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Rough isometries, and combinatorial approximations of geometries of non-compact riemannian manifolds

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1. Introduction.

Among the most elementary relations in mathematics are the equivalence relations. Especially in geometry, equivalence relations between manifolds are usually defined by the existences of maps satisfying some suitable conditions. An isometry is the most fundamental map of this kind and defines the equivalence relation of isometry between riemannian manifolds. Another kind of maps which defines an equivalence relation between riemannian manifolds is the quasi-isometry. Now suppose that X and Y are riemannian manifolds. A diffeomorphism of X onto Y is called a *quasi-isometry* if there is a constant $a \geqq 1$ such that

$$a^{-1}|\xi| \leqq |d\varphi(\xi)| \leqq a|\xi| \qquad \text{for all} \quad \xi \in TX. \tag{1.1}$$

When there is a quasi-isometry from X onto Y, we say that X is *quasi-isometric* to Y: Obviously to be quasi-isometric is an equivalence relation. Another map belonging to a broader class is a pseudo-isometry introduced by Mostow [**15**] for studying discontinuous subgroups of semi-simple Lie groups. A *pseudo-isometry* of X into Y is a continuous map satisfying

$$a^{-1}d(x_1, x_2) - b \leqq d(\varphi(x_1), \varphi(x_2)) \leqq ad(x_1, x_2) \qquad \text{for all} \quad x_1, x_2 \in X \tag{1.2}$$

with suitable constants $a \geqq 1$ and $b \geqq 0$, where d denotes the distance functions of X and Y induced from their riemannian structures. It is evident that a quasi-isometry is a pseudo-isometry. But, as is expected from the observation that (1.2) is not symmetric, the existence of a pseudo-isometry does not define an equivalence relation; in fact there exists a pair of complete riemannian manifolds X and Y such that there is a pseudo-isometry from X onto Y but there are no pseudo-isometries of Y into X.

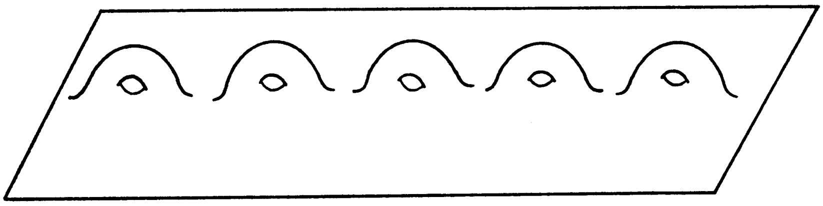
Now we introduce another kind of maps called rough isometries which satisfy a condition weaker than (1.1) and (1.2). Let X be a metric space. For a point x in X, $B_r(x)$ denotes the open r-ball around x: Moreover for a subset

Y of X we denote by $B_r(Y)$ the r-neighborhood of Y; $B_r(Y)=\{x\in X: d(x, Y)<r\}$. A subset Y of X is called *ε-full in* X for $\varepsilon>0$ if $X=B_\varepsilon(Y)$, and is said to be *full* if it is ε-full for some $\varepsilon>0$. A map $\varphi: X\to Y$ between two metric spaces X and Y, *not necessarily continuous*, is called a *rough isometry*, if the image of φ is full in Y and if there are constants $a\geqq 1$ and $b\geqq 0$ such that

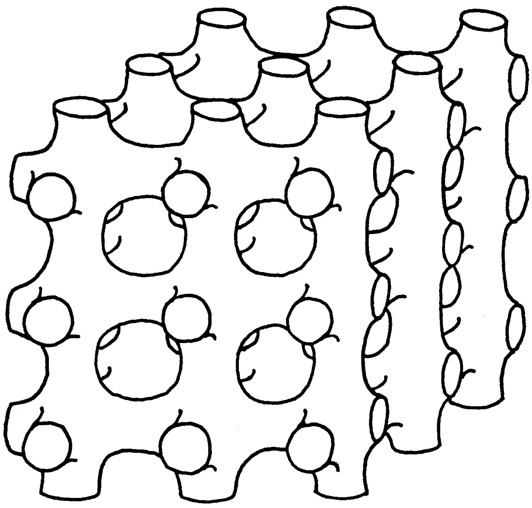
$$a^{-1}d(x_1, x_2)-b\leqq d(\varphi(x_1), \varphi(x_2))\leqq ad(x_1, x_2)+b \quad \text{for all} \quad x_1, x_2\in X. \tag{1.3}$$

Our notion of rough isometry is essentially the same as that of coarse equivalence in the sense of Gromov who discussed, in his article [**10**], the behavior of such a map "at infinity". Quasi-isometries and pseudo-isometries with full images are obviously rough isometries. We can easily show that if $\varphi: X\to Y$ and $\psi: Y\to Z$ are rough isometries then so is the composition $\psi\circ\varphi: X\to Z$. Moreover for a rough isometry $\varphi: X\to Y$, we have a rough isometry $\varphi^-: Y\to X$ such that both $d(\varphi^-\circ\varphi(x), x)$ and $d(\varphi\circ\varphi^-(y), y)$ are bounded in $x\in X$ and in $y\in Y$, respectively. In fact, for each $y\in Y$, choose $x\in X$ so that $d(\varphi(x), y)<\varepsilon$, where we assume that the image of φ is ε-full in Y, and put $\varphi^-(y)=x$. We call such φ^- a *rough inverse* of φ. Two metric spaces are said to be *roughly isometric* if there is a rough isometry between them. As is seen above, to be roughly isometric is an equivalence relation. Note that by weakening the condition (1.2) of pseudo-isometry we are able to make being roughly isometric an equivalence relation: This partially motivates our definition of rough isometry. By definition any two metric spaces of finite diameter are roughly isometric to each other, so the rough isometry makes sense only for non-compact metric spaces. In addition rough isometries neglect "compact factors" of spaces; for example, if X and Y are roughly isometric spaces and K is a compact space, then X is roughly isometric to the product $Y\times K$. Moreover a rough isometry does not preserve topological types because we do not assume a rough isometry to be continuous. For instance, the infinite Loch Ness monster and the infinite jungle gym (these terminologies are due to [**17**]) are roughly isometric to the 2-dimensional and 3-dimensional euclidean spaces, respectively (see Fig. 1).

Here we must mention the work of Milnor [**13**], which motivates our work, gives examples of pairs of riemannian manifolds roughly isometric to each other, and suggests the method of discrete or combinatorial approximation of geometries of riemannian manifolds. First suppose that Γ is a finitely generated group with finite generator system A. For an element $\gamma\neq 1$ of Γ, let $|\gamma|_A$ be the smallest positive integer k such that γ is presented by a product of k elements of $A\cup A^{-1}$, and put $|1|_A=0$. This $|\cdot|_A$ is called the *word norm* of Γ with respect to A, and satisfies the following conditions for all $\beta, \gamma\in\Gamma$: (i) $|\gamma|_A\geqq 0$, and $|\gamma|_A=0$ iff $\gamma=1$; (ii) $|\gamma^{-1}|_A=|\gamma|_A$; (iii) $|\beta\gamma|_A\leqq|\beta|_A+|\gamma|_A$. Also the word norms corresponding to two finite generator systems A and B are equivalent; i.e., there is a constant $a\geqq 1$ such that $a^{-1}|\gamma|_A\leqq|\gamma|_B\leqq a|\gamma|_A$ for all



(a) the infinite Loch Ness monster



(b) the infinite jungle gym

Figure 1.

γ. Now suppose moreover that Γ acts freely and properly discontinuously on a complete riemannian manifold X as isometries and that X/Γ is compact. Fix a point o in X and put $\|\gamma\|=d(o, \gamma o)$ for $\gamma\in\Gamma$. Then obviously the following hold for all $\beta, \gamma\in\Gamma$: (i) $\|\gamma\|\geqq 0$, and $\|\gamma\|=0$ iff $\gamma=1$; (ii) $\|\gamma^{-1}\|=\|\gamma\|$; (iii) $\|\beta\gamma\|\leqq\|\beta\|+\|\gamma\|$. In this situation, Milnor [**13**] has shown the inequalities

$$a^{-1}|\gamma|_A-b\leqq\|\gamma\|\leqq a|\gamma|_A \quad \text{for all } \gamma\in\Gamma, \tag{1.4}$$

where $a\geqq 1$ and $b\geqq 0$ are suitable constants. Now put $\delta_A(\beta, \gamma)=|\beta^{-1}\gamma|_A$. Then δ_A is a left-invariant metric on Γ, called the *word metric* of Γ with respect to A, and the map $\varphi:\Gamma\to X$, $\gamma\to\gamma o$ is a rough isometry (with respect to the word metric δ_A of Γ and the riemannian metric d of X) satisfying the inequality

$$a^{-1}\delta_A(\beta, \gamma)-b\leqq d(\varphi(\beta), \varphi(\gamma))\leqq a\delta_A(\beta, \gamma) \quad \text{for all } \beta, \gamma\in\Gamma. \tag{1.5}$$

Thus we can conclude that if a discrete group Γ acts freely and properly discontinuously on complete riemannian manifolds X and Y as isometries in such a way that both X/Γ and Y/Γ are compact then X is roughly isometric to Y, since to be roughly isometric is an equivalence relation. Also using (1.4), Milnor has shown that the volume growth rate of X is dominated by that of Γ; in fact he proved

$$c^{-1}\cdot\sharp\{\gamma\in\Gamma: |\gamma|_A\leqq a'^{-1}r-b'\}\leqq \mathrm{vol}\, B_r(o)\leqq c\cdot\sharp\{\gamma\in\Gamma: |\gamma|_A\leqq a'r+b'\}, \tag{1.6}$$

where $a'\geqq 1$, $b'\geqq 0$ and $c\geqq 1$ are constants, and, for a set S, $\sharp S$ denotes the cardinality of S. This fact suggests that the geometry of the riemannian manifold X may be approximated by the combinatorial geometry of the discrete group Γ.

Generally speaking, when an equivalence relation is given, what we have to do is, more or less, to find invariants which are preserved by the equivalence relation. This is just what we are going to do in this article for the rough isometry: We will show that some geometric attributes of riemannian manifolds, such as the volume growth rates or the validities of isoperimetric inequalities, are inherited through rough isometries. But, by definition, the local geometry of a manifold does not brought into another manifold by a rough isometry, since we have not assumed a rough isometry even to be continuous. So we need an additional condition on riemannian manifolds which governs local geometries of the manifolds. In fact on a riemannian manifold we will impose the following condition:

(*) the Ricci curvature is bounded from below by a constant $-(n-1)K^2$ $(K>0)$, where n is the dimension of the manifold, and the injectivity radius is positive.

If a complete riemannian n-manifold X satisfies the condition (*), we especially have the following facts:

$$\mathrm{vol}\, B_r(x)\leqq V(r) \quad\text{and}\quad \mathrm{area}\,\partial B_r(x)\leqq A(r) \quad \text{for all } x\in X \text{ and for all } r>0; \tag{1.7}$$

$$\text{the function } r\to \mathrm{vol}\, B_r(x)/V(r) \text{ is monotone non-increasing for all } x\in X; \tag{1.8}$$

$$\mathrm{vol}\, B_r(x)\geqq V_0 r^n \quad \text{for all } x\in X \text{ and for all } r\in(0, \mathrm{inj}\, X/2], \tag{1.9}$$

where $V(r)$ denotes the volume of a geodesic ball in the simply connected complete riemannian n-manifold of constant curvature $-K^2$, $A(r)$ the area of the corresponding geodesic sphere, V_0 a constant depending only on the dimension n, and $\mathrm{inj}\, X$ denotes the injectivity radius of X. The first two facts follow from standard comparison theorems; they hold only under the assumption of lower bound of the Ricci curvature. The inequality (1.9) is a theorem of Croke [8; Proposition 14].

To establish our theorems of invariance of geometric properties of manifolds under rough isometry, we approximate a riemannian manifold by a combinatorial structure, which we call a net. In the case of Milnor's work, the orbit Γo of the action of Γ on X may be considered as a net in our sense, and we have already seen in (1.6) that the geometry of the discrete group Γ reflects that of the riemannian manifold X. This is also the case with a net in a complete riemannian manifold satisfying the condition (*). Moreover a net in a complete riemannian manifold has a canonical metric of combinatorial nature, which corresponds to the word metric in the case of a finitely generated group, and we have an inequality similar to (1.5) for a net in a complete riemannian manifold if the Ricci curvature of the manifold is bounded from below. By this inequality we can see that the net is roughly isometric to the manifold. Now the scheme of the proofs of our theorems of invariance of geometric properties under rough isometries is stated in the following form. Suppose that complete riemannian manifolds X and Y satisfying (*) are roughly isometric to each other. Then a rough isometry between X and Y induces a rough isometry between nets P in X and Q in Y by the aid of the inequality similar to (1.5). On the other hand a discrete approximation lemma such as (1.6) suggests that the geometries of P and Q coincide with those of X and Y, respectively. Thus the rest is to show that two roughly isometric nets P and Q have the same geometry, and this is, in general, easy to prove. So most of our work will be concentrated in the proofs of discrete approximation lemmas similar to (1.6).

The construction of this article is as follows. In § 2, we define a net in a riemannian manifold, and give its elementary properties. In § 3 and § 4, respectively, we will show that the volume growth rate and the validity of isoperimetric inequalities are invariant under rough isometry with the condition (*). Finally, in § 5, we prove the Liouville theorem for harmonic functions on a complete riemannian manifold satisfying (*) which is roughly isometric to the euclidean space. Also this paper contains an appendix where we give a generalization of the local isoperimetric inequality of Buser [**6**], which will be utilized in § 4 and § 5.

2. Nets in riemannian manifolds.

When we attempt to demonstrate that some geometric properties of riemannian manifolds, such as the volume growth rate or the validity of isoperimetric inequalities, are invariant under rough isometry, to approximate riemannian manifolds by discrete or combinatorial structures makes the proofs intelligible. The combinatorial structure which we utilize as an approximation of a riemannian manifold is a net. The purpose of this section is to define it and to state some of its elementary properties.

Let P be a countable set. A family $N=\{N(p): p\in P\}$ is called a *net structure* of P if the following conditions hold for all $p, q\in P$:

(i) $N(p)$ is a finite subset of P;

(ii) $q\in N(p)$ iff $p\in N(q)$.

For a point $p\in P$, each element of $N(p)$ is called a *neighbor* of p. By a *net* we mean a countable set with a net structure. Combining by a segment each pair of two points which are neighbors of each other, we can immediately understand that a net is essentially nothing but a countable 1-dimensional locally finite simplicial complex without orientation, or equivalently, locally finite countable graph. Now suppose that P is a net. A sequence $\boldsymbol{p}=(p_0, \cdots, p_l)$ of points in P is called a *path from* p_0 *to* p_l *of length* l if each p_k is a neighbor of p_{k-1}. A net P is said to be *connected* if any two points in P are combined by a path. For points p and q of a connected net P, $\delta(p, q)$ denotes the minimum of the lengths of paths from p to q. Obviously this δ satisfies the axioms of metric; we call this δ the *combinatorial metric* of P. A net P is said to be *uniform* if $\sup\{\#N(p): p\in P\}<\infty$, where, for a set S, $\#S$ denotes the cardinality of it. By definition the following lemma is immediate.

LEMMA. (i) *If P is a uniform connected net, then, for all $r\geqq 0$ and for all finite subsets S of P, the inequality*

$$\#\{p\in P: \delta(p, S)\leqq r\}\leqq \lambda^r\cdot\#S \tag{2.1}$$

holds, where $\lambda\geqq 1$ is a constant independent of r and S.

(ii) *Suppose that P and Q are connected nets, P uniform, and that $\varphi: P\to Q$ is a rough isometry with respect to the combinatorial metrics of P and Q. Then there is a constant μ such that*

$$\#S\leqq \mu\cdot\#\varphi(S) \tag{2.2}$$

for any finite subset S of P.

Now we give examples of nets. Let Γ be a finitely generated group, and A its finite generator system. Then a net structure $N=\{N(\gamma): \gamma\in\Gamma\}$ on Γ is defined by $N(\gamma)=\gamma(A\cup A^{-1})$. Of course the combinatorial metric coincides with the word metric δ_A with respect to A mentioned in § 1. All finitely generated groups are connected uniform nets in this manner, and this combinatorial structure on a discrete group essentially coincides with the notion of Cayley graph in the combinatorial group theory.

There is another example of nets appearing in riemannian geometry, which plays an important role in our subsequent discussions. Suppose that X is a complete riemannian manifold, and let d be the induced metric. A subset P of X is said to be *ε-separated* for $\varepsilon>0$, if $d(p, q)\geqq\varepsilon$ whenever p and q are distinct points of P, and an ε-separated subset is called *maximal* if it is maximal with respect to the order relation of inclusion. Obviously a maximal ε-separated subset of X is ε-full in X. Let P be a maximal ε-separated subset of X. We define a net structure $N=\{N(p): p\in P\}$ of P by $N(p)=\{q\in P: 0<d(p, q)\leqq 2\varepsilon\}$. A maximal ε-separated subset of a complete riemannian manifold with the net structure described above will be called an *ε-net in X*. It is easy to see that an ε-net in a complete riemannian manifold is connected if the manifold is connected. In our later discussions of this paper, all manifolds and nets are assumed to be connected unless otherwise indicated.

(2.3) LEMMA. *Let X be a complete riemannian n-manifold whose Ricci curvature is bounded from below by $-(n-1)K^2$ $(K>0)$, and let P be an ε-separated subset of X. Then we have*

$$\sharp\{p\in P: x\in B_r(p)\}\leqq\nu \tag{2.4}$$

for all $r>0$ and for all $x\in X$, where $\nu=\nu(n, K, \varepsilon, r)>0$. Consequently every ε-net in a complete riemannian manifold whose Ricci curvature is bounded from below is uniform.

PROOF. Fix $r>0$ and $x\in X$, and put $P_x=\{p\in P: x\in B_r(p)\}$. Obviously $B_{\varepsilon/2}(p)\subset B_{r+\varepsilon/2}(x)\subset B_{2r+\varepsilon/2}(p)$ holds for all $p\in P_x$. Also by (1.8) we have $\operatorname{vol} B_{\varepsilon/2}(p)\geqq \dfrac{V(\varepsilon/2)}{V(2r+\varepsilon/2)}\operatorname{vol} B_{2r+\varepsilon/2}(p)$. Hence with the fact that $B_{\varepsilon/2}(p)$'s are disjoint, we conclude

$$\begin{aligned}\operatorname{vol} B_{r+\varepsilon/2}(x) &\geqq \sum_{p\in P_x}\operatorname{vol} B_{\varepsilon/2}(p)\\ &\geqq \frac{V(\varepsilon/2)}{V(2r+\varepsilon/2)}\sum_{p\in P_x}\operatorname{vol} B_{2r+\varepsilon/2}(p)\\ &\geqq \frac{V(\varepsilon/2)}{V(2r+\varepsilon/2)}\operatorname{vol} B_{r+\varepsilon/2}(x)\cdot\sharp P_x\,;\end{aligned}$$

i.e., $\sharp P_x\leqq V(2r+\varepsilon/2)/V(\varepsilon/2)$. □

The following lemma, which is a generalization of the inequality (1.5) of Milnor, will be a fundamental tool in the later discussions, because this lemma makes it possible to interpret the geometry of a riemannian manifold into the combinatorial geometry of an ε-net in the manifold.

(2.5) LEMMA. *Let X be a complete riemannian n-manifold whose Ricci curvature is bounded from below by $-(n-1)K^2$, and P an ε-net in X. Then P with the combinatorial metric δ is roughly isometric to X: In fact we have*

$$\frac{1}{2\varepsilon}d(p_1, p_2)\leqq\delta(p_1, p_2)\leqq ad(p_1, p_2)+b \qquad \text{for all } p_1, p_2\in P, \tag{2.6}$$

where $a\geqq 1$ and $b\geqq 0$ are constants depending only on n, K and ε, and consequently the inclusion of P into X is a rough isometry.

PROOF. The first inequality in (2.6) trivially holds (without the assumption on the Ricci curvature). We prove the second inequality in (2.6). Suppose that p_1 and p_2 are arbitrary distinct points of P. Let γ be a minimizing geodesic from p_1 to p_2 with unit speed. Put $P_\gamma=\{q\in P: B_\varepsilon(q)\cap\gamma\neq\emptyset\}$. Obviously $\{B_\varepsilon(q): q\in P_\gamma\}$ covers γ, and $\delta(p_1, p_2)\leqq\#P_\gamma$. Moreover take the positive integer k so that $k-1<d(p_1, p_2)/2\varepsilon\leqq k$, and let $x_0\,(=p_1)$, $x_1, \cdots, x_{k-1}, x_k\,(=p_2)$ be the points on γ such that $d(x_{j-1}, x_j)=d(p_1, p_2)/k$ for $j=1, \cdots, k$. Then $q\in B_\varepsilon(\gamma)\subset\bigcup_{j=0}^k B_{2\varepsilon}(x_j)$ for all $q\in P_\gamma$, and therefore $P_\gamma\subset\bigcup_{j=0}^k\{q\in P: x_j\in B_{2\varepsilon}(q)\}$. Hence with (2.4) we have $\#P_\gamma\leqq\sum_{j=0}^k\#\{q\in P: x_j\in B_{2\varepsilon}(q)\}\leqq\nu(k+1)<\nu\{d(p_1, p_2)/2\varepsilon+2\}$, where $\nu=\nu(n, K, \varepsilon, 2\varepsilon)$. Thus we conclude $\delta(p_1, p_2)<\nu\{d(p_1, p_2)/2\varepsilon+2\}$. □

The above lemma especially suggests that any two nets in a complete riemannian manifold whose Ricci curvature is bounded from below are roughly isometric to each other, since the rough isometricity is an equivalence relation.

3. Volume growth rate.

Since the rough isometricity between riemannian manifolds is an equivalence relation, we may expect that a rough isometry preserves some invariants of manifolds. In this section we show that the volume growth rates of geodesic balls in riemannian manifolds are invariant under rough isometries, and the proof of this fact, which is the most elementary and typical in our work, will make one to understand the scheme of the proofs of our theorems. Now let X be a complete riemannian manifold, and o a point in X. Then X is said to be of *polynomial growth of order* m $(\geqq 0)$ if

$$m=\inf\{k>0: \limsup_{r\to\infty} r^{-k}\operatorname{vol}B_r(o)<\infty\} \tag{3.1}$$

holds, and is said to be of *exponential growth* if

$$\liminf_{r\to\infty} r^{-1}\log\operatorname{vol}B_r(o)>0 \tag{3.2}$$

holds. Obviously these definitions do not depend on the choice of a point o in X. It is known that a complete riemannian n-manifold of non-negative Ricci curvature is of polynomial growth of order $\leqq n$, and that a simply connected complete riemannian manifold of negative sectional curvature bounded away from zero is of exponential growth. For other examples of computations of volume growth rates, see [13]: Also in [2], [4], [7] and [12], some relations between the volume growth rate and the other attributes of a riemannian manifold are discussed.

The purpose of this section is to prove the following theorem.

(3.3) THEOREM. *Suppose that X and Y are complete riemannian manifolds satisfying* (∗), *and that X is roughly isometric to Y. Then X is of polynomial growth of order m (resp. of exponential growth) if so is Y.*

As a consequence of this theorem, we obtain pairs of complete riemannian manifolds not roughly isometric to each other; for example, the hyperbolic spaces are not roughly isometric to the euclidean spaces.

We prove Theorem 3.3 by showing that the volume growth rate of a manifold is approximated by that of an ε-net in the manifold. Here, for a net P with a point o in it, the growth rate is defined by replacing $\mathrm{vol}\, B_r(o)$ in (3.1) and (3.2) by $\#\{p \in P : \delta(o, p) < r\}$; e.g., P is said to be *of exponential growth* if $\liminf_{r\to\infty} r^{-1} \log \#\{p \in P : \delta(0, p) < r\} > 0$.

(3.4) LEMMA. *Let P and Q be uniform nets roughly isometric to each other. Then P is of polynomial growth of order m (resp. of exponential growth) if and only if so is Q.*

PROOF. Let $\varphi : P \to Q$ be a rough isometry satisfying

$$a^{-1}\delta(p_1, p_2) - b \leqq \delta(\varphi(p_1), \varphi(p_2)) \leqq a\delta(p_1, p_2) + b \quad \text{for all } p_1, p_2 \in P. \tag{3.5}$$

Fix $o \in P$, and put $o' = \varphi(o)$. Then, with (3.5) and (2.2), we have

$$\#\{p \in P : \delta(o, p) < r\} \leqq \mu \cdot \#\varphi(\{p \in P : \delta(o, p) < r\}) \leqq \mu \cdot \#\{q \in Q : \delta(o', q) < ar + b\},$$

and this implies the lemma. □

The following lemma, which follows from a generalization of the inequality (1.6) of Milnor, claims that the volume growth rate of a manifold is approximated combinatorially.

(3.6) LEMMA. *Suppose that X is a complete riemannian manifold satisfying (*), and that P is an ε-net in X. Then X is of polynomial growth of order m (resp. of exponential growth) if and only if P is of polynomial growth of order m (resp. of exponential growth).*

PROOF. We may consider only the case $0 < \varepsilon \leqq \mathrm{inj}\, X$, because any nets in X are uniform and roughly isometric to each other as Lemma 2.3 and Lemma 2.5 suggest, and consequently, have the same growth rate, by Lemma 3.4. Fix a point o in P. Since the geodesic ball $B_{a^{-1}(r-b)-\varepsilon}(o)$ in X, where a and b are the constants in (2.6), is covered by $\{B_\varepsilon(p) : p \in P, \delta(o, p) < r\}$, we get

$$\mathrm{vol}\, B_{a^{-1}(r-b)-\varepsilon}(o) \leqq V(\varepsilon) \cdot \#\{p \in P : \delta(o, p) < r\},$$

from (1.7). On the other hand, for $p \in P$ with $\delta(o, p) < r$, $B_{\varepsilon/2}(p)$ is contained in $B_{2\varepsilon r + \varepsilon/2}(o)$, and we have

$$\mathrm{vol}\, B_{2\varepsilon r + \varepsilon/2}(o) \geqq \left(\frac{\varepsilon}{2}\right)^n V_0 \cdot \#\{p \in P : \delta(o, p) < r\},$$

by (1.9) and the fact that $B_{\varepsilon/2}(p)$'s are disjoint. Hence we obtain the inequality

$$c^{-1} \mathrm{vol}\, B_{a'^{-1}r-b'}(o) \leqq \#\{p \in P : \delta(o, p) < r\} \leqq c\, \mathrm{vol}\, B_{a'r+b'}(o),$$

where $a' \geqq 1$, $b' \geqq 0$ and $c \geqq 1$ are constants, and this implies the lemma. □

Now Theorem 3.3 follows immediately from Lemma 3.4, Lemma 3.6, Lemma 2.3 and Lemma 2.5. In fact, take X and Y as in Theorem 3.3, and let P and

Q be nets in X and Y, respectively: First note that both P and Q are uniform. Then a rough isometry between X and Y induces a rough isometry between P and Q as Lemma 2.5 suggests, and therefore P and Q have the same growth rate. On the other hand, Lemma 3.6 says that the growth rates of P and Q coincide with those of X and Y, respectively. Hence we conclude that X and Y have the same volume growth rate.

4. Isoperimetric constants.

In this section we show that the validity of some isoperimetric inequalities is inherited through rough isometries. Suppose that X is a complete riemannian n-manifold. For $0<m\leqq\infty$, the *m-dimensional isoperimetric constant* $I_m(X)$ of X is defined by

$$I_m(X)=\inf_{\Omega}\frac{\operatorname{area}\partial\Omega}{(\operatorname{vol}\Omega)^{(m-1)/m}},$$

where Ω runs over all non-empty bounded domains in X with boundaries sufficiently smooth so that their $(n-1)$-dimensional measures $\operatorname{area}\partial\Omega$ are well defined. We also adopt the convention that $(m-1)/m=1$ when $m=\infty$. The classical isoperimetric inequality for the euclidean space $\boldsymbol{R}^n$ suggests that $I_n(\boldsymbol{R}^n)=n^{-(n-1)/n}\alpha_{n-1}^{1/n}$ and $I_m(\boldsymbol{R}_n)=0$ for $m\neq n$, where α_{n-1} denotes the volume of the unit $(n-1)$-sphere. Of course the isoperimetric constant vanishes if X is compact. For any complete riemannian n-manifold X, we also have $I_n(X)\leqq I_n(\boldsymbol{R}^n)$, and $I_m(X)=0$ for $m<n$, by considering the limit of $\operatorname{area}\partial B_r(o)/\{\operatorname{vol}B_r(o)\}^{(m-1)/m}$ when $r\to 0$, where $o\in X$. Moreover it is known that $I_\infty(X)>0$ for any simply connected complete riemannian manifold X whose sectional curvature is bounded from above by a negative constant, and that $I_m(X)=0$ $(m>n)$ for a complete riemannian n-manifold X of non-negative Ricci curvature. For other results concerned with isoperimetric constants, see [**6**], [**16**] and [**19**], and papers cited in them.

The main result in this section is the following:

(4.1) THEOREM. *Let X and Y be complete riemannian manifolds satisfying* (*) *which are roughly isometric to each other, and suppose that* $\max\{\dim X, \dim Y\}\leqq m\leqq\infty$. *Then $I_m(X)>0$ if and only if $I_m(Y)>0$.*

The scheme of the proof is the same as that of Theorem 3.3. First we define the isoperimetric constants for a net, and next, we show the discrete approximation holds: Then Theorem 4.1 will follow immediately.

Now suppose that P is a net. For a subset S of P we define its *boundary* ∂S by

$$\partial S=\{p\in P:\delta(p, S)=1\}.$$

Then the *m-dimensional isoperimetric constant* of P is defined by

$$I_m(P)=\inf_S \frac{\#\partial S}{(\#S)^{(m-1)/m}},$$

where S ranges over all the non-empty finite subsets of P.

(4.2) LEMMA. *Let P and Q be uniform nets roughly isometric to each other. Then $I_m(P)>0$ if and only if $I_m(Q)>0$.*

PROOF. Let φ be a rough isometry of P into Q satisfying (3.5) with $(\kappa+1)$-full image. Take an arbitrary non-empty finite subset S of P, and put $T=\{q\in Q:\delta(q, \varphi(S))\leqq\kappa\}$. By (2.2) we have

$$\#T\geqq\#\varphi(S)\geqq\mu^{-1}\cdot\#S. \tag{4.3}$$

Now let $q\in\partial T$. There is $p\in P$ such that $\delta(q, \varphi(p))\leqq\kappa$. Then $1\leqq\delta(\varphi(p), \varphi(S))\leqq 2\kappa+1$, since $\delta(q, \varphi(S))=\kappa+1$, and this implies, with (3.5), that $1\leqq\delta(p, S)\leqq\sigma+1$ with $\sigma=a(b+2\kappa+1)-1$. Especially we get $\delta(p, \partial S)\leqq\sigma$. Hence we obtain $\partial T\subset\{q\in Q:\delta(q, \varphi(\{p\in P:\delta(p, \partial S)\leqq\sigma\}))\leqq\kappa\}$. Then, by (2.1) we obtain

$$\begin{aligned}\#\partial T&\leqq\#\{q\in Q:\delta(q, \varphi(\{p\in P:\delta(p, \partial S)\leqq\sigma\}))\leqq\kappa\}\\ &\leqq\lambda_Q^\kappa\cdot\#\varphi(\{p\in P:\delta(p, \partial S)\leqq\sigma\})\\ &\leqq\lambda_Q^\kappa\cdot\#\{p\in P:\delta(p, \partial S)\leqq\sigma\}\\ &\leqq\lambda_P^\sigma\lambda_Q^\kappa\cdot\#\partial S,\end{aligned} \tag{4.4}$$

where λ_P and λ_Q are the constants in (2.1) for P and for Q, respectively. From (4.3) and (4.4), we get

$$\lambda_P^\sigma\lambda_Q^\kappa\mu^{(m-1)/m}\frac{\#\partial S}{(\#S)^{(m-1)/m}}\geqq\frac{\#\partial T}{(\#T)^{(m-1)/m}}\geqq I_m(Q).$$

Letting $\#\partial S/(\#S)^{(m-1)/m}\to I_m(P)$, we obtain the inequality

$$\lambda_P^\sigma\lambda_Q^\kappa\mu^{(m-1)/m}I_m(P)\geqq I_m(Q),$$

and this implies the lemma. □

(4.5) LEMMA. *Suppose that X is a complete riemannian n-manifold satisfying $(*)$ and P is an ε-net in X. Then, for every $m\in[n, \infty]$, $I_m(X)>0$ if and only if $I_m(P)>0$.*

The special case of the lemma with $m=\infty$, X the universal covering of a compact riemannian manifold M, and P the fundamental group of M with a net structure described in §2, is due to Gromov [**11**; Theorem 6.19]; see also [**10**] and [**5**].

PROOF. It is sufficient to consider only the case when $\varepsilon\in(0, \operatorname{inj}X/2]$. First we prove that $I_m(X)>0$ implies $I_m(P)>0$. Let S be an arbitrary non-empty finite subset of P. Put $\Omega=\bigcup_{p\in S}B_\varepsilon(p)$. Immediately we have $(\varepsilon/2)^n V_0\cdot\#S\leqq\operatorname{vol}\Omega$, by (1.9). Also since $\partial\Omega\subset\bigcup_{p\in\partial' S}\partial B_\varepsilon(p)$, where $\partial' S=\partial(P-S)$, we have area $\partial\Omega\leqq A(\varepsilon)\cdot\#\partial' S$ from (1.7). In addition, by (2.1), $\#\partial' S\leqq\lambda\cdot\#\partial S$ holds, and we get

$\operatorname{area}\partial\Omega \leqq \lambda A(\varepsilon)\cdot\#\partial S$. Thus we obtain the inequality $\operatorname{area}\partial\Omega/(\operatorname{vol}\Omega)^{(m-1)/m} \leqq c_1\cdot\#\partial S/(\#S)^{(m-1)/m}$ with $c_1=\lambda A(\varepsilon)(\varepsilon/2)^{-n(m-1)/m}V_0^{-(m-1)/m}$, and this implies

$$I_m(X) \leqq c_1 I_m(P).$$

Next we prove that $I_m(P)>0$ implies $I_m(X)>0$, by the method of Buser [6]. Let Ω be an arbitrary bounded domain in X with smooth boundary. Put $S=\{p\in P:\operatorname{vol}(\Omega\cap B_\varepsilon(p))>\operatorname{vol}B_\varepsilon(p)/2\}$ and $P_0=(P-S)\cap B_\varepsilon(\Omega)$, where $B_\varepsilon(\Omega)$ denotes the ε-neighborhood of Ω. Evidently $\{B_\varepsilon(p):p\in S\cup P_0\}$ covers Ω. By the local isoperimetric inequality (see Appendix), we have

$$\frac{\operatorname{area}(\partial\Omega\cap B_\varepsilon(p))}{\{\operatorname{vol}(\Omega\cap B_\varepsilon(p))\}^{(m-1)/m}} \geqq j_m \qquad \text{for all} \quad p\in P_0.$$

Thus with (2.4) we get

$$\begin{aligned}(4.6)\qquad \sum_{p\in P_0}\operatorname{vol}(\Omega\cap B_\varepsilon(p)) &\leqq \Big[\sum_{p\in P_0}\{\operatorname{vol}(\Omega\cap B_\varepsilon(p))\}^{(m-1)/m}\Big]^{m/(m-1)}\\ &\leqq \Big[j_m^{-1}\sum_{p\in P_0}\operatorname{area}(\partial\Omega\cap B_\varepsilon(p))\Big]^{m/(m-1)}\\ &\leqq \{\nu j_m^{-1}\operatorname{area}\partial\Omega\}^{m/(m-1)},\end{aligned}$$

where $\nu=\nu(n, K, \varepsilon, \varepsilon)$. We proceed dividing the proof into two cases according to whether S is empty or not.

Case 1: $S=\emptyset$. In this case we have from (4.6) that

$$\operatorname{vol}\Omega \leqq \sum_{p\in P_0}\operatorname{vol}(\Omega\cap B_\varepsilon(p)) \leqq \{\nu j_m^{-1}\operatorname{area}\partial\Omega\}^{m/(m-1)},$$

i. e.,

$$(4.7)\qquad \frac{\operatorname{area}\partial\Omega}{(\operatorname{vol}\Omega)^{(m-1)/m}} \geqq \nu^{-1}j_m.$$

Case 2: $S\neq\emptyset$. Put $i=\operatorname{area}\partial\Omega/(\operatorname{vol}\Omega)^{(m-1)/m}$. Then by (4.6) we have

$$\begin{aligned}\operatorname{vol}\Omega &\leqq \sum_{p\in S\cup P_0}\operatorname{vol}(\Omega\cap B_\varepsilon(p))\\ &\leqq \sum_{p\in S}\operatorname{vol}(\Omega\cap B_\varepsilon(p))+\{\nu j_m^{-1}\operatorname{area}\partial\Omega\}^{m/(m-1)}\\ &\leqq V(\varepsilon)\cdot\#S+(\nu j_m^{-1}i)^{m/(m-1)}\operatorname{vol}\Omega.\end{aligned}$$

Thus we obtain

$$(4.8)\qquad \operatorname{vol}\Omega \leqq 2V(\varepsilon)\cdot\#S \qquad \text{if} \quad i\leqq 2^{-(m-1)/m}\nu^{-1}j_m.$$

Now put $H=\{x\in X:\operatorname{vol}(\Omega\cap B_\varepsilon(x))=\operatorname{vol}B_\varepsilon(x)/2\}$. Let $p\in\partial S$. Then there is $p_1\in S$ such that $d(p, p_1)\leqq 2\varepsilon$. Note that $\operatorname{vol}(\Omega\cap B_\varepsilon(p))\leqq\operatorname{vol}B_\varepsilon(p)/2$ and $\operatorname{vol}(\Omega\cap B_\varepsilon(p_1))>\operatorname{vol}B_\varepsilon(p_1)/2$. Hence on a minimizing geodesic from p to p_1 of length $\leqq 2\varepsilon$, we find a point of H, since the function $x\to\operatorname{vol}(\Omega\cap B_\varepsilon(x))$ on X is continuous. This shows that $\partial S\subset B_{2\varepsilon}(H)$. Now let Q be a maximal ε-separated subset of H. Then we have $\bigcup_{p\in\partial S}B_{\varepsilon/2}(p)\subset B_{5\varepsilon/2}(H)\subset\bigcup_{q\in Q}B_{7\varepsilon/2}(q)$. This implies, with (1.9), (1.8), the local isoperimetric inequality for $m=\infty$ (see Appendix) and (2.4) that

$$(4.9)\qquad \begin{aligned} \left(\frac{\varepsilon}{2}\right)^n V_0\cdot\#\partial S &\leqq \sum_{p\in\partial S}\mathrm{vol}\,B_{\varepsilon/2}(p) \\ &\leqq \sum_{q\in Q}\mathrm{vol}\,B_{7\varepsilon/2}(q) \\ &\leqq \frac{V(7\varepsilon/2)}{V(\varepsilon)}\sum_{q\in Q}\mathrm{vol}\,B_\varepsilon(q) \\ &= 2\frac{V(7\varepsilon/2)}{V(\varepsilon)}\sum_{q\in Q}\mathrm{vol}(\Omega\cap B_\varepsilon(q)) \\ &\leqq 2j_\infty^{-1}\frac{V(7\varepsilon/2)}{V(\varepsilon)}\sum_{q\in Q}\mathrm{area}(\partial\Omega\cap B_\varepsilon(q)) \\ &\leqq 2\nu j_\infty^{-1}\frac{V(7\varepsilon/2)}{V(\varepsilon)}\mathrm{area}\,\partial\Omega\,. \end{aligned}$$

Now from (4.8) and (4.9) we obtain

$$(4.10)\qquad \frac{\mathrm{area}\,\partial\Omega}{(\mathrm{vol}\,\Omega)^{(m-1)/m}} \geqq c_2\frac{\#\partial S}{(\#S)^{(m-1)/m}} \qquad \text{if}\quad i\leqq 2^{-(m-1)/m}\nu^{-1}j_m\,,$$

where $c_2=2^{-(2m-1)/m}\nu^{-1}(\varepsilon/2)^n V_0 V(\varepsilon)^{1/m}V(7\varepsilon/2)^{-1}j_\infty$.

Hence by (4.7) and (4.10) we conclude

$$I_m(X)\geqq \min\{2^{-(m-1)/m}\nu^{-1}j_m,\ c_2 I_m(P)\}\,. \qquad \square$$

Now Theorem 4.1 follows from Lemma 4.2 and Lemma 4.5.

5. Liouville's theorem.

The classical theorem of Liouville, which suggests that all positive harmonic functions on the euclidean space are constant, has a lot of geometric generalizations. Among such works, Yau [**18**] showed that this also holds for complete riemannian manifolds of non-negative Ricci curvature by the gradient estimate of positive harmonic functions. Also a geometric interpretation of the work of Moser [**14**] says that on a riemannian manifold quasi-isometric to the euclidean space there are no positive harmonic functions other than constants (cf. [**9**; § 4]), and his proof relies on the validity of analytic inequalities such as the Sobolev inequality. It is well known that such inequalities are closely related to isoperimetric inequalities: On the other hand we have seen in the previous section that the validity of some isoperimetric inequalities is invariant under rough isometry. So we expect that we can extend Moser's theorem to complete riemannian manifolds roughly isometric to the euclidean space. The purpose of this section is to show that this is in fact true under the additional condition (*).

(5.1) THEOREM. *Suppose that X is a complete riemannian n-manifold satisfying* (*), *and that X is roughly isometric to the euclidean m-space $\boldsymbol{R}^m$ with $m \geqq n$. Then every positive harmonic function on X is constant.*

By this theorem, we know that the infinite Loch Ness monster, the infinite jungle gym, and their higher dimensional versions have no positive harmonic functions other than constants. In this manner we can construct many new examples of riemannian manifolds for which the Liouville theorem holds.

Theorem 5.1 is an immediate consequence of the following two propositions.

(5.2) PROPOSITION. *Let X be a complete riemannian manifold satisfying* (*) *which is roughly isometric to the euclidean plane $\boldsymbol{R}^2$. Then every positive superharmonic function on X is constant.*

PROOF. From Theorem 3.3, X is of polynomial growth of order 2. Then by a theorem of Cheng and Yau [**7**; § 1], we conclude the proposition. □

(5.3) PROPOSITION. *Suppose that X is a complete riemannian n-manifold satisfying* (*), *and that X is roughly isometric to the euclidean m-space with $m \geqq \max\{3, n\}$. Then there are bounded domains Ω_k $(k=1, 2, \cdots)$ in X satisfying the following conditions:*

(i) $\bar{\Omega}_k \subset \Omega_{k+1}$, *and* $X = \bigcup_{k=1}^{\infty} \Omega_k$;

(ii) *for each k and for every positive harmonic function u on Ω_{4k}, we have the Harnack inequality*

$$\sup_{\Omega_k} u \leqq h \cdot \inf_{\Omega_k} u , \tag{5.4}$$

where h is a constant independent of u and k.

In the rest of this section we prove Proposition 5.3 step by step. We follow the work of Bombieri and Guisti [**3**] who have modified arguments of Moser [**14**]. Now let X be a complete riemannian n-manifold satisfying (*) and roughly isometric to $\boldsymbol{R}^m$ $(m \geqq \max\{3, n\})$. Also denote by $\boldsymbol{Z}^m$ the set of lattice points in $\boldsymbol{R}^m$, which we consider as a 1-net in $\boldsymbol{R}^m$, and let P be an ε-net in X with $\varepsilon \in (0, \operatorname{inj} X/2]$. Then there is a rough isometry $\varphi: \boldsymbol{Z}^m \to P$. We assume that the image of φ is $(\kappa+1)$-full in P and φ satisfies the inequality

$$a^{-1}\delta(z_1, z_2) - b \leqq \delta(\varphi(z_1), \varphi(z_2)) \leqq a\delta(z_1, z_2) + b \qquad \text{for all } z_1, z_2 \in \boldsymbol{Z}^m. \tag{5.5}$$

Put

$$C_k = \{z = (z_1, \cdots, z_m) \in \boldsymbol{Z}^m : |z_l| \leqq \sigma k \text{ for } l = 1, \cdots, m\},$$

$$S_k = \{p \in P : \delta(p, \varphi(C_k)) \leqq \rho\},$$

$$\Omega_k = \bigcup_{p \in S_k} B_\varepsilon(p),$$

with constants $\rho \geqq \kappa$ and $\sigma \geqq 1$ which will be suitably chosen in the following discussions. Obviously $X = \bigcup_{k=1}^{\infty} \Omega_k$ holds. Also by definition we obtain

$$c_1^{-1} k^m \leqq \operatorname{vol} \Omega_k \leqq c_1 k^m, \tag{5.6}$$

where $c_1 \geqq 1$ is a constant independent of k. Now denote by $B_\varepsilon(\Omega_k)$ the ε-neighborhood of Ω_k. From (2.6) and (5.5) we can choose the constant σ so that

$$\overline{B_\varepsilon(\Omega_k)} \subset \Omega_{k+1} \tag{5.7}$$

holds. Note that (5.7) implies especially that $\bar{\Omega}_k \subset \Omega_{k+1}$ and $\overline{B_r(\Omega_k)} \subset \Omega_{2k}$ for all $r \in (0, \varepsilon k]$.

To prove Harnack's inequality (5.4), first we show a Sobolev-type inequality for functions on X with compact supports: Put $M = m/(m-2)$.

CLAIM 1. *There is a constant c_2 such that*

$$\left\{\int_X |u|^{2M} dx\right\}^{1/M} \leqq c_2 \int_X |\nabla u|^2 dx \qquad \textit{for all} \quad u \in H_{1,1}(X). \tag{5.8}$$

PROOF. By Theorem 4.1, we have $I_m(X) > 0$. Moreover it is known that

$$I_m(X) \cdot \left\{\int_X |v|^{m/(m-1)} dx\right\}^{(m-1)/m} \leqq \int_X |\nabla v| \, dx$$

for all $v \in H_{1,1}(X)$ (cf. [**16**; § 3]). Apply this for $v = |u|^{2(m-1)/(n-2)}$: Then with the Schwartz inequality we get (5.8) with $c_2 = 4\{(m-1)/(m-2)\}^2 I_m(X)^{-2}$. □

Next we prove the following fact by the inequality (5.8), the iteration argument and the abstract John-Nirenberg theorem due to Bombieri and Guisti. We use the conventions that $⨍_\Omega u \, dx = \frac{1}{\operatorname{vol} \Omega} \int_\Omega u \, dx$, and that $B_0(\Omega) = \Omega$, for a domain Ω in X and an integrable function u on Ω.

CLAIM 2. *For each k and for each positive harmonic function u on Ω_{2k}, we have*

$$\sup_{\Omega_k} u \leqq e^{c_3 g(u)} \cdot \inf_{\Omega_k} u \tag{5.9}$$

with

$$g(u) = \sup_{0 \leqq r \leqq \varepsilon k} \inf_{\alpha \in R} ⨍_{B_r(\Omega_k)} |\log u - \alpha| \, dx, \tag{5.10}$$

where c_3 is a positive constant independent of k and u.

PROOF. (We shall only give the outline of the proof; see, for detail, [**14**; § 4] and [**3**; § 5].) Let $v = u^{p/2}$ ($p \neq 0, 1$; note that we do not restrict p to being positive) and η an arbitrary Lipschitz continuous function on X with compact support. Then we obtain

$$\int_X \eta^2 |\nabla v|^2 dx \leqq \left(\frac{p}{p-1}\right)^2 \int_X v^2 |\nabla \eta|^2 dx.$$

On the other hand, from (5.8), we have

$$\left\{\int_X |\eta v|^{2M} dx\right\}^{1/M} \leqq 2c_2 \int_X (\eta^2 |\nabla v|^2 + v^2 |\nabla \eta|^2) dx,$$

and hence we get the inequality

$$\left\{\int_X |\eta v|^{2M} dx\right\}^{1/M} \leqq 2c_2\left\{\left(\frac{p}{p-1}\right)^2+1\right\}\int_X |\nabla\eta|^2 v^2 dx\,.$$

For $0 \leqq r < r' \leqq \varepsilon k$, applying the above inequality to

$$\eta(x)=\begin{cases} 1 & \text{if } x\in B_r(\Omega_k) \\ (r'-r)^{-1}\{r'-d(x,\,\Omega_k)\} & \text{if } x\in B_{r'}(\Omega_k)\ \ B_r(\Omega_k) \\ 0 & \text{otherwise,} \end{cases}$$

we obtain

$$\left\{\int_{B_r(\Omega_k)} u^{Mp} dx\right\}^{1/M} \leqq 2c_2\left\{\left(\frac{p}{p-1}\right)^2+1\right\}(r'-r)^{-2}\int_{B_{r'}(\Omega_k)} u^p dx \tag{5.11}$$

for all $p\neq 0,\ 1$. Now a standard iteration argument yields with (5.11) that

$$\sup_{B_r(\Omega_k)} u^p \leqq c_4 \frac{\operatorname{vol} B_{r'}(\Omega_k)}{(r'-r)^m}\int_{B_{r'}(\Omega_k)} u^p dx \tag{5.12}$$

if $0\leqq r<r'\leqq \varepsilon k$, $p\neq 0$ and $|M^i p-1|\geqq 1/m$ for all $i=0, 1, \cdots$, where c_4 is a constant independent of k, u, p, r and r'. From (5.6), (5.7) and (5.12) we have

$$\sup_{B_r(\Omega_k)} u^p \leqq c_5\left(\frac{r'}{\varepsilon k}-\frac{r}{\varepsilon k}\right)^{-mM}\int_{B_{r'}(\Omega_k)} u^p dx \tag{5.13}$$

for all $p\neq 0$ and $0\leqq r<r'\leqq\varepsilon k$, where c_5 is a constant independent of k, u, p, r and r'. Applying the abstract John-Nirenberg inequality (see [**3**; §4]) to (5.13) we conclude (5.9). □

Now, for the proof of the Harnack inequality it is sufficient to estimate $g(u)$. To do this we need the Sobolev inequality of Neumann type (Claim 6), and it is derived from an isoperimetric inequality. So for a while we concentrate ourselves on the proof of the isoperimetric inequality. First we prove a combinatorial isoperimetric inequality for C_k (Claim 3), and next for S_k (Claim 4). Then we will be able to prove an isoperimetric inequality for Ω_k by the discrete approximation method (Claim 5). In the following discussion, $c_6, \cdots, c_{11}$ are positive constants independent of k.

CLAIM 3. *For any non-empty disjoint subsets D_1 and D_2 of C_k such that $C_k=D_1\cup D_2$, we have*

$$\max\left\{\frac{\#(\partial D_1\cap C_k)}{(\# D_1)^{(m-1)/m}},\ \frac{\#(\partial D_2\cap C_k)}{(\# D_2)^{(m-1)/m}}\right\}\geqq c_6\,. \tag{5.14}$$

PROOF. For $z=(z_1,\cdots,z_m)\in \boldsymbol{Z}^m$, let $\Gamma_z=\{x=(x_1,\cdots,x_m)\in\boldsymbol{R}^m : |x_l-z_l|\leqq 1/2$ for $l=1,\cdots,m\}$. Define a domain Ω in $\boldsymbol{R}^m$ by $\Omega=\operatorname{int}\bigcup_{z\in C_k}\Gamma_z$, and two disjoint open subsets Ω_l $(l=1,\ 2)$ of Ω by $\Omega_l=\operatorname{int}\bigcup_{z\in D_l}\Gamma_z$, where the notation int denotes the interior. Also denotes by H the piecewise smooth hypersurface $\partial\Omega_1\cap\Omega=\partial\Omega_2\cap\Omega$ in Ω. Then the isoperimetric inequality for Ω suggests that

$$\frac{\text{area}\, H}{[\min\{\text{vol}\,\Omega_1,\ \text{vol}\,\Omega_2\}]^{(m-1)/m}} \geqq c_7 .$$

By definition, obviously $\text{vol}\,\Omega_l = \# D_l$ $(l=1, 2)$ holds. On the other hand, $\text{area}\, H \leqq 2m \cdot \#(\partial D_l \cap C_k)$ $(l=1, 2)$ since $H \subset \bigcup_{z \in \partial D_l \cap C_k} \partial \Gamma_z$. Hence we get (5.14) with $c_6 = c_7/2m$. □

CLAIM 4. *For all non-empty disjoint subsets T_1 and T_2 of S_k with $S_k = T_1 \cup T_2$, the isoperimetric inequality*

$$\max\left\{\frac{\#(\partial T_1 \cap S_k)}{(\# T_1)^{(m-1)/m}},\ \frac{\#(\partial T_2 \cap S_k)}{(\# T_2)^{(m-1)/m}}\right\} \geqq c_8 \tag{5.15}$$

holds.

PROOF. Put $D_l = \varphi^{-1}(T_l) \cap C_k$ $(l=1, 2)$. The proof is divided into two cases.

Case 1: $D_1 = \emptyset$ or $D_2 = \emptyset$. We consider the case when $D_1 = \emptyset$. In this case $\varphi(C_k) \subset T_2$. For $p_1 \in T_1$ there is $p_2 \in \varphi(C_k) \subset T_2$ such that $\delta(p_1, p_2) \leqq \rho$ by the definition of S_k. Now let $\boldsymbol{p}$ be a path from p_1 to p_2 in the net P with length $\leqq \rho$. Then, again by the definition of S_k, $\boldsymbol{p}$ is contained in S_k, and therefore, there is a point of $\partial T_1 \cap S_k$ on $\boldsymbol{p}$. Hence $p_1 \in \{p \in P : \delta(p, \partial T_1 \cap S_k) \leqq \rho\}$. This implies $T_1 \subset \{p \in P : \delta(p, \partial T_1 \cap S_k) \leqq \rho\}$, and with (2.1), we get $(\# T_1)^{(m-1)/m} \leqq \# T_1 \leqq \# \{p \in P : \delta(p, \partial T_1 \cap S_k) \leqq \rho\} \leqq \lambda^{\rho} \cdot \#(\partial T_1 \cap S_k)$. Thus in the case when $D_1 = \emptyset$ or $D_2 = \emptyset$ we have

$$\max\left\{\frac{\#(\partial T_1 \cap S_k)}{(\# T_1)^{(m-1)/m}},\ \frac{\#(\partial T_2 \cap S_k)}{(\# T_2)^{(m-1)/m}}\right\} \geqq \lambda^{-\rho}. \tag{5.16}$$

Case 2: $D_1, D_2 \neq \emptyset$. Let $z \in \partial D_1 \cap C_k$. Then we have $\delta(\varphi(z), T_1 \cap \varphi(C_k)) \leqq a+b$ by (5.5), and $\varphi(z) \in T_2 \cap \varphi(C_k)$ since $z \in D_2$. Thus we can find a path $\boldsymbol{p}$ in P of length $\leqq a+b$ from a point in $T_1 \cap \varphi(C_k)$ to $\varphi(z)$. Now choose the constant ρ in the definition of S_k as $\rho \geqq (a+b)/2$, then $\boldsymbol{p}$ is contained in S_k. Thus an element of $\partial T_1 \cap S_k$ lies in $\boldsymbol{p}$, and consequently $\varphi(z) \in \{p \in P : \delta(p, \partial T_1 \cap S_k) \leqq a+b\}$; that is, $\varphi(\varphi D_1 \cap C_k) \subset \{p \in P : \delta(p, \partial T_1 \cap S_k) \leqq a+b\}$. This shows with (2.1) and (2.2) that

$$\#(\partial D_l \cap C_k) \leqq \lambda^{a+b} \mu \cdot \#(\partial T_l \cap S_k) \tag{5.17}$$

for $l=1, 2$. Now let p_1 be a point in T_1. Then there is a point $q \in \varphi(C_k)$ such that $\delta(p_1, q) \leqq \rho$, and combining p_1 and q by a path of length $\leqq \rho$, we can show that

$$p_1 \in \{p \in P : \delta(p, \partial T_1 \cap S_k) \leqq \rho\} \qquad \text{if} \quad q \in T_2 \cap \varphi(C_k),$$

and that

$$p_1 \in \{p \in P : \delta(p, T_1 \cap \varphi(C_k)) \leqq \rho\} \qquad \text{if} \quad q \in T_1 \cap \varphi(C_k).$$

Hence we obtain

$$T_1 \subset \{p \in P : \delta(p, \partial T_1 \cap S_k) \leqq \rho\} \cup \{p \in P : \delta(p, T_1 \cap \varphi(C_k)) \leqq \rho\}.$$

Thus we have

$$\# T_l \leqq \lambda^\rho \{\#(\partial T_l \cap S_k) + \#(T_l \cap \varphi(C_k))\} \leqq \lambda^\rho \{\#(\partial T_l \cap S_k) + \# D_l\}$$

for $l=1, 2$, and putting $i_l = \#(\partial T_l \cap S_k)/(\# T_l)^{(m-1)/m}$ $(l=1, 2)$, we obtain

$$\# T_l \leqq \lambda^\rho \{i_l \cdot (\# T_l)^{(m-1)/m} + \# D_l\} \leqq \lambda^\rho \{i_l \cdot \# T_l + \# D_l\}.$$

This shows that

$$\# T_l \leqq 2\lambda^\rho \cdot \# D_l \quad \text{if} \quad i_l \leqq 2^{-1}\lambda^{-\rho} \tag{5.18}$$

for $l=1, 2$. Then, by (5.17), (5.18), and the isoperimetric inequality (5.14) for C_k, we obtain in the case of $D_1, D_2 \neq \emptyset$ that

$$\max\left\{\frac{\#(\partial T_1 \cap S_k)}{(\# T_1)^{(m-1)/m}}, \frac{\#(\partial T_2 \cap S_k)}{(\# T_2)^{(m-1)/m}}\right\} \geqq c_9 \quad \text{if} \quad \max\{i_1, i_2\} \leqq 2^{-1}\lambda^{-\rho}, \tag{5.19}$$

where $c_9 = 2^{-(m-1)/m}\lambda^{-a-b-\rho(m-1)/m}\mu^{-1}c_6$. Now from (5.16) and (5.19) we complete the proof of (5.15) with $c_8 = \min\{2^{-1}\lambda^{-\rho}, c_9\}$. □

The following isoperimetric inequality for Ω_k will be proved in a way similar to Lemma 4.5 by the local isoperimetric inequality. Denote by Ω_k' the ε-neighborhood of Ω_k.

CLAIM 5. *Suppose that H is a smooth hypersurface in Ω_k' which divides Ω_k' into two disjoint domains D_1' and D_2'. If both $D_1 = D_1' \cap \Omega_k$ and $D_2 = D_2' \cap \Omega_k$ are non-empty, then we have*

$$\frac{\operatorname{area} H}{[\min\{\operatorname{vol} D_1, \operatorname{vol} D_2\}]^{(m-1)/m}} \geqq c_{10}. \tag{5.20}$$

PROOF. Without loss of generality we may assume $\operatorname{vol}(D_1 \cap B_\varepsilon(p)) \neq \operatorname{vol}(D_2 \cap B_\varepsilon(p))$ for all $p \in S_k$. Put $T_l = \{p \in S_k : \operatorname{vol}(D_l \cap B_\varepsilon(p)) > \operatorname{vol} B_\varepsilon(p)/2\}$ $(l=1, 2)$; then the assumption says that S_k is the disjoint union of T_1 and T_2. As (4.6), we get

$$\sum_{p \in T_2} \operatorname{vol}(D_1 \cap B_\varepsilon(p)), \sum_{p \in T_1} \operatorname{vol}(D_2 \cap B_\varepsilon(p)) \leqq \{\nu j_m^{-1} \operatorname{area} H\}^{m/(m-1)}, \tag{5.21}$$

where $\nu = \nu(n, K, \varepsilon, \varepsilon)$ is the constant (see Appendix). We proceed by dividing the proof into two cases.

Case 1: $T_1 = \emptyset$ or $T_2 = \emptyset$. When $T_1 = \emptyset$, we have

$$\begin{aligned}(\operatorname{vol} D_1)^{(m-1)/m} &\leqq \left\{\sum_{p \in T_2} \operatorname{vol}(D_1 \cap B_\varepsilon(p))\right\}^{(m-1)/m} \\ &\leqq \nu j_m^{-1} \operatorname{area} H\end{aligned}$$

from (5.21), since $S_k = T_2$. Hence in this case we obtain

$$\frac{\operatorname{area} H}{[\min\{\operatorname{vol} D_1, \operatorname{vol} D_2\}]^{(m-1)/m}} \geqq \nu^{-1} j_m. \tag{5.22}$$

Case 2: $T_1, T_2 \neq \emptyset$. Put $i_l = \operatorname{area} H/(\operatorname{vol} D_l)^{(m-1)/m}$ $(l=1, 2)$. Then from (5.21) we get

$$\operatorname{vol} D_1 \leqq \sum_{p \in T_1} \operatorname{vol}(D_1 \cap B_\varepsilon(p)) + \sum_{p \in T_2} \operatorname{vol}(D_1 \cap B_\varepsilon(p))$$
$$\leqq V(\varepsilon) \cdot \# T_1 + (\nu j_m^{-1} i_1)^{m/(m-1)} \operatorname{vol} D_1 ,$$

and consequently we have

$$\text{(5.23)} \qquad \operatorname{vol} D_l \leqq 2V(\varepsilon) \cdot \# T_l \quad \text{if} \quad i_l \leqq 2^{-(m-1)/m} \nu^{-1} j_m ,$$

for $l=1, 2$. On the other hand put $G = \{x \in \overline{\Omega_k} : \operatorname{vol}(D_1' \cap B_\varepsilon(p)) = \operatorname{vol}(D_2' \cap B_\varepsilon(p))\}$; then as in the proof of Lemma 4.5, we obtain that $B_{\varepsilon/2}(p) \subset B_{5\varepsilon/2}(G)$ if $p \in (\partial T_1 \cap S_k) \cup (\partial T_2 \cap S_k)$. Let Q be a maximal ε-separated subset of G. Then again as (4.9) we have

$$\text{(5.24)} \qquad \left(\frac{\varepsilon}{2}\right)^n V_0 \{\#(\partial T_1 \cap S_k) + \#(\partial T_2 \cap S_k)\} \leqq 2\nu j_\infty^{-1} \frac{V(7\varepsilon/2)}{V(\varepsilon)} \operatorname{area} H .$$

From (5.23), (5.24), and (5.15) we have

$$\text{(5.25)} \qquad \frac{\operatorname{area} H}{[\min\{\operatorname{vol} D_1, \operatorname{vol} D_2\}]^{(m-1)/m}} \geqq c_{11} \quad \text{if} \quad \max\{i_1, i_2\} \leqq 2^{-(m-1)/m} \nu^{-1} j_m ,$$

where $c_{11} = 2^{-(2m-1)/m} (\varepsilon/2)^n \nu^{-1} V_0 V(\varepsilon)^{1/m} V(7\varepsilon/2)^{-1} j_\infty c_8$. Now (5.20) follows from (5.22) and (5.25). □

CLAIM 6. *For every $v \in H_{1,1}(\overline{\Omega_k'})$, the following inequality holds;*

$$\text{(5.26)} \qquad \inf_{\alpha \in \boldsymbol{R}} \left\{ \int_{\Omega_k} |v - \alpha|^{m/(m-1)} dx \right\}^{(m-1)/m} \leqq c_{10}^{-1} \int_{\Omega_k'} |\nabla v| \, dx .$$

PROOF (cf. [**16**; §3], [**3**; §2]). It is sufficient to consider only the case when u is a smooth Morse function on $\overline{\Omega_k'}$ (cf. [**1**; §3]). Put $\Omega_+(t) = \{x \in \Omega_k : v(x) > t\}$, $\Omega_-(t) = \{x \in \Omega_k : v(t) < t\}$ and $H(t) = \{x \in \Omega_k' : v(x) = t\}$. Then $H(t)$'s are smooth hypersurface for all but finite number of t. Moreover by adding a suitable constant to v, we may assume $\operatorname{vol} \Omega_+(0) = \operatorname{vol} \Omega_-(0)$. Then we have

$$\text{(5.27)} \qquad \int_{\Omega_+(0)} v^{m/(m-1)} dx = \frac{m}{m-1} \int_{\Omega_+(0)} dx \int_0^{u(x)} t^{1/(m-1)} dt$$
$$= \frac{m}{m-1} \int_0^\infty dt \int_{\Omega_+(t)} t^{1/(m-1)} dx$$
$$= \frac{m}{m-1} \int_0^\infty t^{1/(m-1)} \operatorname{vol} \Omega_+(t) dt .$$

On the other hand, since the function $t \to \operatorname{vol} \Omega_+(t)$ is monotone decreasing, we have $t \{\operatorname{vol} \Omega_+(t)\}^{(m-1)/m} \leqq \int_0^t \{\operatorname{vol} \Omega_+(s)\}^{(m-1)/m} ds$, and consequently we get

$$\frac{m}{m-1} t^{1/(m-1)} \operatorname{vol} \Omega_+(t)$$
$$\leqq \frac{m}{m-1} \{\operatorname{vol} \Omega_+(t)\}^{(m-1)/m} \left[\int_0^t \{\operatorname{vol} \Omega_+(s)\}^{(m-1)/m} ds \right]^{1/(m-1)}$$

$$=\frac{d}{dt}\left[\int_0^t \{\mathrm{vol}\,\Omega_+(s)\}^{(m-1)/m}ds\right]^{m/(m-1)}$$

This implies, with (5.27) and the isoperimetric inequality (5.20), that

$$\int_{\Omega_+(0)} v^{m/(m-1)}dx \leqq \left[\int_0^\infty \{\mathrm{vol}\,\Omega_+(t)\}^{(m-1)/m}dt\right]^{m/(m-1)}$$

$$\leqq \left[c_{10}^{-1}\int_0^\infty \mathrm{area}\,H(t)dt\right]^{m/(m-1)},$$

since $\mathrm{vol}\,\Omega_+(t) \leqq \mathrm{vol}\,\Omega_-(t)$ for $t \geqq 0$. Also in the same way we get

$$\int_{\Omega_-(0)} (-v)^{m/(m-1)}dx \leqq \left[c_{10}^{-1}\int_{-\infty}^0 \mathrm{area}\,H(t)dt\right]^{m/(m-1)}$$

Thus we conclude

$$\left\{\int_{\Omega_k} |v|^{m/(m-1)}dx\right\}^{(m-1)/m} \leqq c_{10}^{-1}\int_{-\infty}^\infty \mathrm{area}\,H(t)dt$$

$$=c_{10}^{-1}\int_{\Omega_k'} |\nabla v|\,dx\,. \qquad \square$$

Now we are in position to complete the proof of Proposition 5.3. Let u be a positive harmonic function defined on Ω_{4k}, and $g(u)$ be the quantity defined by (5.10). In the discussion below $c_{12}, \cdots, c_{15}$ denote positive constants independent of k and u. Put $v=\log u$. Then from (5.6), the Hölder inequality, (5.26), (5.7) and Schwarz's inequality we get

$$g(u) \leqq c_1 k^{-m} \inf_{\alpha\in \mathbf{R}} \int_{\Omega_{2k}} |v-\alpha|\,dx \tag{5.28}$$

$$\leqq c_1 k^{-m} (\mathrm{vol}\,\Omega_{2k})^{1/m} \inf_{\alpha\in \mathbf{R}} \left\{\int_{\Omega_{2k}} |v-\alpha|^{m/(m-1)}dx\right\}^{(m-1)/m}$$

$$\leqq c_{12} k^{1-m} \int_{\Omega_{2k}'} |\nabla v|\,dx$$

$$\leqq c_{13}\left\{k^{2-m}\int_{\Omega_{3k}} |\nabla v|^2 dx\right\}^{1/2}.$$

Moreover we have

$$\int_X \eta^2 |\nabla v|^2 \leqq 4\int_X |\nabla \eta|^2 dx$$

for an arbitrary Lipschitz function η with compact support, since u is harmonic (cf. [**14**]). Especially applying this inequality for

$$\eta(x)=\begin{cases} 1 & \text{if } x\in \Omega_{3k} \\ 1-(\varepsilon k)^{-1}d(x, \Omega_{3k}) & \text{if } x\in B_{\varepsilon k}(\Omega_{3k}) \\ 0 & \text{otherwise,} \end{cases}$$

we obtain with (5.6) and (5.7) that

$$\int_{\Omega_{3k}} |\nabla v|^2 dx \leqq 4(\varepsilon k)^{-2} \operatorname{vol} \Omega_{4k} \leqq c_{14} k^{m-2}.$$

Thus with (5.28) we have $g(u) \leqq c_{15}$, and consequently we conclude the Harnack inequality (5.4) by (5.9). This completes the proof of Proposition 5.3.

Appendix. Local isoperimetric inequality.

Here we generalize the local isoperimetric inequality of Buser [**6**] to the following form, which has been used in the proofs of Lemma 4.5 and Claim 5 in §5. Recall that we have adopt the convention that $(m-1)/m=\infty$ when $m=\infty$.

LEMMA. *Suppose that X is a complete riemannian n-manifold satisfying* (*), *and that $p \in X$, $\varepsilon \in (0, \operatorname{inj} X/2]$ and $m \in [n, \infty]$. If H is a smooth hypersurface in the geodesic ball $B_\varepsilon(p)$ dividing it into two non-empty disjoint domains D_1 and D_2, then the isoperimetric inequality*

$$\frac{\operatorname{area} H}{[\min\{\operatorname{vol} D_1, \operatorname{vol} D_2\}]^{(m-1)/m}} \geqq j_m \tag{6.1}$$

holds, where $j_m = j_m(n, K, \varepsilon)$ is a positive constant.

PROOF. We consider the case when $\operatorname{vol} D_1 \leqq \operatorname{vol} D_2$. Also since $\operatorname{vol} D_1 \leqq \operatorname{vol} B_\varepsilon(p)/2 \leqq V(\varepsilon)/2$, we get $(\operatorname{vol} D_1)^{(m-1)/m} \leqq (V(\varepsilon)/2)^{(m-1)/m-(n-1)/n} (\operatorname{vol} D_1)^{(n-1)/n}$ for $m \in [n, \infty]$. This guarantees that we may only give the proof in the case $m=n$ and that $j_m \geqq (V(\varepsilon)/2)^{(n-1)/n-(m-1)/m} j_n$ for $m \in [n, \infty]$.

First, by Buser's local isoperimetric inequality [**6**; §5], we get

$$\operatorname{vol} D_1 \leqq c_1 \cdot \operatorname{area} H, \tag{6.2}$$

where $c_1 = c_1(n, K, \varepsilon) > 0$. Next, apply Croke's isoperimetric inequality [**8**; Theorem 11] to D_1: Then we have

$$(\operatorname{vol} D_1)^{(n-1)/n} \leqq c_2 \{\operatorname{area} H + \operatorname{area} \partial' D_1\}, \tag{6.3}$$

where $\partial' D_1 = \partial D_1 - H$ and $c_2 = c_2(n) > 0$.

Now let $\Theta = \Theta(\xi, r) : U_p X \times [0, \varepsilon] \to \boldsymbol{R}$ be the positive function defined by $\Psi^* dx = \Theta(\xi, r) d\xi \wedge dr$, where $U_p X = \{\xi \in T_p X : |\xi| = 1\}$, $\Psi : U_p X \times [0, \varepsilon] \to B_\varepsilon(p)$, $(\xi, r) \to \exp_p r\xi$, dx and $d\xi$ are the volume forms of X and $U_p X$, respectively. Put $U_0 = \{\xi \in U_p X : \exp_p \varepsilon\xi \in \partial' D_1\}$, and for $\xi \in U_0$ let $\rho(\xi)$ be the supremum of the distances between p and the points of the intersection of H with the geodesic $\exp_p r\xi$ $(0 \leqq r \leqq \varepsilon)$; here we define $\rho(\xi) = 0$ when the geodesic $\exp_p r\xi$ does not intersect with H.

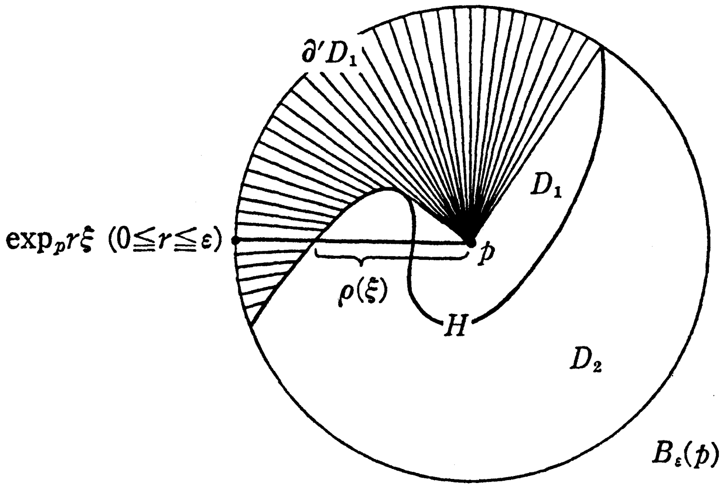


Figure 2.

By a comparison argument as (1.8), we get two inequalities

$$\int_{\rho(\xi)}^{\varepsilon} \Theta(\xi, r)dr \geqq \frac{V(\varepsilon)-V(\rho(\xi))}{A(\varepsilon)} \Theta(\xi, \varepsilon),$$

$$\Theta(\xi, \varepsilon) \leqq \frac{A(\varepsilon)}{A(\rho(\xi))} \Theta(\xi, \rho(\xi))$$

for $\xi \in U_0$. Hence we have

$$\begin{aligned}\int_{\rho(\xi)}^{\varepsilon} \Theta(\xi, r)dr &\geqq \frac{V(\varepsilon)}{A(\varepsilon)} \Theta(\xi, \varepsilon) - \frac{V(\rho(\xi))}{A(\varepsilon)} \Theta(\xi, \varepsilon) \\ &\geqq \frac{V(\varepsilon)}{A(\varepsilon)} \Theta(\xi, \varepsilon) - \frac{V(\rho(\xi))}{A(\rho(\xi))} \Theta(\xi, \rho(\xi)) \\ &\geqq \frac{V(\varepsilon)}{A(\varepsilon)} \{\Theta(\xi, \varepsilon) - \Theta(\xi, \rho(\xi))\} ;\end{aligned}$$

here the last inequality follows from the fact that the function $r \to V(r)/A(r)$ is increasing. Thus we have

$$\begin{aligned}\mathrm{vol}\, D_1 &\geqq \int_{U_0} d\xi \int_{\rho(\xi)}^{\varepsilon} \Theta(\xi, r)dr \\ &\geqq \frac{V(\varepsilon)}{A(\varepsilon)} \int_{U_0} \{\Theta(\xi, \varepsilon) - \Theta(\xi, \rho(\xi))\} d\xi \\ &\geqq \frac{V(\varepsilon)}{A(\varepsilon)} \{\mathrm{area}\, \partial' D_1 - \mathrm{area}\, H\},\end{aligned}$$

i.e.,

$$\mathrm{area}\, \partial' D_1 - \mathrm{area}\, H \leqq c_3 \cdot \mathrm{vol}\, D_1, \tag{6.4}$$

where $c_3 = c_3(n, K, \varepsilon) > 0$.

Now (6.2) and (6.4) yield area $\partial' D_1 \leqq (1+c_1c_3)$area H; then (6.3) and this inequality imply $(\mathrm{vol}\, D_1)^{(n-1)/n} \leqq c_2(2+c_1c_3)$area H. □

References

[1] T. Aubin, Problèmes isopérimétriques et espaces de Sobolev, J. Diff. Geom., **11** (1976), 573-598.

[2] A. Avez, Variétés riemanniennes sans points focaux, C. R. Acad. Sci. Paris Ser. A, **270** (1970), 188-191.

[3] E. Bombieri and E. Guisti, Harnack's inequality for elliptic differential equations on minimal surfaces, Invent. Math., **15** (1972), 24-46.

[4] R. Brooks, A relation between growth and the spectrum of the laplacian, Math. Z., **178** (1981), 501-508.

[5] R. Brooks, The fundamental group and the spectrum of the laplacian, Comment. Math. Helv., **56** (1981), 581-598.

[6] P. Buser, A note on the isoperimetric constant, Ann. Sci. École Norm. Sup., **15** (1982), 213-230.

[7] S.-Y. Cheng and S.-T. Yau, Differential equations on riemannian manifolds and their geometric applications, Comm. Pure Appl. Math., **28** (1975), 333-354.

[8] C. B. Croke, Some isoperimetric inequalities and eigenvalue estimates, Ann. Sci. École Norm. Sup., **13** (1980), 419-435.

[9] R. E. Greene and H. Wu, Function theory on manifolds which possess a pole, Lecture Notes in Math., **699**, Springer, 1979.

[10] M. Gromov, Hyperbolic manifolds, groups and actions, Ann. of Math. Studies, **97**, Princeton Univ. Press, Princeton, 1981, 183-213.

[11] M. Gromov, Structures métriques pour les variétés riemanniennes, Cedic, Paris, 1981.

[12] A. Manning, Topological entropy for geodesic flows, Ann. of Math., **110** (1979), 567-573.

[13] J. Milnor, A note on curvature and fundamental group, J. Diff. Geom., **2** (1968), 1-7.

[14] J. Moser, On Harnack's theorem for elliptic differential equations, Comm. Pure Appl. Math., **14** (1961), 577-591.

[15] G. D. Mostow, Strong rigidity of locally symmetric spaces, Ann. of Math. Studies, **78**, Princeton Univ. Press, Princeton, 1973.

[16] R. Osserman, The isoperimetric inequality, Bull. Amer. Math. Soc., **84** (1978), 1182-1238.

[17] A. Phillips and D. Sullivan, Geometry of leaves, Topology, **20** (1981), 209-218.

[18] S.-T. Yau, Harmonic functions on complete riemannian manifolds, Comm. Pure Appl. Math., **28** (1975), 201-228.

[19] S.-T. Yau, Survey on partial differential equations in differential geometry, Ann. of Math. Studies, **102**, Princeton Univ. Press, Princeton, 1982, 3-71.

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