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Exceptional manifolds for generalized Schoenflies theorem

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In Tamura [2] the generalized Schoenflies theorem for spheres was proved. The statement is as follows:

THEOREM. *Let M be a connected orientable smooth n-manifold satisfying one of the following conditions:*

i) *M is noncompact or with nonempty boundary,*

ii) *M has a non-zero j-th Betti number for some $j\neq 0$, n,*

iii) *The fundamental group of M is an infinite group,*

iv) *M is a homology sphere.*

Then every inessential $(n-1)$-sphere embedded in M bounds an embedded n-disk.

Also in [3] the generalized Schoenflies theorem for $S^{p}\times S^{q}$ was proved:

THEOREM. *Let M be a manifold as in the above theorem. Let $p+q=n-1$. Then every inessential $S^{p}\times S^{q}$ embedded in M bounds an embedded $D^{p+1}\times S^{q}$ or an embedded $S^{p}\times D^{q+1}$.*

Following Tamura [2], a manifold in which some inessential embedded sphere (resp. $S^{p}\times S^{q}$) does not bounds an embedded disk (resp. $D^{p+1}\times S^{q}$ or $S^{p}\times D^{q+1}$) is said to be *exceptional*. In this paper we prove that exceptionality for sphere is equivalent to that for $S^{p}\times S^{q}$ in most cases. Theorem 1 below also shows that the Schoenflies theorem for spheres implies the Schoenflies theorem for $S^{p}\times S^{q}$.

Throughout this paper we work in PL category. Although [2], [3] deal with smooth manifolds, the PL versions of the theorems can be proved in the same way. All manifolds are assumed to be connected and orientable. S^{m} denotes the m-sphere, and D^{m} denotes the m-disk centered at 0. "Link" denotes the linking number.

THEOREM 1. *If $1<q<p-1$, every $(p+q+1)$-dimensional manifold which is exceptional for $S^{p}\times S^{q}$ is exceptional for S^{p+q}.*

PROOF OF THEOREM 1. Let M be a $(p+q+1)$-dimensional manifold which is not exceptional for S^{p+q}; i.e. every inessential $(p+q)$-sphere embedded in M bounds an embedded $(p+q+1)$-disk. We will prove that every inessential $S^{p}\times S^{q}$

embedded in M bounds an embedded $S^p\times D^{q+1}$ or an embedded $D^{p+1}\times S^q$.

Let W be a $(p+q)$-dimensional inessential submanifold in M which is homeomorphic to $S^p\times S^q$, and let A, B be closures of two components of $M-W$. Let $h: S^p\times S^q\to W$ be a homeomorphism which is nullhomotopic in M, and let $S=h(\{*\}\times S^q)$ for $*\in S^p$. We will show either A or B is homeomorphic to $S^p\times D^{q+1}$ or $D^{p+1}\times S^q$.

CLAIM 1. *S bounds a singular disk Δ contained in A or B.*

PROOF. Since h is nullhomotopic, S is contractible. So there exists an immersion $f: D^{q+1}\to M$ such that $f|\partial D^{q+1}$ is a homeomorphism to S, and that $f|\operatorname{int} D^{q+1}$ is transverse with respect to W. Then $f^{-1}(W)$ is a q-dimensional submanifold in D^{q+1}. We will show that f can be altered preserving $f|\partial D^{q+1}$ so that $f^{-1}(W)=\partial D^{q+1}$.

Let E_i $(i=1, 2, \cdots)$ denote closures of components of $D^{q+1}-f^{-1}(W)$, and let $\eta_i=(f|f^{-1}(W))_*[\partial E_i]\in H_q(W)$. If there exists i such that $\eta_i=0$, we can reduce the number of components of $f^{-1}(W)$ as follows. Firstly we connect components of ∂E_i by disjoint tubular neighbourhoods of arcs $\{\alpha_j\}$ in E_i, and define $f': D^{q+1}\to M$ so that $f'|(D^{q+1}-E_i)=f|(D^{q+1}-E_i)$, $\operatorname{Im}(f'|\alpha_j)\subset W$. Denote $E_i-\bigcup\alpha_j$ by E_i'. Then $\partial E_i'$ is a connected sum of all components of ∂E_i. By assumption $f'|\partial E_i'$: $\partial E_i'\to W$ induces a zero-map between q-dimensional homology groups. So $f'|\partial E_i'$ can be extended to a map $g: E_i'\to W$ because the only obstruction lies in $H^{q+1}(E_i', \partial E_i'; \pi_q(W))$, and by Hurewicz's theorem it is represented by $(f'|\partial E_i')_*[\partial E_i']$ which is equal to η_i. Let $f'': D^{q+1}\to M$ be a map such that $f''|(D^{q+1}-U(E_i))=f|(D^{q+1}-U(E_i))$ and that $f''|E_i$ is a map made by pushing g off W in the regular neighbourhood of W, where $U(E_i)$ denotes a regular neighbourhood of E_i in D^{q+1}. Then components of $f''^{-1}(W)$ is less than that of $f^{-1}(W)$. So make f'' be a new f.

Repeating this process, we may assume that every η_i is not 0. If $f^{-1}(W)$ consists of only one component, the proof of Claim 1 terminates. So we may assume that there are at least two E_i's.

Let $K_A=\operatorname{Ker}\{(i_A)_*: H_q(W)\to H_q(A)\}$, and let $K_B=\operatorname{Ker}\{(i_B)_*: H_q(W)\to H_q(B)\}$, where i_A, i_B are inclusions. As $f(E_i)$ lies in A for some E_i, $K_A\neq 0$. Similarly $K_B\neq 0$. Let x be the element of $H_q(W)$ represented by $h(\{*\}\times S^q)$, and let y be the element of $H_p(W)$ represented by $h(S^p\times\{*\})$.

As $H_q(W)=\langle x\rangle$, $K_A=\langle m_A x\rangle$ and $K_B=\langle m_B x\rangle$ for some integers m_A, m_B different from 0. So there exist $(q+1)$-chains $C_A\in C_{q+1}(A)$ and $C_B\in C_{q+1}(B)$ such that $[\partial C_A]=m_A x$, $[\partial C_B]=m_B x$, and that $\operatorname{int} C_A\subset A$, $\operatorname{int} C_B\subset B$. Since $x\cdot y=1$, $[C_A, \partial C_A]\cdot(i_A)_* y=m_A\neq 0$, and $[C_B, \partial C_B]\cdot(i_B)_* y=m_B\neq 0$. Thus neither $(i_A)_* y$ nor $(i_B)_* y$ is a torsion element by duality theorem.

Let $T=h(S^p\times\{*\})$. Similarly to S, T is inessential in M. Hence there exists an immersion $g: D^{p+1}\to M$ such that $g|\partial D^{p+1}$ is a homeomorphism to T, and that g is transverse to W. Let F_i be closures of components of $D^{p+1}-g^{-1}(W)$.

Then for some F_i, $(g|\partial F_i)_*[\partial F_i]\in H_p(W)$ is not 0 because $(g|\partial D^{p+1})_*[\partial D^{p+1}]\neq 0$ in $H_p(W)$. So some nonzero multiple of $(i_A)_*y$ or $(i_B)_*y$ equals zero. This is a contradiction. Thus the proof of Claim 1 is completed.

Since $q<p-1$, we have $2(q+1)+1\leqq p+q+1$. This implies that Δ can be homotoped to an embedded disk preserving $\partial\Delta$. So we may assume that Δ is an embedded disk.

Let $U(\Delta)$ be a regular neighbourhood of Δ such that $U(\Delta)\cap W$ is also a regular neighbourhood of $\partial\Delta=S$. Let $\phi: D^p\times D^{q+1}\to U(\Delta)$ be a homeomorphism such that $\phi(\{0\}\times D^{q+1})=\Delta$, $\phi(D^p\times\partial D^{q+1})=U(\Delta)\cap W$. (The framing is arbitrary.) Let W' be $(W-\phi(D^p\times\partial D^{q+1}))\cup(\partial D^p\times D^{q+1})$. Clearly W' is homeomorphic to a $(p+q)$-sphere.

CLAIM 2. *W' is inessential in M.*

PROOF. Let V be $h^{-1}(U(\Delta)\cap W)$ in $S^p\times S^q$, and let V' be the closure of $S^p\times S^q-V$. Then both of them are homeomorphic to $D^p\times S^q$. Because h is nullhomotopic, considering the cone of $S^p\times S^q$ we can see that $h|V'$ is homotopic to $h|V$ relative to $\partial V=\partial V'$. So W' can be deformed homotopically to $(W'-h(V'))\cup h(V)=\phi(D^p\times\partial D^{q+1})\cup\phi(\partial D^p\times D^{q+1})$. Clearly it is inessential.

Now by assumption W' bounds an embedded $(p+q+1)$-disk D in M. In the case that D is the opposite side of $M-W'$ to Δ, $W=\partial(D\cup\phi(D^p\times D^{q+1}))$. Clearly $D\cup\phi(D^p\times D^{q+1})$ is homeomorphic to $S^p\times D^{q+1}$. It is not twisted because its boundary W is homeomorphic to $S^p\times S^q$. In this case the assumption that $q<p-1$ is unnecessary. In the case that D is in the same side of $M-W'$ as Δ, $W=\partial(D-\phi(D^p\times\operatorname{int}D^{q+1}))$. As $q>1$, $p+3\leqq p+q+1$. So by Zeeman's theorem ([4]), $\phi(D^p\times D^{q+1})$ is unknotted in D. Thus $D-\phi(D^p\times\operatorname{int}D^{q+1})$ is homeomorphic to $D^{p+1}\times S^q$. This completes the proof of Theorem 1.

REMARK 1. Theorem 1 does not hold if $q=1$. A counterexample can be constructed as follows.

Let M be a $(p+2)$-dimensional manifold different from a sphere, and let $D\subset M$ be an embedded $(p+2)$-disk. There exists a knotted p-handle H embedded in D. Since H is knotted, $D-\operatorname{int}(H)$ is not homeomorphic to $D^{p+1}\times S^1$. As $M-D$ is not a disk, $(M-\operatorname{int}D)\cup H$ is not homeomorphic to $S^p\times D^2$. So $W=\partial(D-H)$ which is homeomorphic to $S^p\times S^1$ bounds neither an embedded $S^p\times D^2$ nor an embedded $D^{p+1}\times S^1$. Moreover as W is contained in D, it is clearly inessential. This shows that W is a counterexample to Theorem 1 in case of $q=1$.

REMARK 2. Claim 1 holds without the assumption on q. If we assume the simply-connectedness of M, Theorem 1 holds for $q=p-1$. It can be proved using Whitney's theorem. If M is 2-connected, Theorem 1 holds for $q=p$. It is a consequence of Irwin's theorem ([**1**]).

THEOREM 2. (*The converse of Theorem* 1.) *If* $p>q$, $m=p+q$, *then every* $(m+1)$*-dimensional manifold which is exceptional for* S^m *is exceptional for* $S^p\times S^q$.

PROOF OF THEOREM 2. Let M be an $(m+1)$-dimensional manifold which is not exceptional for $S^p\times S^q$. We will prove that every inessential m-sphere embedded in M bounds an embedded $(m+1)$-disk.

Let W be an m-dimensional inessential submanifold of M which is homeomorphic to S^m. Let U be an open $(m+1)$-disk embedded in M such that $(U, U\cap W)$ is a standard disk pair. Let $\phi: D^{p+1}\times D^q\to U$ be an embedding such that $\phi(D^{p+1}\times D^q)\cap W=\phi(D^{p+1}\times\partial D^q)$, and $\mathrm{Link}(\{x\}\times\partial D^q, \{0\}\times\partial D^q)=0$ for $x\in\partial D^{p+1}$. Then $W'=(W-\mathrm{int}\, D^{p+1}\times\partial D^q)\cup\phi(\partial D^{p+1}\times D^q)$ is homeomorphic to $S^p\times S^q$. By the definitions of U and ϕ, W' can be homotoped into W. Since W is inessential, so is W'. Hence W' bounds V which is homeomorphic to $S^p\times D^{q+1}$, or V' which is homeomorphic to $D^{p+1}\times S^q$. If W' bounds V, W bounds $V\cup\phi(D^{p+1}\times D^q)$ which is homeomorphic to D^{m+1}. If W bounds V', W' bounds $V-\mathrm{int}\,\phi(D^{p+1}\times D^q)$ which is homeomorphic to D^{m+1}. So the proof of Theorem 2 is completed.

References

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