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Rings of automorphic forms which are not Cohen-Macaulay, I

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By Noether's normalization theorem, a noetherian graded algebra R has a polynomial subring S generated by homogeneous elements such that R is finite over S. It is known (see, for instance Stanley [**24**], § 3) that R is Cohen-Macaulay (C.-M., for short) if and only if R is free over any (equivalently some) such S. Thus it is meaningful to ask which of the graded rings of automorphic forms are C.-M. This is a problem posed by Eichler [**4**], [**5**]. Igusa [**16**] determined the structure of the graded rings of Siegel modular forms of degree two for groups containing $\Gamma_2(2)$, and Resnikoff and Tai [**20**], [**26**] determined the structure of the graded rings of automorphic forms on the complex 2-ball for some arithmetic group. These rings turn out to be C.-M. However Freitag [**6**] showed that the ring of Hilbert modular forms of degree $\geqq 3$ is not C.-M., while in our previous paper [**27**], we proved that the ring of Hilbert modular forms of even weight and of degree two is C.-M. In this paper we show that the ring of automorphic forms fails to be C.-M. for a large class of neat arithmetic groups as well as for the Siegel modular group $\Gamma_g = Sp_{2g}(\boldsymbol{Z})$, $g \geqq 4$.

Samuel [**22**] stated "All the examples of U. F. D.'s I know are C.-M. Is it true in general?" (see Lipman [**19**] for the history of this question). In the case of characteristic zero, Freitag and Kiehl [**9**] gave a negative answer to this question of Samuel by constructing analytic local rings which are U. F. D.'s of dimension 60 and depth 3, hence not C.-M. As far as we know these are the only previously known examples. As Freitag [**7**], [**8**] has shown, the ring of Siegel modular forms for Γ_g ($g \geqq 3$) is U. F. D. Hence our result shows that they furnish negative examples to Samuel's question in arbitrary high dimension.

To prove our assertion it is enough to prove that the Baily-Borel compactification of the corresponding quotient space is not a C.-M. scheme, where a C.-M. scheme is a scheme whose local rings are all C.-M. This gives some generalization of the result of Igusa [**17**], where he shows that the Baily-Borel compactification does not admit a finite nonsingular covering under some condition on the bounded symmetric domain and the arithmetic group.

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§1. Main result.

1.1. Let H_g be the Siegel space of degree g, i.e., $\{Z\in \boldsymbol{M}_g(\boldsymbol{C}) \mid {}^tZ=Z, \operatorname{Im} Z>0\}$. The symplectic group $Sp_{2g}(\boldsymbol{R})=\left\{M=\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in \boldsymbol{M}_{2g}(\boldsymbol{R}) \,\middle|\, A{}^tD-B{}^tC=1_g, A{}^tB=B{}^tA, C{}^tD=D{}^tC\right\}$ acts on H_g by the *symplectic transformation*

$$Z \longrightarrow MZ=(AZ+B)(CZ+D)^{-1} \qquad M=\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in Sp_{2g}(\boldsymbol{R}).$$

Let Γ_g denote the Siegel modular group $Sp_{2g}(\boldsymbol{Z})$. A holomorphic function f on H_g is called a *Siegel modular form* of weight k if it satisfies

$$f(MZ)=|CZ+D|^k f(Z) \qquad M=\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in \Gamma_g$$

and if it is holomorphic also at cusps (the last condition is automatic if $g>1$). Let $A(\Gamma_g)=\bigoplus_{k\geq 0} A(\Gamma_g)_k$ denote the graded ring of Siegel modular forms. The quotient space H_g/Γ_g is the coarse moduli space of the principally polarized abelian varieties over $\boldsymbol{C}$ of dimension g. It has the natural compactification $(H_g/\Gamma_g)^*$ called the *Satake compactification* which is isomorphic to $\operatorname{Proj}(A(\Gamma_g))$, and set-theoretically equals

$$H_g/\Gamma_g \cup H_{g-1}/\Gamma_{g-1} \cup \cdots \cup H_1/\Gamma_1 \cup \{\text{a point}\}.$$

THEOREM 1. *Let $g\geqq 4$. Then the Satake compactification $(H_g/\Gamma_g)^*$ is not a Cohen-Macaulay scheme. For the graded ring $A(\Gamma_g)$ of Siegel modular forms, the ring $A(\Gamma_g)^{(r)}=\bigoplus_{k\equiv 0(r)} A(\Gamma_g)_k$ is not Cohen-Macaulay for any integer r.*

Let $\mathcal{D}$ be a bounded symmetric domain, and Γ an arithmetic group acting on $\mathcal{D}$. The quotient space $\mathcal{D}/\Gamma$ has the natural compactification $(\mathcal{D}/\Gamma)^*$, which is called the Baily-Borel compactification [2]. Let $j(\gamma, z)$ be the Jacobian of $\gamma\in\Gamma$, at a point $z\in\mathcal{D}$, which is an automorphy factor. Let us fix some automorphy factor ρ such that $\rho^{k_0}=j^{-1}$ for a positive integer k_0. A holomorphic function f on $\mathcal{D}$ is called an *automorphic form* for Γ of weight k if it satisfies

$$f(\gamma z)=\rho(\gamma, z)^k f(z) \qquad \text{for} \quad \gamma\in\Gamma$$

and if f is holomorphic also at cusps (the last condition is automatic if

$\operatorname{codim}((\mathcal{D}/\Gamma)^*-(\mathcal{D}/\Gamma))\geqq 2)$. Then the compactification $(\mathcal{D}/\Gamma)^*$ is isomorphic to the projective spectrum of the graded ring of automorphic forms for Γ. Γ is said to be *neat* if, taking some (equivalently any) faithful representation ϕ of Γ to $GL_m(\boldsymbol{C})$, the algebra generated over $\boldsymbol{Q}$ by all the eigenvalues of $\phi(\gamma)$ is torsion free for every $\gamma\in\Gamma$. Any arithmetic group has a neat arithmetic subgroup of finite index.

THEOREM 2. *Let $\mathcal{D}$ be a bounded symmetric domain, and Γ a neat arithmetic group acting on $\mathcal{D}$. Let $\mathcal{D}'$ (resp. $\mathcal{D}''$) be the highest (resp. the second highest) dimensional rational boundary component. Suppose* $\operatorname{rank}\mathcal{D}'=\operatorname{rank}\mathcal{D}-1$ *and suppose one of the following holds;*

(i) $m:=\dim\mathcal{D}-\dim\mathcal{D}'\geqq 3$ *is odd,*

(ii) $m\geqq 4$ *is even, and* $\dim\mathcal{D}''\geqq\dim\mathcal{D}'-2$.

Then the Baily-Borel compactification $(\mathcal{D}/\Gamma)^*$ is not a Cohen-Macaulay scheme. If $A(\Gamma)$ denotes the ring of automorphic forms, then $A(\Gamma)^{(r)}$ is not Cohen-Macaulay for any r.

REMARKS. (i) Let $R=\bigoplus_k R_k$ be a graded algebra, and let $R^{(r)}=\bigoplus_{k\equiv 0(r)} R_k$. Then it is standard that $\operatorname{Proj}(R)\simeq\operatorname{Proj}(R^{(r)})$ for any r, and that $\operatorname{Proj}(R)$ is a C.-M. scheme if R is C.-M. (cf. [**27**], §1). So the first assertion implies the second both in Theorem 1 and in Theorem 2.

(ii) In the case of characteristic zero, an invariant subring of a C.-M. ring under an action of a finite group is also C.-M. by Hochster and Eagon [**15**]. It follows from this and from Theorem 1 that the ring of Siegel modular forms for any normal subgroup of Γ_g of finite index is not C.-M. if $g\geqq 4$.

(iii) The proof of Theorem 1 is given in §4, which is easily generalized to the following case. For a diagonal matrix

$$T=\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_g\end{pmatrix},\qquad t_i\,|\,t_{i+1}\quad (i=1,\cdots,g-1),$$

let

$$\Gamma_g(T)=\left\{M=\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in\boldsymbol{M}_{2g}(\boldsymbol{Z})\;\middle|\;{}^tM\begin{pmatrix} 0 & T\\ -T & 0\end{pmatrix}M=\begin{pmatrix} 0 & T\\ -T & 0\end{pmatrix}\right\}.$$

$\Gamma_g(T)$ acts on H_g by

$$Z\longrightarrow MZ=(TAT^{-1}Z+TB)(CT^{-1}Z+D)^{-1},\qquad M=\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in\Gamma_g(T).$$

Then the Satake compactification $(H_g/\Gamma_g(T))^*$ is not a C.-M. scheme for $g\geqq 4$, and $A(\Gamma_g(T))^{(r)}$ is not C.-M. for any r, $A(\Gamma_g(T))$ denoting the graded ring of Siegel modular forms for $\Gamma_g(T)$.

(iv) The rings $A(\Gamma_1)$, $A(\Gamma_2)$ of Siegel modular forms of degree 1, 2 are known to be C.-M. (cf. Igusa [**16**]). On the other hand, the graded ring $A(\Gamma_3)$

of Siegel modular forms of degree three is believed to be not C.-M. However our method does not work in this case. It will be investigated in a later paper [28].

§ 2. Proof of Theorem 2.

2.1. Let $\mathcal{D}$, $\mathcal{D}'$, $\mathcal{D}''$, Γ be as in Theorem 2. Let $X=\mathcal{D}/\Gamma$, and let X^* be its Baily-Borel compactification. Set-theoretically, $D^*:=X^*-X$ is the union of lower dimensional pieces

$$\mathcal{D}'/\Gamma'_1, \cdots, \mathcal{D}'/\Gamma'_s, \mathcal{D}''/\Gamma''_1, \cdots, \mathcal{D}''/\Gamma''_t, \mathcal{D}'''/\Gamma'''_1, \cdots$$

similar to $\mathcal{D}/\Gamma$. We denote by X' the highest dimensional cusp $\mathcal{D}'/\Gamma'_1\cup\cdots\cup\mathcal{D}'/\Gamma'_s$. Let $\bar{X}$, together with the morphism $\pi:\bar{X}\to X^*$, be a toroidal compactification which was constructed by [1] and which is determined by a projective regular Γ-admissible decomposition of the associated polyhedral cone. π coincides with the normalization of the blowing up of X^* along some sheaf $\mathcal{I}^*$ of ideals with the support of $\mathcal{O}_{X^*}/\mathcal{I}^*$ contained in D^*. Hence X is canonically contained in $\bar{X}$ on which π induces the identity map. $D:=\bar{X}-X$ is known to be a divisor with only normal crossings. The following is a direct consequence of the construction of $\bar{X}$, where it is essential that $\operatorname{rank}\mathcal{D}'=\operatorname{rank}\mathcal{D}-1$.

LEMMA 1. i) *The fibre* $\pi^{-1}(x)$, $x\in X'$, *is an abelian variety of dimension* $m-1$, *where* $m=\dim\mathcal{D}-\dim\mathcal{D}'$.

ii) *Let* Γ' *be any arithmetic group having* Γ *as a normal subgroup, and let* $\mathcal{I}_{D^*}$ *be the sheaf of ideals determining the reduced subscheme* D^*. *Then by a suitable choice of a* Γ*-admissible decomposition,* Γ'/Γ *acts naturally on* $\bar{X}$, *and* $\mathcal{I}_{D^*}^r$ *equals* $\mathcal{I}^*$ *on* $X\cup X'$ *for a positive integer* r. *In this case, the quotient space* $\bar{X}/(\Gamma'/\Gamma)$ *gives a toroidal compactification of* $\mathcal{D}/\Gamma'$, *and* $\pi^{-1}(X\cup X')$ *is the blowing up of* $X\cup X'$ *with respect to the sheaf of ideals defining the reduced subscheme* X'.

2.2. Let j, ρ, k_0 be as in § 1. There is a coherent sheaf $\mathcal{L}(\rho^k)'$ on X defined by

$$H^0(U, \mathcal{L}(\rho^k)')=\{f\in\mathcal{O}_{p^{-1}}(U)\mid f(\gamma z)=\rho(\gamma, z)^k f(z), \gamma\in\Gamma, z\in p^{-1}(U)\},$$

where p is the projection of $\mathcal{D}$ onto X, and U is any open subset of X. Baily and Borel showed that $\mathcal{L}(j^{-1})'$ canonically extends to an ample invertible sheaf $\mathcal{L}(j^{-1})$ on X^*, since Γ is neat (cf. Mumford [18], the proof of Proposition 3.4). At any rate there is an integer k_1 such that $k_1|k_0$ and $\mathcal{L}(\rho^{k_1})'$ extends to an invertible sheaf $\mathcal{L}(\rho^{k_1})$ on X^* satisfying $\mathcal{L}(\rho^{k_1})^{\otimes k_0/k_1}=\mathcal{L}(j^{-1})$ (for instance, take k_0 as k_1). Since X^* is normal and projective, X^* is isomorphic to

$\mathrm{Proj}(\bigoplus_{k_1|k} H^0(X^*, \mathcal{L}(\rho)^k))$. Our purpose is to show that this is not a C.-M. scheme, and so we may replace ρ by ρ^{k_1}. In other words, we may assume

(i) $\mathcal{L} := \mathcal{L}(\rho)$ is an ample invertible sheaf,

(ii) $\mathcal{L}^{\otimes k} = \mathcal{L}(\rho^k)$, especially $\mathcal{L}^{\otimes k_0} = \mathcal{L}(j^{-1})$.

If $\mathrm{codim}(X^* - X) \geqq 2$, then $\mathcal{L}^{\otimes k}$ equals the direct image $i_* \mathcal{L}(\rho^k)'$, i being the inclusion of X to X^*. $H^0(X^*, \mathcal{L}^{\otimes k})$ is just the space of automorphic forms of weight k. A global section of $H^0(X^*, \mathcal{L}^{\otimes k} \otimes \mathcal{I}_{D^*})$ is called a cusp form of weight k. When $\mathcal{D}$ is a point, the space of automorphic forms, or cusp forms of weight $k \geqq 0$ is just $\boldsymbol{C}$. It is well-known that if $k \gg 0$, then $\dim_C H^0(X^*, \mathcal{L}^{\otimes k})$ equals the sum of the dimensions of the spaces of cusp forms of weight k on X^*, $(\mathcal{D}'/\Gamma'_1)^*$, $(\mathcal{D}'/\Gamma'_2)^*$, $\cdots$, where $\mathcal{D}'/\Gamma'_1$, $\mathcal{D}'/\Gamma'_2$, $\cdots$ are all the members appearing in the cusps of X^*.

Let us put $Q(k) := \chi(X^*, \mathcal{L}^{\otimes k})$, the Euler-Poincaré characteristic, which equals $\dim_C H^0(X^*, \mathcal{L}^{\otimes k})$ for $k \gg 0$, since $\mathcal{L}$ is ample. Let $\mathcal{M} = \pi^* \mathcal{L}$. Then the canonical invertible sheaf $K_{\bar{X}}$ on $\bar{X}$ is isomorphic to $\mathcal{M}^{\otimes k_0} \otimes \mathcal{O}_{\bar{X}}(-D)$ (cf. [1], Chap. IV, §1, Theorem 1). We put $P(k) := \chi(\bar{X}, \mathcal{M}^{\otimes k} \otimes \mathcal{O}_{\bar{X}}(-D))$. It equals $\dim_C H^0(\bar{X}, \mathcal{M}^{\otimes k} \otimes \mathcal{O}_{\bar{X}}(-D))$ for $k \gg 0$ by the vanishing theorem of Kodaira type (Grauert and Riemenschneider [11]). Hence $P(k)$ for $k \gg 0$ is equal to the dimension of the space of cusp forms of weight k, since $H^0(\bar{X}, \mathcal{M}^{\otimes k} \otimes \mathcal{O}_{\bar{X}}(-D)) \simeq H^0(X^*, \mathcal{L}^{\otimes k} \otimes \mathcal{I}_{D^*})$.

PROPOSITION 1. *Let Γ be a neat arithmetic group acting on $\mathcal{D}$. If we denote $n = \dim \mathcal{D}$, $n' = \dim \mathcal{D}'$, $n'' = \dim \mathcal{D}''$, then we have*

$$P(k+k_0) = (-1)^n P(-k) + O(k^{n'}).$$

Under the additional assumption $\mathrm{rank}\, \mathcal{D}' = \mathrm{rank}\, \mathcal{D} - 1$, *we have*

$$P(k+k_0) = (-1)^n P(-k) + O(k^{\max(n'', n'-m+1)}).$$

2.3. PROOF OF PROPOSITION 1. Tensoring $\mathcal{M}^{\otimes (k+k_0)}$ with the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\bar{X}}(-D) \longrightarrow \mathcal{O}_{\bar{X}} \longrightarrow \mathcal{O}_D \longrightarrow 0,$$

we get

$$\chi(\bar{X}, \mathcal{M}^{\otimes (k+k_0)} \otimes \mathcal{O}_{\bar{X}}(-D)) = \chi(\bar{X}, \mathcal{M}^{\otimes (k+k_0)}) - \chi(D, \mathcal{M}^{\otimes (k+k_0)} \otimes \mathcal{O}_D).$$

Since $\chi(\bar{X}, \mathcal{M}^{\otimes (k+k_0)}) = (-1)^n P(-k)$ by the Serre duality theorem, we have

$$P(k+k_0) = (-1)^n P(-k) - \chi(D, \mathcal{M}^{\otimes (k+k_0)} \otimes \mathcal{O}_D).$$

We have the Leray spectral sequence

$$E_2^{p,q} = H^p(D^*, R^q \pi_* (\mathcal{M}^{\otimes (k+k_0)} \otimes \mathcal{O}_D)) \Longrightarrow H^{p+q}(D, \mathcal{M}^{\otimes (k+k_0)} \otimes \mathcal{O}_D).$$

By the projection formula, we have $H^p(D^*, R^q \pi_* (\mathcal{M}^{\otimes (k+k_0)} \otimes \mathcal{O}_D)) = H^p(D^*, \mathcal{L}^{\otimes (k+k_0)}$

$\otimes R^q\pi_*\mathcal{O}_D)$, which vanishes for $p>0$ and $k\gg 0$, because $\mathcal{L}$ is ample. Thus

$$H^0(D^*, \mathcal{L}^{\otimes(k+k_0)}\otimes R^i\pi_*\mathcal{O}_D)\simeq H^i(D, \mathcal{M}^{\otimes(k+k_0)}\otimes\mathcal{O}_D), \qquad k\gg 0$$

and

$$\chi(D, \mathcal{M}^{\otimes(k+k_0)}\otimes\mathcal{O}_D)=\sum_{i=0}^{n-1}(-1)^i\dim_C H^0(D^*, \mathcal{L}^{\otimes(k+k_0)}\otimes R^i\pi_*\mathcal{O}_D), \qquad k\gg 0.$$

Since the dimension of D^* equals n', we immediately see the first assertion. Let us suppose $\operatorname{rank}\mathcal{D}'=\operatorname{rank}\mathcal{D}-1$. As we recalled in Lemma 1, $\pi^{-1}(X')$ is flat over X', and moreover its fibres are abelian varieties of dimension $m-1$. By the base change theorem $R^i\pi_*\mathcal{O}_D$ is locally free on X'. By cup product we have a canonical homomorphism

$$\bigwedge^i R^1\pi_*\mathcal{O}_D \longrightarrow R^i\pi_*\mathcal{O}_D$$

on X' (cf. [**12**], Chap. 0, 12.2). It is an isomorphism since so is the induced homomorphism on the fibre at each point. Since $\dim\pi^{-1}(x)=m-1$ for $x\in X'$, the sheaf $R^i\pi_*\mathcal{O}_D$, $i>m-1$, is supported on D^*-X'. So

$$\chi(D, \mathcal{M}^{\otimes(k+k_0)}\otimes\mathcal{O}_D)=\sum_{i=0}^{m-1}(-1)^i\dim_C H^0(D^*, \mathcal{L}^{\otimes(k+k_0)}\otimes R^i\pi_*\mathcal{O}_D)+O(k^{n''}).$$

Now our assertion follows from the following lemma;

LEMMA. *Let Y' be a normal irreducible projective variety of dimension n' with an ample invertible sheaf $\mathcal{L}'$, and Y'' its subvariety of dimension n''. Let $Y^0=Y'-Y''$.*

(i) *Let $\mathcal{F}$, $\mathcal{G}$ be coherent sheaves on Y' such that $\mathcal{F}|_{Y^0}\simeq\mathcal{G}|_{Y^0}$. Then*

$$\dim_C H^0(Y', \mathcal{L}'^{\otimes k}\otimes\mathcal{F})=\dim_C H^0(Y', \mathcal{L}'^{\otimes k}\otimes\mathcal{G})+O(k^{n''}).$$

(ii) *Suppose Y^0 is nonsingular. Let $\mathcal{E}_1, \cdots, \mathcal{E}_{m-1}$ be coherent sheaves on Y' such that $\mathcal{E}_1|_{Y^0}$ is locally free of rank $m-1$, and $\mathcal{E}_i|_{Y^0}\simeq\bigwedge^i\mathcal{E}_1|_{Y^0}$. Then*

$$\sum_{i=0}^{m-1}(-1)^i\dim_C H^0(Y', \mathcal{L}'^{\otimes k}\otimes\mathcal{E}_i)=O(k^{\max(n'', n'-m+1)}),$$

$\mathcal{E}_0$ being the structure sheaf $\mathcal{O}_{Y'}$.

PROOF. To prove (i) we may assume that $\mathcal{F}$, $\mathcal{G}$ have no coherent subsheaves supported on Y'', and that $\mathcal{F}$, $\mathcal{G}$ are generated by their global sections. Let $\{s_i\}$ be global sections of $\mathcal{F}$. Then $s_i|_{Y^0}$ can be regarded as sections of $H^0(Y^0, \mathcal{G})$ via the isomorphism. Let $\{s_i'\}$ be the rational sections of $\mathcal{G}$ given as their extensions, and let $\tilde{\mathcal{G}}$ be the coherent sheaf generated by $\mathcal{G}$ and $\{s_i'\}$. Then we have two short exact sequences

$$0\longrightarrow\mathcal{F}\longrightarrow\tilde{\mathcal{G}}\longrightarrow\tilde{\mathcal{G}}/\mathcal{F}\longrightarrow 0$$

$$0 \longrightarrow \mathcal{G} \longrightarrow \tilde{\mathcal{G}} \longrightarrow \tilde{\mathcal{G}}/\mathcal{G} \longrightarrow 0,$$

where $\tilde{\mathcal{G}}/\mathcal{F}$, $\tilde{\mathcal{G}}/\mathcal{G}$ are supported on Y''. Then

$$\dim_C H^0(Y', \mathcal{L}'^{\otimes k}\otimes\mathcal{F}) = \dim_C H^0(Y', \mathcal{L}'^{\otimes k}\otimes\tilde{\mathcal{G}}) + O(k^{n''}),$$
$$\dim_C H^0(Y', \mathcal{L}'^{\otimes k}\otimes\mathcal{G}) = \dim_C H^0(Y', \mathcal{L}'^{\otimes k}\otimes\tilde{\mathcal{G}}) + O(k^{n''}).$$

This shows (i). Now let us prove (ii). We may assume $\mathcal{E}_i$'s are torsion free. There is a proper modification $\phi:\tilde{Y}\to Y$ such that $\tilde{Y}$ is a compact complex manifold with $\phi:\phi^{-1}(Y^0)\xrightarrow{\sim} Y^0$, and $\mathcal{E}'_1 := \phi^*\mathcal{E}_1$ is locally free of rank $m-1$ (Riemenschneider [21]). By the Riemann-Roch theorem we easily see that

$$\sum_{i=0}^{m-1}(-1)^i\chi(Y, \phi^*\mathcal{L}'^{\otimes k}\otimes\overset{i}{\wedge}\mathcal{E}'_1) = O(k^{n'-m+1}).$$

Then by the same argument as in the proof of Proposition 1 we have

$$\sum_{i=0}^{m-1}(-1)^i\dim_C H^0(Y', \mathcal{L}'^{\otimes k}\otimes\phi_*\overset{i}{\wedge}\mathcal{E}'_1) = O(k^{\max(n'', n'-m+1)}).$$

We are done, since $\mathcal{E}_i$ and $\phi_*\overset{i}{\wedge}\mathcal{E}'_1$ satisfy the condition in (i). q. e. d.

2.4. PROOF OF THEOREM 2. By Remark (i) of §1, it is enough to show that $X^* = \mathrm{Proj}(\bigoplus_{k\geq 0} H^0(X^*, \mathcal{L}^{\otimes k}))$ cannot be a C.-M. scheme. The dualizing sheaf ω_{X^*} is the uniquely determined coherent sheaf on X^* which gives rise to a functorial isomorphism $\mathrm{Hom}(\mathcal{F}, \omega_{X^*}) \simeq H^n(X^*, \mathcal{F})^\vee$ for any coherent sheaf $\mathcal{F}$ (cf. Hartshorne [14]). By Grauert and Riemenschneider [11], ω_{X^*} coincides with i_*K_X, where i is the canonical inclusion of X into X^*, and K_X is the canonical invertible sheaf on X. Obviously $K_X = \mathcal{L}^{\otimes k_0}|_X$, and hence $\omega_{X^*} = i_*(\mathcal{L}^{\otimes k_0}|_X) = \mathcal{L}^{\otimes k_0}$ by Koecher's principle (cf. Serre [23]).

We suppose that X^* is a C.-M. scheme. Then by [14], for instance, we have the Serre duality $H^i(X^*, \mathcal{L}^{\otimes(k+k_0)}) \simeq H^{n-i}(X^*, \mathcal{L}^{\otimes -k})^\vee$, and hence $Q(k+k_0) = (-1)^n Q(-k)$. If $P'(k)$ denotes the Hilbert polynomial for the space of cusp forms of weight k on X', then $Q(k) = P(k) + P'(k) + O(k^{n''})$. Now we can apply to X' and P' the first assertion of Proposition 1, and we get $P'(k+k_0) = (-1)^{n'}P'(-k) + O(k^{n''})$, where k'_0 is an integer such that $0 < k'_0 < k_0$ (Baily and Borel [2], Proposition 1.11). Hence $P'(k)$ is of the form

$$P'(k) = c_0(k-k'_0/2)^{n'} + c_2(k-k'_0/2)^{n'-2} + \cdots + O(k^{n''}), \qquad c_0 \neq 0.$$

Then, applying to $P(k)$ the second assertion of Proposition 1, we get $Q(k+k_0) - (-1)^n Q(-k) = \{P(k+k_0) - (-1)^n P(-k)\} + \{P'(k+k_0) - (-1)^n P'(-k)\} + O(k^{n''}) = P'(k+k_0) - (-1)^n P'(-k) + O(k^{\max(n'', n'-m+1)})$. Hence $Q(k+k_0) - (-1)^n Q(-k) = c_0\{1-(-1)^m\}k^{n'} + n'c_0\{k_0 + (-1+(-1)^{m-1})k'_0/2\}k^{n'-1} + \cdots + O(k^{\max(n'', n'-m+1)})$, which cannot vanish by our assumption, hence we have a contradiction. So X^*

is not a C.-M. scheme. q. e. d.

§ 3. Siegel modular forms.

3.1. Let $X:=X_g:=H_g/\Gamma_g$, and X^* be its Satake compactification, which is set-theoretically equal to $X_g\cup X_{g-1}^*=X_g\cup X_{g-1}\cup\cdots\cup X_0$. The dimension n of X^* equals $g(g+1)/2$. For an integer k such that kg is even, let $\mathcal{L}(k)$ denote the coherent sheaf on X^* corresponding to Siegel modular forms of weight k, i. e., the sheaf defined by

$$H^0(U, \mathcal{L}(k))$$
$$=\left\{f\in\mathcal{O}_{p^{-1}(U)}\ \middle|\ \begin{array}{l}\text{(i) } f(MZ)=|CZ+D|^k f(Z) \text{ for } M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\Gamma_g,\ Z\in p^{-1}(U\cap X),\\ \text{(ii) } f \text{ extends holomorphically to the intersection of } U \text{ and}\\ \quad\text{of the cusps}\end{array}\right\}$$

for open sets U of X^*, p being the canonical projection of H_g to X, where the second condition is automatic if $g>1$. $H^0(X^*, \mathcal{L}(k))$ is the space of Siegel modular forms of weight k. It is easy to verify that $\mathcal{L}(k)$ is reflexive and that if k is even, then $\mathcal{L}(k)|_{X_{g-1}^*}$ is the coherent sheaf on X_{g-1}^* corresponding to Siegel modular forms of weight k.

H_r, $0\leqq r\leqq g-1$, can be regarded as a rational boundary component of H_g. Let Z be a point of H_r where $0\leqq r\leqq g$. Then the group of matrices of the form

$$M=\left(\begin{array}{cc|cc} A' & 0 & B' & * \\ * & {}^tU^{-1} & * & * \\ \hline C' & 0 & D' & * \\ 0 & 0 & 0 & U\end{array}\right)\in\Gamma_g,\qquad M'=\begin{pmatrix}A' & B'\\ C' & D'\end{pmatrix}\in\Gamma_r,\qquad M'Z=Z,$$

is equal to the stabilizer group at Z in Γ_g up to conjugacy. Then the following is standard;

LEMMA 2. *Let $Z\in H_r$, $0\leqq r\leqq g$. Then $\mathcal{L}(k)$ is invertible at the image of Z in X_g^*, if $|C'Z+D'|^k|U|^k$ equals one for any $M\in\Gamma_g$ stabilizing Z, where C', D', U are as above.*

COROLLARY. *There is a positive integer N_0 satisfying the following;*
(i) *$\mathcal{L}_0:=\mathcal{L}(N_0)$ is an ample invertible sheaf,*
(ii) *$\mathcal{L}(k+N_0)=\mathcal{L}(k)\otimes\mathcal{L}_0$ for all k,*
(iii) *the algebra $\bigoplus_{s\geqq 0}H^0(X^*, \mathcal{L}_0^{\otimes s})$ is generated by $H^0(X^*, \mathcal{L}_0)$.*

Since Γ_g has fixed points of even order, N_0 must be an even integer.
Let X^0 denote the Zariski open subset of X consisting of the images of

points in H_g whose stabilizer in Γ_g is trivial, i.e., $\{\pm 1_{2g}\}$. When $g\geqq 3$, X_g^0 is just the smooth locus of X_g.

LEMMA 3. *The codimension of $X_g - X_g^0$ in X_g is $g-1$. Moreover, the image in X_g of the fixed point set under the action of $M\in\Gamma_g$ with $M^2\neq 1_{2g}$ is of codimension $\geqq g$. Especially, $\mathcal{L}(k)$ for even k is invertible except on a subvariety of codimension $\geqq g$.*

PROOF. Let M be a torsion element of Γ_g. M can be diagonalized as

$$ {}^tUMU=\begin{pmatrix} \zeta^{t_1} & & & & & 0 \\ & \ddots & & & & \\ & & \zeta^{t_n} & & & \\ & & & \bar\zeta^{t_1} & & \\ & & & & \ddots & \\ 0 & & & & & \bar\zeta^{t_n} \end{pmatrix} $$

where ζ is a root of unity and U is a unitary matrix. Then the dimension of the fixed point set in H_g under M is given by

$$ \#\{(t_i, t_j) \mid 1\leqq i\leqq j\leqq g,\ \zeta^{t_i+t_j}=1\} $$

(cf. Gottschling [**10**]). The first assertion follows immediately from this. If $M^2\neq 1_{2g}$, then some $\zeta^{t_i}\neq\pm 1$, hence the second follows. $\mathcal{L}(k)$ is invertible at a point Z_0 fixed only by M's such that $M^2=1_{2g}$. Indeed, letting $M=\begin{pmatrix} A & B \\ C & D\end{pmatrix}$, we get $(CZ_0+D)(C(MZ_0)+D)=1_g$. Hence $(CZ_0+D)^2=1_g$, thus $|CZ_0+D|^k=1$. q.e.d.

3.2. Let $\pi:\bar X\to X^*$ be a toroidal compactification. Let $D:=\bar X - X$, and $D^*:=X^*-X=X_{g-1}\cup\cdots\cup X_0$. π induces a map of D to D^*, which we shall also denote by π.

PROPOSITION 2. *If $i>0$, then*

$$ R^i\pi_*\mathcal{O}_{\bar X}\simeq R^i\pi_*\mathcal{O}_D $$

on X_{g-1}.

PROOF. Let $\Gamma_g(l):=\{M\in\Gamma_g \mid M\equiv 1_{2g} \bmod l\}$ be the principal congruence subgroup of level l. Let $X(l)=H_g/\Gamma_g(l)$. Denote by $X^*(l)$ and $\bar X(l)$ its Satake compactification and its toroidal compactification, respectively. Let $D^*(l):=X^*(l)-X(l)$, and $D(l):=\bar X(l)-X(l)$. We shall denote also by π the morphism of $\bar X(l)$ to $X^*(l)$. Moreover, $X'(l)$ donotes the union of the highest dimensional cusps in $D^*(l)$, which is a disjoint union of copies of $H_{g-1}/\Gamma_{g-1}(l)$.

We first show $R^i\pi_*\mathcal{O}_{\bar X(l)}\simeq R^i\pi_*\mathcal{O}_{D(l)}$ on $X'(l)$ for $i>0$, provided that $l\geqq 3$. Let $\mathcal{J}_{D^*(l)}$ be the sheaf of ideals of $D^*(l)$ with the reduced structure in $\mathcal{O}_{X^*(l)}$, and let $\mathcal{J}:=\mathcal{J}_{D^*(l)}\mathcal{O}_{\bar X(l)}$. Here we note that we have the canonical injection of $\mathcal{O}_{X^*(l)}$ to $\mathcal{O}_{\bar X(l)}$. Since $l\geqq 3$, we can apply Lemma 1 to our argument. So $\pi^{-1}(X(l)\cup X'(l))$ is the blowing up with respect to $\mathcal{J}_{D^*(l)}|_{X(l)\cup X'(l)}$, and hence

$\mathcal{J}|_{\pi^{-1}(X(l)\cup(X'(l))}$ is an invertible sheaf of ideals on $\pi^{-1}(X(l)\cup X'(l))$ defining $\pi^{-1}(X'(l))$. We have a short exact sequence

$$0 \longrightarrow \mathcal{J}^{j+1} \longrightarrow \mathcal{J}^{j} \longrightarrow \mathcal{J}^{j}/\mathcal{J}^{j+1} \longrightarrow 0,$$

where $\mathcal{J}^{j}/\mathcal{J}^{j+1}$ is an invertible sheaf on $\pi^{-1}(X'(l))$. We thus have a long exact sequence

$$\longrightarrow R^{i}\pi_{*}\mathcal{J}^{j+1} \overset{\alpha_{i,j}}{\longrightarrow} R^{i}\pi_{*}\mathcal{J}^{j} \longrightarrow R^{i}\pi_{*}\mathcal{J}^{j}/\mathcal{J}^{j+1} \longrightarrow R^{i+1}\pi_{*}\mathcal{J}^{j+1} \longrightarrow .$$

For a point $x\in X'(l)$ and for $j>0$, $(\mathcal{J}^{j}/\mathcal{J}^{j+1})_{\pi^{-1}(x)}$ is ample on $\pi^{-1}(x)$ by the definition of the blowing up, and hence the higher cohomology groups $H^{i}(\pi^{-1}(x), (\mathcal{J}^{j}/\mathcal{J}^{j+1})_{\pi^{-1}(x)})$, $i>0$, vanish since $\pi^{-1}(x)$ is an abelian variety. By the base change theorem $R^{i}\pi_{*}\mathcal{J}^{j}/\mathcal{J}^{j+1}$ vanishes at $x\in X'(l)$ if $i>0$, $j>0$, and hence $\alpha_{i,j}$ is surjective at x for $i>0$, $j>0$. Since $R^{i}\pi_{*}\mathcal{J}^{j}=0$ for $i>0$, $j\gg 0$ ([**12**], Théorème (2.2.1), (ii)), it follows that $R^{i}\pi_{*}\mathcal{J}$, $i>0$, vanishes on $X'(l)$. Considering the long exact sequence in the case $j=0$, we get $R^{i}\pi_{*}\mathcal{O}_{\tilde{X}(l)} \simeq R^{i}\pi_{*}\mathcal{O}_{D(l)}$ on $X'(l)$ for $i>0$.

To prove the proposition we note that by general theory (cf. Grothendieck [**13**], Théorème 5.3.1, the proof of its corollary and Corollaire to Proposition 5.2.3), $H^{i}(Y/G, (\phi_{*}\mathcal{F})^{G})$ and $H^{i}(Y, \mathcal{F})^{G}$ are canonically isomorphic, where Y is a separated scheme over $\boldsymbol{C}$ with an action of a finite group G, and $\mathcal{F}$ is a coherent sheaf on Y having an action of G compatible with the action on Y, and $\phi: Y\to Y/G$ is the quotient morphism. Now let U be an affine open subset of $X_{g-1}\subset X_{g}^{*}$ and let $\tilde{U}$ be the inverse image of U in $X_{g-1}(l)=H_{g-1}/\Gamma_{g-1}(l)\subset X'(l)$. Then, letting G be the subgroup of $\Gamma_{g}/\Gamma_{g}(l)$ stabilizing $\tilde{U}$, we have

$$H^{0}(U, R^{i}\pi_{*}\mathcal{O}_{\tilde{X}})=H^{i}(\pi^{-1}(U), \mathcal{O}_{\tilde{X}})=H^{i}(\pi^{-1}(\tilde{U}), \mathcal{O}_{\tilde{X}(l)})^{G}=H^{0}(\tilde{U}, R^{i}\pi_{*}\mathcal{O}_{\tilde{X}(l)})^{G},$$

$$H^{0}(U, R^{i}\pi_{*}\mathcal{O}_{D})=H^{i}(\pi^{-1}(U), \mathcal{O}_{D})=H^{i}(\pi^{-1}(\tilde{U}), \mathcal{O}_{D(l)})^{G}=H^{0}(\tilde{U}, R^{i}\pi_{*}\mathcal{O}_{D(l)})^{G}.$$

By what we saw above, the terms on the extreme right hand side are canonically isomorphic if $i>0$. Hence we are done. q. e. d.

REMARKS. (i) As we easily see, Proposition 2 is true for a toroidal compactification of a quotient space $\mathcal{D}/\Gamma$ of a bounded symmetric domain $\mathcal{D}$ by an arithmetic group Γ provided that $\operatorname{rank}\mathcal{D}'=\operatorname{rank}\mathcal{D}-1$, $\mathcal{D}'$ being the highest dimensional rational boundary component of $\mathcal{D}$.

(ii) Let Z be a point of H_{g-1} whose stabilizer in $\Gamma_{g-1}/\{\pm 1_{2g-2}\}$ is trivial, and let $y\in H_{g-1}/\Gamma_{g-1}(l)\subset X_{g}^{*}(l)$ be the corresponding point. Then the stabilizer subgroup P at y of Γ_{g} is generated by $\Gamma_{g}(l)$ and matrices M of the form

$$M=\pm\left(\begin{array}{cc|cc} 1_{g-1} & 0 & 0 & b \\ {}^t v & \pm 1 & {}^t b & e \\ \hline 0 & & 1_{g-1} & -v \\ & & 0 & \pm 1 \end{array}\right)\in \Gamma_g .$$

Let W (resp. U) be the group generated by $\Gamma_g(l)$ and matrices M of the form

$$M=\left(\begin{array}{cc|cc} 1_{g-1} & 0 & 0 & b \\ {}^t v & 1 & {}^t b & e \\ \hline 0 & & 1_{g-1} & -v \\ & & 0 & 1 \end{array}\right) \quad \left(\text{resp. } \left(\begin{array}{c|cc} 1_g & 0 & 0 \\ & 0 & e \\ \hline 0 & 1_g & \end{array}\right)\right).$$

Then we have inclusions of normal subgroups

$$\Gamma_g(l)\subset U\subset W\subset P.$$

U acts trivially on the fibre $\pi^{-1}(y)$. Let us suppose $l\geqq 3$. Then $\pi^{-1}(y)$ is isomorphic to an abelian variety $\boldsymbol{C}^{g-1}/(Z, 1_{g-1})(l\boldsymbol{Z})^{g-1}$. Regarding z as an element of $\boldsymbol{C}^{g-1}$, an element M of W/U acts on $\pi^{-1}(y)$ as $z\to z+Zv+b$. So the quotient of $\pi^{-1}(y)$ by W is isomorphic to itself. Finally P/W acts on the abelian variety as $z\to\pm z$. It follows that the fibre $\pi^{-1}(x)$ for $x\in X_{g-1}^0$ is a $(g-1)$-dimensional Kummer variety, i.e., the quotient of a $(g-1)$-dimensional abelian variety by the group $\{\pm\mathrm{id}\}$, where π is the morphism of $\bar{X}_g$ to X_g^*.

3.3. Let N_0 be as in Corollary to Lemma 2. Then the Euler-Poincaré characteristic $\chi(X^*, \mathcal{L}(k+sN_0))$ is a numerical polynomial of s, since $\mathcal{L}(k+sN_0)=\mathcal{L}(k)\otimes\mathcal{L}_0^{\otimes s}$ and $\mathcal{L}_0$ is invertible. Let $Q(k):=\chi(X^*, \mathcal{L}(k))$. If k is large enough, then $Q(k)$ gives the dimension of the space of Siegel modular forms of weight k, which equals $\sum_{r=0}^{g}\dim_{\boldsymbol{C}}\{$cusp forms of weight k for $\Gamma_r\}$ if $k>2g+1$ is even (cf. Cartan [3]).

We shall define $P(k)$ as follows. Fix an integer k_1 with $0\leqq k_1<N_0$. Then $\bigoplus_{s\geqq 0}\{$cusp forms of weight $k_1+sN_0\}$ is a graded module over the ring $\bigoplus_{s\geqq 0} H^0(X^*, \mathcal{L}_0^{\otimes s})$. $P(k_1+sN_0)$ is defined to be the Hilbert polynomial in s for the graded module. Then $P(k)$ is well-defined for any k and equals the dimension of the space of cusp forms of weight k for $k\gg 0$ by definition.

COROLLARY TO PROPOSITION 2. *Let $\mathcal{I}_D$ be the sheaf of ideals of D, and let $\mathcal{M}(k)=\pi^*\mathcal{L}(k)$. Under the condition $g\geqq 3$ and k even, we have*

$$\chi(\bar{X}, \mathcal{M}(k)\otimes\mathcal{I}_D)=P(k)+O(k^{(g-1)(g-2)/2}).$$

PROOF. Tensoring $\mathcal{M}(k)$ with the short exact sequence

$$0\longrightarrow \mathcal{I}_D\longrightarrow \mathcal{O}_{\bar{X}}\longrightarrow \mathcal{O}_D\longrightarrow 0,$$

we have a short exact sequence

$$0 \longrightarrow \mathcal{M}(k)\otimes\mathcal{I}_D \longrightarrow \mathcal{M}(k) \longrightarrow \mathcal{M}(k)\otimes\mathcal{O}_D \longrightarrow 0.$$

Hence

$$\chi(\bar{X}, \mathcal{M}(k)\otimes\mathcal{I}_D)=\chi(\bar{X}, \mathcal{M}(k))-\chi(D, \mathcal{M}(k)\otimes\mathcal{O}_D).$$

Now let us put $\mathcal{L}'(k):=\mathcal{L}(k)\otimes\mathcal{O}_{X^*_{g-1}}$, which equals the coherent sheaf corresponding to Siegel modular forms of weight k on X^*_{g-1}, since k is even. We have the Leray spectral sequence

$$E_2^{p,q}=H^p(X^*, R^q\pi_*\mathcal{M}(k)) \Longrightarrow H^{p+q}(\bar{X}, \mathcal{M}(k))$$

$$E_2^{p,q}=H^p(X^*_{g-1}, R^q\pi_*(\mathcal{M}(k)\otimes\mathcal{O}_D)) \Longrightarrow H^{p+q}(D, \mathcal{M}(k)\otimes\mathcal{O}_D).$$

By the same argument as in the proof of Proposition 1, we get

$$H^0(X^*, R^i\pi_*\mathcal{M}(k))=H^i(\bar{X}, \mathcal{M}(k)), \qquad k\gg 0,$$

$$H^0(X^*_{g-1}, R^i\pi_*(\mathcal{M}(k)\otimes\mathcal{O}_D))=H^i(D, \mathcal{M}(k)\otimes\mathcal{O}_D), \qquad k\gg 0.$$

Now by Lemma 3 and Proposition 2, both $R^i\pi_*\mathcal{M}(k)$ and $R^i\pi_*(\mathcal{M}(k)\otimes\mathcal{O}_D)$ are isomorphic to $\mathcal{L}'(k)\otimes R^i\pi_*\mathcal{O}_D$ on X^*_{g-1} minus a subvariety of codimension $\geqq g-1$ if $i>0$. Thus

$$\dim_C H^0(X^*, R^i\pi_*\mathcal{M}(k))=\dim_C H^0(X^*_{g-1}, R^i\pi_*(\mathcal{M}(k)\otimes\mathcal{O}_D))+O(k^{(g-1)(g-2)/2})$$

for $i>0$, hence

$$\chi(\bar{X}, \mathcal{M}(k))-\chi(D, \mathcal{M}(k)\otimes\mathcal{O}_D)$$
$$=\dim_C H^0(X^*, \mathcal{L}(k))-\dim_C H^0(X^*_{g-1}, \mathcal{L}'(k))+O(k^{(g-1)(g-2)/2}), \qquad k\gg 0.$$

We are done, since $P(k)$ equals $\dim_C H^0(X^*, \mathcal{L}(k))-\dim_C H^0(X^*_{g-1}, \mathcal{L}'(k))$ for $k\gg 0$. q. e. d.

Since $\bar{X}$ has only quotient singularities, the canonical coherent sheaf $K_{\bar{X}}$ (in the sense of Grauert-Riemenschneider [**11**]) and the dualizing sheaf coincide. Let $\bar{X}^0$ be the open subset of $\bar{X}$ whose points are not ramification points of the quotient morphism of $\bar{X}(l)$ to $\bar{X}$ for some $l\geqq 3$. Then $\bar{X}-\bar{X}^0$ is just the singular locus, when $g\geqq 3$ (Tai [**25**]).

LEMMA 4. *Let* $g\geqq 3$.
(i) *For the canonical injection* i *of* X^0 *to* X, *we have* $i_*(\mathcal{L}(k)|_{X^0})=\mathcal{L}(k)|_X$.
(ii) *For the canonical injection* $\bar{i}$ *of* $\bar{X}^0$ *to* $\bar{X}$, *we have*

$$K_{\bar{X}}=\bar{i}_*((\mathcal{M}(g+1)\otimes\mathcal{I}_D)|_{\bar{X}^0}) \qquad \textit{if } g \textit{ is odd},$$

$$K_{\bar{X}}=\bar{i}_*((\mathcal{M}(g+1)|_{\bar{X}^0}) \qquad \textit{if } g \textit{ is even}.$$

PROOF. Since $g\geqq 3$, $\operatorname{codim}(X-X^0)$ is greater than one. Then (i) is an easy consequence of the extendability of holomorphic functions across a subvariety of codimension two. In the case (ii) with odd g, we have $K_{\bar{X}^0}=(\mathcal{M}(g+1)\otimes\mathcal{J}_D)|_{X^0}$ by Tai [25], Theorem 1.1. If g is even, then any section in $H^0(U, \mathcal{M}(g+1))$, for an open set U with $U\cap D\neq\emptyset$, vanishes automatically at a point of D, so $K_{\bar{X}^0}=\mathcal{M}(g+1)|_{\bar{X}^0}$ (loc. cit). Then our assertion follows from Grauert and Riemenschneider [11]. q. e. d.

§4. Proof of Theorem 1.

4.1. We shall prove Theorem 1 for even $g\geqq 4$.

PROPOSITION 3. *Let $g\geqq 4$ be even, and N_0 be as in Corollary to Lemma 2. If k is divisible by N_0, then*

$$Q(k+g+1)=(-1)^n Q(-k)+(2^{g-2}-1)\chi(X^*_{g-1}, \mathcal{L}'(k))+O(k^{n-g-1}),$$

where $\mathcal{L}'(k)$ is $\mathcal{L}(k)\otimes\mathcal{O}_{X^*_{g-1}}$.

PROOF. N_0 is an even integer, so $k+g+1$ is odd. Since any modular form for Γ_g of odd weight is a cusp form, we have $Q(k+g+1)=P(k+g+1)$. $\mathcal{M}(g+1)$ and $K_{\bar{X}}$ are isomorphic on $\bar{X}^0$ by Lemma 4, (ii), and $H^0(\bar{X}, \mathcal{M}(k+g+1))=H^0(\bar{X}^0, \mathcal{M}(k+g+1))=H^0(\bar{X}, \mathcal{M}(k)\otimes K_{\bar{X}})$, since $\operatorname{codim}(\bar{X}-\bar{X}^0)\geqq 2$. Thus

$$P(k+g+1)=\chi(X, \mathcal{M}(k)\otimes K_{\bar{X}})$$

by the vanishing theorem of Kodaira type [11]. By the Serre duality we have

$$\chi(\bar{X}, \mathcal{M}(k)\otimes K_{\bar{X}})=(-1)^n\chi(\bar{X}, \mathcal{M}(-k)).$$

On the other hand, we have

$$\begin{aligned}\chi(\bar{X}, \mathcal{M}(k))&=\sum_{i=0}^{n}(-1)^i\dim_C H^i(\bar{X}, \mathcal{M}(k))\\ &=\sum_{i=0}^{n}(-1)^i\dim_C H^0(X^*, \mathcal{L}(k)\otimes R^i\pi_*\mathcal{O}_{\bar{X}}), \qquad k\gg 0,\end{aligned}$$

by the same argument as in the proof of Proposition 1. Since the fibre $\pi^{-1}(x)$ for $x\in X^*_{g-1}$ is of dimension $g-1$, $R^i\pi_*\mathcal{O}_{\bar{X}}$ is supported on X^*_{g-2} for $i\geqq g$. Hence by Proposition 2,

$$\chi(\bar{X}, \mathcal{M}(k))=Q(k)+\sum_{i=1}^{g-1}(-1)^i\dim_C H^0(X^*, \mathcal{L}(k)\otimes R^i\pi_*\mathcal{O}_D)+O(k^{(g-1)(g-2)/2}).$$

$\pi:\pi^{-1}(X^0_{g-1})\to X^0_{g-1}$ is a fibre space of Kummer varieties, so $R^i\pi_*\mathcal{O}_D|_{X^0_{g-1}}$ is 0 if i is odd or $i\geqq g$, and it is a vector bundle of rank $\binom{g-1}{i}$ if $i<g$ is even. So by the Riemann-Roch theorem $\chi(\bar{X}, \mathcal{M}(k))=Q(k)+(2^{g-2}-1)\chi(X^*_{g-1}, \mathcal{L}'(k))+O(k^{n-g-1})$, since the sum of $\binom{g-1}{i}$ for $i=2, 4, \cdots, g-2$ is equal to $2^{g-2}-1$. Now

$$\begin{aligned} Q(k+g+1) &= (-1)^n \chi(\bar{X}, \mathcal{M}(-k)) \\ &= (-1)^n \{Q(-k)+(2^{g-2}-1)\chi(X^*_{g-1}, \mathcal{L}'(-k))+O(k^{n-g-1})\} \\ &= (-1)^n \{Q(-k)+(-1)^{n-g}(2^{g-2}-1)\chi(X^*_{g-1}, \mathcal{L}'(k))\}+O(k^{n-g-1}), \end{aligned}$$

and we are done. q. e. d.

By Grauert and Riemenschneider **[11]**, $i_*K_{X^0}$ gives the dualizing sheaf, i being the inclusion of X^0 to X^*, as we saw in § 2.4. Since $K_{X^0} \simeq \mathcal{L}(g+1)|_{X^0}$, and since $\operatorname{codim}(X^*-X^0) \geqq n-g+1>2$, $i_*K_{X^0}=i_*(\mathcal{L}(g+1)|_{X^0})=\mathcal{L}(g+1)$ by the extendability of holomorphic functions across a subvariety of codimension 2.

Now let us show that X^* is not C.-M. If X^* is C.-M., then we have for k divisible by N_0

$$\begin{aligned} H^i(X^*, \mathcal{L}(k+g+1)) &= H^i(X^*, \mathcal{L}(k) \otimes \mathcal{L}(g+1)) \\ &\simeq H^{n-i}(X^*, \mathcal{L}(-k))^\vee \end{aligned}$$

by the Serre duality, and so $Q(k+g+1)=(-1)^n Q(-k)$. This contradicts Proposition 3. Hence X^* is not C.-M.

4.2. Let us prove Theorem 1 for odd $g \geqq 5$. The above argument works also for this case, so it is enough to show the following;

PROPOSITION 4. *Let $g \geqq 3$ be odd. If k is divisible by N_0, then*

$$\begin{aligned} P(k) &= (-1)^n P(-k+g+1) - 2^{g-2}\chi(X^*_{g-1}, \mathcal{L}'(k))+O(k^{n-g-1}), \\ Q(k) &= (-1)^n Q(-k+g+1) - (2^{g-2}-2)\chi(X^*_{g-1}, \mathcal{L}'(k))+O(k^{n-g-1}). \end{aligned}$$

PROOF. By the short exact sequence

$$0 \longrightarrow \mathcal{M}(k) \otimes \mathcal{I}_D \longrightarrow \mathcal{M}(k) \longrightarrow \mathcal{M}(k) \otimes \mathcal{O}_D \longrightarrow 0,$$

we get

$$\chi(\bar{X}, \mathcal{M}(k) \otimes \mathcal{I}_D) = \chi(\bar{X}, \mathcal{M}(k)) - \chi(D, \mathcal{M}(k) \otimes \mathcal{O}_D).$$

Then

$$\chi(\bar{X}, \mathcal{M}(k) \otimes \mathcal{I}_D) = P(k) + O(k^{(g-1)(g-2)/2})$$

by Corollary to Proposition 2, and

$$\begin{aligned} \chi(\bar{X}, \mathcal{M}(k)) &= (-1)^n \chi(\bar{X}, \mathcal{M}(-k) \otimes K_{\bar{X}}) \\ &= (-1)^n P(-k+g+1) \end{aligned}$$

$$\begin{aligned} \chi(D, \mathcal{M}(k) \otimes \mathcal{O}_D) &= \sum_{i=0}^{n-1} (-1)^i \dim_C H^i(D, \mathcal{M}(k) \otimes \mathcal{O}_D) \\ &= \sum_{i=0}^{g-1} (-1)^i \dim_C H^0(X^*_{g-1}, \mathcal{L}(k) \otimes R^i \pi_* \mathcal{O}_D) + O(k^{(g-1)(g-2)/2}) \\ &= 2^{g-2}\chi(X^*_{g-1}, \mathcal{L}'(k)) + O(k^{n-g-1}) \end{aligned}$$

by an argument similar to that in Proposition 3. This gives the first assertion. Since $Q(k)=P(k)+\chi(X_{g-1}^{*}, \mathcal{L}'(k))$, we have

$$Q(k)-(-1)^{n}Q(-k+g+1)$$
$$=\{P(k)-(-1)^{n}P(-k+g+1)\}+\{\chi(X_{g-1}^{*}, \mathcal{L}'(k))-(-1)^{n}\chi(X_{g-1}^{*}, \mathcal{L}'(-k+g+1))\}$$
$$=-(2^{g-2}-1)\chi(X_{g-1}^{*}, \mathcal{L}'(k))-(-1)^{n}\chi(X_{g-1}^{*}, \mathcal{L}'(-k+g+1))+O(k^{n-g-1}).$$

Here we note that $-(-1)^{n}\chi(X_{g-1}^{*}, \mathcal{L}'(-k+g+1))=-(-1)^{n}(-1)^{n-g}\chi(X_{g-1}^{*}, \mathcal{L}'(k))+O(k^{n-g-1})=\chi(X_{g-1}^{*}, \mathcal{L}'(k))+O(k^{n-g-1})$, because g is odd. Then the second assertion follows immediately from this. q. e. d.

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