi)\hat{u}(\xi)](x)$.

Then, since $\sum_{A\subset U}\Phi_{K,A}(\xi)=1$, we have $a_{K,h}(x)=\sum_{A\subset U}a_{K,h,A}(x)$. It follows that

$$\begin{aligned}P_h(x,D)u(x)&=\sum_{K\in\boldsymbol{N}^N}a_{K,h}(x)\mathcal{F}^{-1}[b_{K,h}(\xi)\hat{u}(\xi)](x)\\&=\sum_{A\subset U}\sum_{K\in\boldsymbol{N}^N}a_{K,h,A}(x)u_{K,h}(x).\end{aligned}$$

Hence, (5.4) will follow from

$$\Big\|\sum_{K\in\boldsymbol{N}^N}a_{K,h,A}(x)u_{K,h}(x)\Big\|_{L^p}\leqq C\frac{\log(2+|h^{(1)}|)\cdots\log(2+|h^{(N)}|)}{1+|h_1|^{n+1}+\cdots+|h_n|^{n+1}}\|u(x)\|_{L^p}. \tag{5.5}$$

To prove (5.5), we may assume $A=\{\nu+1, \nu+2, \cdots, N\}$ $(0\leqq\nu\leqq N)$ without loss of generality. We would like to show that there exists a constant $C$ which is independent of $K$, $h$ and $A$ and satisfies the estimate

(5.6) $$|a_{K,h,A}(x)|\leqq C\omega_\nu(2^{-k(1)}, \cdots, 2^{-k(\nu)})(1+|h_1|^{n+1}+\cdots+|h_n|^{n+1})^{-1}.$$

First, the condition $(*\nu)$ and the facts (5.1), together with the equality

$$h_1^{n+1}a_{K,h}(x)=\int Q_K(x, \xi)\cdot\Big(\frac{i}{\pi}\cdot\frac{\partial}{\partial\xi_1}\Big)^{n+1}\exp(-\pi ih\cdot\xi)d\xi$$

$$=\Big(\frac{1}{\pi i}\Big)^{n+1}\int\exp(-\pi ih\cdot\xi)\Big(\frac{\partial}{\partial\xi_1}\Big)^{n+1}\{P(x, 2^{(k(1)+2)M^{(1)}}\xi^{(1)}, \cdots, 2^{(k(N)+2)M^{(N)}}\xi^{(N)})$$
$$\cdot\phi_{1,k(1)}(2^{(k(1)+2)M^{(1)}}\xi^{(1)})\cdots\phi_{N,k(N)}(2^{(k(N)+2)M^{(N)}}\xi^{(N)})\}d\xi,$$

imply the estimate

$$|h_1|^{n+1}|\Delta^1_{y^{(1)}}(\cdots\{\Delta^\nu_{y^{(\nu)}}a_{K,h}(x)\}\cdots)|$$
$$\leqq C\cdot\sup_{0\leq j\leq n+1}2^{(k(1)+2)m_1 j}\omega_\nu(|y^{(1)}|, \cdots, |y^{(\nu)}|)\cdot\sup_{\eta\in\operatorname{supp}\phi_{1,k(1)}}(1+[\eta]_1)^{-jm_1}$$
$$\leqq C\omega_\nu(|y^{(1)}|, \cdots, |y^{(\nu)}|)$$

with a constant $C$ independent of $K$, $h$ and $y^{(1)}, \cdots, y^{(\nu)}$.

In the same manner we obtain the estimates

$$|h_l|^{n+1}|\Delta^1_{y^{(1)}}(\cdots\{\Delta^\nu_{y^{(\nu)}}a_{K,h}(x)\}\cdots)|\leqq C\omega_\nu(|y^{(1)}|, \cdots, |y^{(\nu)}|)$$

for $l=2, \cdots N$ and

$$|\Delta^1_{y^{(1)}}(\cdots\{\Delta^\nu_{y^{(\nu)}}a_{K,h}(x)\}\cdots)|\leqq C\omega_\nu(|y^{(1)}|, \cdots, |y^{(\nu)}|).$$

It follows that

(5.7) $$|\Delta^1_{y^{(1)}}(\cdots\{\Delta^\nu_{y^{(\nu)}}a_{K,h}(x)\}\cdots)|$$
$$\leqq C\omega_\nu(|y^{(1)}|, \cdots, |y^{(\nu)}|)(1+|h_1|^{n+1}+\cdots+|h_n|^{n+1})^{-1}.$$

From (5.7) and the equality

$$a_{K,h,A}(x)=\mathcal{F}^{-1}\Big[\prod_{j=1}^{\nu}(1-\Psi(2^{2-k(j)}[\xi^{(j)}]_j))\cdot\prod_{j=\nu+1}^{N}\Psi(2^{2-k(j)}[\xi^{(j)}]_j)\hat{a}_{K,h}(\xi)\Big](x)$$
$$=\int\prod_{j=1}^{N}\mathcal{F}^{-1}[\Psi(2^{2-k(j)}[\xi^{(j)}]_j)](y^{(j)})\cdot(-1)^\nu\cdot\Delta^1_{y^{(1)}}(\cdots\{\Delta^\nu_{y^{(\nu)}}a_{K,h}(x^{(1)},$$
$$\cdots, x^{(\nu)}, x^{(\nu+1)}-y^{(\nu+1)}, \cdots, x^{(N)}-y^{(N)})\}\cdots)dy$$
$$=\int\prod_{j=1}^{N}(2^{k(j)|M^{(j)}|}\mathcal{F}^{-1}[\Psi(4[\xi^{(j)}]_j)](2^{k(j)M^{(j)}}y^{(j)}))$$
$$\cdot(-1)^\nu\cdot\Delta^1_{y^{(1)}}(\cdots\{\Delta^\nu_{y^{(\nu)}}a_{K,h}(x^{(1)}, \cdots, x^{(\nu)},$$
$$x^{(\nu+1)}-y^{(\nu+1)}, \cdots, x^{(N)}-y^{(N)})\}\cdots)dy,$$

we conclude

$$\begin{aligned}|a_{K,h,A}(x)| &\leqq C(1+|h_1|^{n+1}+\cdots+|h_n|^{n+1})^{-1}\\ &\quad\cdot\int\omega_\nu(|2^{-k(1)M^{(1)}}y^{(1)}|, \cdots, |2^{-k(\nu)M^{(\nu)}}y^{(\nu)}|)\\ &\qquad\cdot\prod_{j=1}^{N}|\mathcal{F}^{-1}[\Psi(4[\xi^{(j)}]_j)](y^{(j)})|dy\\ &\leqq C(1+|h_1|^{n+1}+\cdots+|h_n|^{n+1})^{-1}\omega_\nu(2^{-k(1)}, \cdots, 2^{-k(\nu)})\\ &\quad\cdot\int(1+|y^{(1)}|)\cdots(1+|y^{(N)}|)\prod_{j=1}^{N}|\mathcal{F}^{-1}[\Psi(4[\xi^{(j)}]_j)](y^{(j)})|dy\\ &\leqq C\omega_\nu(2^{-k(1)}, \cdots, 2^{-k(\nu)})(1+|h_1|^{n+1}+\cdots+|h_n|^{n+1})^{-1}.\end{aligned}$$

Thus we have (5.6).

On the other hand, applying Proposition 3.1 to $u_{K,h}(x)$ by putting $\nu=N$ and $a^{(j)}=2^{-2M^{(j)}}h^{(j)}$, we obtain

$$\|(\sum_{K\in N^N}|u_{K,h}(x)|^2)^{1/2}\|_{L^p}\leqq C\prod_{j=1}^{N}\log(2+|h^{(j)}|)\|u(x)\|_{L^p} \tag{5.8}$$

with a constant $C$ independent of $h$ and $u(x)$.

Now we put $K'=(k(1), \cdots, k(\nu))$, $K''=(k(\nu+1), \cdots, k(N))$ and $K=(K', K'')$. Then it follows from (5.6) and (5.8) that

$$\begin{aligned}&\|(\sum_{K''\in N^{N-\nu}}|\sum_{K'\in N^\nu}a_{K,h,A}(x)u_{K,h}(x)|^2)^{1/2}\|_{L^p}\\ &\leqq\|(\sum_{K''\in N^{N-\nu}}\{\sum_{K'\in N^\nu}|a_{(K',K''),h,A}(x)|^2\sum_{K'\in N^\nu}|u_{(K',K''),h}(x)|^2\})^{1/2}\|_{L^p}\\ &\leqq(\sup_{K''\in N^{N-\nu}}\sup_{x\in R^n}(\sum_{K'\in N^\nu}|a_{(K',K''),h,A}(x)|^2)^{1/2})\cdot\|(\sum_{K\in N^\nu}|u_{K,h}(x)|^2)^{1/2}\|_{L^p}\\ &\leqq C(\sum_{K'\in N^\nu}\omega_\nu(2^{-k(1)}, \cdots, 2^{-k(\nu)})^2)^{1/2}(1+|h_1|^{n+1}+\cdots+|h_n|^{n+1})^{-1}\\ &\quad\cdot\prod_{j=1}^{N}\log(2+|h^{(j)}|)\cdot\|u(x)\|_{L^p}.\end{aligned}$$

On the other hand, suppose that

$$\xi\in\operatorname{supp}\hat{u}_{K,h}(\xi)\subset\operatorname{supp}b_{K,h}(\xi).$$

Then we have

$$\begin{cases}[\xi^{(j)}]_j\leqq 4/3 & \text{if } k(j)=0,\\ 2^{k(j)-1}\leqq[\xi^{(j)}]_j\leqq 2^{k(j)+2}/3 & \text{if } k(j)>0.\end{cases}$$

We also have $[\xi^{(j)}]_j\leqq 2^{k(j)}/3$ for $j=\nu+1, \nu+2, \cdots, N$ if

$$\xi\in\operatorname{supp}\hat{a}_{K,h,A}(\xi)\subset\operatorname{supp}\prod_{j=\nu+1}^{N}\Psi(2^{2-k(j)}[\xi^{(j)}]_j).$$

Hence, if

$$\xi \in \operatorname{supp} \mathcal{F}[a_{K,h,A}(x)u_{K,h}(x)](\xi) \subset \operatorname{supp}(\hat{a}_{K,h,A} * \hat{u}_{K,h})(\xi),$$

we have

$$\begin{cases} [\xi^{(j)}]_j \leqq 5/3 & \text{if } j>\nu \text{ and } k(j)=0, \\ 2^{k(j)-1}/3 \leqq [\xi^{(j)}]_j \leqq 5 \cdot 2^{k(j)}/3 & \text{if } j>\nu \text{ and } k(j)>0. \end{cases} \tag{5.9}$$

So we can apply Proposition 3.3 to the sequence

$$\{ \sum_{K' \in N^\nu} a_{(K',K''),h,A}(x) \cdot u_{(K',K''),h}(x) \}_{K'' \in N^{N-\nu}}$$

to obtain the estimate

$$\begin{aligned} &\| \sum_{K'' \in N^{N-\nu}} \sum_{K' \in N^\nu} a_{(K',K''),h,A}(x) \cdot u_{(K',K''),h}(x) \|_{L^p} \\ &\leqq C(1+|h_1|^{n+1}+\cdots+|h_n|^{n+1})^{-1} \cdot \prod_{j=1}^{N} \log(2+|h^{(j)}|) \\ &\quad \cdot (\sum_{K' \in N^\nu} \omega_\nu(2^{-k(1)}, \cdots, 2^{-k(\nu)})^2)^{1/2} \cdot \|u(x)\|_{L^p}. \end{aligned}$$

Since

$$\sum_{K' \in N^\nu} \omega_\nu(2^{-k(1)}, \cdots, 2^{-k(\nu)})^2 \leqq C \int_0^1 \frac{\omega_\nu(t_1, \cdots, t_\nu)^2}{t_1 \cdots t_\nu} dt_1 \cdots dt_\nu < \infty,$$

we have (5.5). This completes the proof of Theorem 2.

## § 6. Some generalizations.

In this section we give two slight generalizations of Theorem 2.

First, let $L$ be an integer greater than 1. In this section we change the definition of $\Delta_y^\nu$. For $\nu=1, 2, \cdots, N$ and $y \in \boldsymbol{R}^{n(\nu)}$, we denote by $\Delta_y^\nu$ the difference of $L$-th order with respect to the $\nu$-th part; that is, we write

$$\Delta_y^\nu f(x) = \sum_{k=0}^{L} (-1)^k \binom{L}{k} f(x^{(1)}, \cdots, x^{(\nu)}-ky, \cdots, x^{(N)}).$$

Then the conditions $(*\mu)$ $(\mu=1, \cdots, N)$ become weaker than the original ones, but we still have the following

THEOREM 3. *If the modulus of continuity* $\{\omega_1, \cdots, \omega_N\}$ *satisfies the condition* 1) *of the main theorem, then every symbol satisfying the weakened conditions* $(*\mu)$ *for* $\mu=0, 1, \cdots, N$ *defines an* $L^p$*-bounded operator.*

SKETCH OF THE PROOF. We proceed as in Section 5, and here we remark only the difference.

For $K \in \boldsymbol{N}^N$ and every element $A=(a(1), \cdots, a(N))$ of $\boldsymbol{I}=\{0, 1, \cdots, L\}^N$, we put

$$\Phi_{K,A}(\xi) = \prod_{a(j)=0} \left(1 + \sum_{r=0}^{L} (-1)^r \binom{L}{r} \Psi(2^{2-k(j)}[r\xi^{(j)}]_j)\right)$$

$$\times \prod_{a(j)\neq 0}(-1)^{a(j)-1}\binom{L}{a(j)}\Psi(2^{2-k(j)}[a(j)\xi^{(j)}]_j)$$

and

$$a_{K,h,A}(x)=\mathcal{F}^{-1}[\Phi_{K,A}(\xi)\hat{a}_{K,h}(\xi)](x).$$

Then, since $\sum_{A\in I}\Phi_{K,A}(\xi)\equiv 1$, we have only to prove (5.5) for this $a_{K,h,A}(x)$. We assume $a(1)=\cdots=a(\nu)=0$, $a(\nu+1), \cdots, a(N)>0$. Then we have (5.7) as in Section 5.

On the other hand, we have the equality

$$\begin{aligned}
a_{K,h,A}(x)&=\mathcal{F}^{-1}\Big[\prod_{j=1}^{\nu}\Big(1+\sum_{r=0}^{L}(-1)^r\binom{L}{r}\Psi(2^{2-k(j)}[r\xi^{(j)}]_j)\Big)\\
&\qquad\times\prod_{j=\nu+1}^{N}(-1)^{a(j)-1}\binom{L}{a(j)}\Psi(2^{2-k(j)}[a(j)\xi^{(j)}])\hat{a}_{K,h}(\xi)\Big](x)\\
&=\int\prod_{j=1}^{\nu}\Big(\delta(y^{(j)})+\sum_{r=0}^{L}(-1)^r\binom{L}{r}\mathcal{F}^{-1}[\Psi(2^{2-k(j)}[r\xi^{(j)}]_j)](y^{(j)})\Big)\\
&\qquad\times\prod_{j=\nu+1}^{N}(-1)^{a(j)-1}\binom{L}{a(j)}\mathcal{F}^{-1}[\Psi(2^{2-k(j)}[a(j)\xi^{(j)}]_j)](y^{(j)})\\
&\qquad\times a_{K,h}(x^{(1)}-y^{(1)}, \cdots, x^{(N)}-y^{(N)})dy\\
&=\int\prod_{j=1}^{N}\Big(2^{k(j)|M(j)|}\mathcal{F}^{-1}[\Psi(4[\xi^{(j)}]_j)](2^{k(j)M(j)}y^{(j)})\\
&\qquad\cdot(-1)^{a(\nu+1)+\cdots+a(N)-N+\nu}\binom{L}{a(j)}\Big)\times\{\Delta^1_{y^{(1)}}(\cdots\{\Delta^{\nu}_{y^{(\nu)}}a_{K,h}(x^{(1)}, \cdots, x^{(\nu)},\\
&\qquad x^{(\nu+1)}-a(\nu+1)y^{(\nu+1)}, x^{(N)}-a(N)y^{(N)})\}\cdots)\}dy,
\end{aligned}$$

since

$$\begin{aligned}
&\mathcal{F}^{-1}[\Psi(2^{2-k(j)}[r\xi^{(j)}]_j)](y^{(j)})\\
&=\mathcal{F}^{-1}[\Psi(4[2^{-k(j)M(j)}\cdot r\xi^{(j)}]_j)](y^{(j)})\\
&=\frac{2^{m(j)}}{r^{n(j)}}\mathcal{F}^{-1}[\Psi(4[\xi^{(j)}]_j)]\Big(\frac{1}{r}2^{k(j)M(j)}y^{(j)}\Big),
\end{aligned}$$

where $m(j)=\sum_{l\in A(j)}m_l\cdot k(j)$. Combining these facts, we obtain (5.6). On the other hand, we have (5.9) also in this case. Hence this theorem can be proved in the same way as in Section 5.

Putting $N=1$ in this theorem, we obtain Theorem 7 in [**13**].

Next, to treat symbols having less regularity with respect to $x$, we modify the conditions $(*\mu)$. Let $\Omega(t)$ be a bounded, monotone-decreasing function of $\boldsymbol{R}^+$ into $\boldsymbol{R}^+$, and consider the condition $(*\mu\Omega)$ ($\mu=0, 1, \cdots, N$) as follows:

($*0\Omega$) For every $\nu=1, 2, \cdots, N$, $l\in\Lambda(\nu)$ and $k=0, 1, \cdots, n+1$ we have

$$|\partial_{\xi_l}^k P(x, \xi)|\leqq C(1+[\xi^{(\nu)}]_\nu)^{-m_l k}\prod_{j=1}^N \Omega([\xi^{(j)}]_j).$$

($*\mu\Omega$) ($\mu=1, \cdots, N$) For every $\nu=1, 2, \cdots N$, $l\in\Lambda(\nu)$, $k=0, 1, \cdots, n+1$, $1\leqq\nu(1)<\cdots<\nu(\mu)\leqq N$ and $y(1)\in\boldsymbol{R}^{n(\nu(1))}, \cdots, y(\mu)\in\boldsymbol{R}^{n(\nu(\mu))}$ we have

$$|\Delta_{y(1)}^{\nu(1)}(\cdots\{\Delta_{y(\mu)}^{\nu(\mu)}(\partial_{\xi_l}^k P(x, \xi))\}\cdots)|$$
$$\leqq C\omega_\mu(|y(1)|, \cdots, |y(\mu)|)(1+[\xi^{(\nu)}]_\nu)^{-m_l k}\prod_{j=1}^N \Omega([\xi^{(j)}]_j).$$

Here $\Delta_y^\nu$ denotes the difference of $L$-th order defined in this section.

Then our final result is as follows:

THEOREM 4. *Suppose that* $\{\omega_1(t_1), \cdots, \omega_N(t_1, \cdots, t_N)\}$ *is a modulus of continuity satisfying*

$$\int_0^1\cdots\int_0^1\frac{\omega_\nu(t_1, \cdots, t_\nu)^2}{t_1\cdots t_\nu}\Omega\left(\frac{1}{t_1}\right)^2\cdots\Omega\left(\frac{1}{t_\nu}\right)^2 dt_1\cdots dt_\nu<\infty$$

*for all* $\nu=1, \cdots, N$. *If a symbol* $P(x, \xi)$ *satisfies the conditions* ($*\mu\Omega$) *for all* $\mu=0, 1, \cdots, N$, *then the associated operator* $P(x, D)$ *is* $L^p$*-bounded.*

REMARK. Especially, if $\Omega(t)$ satisfies $\int_1^\infty \frac{\Omega(t)^2}{t}dt<\infty$, then no regularity conditions with respect to $x$ other than ($*0\Omega$) are needed, since we can take $\omega_\nu\equiv 1$ for all $\nu=1, \cdots, N$.

SKETCH OF THE PROOF. We may assume $\Omega(0)=\Omega(1)$. In the same way as we have obtained the estimate (5.7), we obtain the estimate

$$|\Delta_{y(1)}^1\cdots\Delta_{y(\nu)}^\nu a_{K,h}(x)|$$
$$\leqq C\omega_\nu(|y(1)|, \cdots, |y(\nu)|)(1+|h_1|^{n+1}+\cdots+|h_n|^{n+1})^{-1}\Omega_{k(1)}\cdots\Omega_{k(N)},$$

where $\Omega_k=\Omega(2^{k-2})$. It follows that

$$|a_{K,h,A}(x)|\leqq C\omega_\nu(2^{-k(1)}, \cdots, 2^{-k(\nu)})\Omega_{k(1)}\cdots\Omega_{k(\nu)}\Omega(0)^{N-\nu}$$
$$\times(1+|h_1|^{n+1}+\cdots+|h_n|^{n+1})^{-1}$$

for $A\in\boldsymbol{I}$ such that $a(j)=0$ ($j\leqq\nu$) and $a(j)>0$ ($j>\nu$). Hence we have only to show

$$\sum_{K'\in N^\nu}\omega_\nu(2^{-k(1)}, \cdots, 2^{-k(\nu)})^2\Omega_{k(1)}^2\cdots\Omega_{k(\nu)}^2<\infty.$$

Putting

$$I_A=\{(t_1, \cdots, t_\nu);\ 0\leqq t_j\leqq 1\ (j\in A),\ 1\leqq t_j\leqq 8\ (j\notin A)\}$$

for $A\in\boldsymbol{I}$, we have

$$\sum_{K'\in N^\nu}\omega_\nu(2^{-k(1)},\cdots,2^{-k(\nu)})^2\Omega_{k(1)}^2\cdots\Omega_{k(\nu)}^2$$

$$\leqq C\int_0^1\cdots\int_0^1\frac{\omega_\nu(t_1,\cdots,t_\nu)^2}{t_1\cdots t_\nu}\Omega\left(\frac{1}{8t_1}\right)^2\cdots\Omega\left(\frac{1}{8t_\nu}\right)^2dt_1\cdots dt_\nu$$

$$\leqq C\sum_{A\in I}\int\cdots\int_{I_A}\frac{\omega_\nu(t_1/8,\cdots,t_\nu/8)^2}{t_1\cdots t_\nu}\Omega\left(\frac{1}{t_1}\right)^2\cdots\Omega\left(\frac{1}{t_\nu}\right)^2dt_1\cdots dt_\nu.$$

If $A=\{1,\cdots,j\}$, then we have

$$\int\cdots\int_{I_A}\frac{\omega_\nu(t_1/8,\cdots,t_\nu/8)^2}{t_1\cdots t_\nu}\Omega\left(\frac{1}{t_1}\right)^2\cdots\Omega\left(\frac{1}{t_\nu}\right)^2dt_1\cdots dt_\nu$$

$$\leqq\int_1^8\cdots\int_1^8\Omega(0)^{2(\nu-j)}4^{\nu-j}\frac{dt_{j+1}\cdots dt_\nu}{t_{j+1}\cdots t_\nu}$$

$$\times\int_0^1\cdots\int_0^1\frac{\omega_j(t_1,\cdots,t_j)^2}{t_1\cdots t_j}\Omega\left(\frac{1}{t_1}\right)^2\cdots\Omega\left(\frac{1}{t_j}\right)^2dt_1\cdots dt_j$$

$$\leqq C.$$

Similar estimates for general $A\in \boldsymbol{I}$ lead to the desired estimate. This completes the proof.

ACKNOWLEDGEMENT. The author expresses his sincere gratitude to Professor H. Komatsu for his valuable encouragements.

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