J. Math. Soc. Japan
Vol. 37, No. 2, 1985

Helical minimal imbeddings of order 4 into spheres

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(Received May 21, 1984)

§ 0. Introduction.

Let $f: M \to \bar{M}$ be an isometric immersion of a Riemannian manifold M into a Riemannian manifold $\bar{M}$. If for each geodesic γ of M the curve $f\circ\gamma$ in $\bar{M}$ is of osculating order d and has constant curvatures which are independent of the choice of γ, then f is called a *helical immersion of order d*. In this paper we shall study helical minimal immersions of order 4 into a unit sphere $S(1)$.

Besse [**1**] showed that a strongly harmonic manifold admits a helical minimal immersion into a sphere. As is well-known, making use of eigenfunctions of the Laplace operator, we obtain minimal immersions of compact rank one symmetric spaces into spheres (cf. [**14**]). Similarly we have the α-th standard minimal immersions of strongly harmonic manifolds into spheres. Let m_α be the multiplicity of the α-th eigenvalue of the Laplace operator and ϕ_i $(i=1, \cdots, m_\alpha)$ an orthonormal base for its eigenspace. Then we define Φ_α by $\Phi_\alpha(x)=(\phi_1(x), \cdots, \phi_{m_\alpha}(x))\in \boldsymbol{R}^{m_\alpha}$. If we change homothetically the metric on the strongly harmonic manifold, then Φ_α becomes a helical minimal immersion into a hypersphere of $\boldsymbol{R}^{m_\alpha}$. We call Φ_α the *α-th standard minimal immersion* of strongly harmonic manifolds. Tsukada [**13**] proved that if $f: M\to S(1)$ is a helical minimal immersion of a strongly harmonic manifold M, then f is equivalent to some Φ_α, that is, $f=\Psi\circ\Phi_\alpha$ with some isometry Ψ of $S(1)$.

Let $f: M\to S(1)$ be a helical minimal imbedding of a compact n-dimensional Riemannian manifold M. If the order d of f is equal to 1, then f is totally geodesic. In the case $d=2$, Little [**5**] and the author [**8**] showed that M is isometric to one of real projective space $\boldsymbol{RP}^n$, complex projective space $\boldsymbol{CP}^m$ $(n=2m)$, quaternion projective space $\boldsymbol{QP}^m$ $(n=4m)$ and Cayley projective space $\boldsymbol{Cay}P^2$ $(n=16)$ with canonical metrics and f is equivalent to Φ_1. If $d=3$, then M is isometric to S^n and $f\approx\Phi_3$. This result was given by Nakagawa [**6**] (see also [**10**], [**11**]). The case $d=4$ was studied in [**11**] and proved that M is isometric to one of projective spaces $\boldsymbol{RP}^n$, $\boldsymbol{CP}^m$, $\boldsymbol{QP}^m$ and $\boldsymbol{Cay}P^2$ under the condition that $a=\langle\dot\gamma(0), \dot\gamma(L)\rangle>0$ for any unit speed geodesic γ where L is the diameter of M and $\langle\ ,\ \rangle$ denotes the inner product of the Euclidean space in which $S(1)$ is naturally imbedded (also was proved that $f\approx\Phi_2$). Furthermore if $d=5$, then

$M \approx S^n$ and $f \approx \Phi_5$ (see [12]).

For a helical immersion $f: M \to S(1)$ there exists a function F such that $\langle f(x), f(y) \rangle = F(\delta(x, y))$, δ being the distance function on M. For instance, if $f = \Phi_\alpha: M \to S(1)$ is the α-th standard minimal immersion of a compact rank one symmetric space M, then F is a zonal spherical function and it is easily shown that the order of f is not greater than 2 if and only if F is monotone decreasing on $(0, L)$ (cf. [12]). Moreover in [12] the author showed that for a helical minimal imbedding $f: M \to S(1)$ of order d of a compact Riemannian manifold M into $S(1)$, if F is monotone decreasing on $(0, L)$ and f is not totally geodesic, then d is an even integer. Thus it seems very important to study the case $d=4$. In fact, the condition $a<0$ in the case $d=4$ is equivalent to that F is monotone decreasing on $(0, L)$ (cf. (1.8)). In the present paper, we shall show that if $d=4$, then $a<0$ does not occur.

Well we give the organization of this paper. In §1, we summarize the results obtained in [11]. We give in §2 all normal Jacobi fields in terms of the second fundamental form and using them we obtain many equations satisfied by the second fundamental form. Also we define a one parameter family $S_X(s)$ of symmetric transformations acting on the subspace $\{X\}^\perp$ in the tangent space T_xM where $X \in T_xM$. In Lemmas 2.4, 2.5 and Corollary 2.5, good properties possessed by $S_X(s)$ will be given. Since M is a Blaschke manifold (cf. [9]), all geodesics from a point x of M to y of its cut-locus form a submanifold in M. §3 is devoted to studying geodesics from $\gamma(L/2)$ to $\gamma(3L/2)$ where γ is a geodesic such that $\gamma(0)=x$ and $\gamma(L)=y$. We shall prove that such geodesics lie on the submanifold. In §4, we shall show $a<0$ does not occur. The result is stated in Theorem 4.4.

§1. Notations and preliminaries.

Let $f: M \to \overline{M}$ be an isometric immersion of a Riemannian manifold M into a Riemannian manifold $\overline{M}$ and γ an arbitrary geodesic of M. If the curve $f \circ \gamma$ in $\overline{M}$ has constant curvatures $\kappa_1, \cdots, \kappa_{d-1}$ $(\neq 0, \kappa_d = 0)$ which are independent of the choice of γ, then f is called a *helical immersion of order d*. In this paper, the ambient space $\overline{M}$ will be a unit sphere $S(1)$.

In [9] the author showed that if a compact Riemannian manifold M admits a helical immersion into a unit sphere, then M is a Blaschke manifold (for the definition, see [1]). In particular, all geodesics of M are simply closed geodesics with the same length, which will be denoted by $2L$. Thus the diameter of M is equal to L. Let x be an arbitrarily fixed point of M and $X \in U_xM$ (unit tangent sphere at x). Let γ be the unit speed geodesic such that $\gamma(0)=x$ and $\dot\gamma(0) = X$. The cut-locus $\mathrm{Cut}(x)$ of x is a submanifold in M whose dimension is independent of x. Let $\mathcal{H}_x(X)$ denote the linear subspace Span$\{\dot\sigma(0) : \sigma$ is a minimal geodesic from x to $y\}$ in T_xM where $y=\gamma(L)$. Then we see that $T_xM=T_x\mathrm{Cut}(y)\oplus\mathcal{H}_x(X)$ (orthogonal direct sum). Let $e=\dim\mathcal{H}_x(X)$. It is well-known that e is equal to 1, 2, 4, 8 or n $(=\dim M)$ (cf. [**1**]). The orthogonal complement of X in $\mathcal{H}_x(X)$ will be denoted by $\mathcal{H}_x^*(X)$.

In the sequel, we assume that $f: M\to S(1)$ is a *helical minimal imbedding of order* 4 *of a compact Riemannian manifold M.* Here we remark the following. If a helical immersion $M\to S(1)$ is not injective, then we see from Corollary 6.3 [**9**] that M is diffeomorphic to a sphere S^n and the immersion is the composite of the covering map $S^n\to \boldsymbol{R}P^n$ and a helical imbedding $\boldsymbol{R}P^n\to S(1)$. Thus we may always assume that a helical immersion into $S(1)$ is an imbedding. Let $\iota: S(1)\to E$ be the canonical inclusion of $S(1)$ into the Euclidean space E whose origin coincides with the center of $S(1)$. The imbedding $\phi=\iota\circ f$ is also a helical imbedding of order 4. Let γ be a unit speed geodesic in M. The curvatures of $f\circ\gamma$ will be denoted by κ_1, κ_2 and κ_3. Then the curvatures λ_1, λ_2 and λ_3 of $\tau=\phi\circ\gamma$ are given by

$$\lambda_1^2=1+\kappa_1^2,\qquad \lambda_1^2\lambda_2^2=\kappa_1^2\kappa_2^2,\qquad \lambda_2^2+\lambda_3^2=\kappa_2^2+\kappa_3^2 \tag{1.1}$$

(see Corollary 4.2 [**9**]). Let $x=\gamma(0)$ and $X=\dot{\gamma}(0)$. Let H denote the second fundamental form of the imbedding f. Frenet vectors of τ at x are given by

$$\begin{aligned}
&\tau^{(1)}(X)=X\,,\\
&\tau^{(2)}(X)=\lambda_1^{-1}\{-x+H(X,X)\}\,,\\
&\tau^{(3)}(X)=(\lambda_1\lambda_2)^{-1}(DH)(X^3)\,,\\
&\tau^{(4)}(X)=(\lambda_1\lambda_2\lambda_3)^{-1}\{-\lambda_2^2x+\lambda_2^2H(X,X)+(D^2H)(X^4)\}
\end{aligned} \tag{1.2}$$

where D denotes the van der Waerden-Bortolotti covariant differential operator (cf. Theorem 4.1 [**9**]). Define functions $f_1,\cdots,f_4$ on $\boldsymbol{R}$ by the differential equation

$$\begin{aligned}
&f_1'=1-\lambda_1f_2\\
&f_2'=\lambda_1f_1-\lambda_2f_3\\
&f_3'=\lambda_2f_2-\lambda_3f_4\\
&f_4'=\lambda_3f_3
\end{aligned} \tag{1.3}$$

with initial conditions $f_1(0)=\cdots=f_4(0)=0$. Furthermore define $\xi(s;X)$, $\zeta(s;X)$ and F by

$$\begin{aligned}
&\xi(s;X)=f_2(s)\tilde{\tau}^{(2)}(X)+f_4(s)\tilde{\tau}^{(4)}(X)\,,\\
&\zeta(s;X)=f_3(s)\tau^{(3)}(X)\,,\\
&F(s)=1-\lambda_1^{-1}f_2(s)-\lambda_2(\lambda_1\lambda_3)^{-1}f_4(s)
\end{aligned}$$

respectively where

$$\tilde{\tau}^{(2)}(X)=\lambda_1^{-1}H(X, X),$$

$$\tilde{\tau}^{(4)}(X)=(\lambda_1\lambda_2\lambda_3)^{-1}\{\lambda_2^2H(X, X)+(D^2H)(X^4)\}.$$

Then $\xi(s; X)$ and $\zeta(s; X)$ are normal to M (and tangent to $S(1)$). Equation (1.3) implies that $F'=-f_1$. If we solve Frenet differential equation, then we have (omitting ϕ)

$$\tau(s)=x+f_1(s)X+f_2(s)\tau^{(2)}(X)+f_3(s)\tau^{(3)}(X)+f_4(s)\tau^{(4)}(X)$$

which is rewritten as

$$\tau(s)=F(s)x+f_1(s)X+\xi(s; X)+\zeta(s; X). \tag{1.4}$$

It follows that

$$\langle z, w\rangle=F(\delta(z, w)) \tag{1.5}$$

for every $z, w\in M$ where $\langle\ ,\ \rangle$ denotes the inner product of E and δ the distance function of M.

Since τ is a periodic curve with period $2L$, we see from (1.3) that f_1 and f_3 (resp. f_2 and f_4) are odd (resp. even) functions with period $2L$. Hence we have $f_1(L)=f_3(L)=0$. Let $a=f_1'(L)$, $a_3'=f_3'(L)$, $a_2=f_2(L)$, $a_4=f_4(L)$ and $b=F(L)$. We should remark $a\neq 0$ which is derived from the assumption f is minimal. Since s is the arc-length parameter, we have $a^2+(a_3')^2=1$ and moreover from $\tau(L)\in S(1)$, $a_2^2+a_4^2=2(1-b)$. Making use of (1.3), we see from these equations that (I): $a_3'=0$ or (II): $a_3'=2\lambda_1\lambda_2 a/(\lambda_2^2+\lambda_3^2-\lambda_1^2)$. However we have shown in [**11**] that the case (I) does not occur. In the case (II), λ_1, λ_2 and λ_3 are given by

$$\lambda_1^2=\frac{\nu^2}{2}(3a+5),\qquad \lambda_2^2=\frac{9}{2}\nu^2\frac{1-a^2}{3a+5},\qquad \lambda_3^2=\frac{8\nu^2}{3a+5} \tag{1.6}$$

where $\nu=\pi/L=((1-a)/(1-b))^{1/2}$. Furthermore we obtain

$$f_1(s)=\frac{1}{4\nu}\{2(1-a)\sin\nu s+(1+a)\sin 2\nu s\}, \tag{1.7}$$

$$F(s)=1+\frac{3a-5}{8\nu^2}+\frac{1}{8\nu^2}\{4(1-a)\cos\nu s+(1+a)\cos 2\nu s\} \tag{1.8}$$

(cf. [**11**]). Let $h(s)$, $k(s)$ and $l(s)$ be defined by

$$h(s)=1-F(s)-\frac{1}{a_4}(1-b)f_4(s),$$

$$k(s)=\frac{1}{a_4}f_4(s),$$

$$l(s)=\frac{1}{a_3'}f_3(s)$$

respectively. Using (1.3), (1.6), (1.7) and (1.8), we have

(1.9)
$$h(s)=\frac{1}{4\nu^2}(1-\cos 2\nu s)\,,$$
$$k(s)=\frac{1}{8}(3-4\cos\nu s+\cos 2\nu s)\,,$$
$$l(s)=-\frac{1}{4\nu}(2\sin\nu s-\sin 2\nu s)\,.$$

Define $(D\xi)(s\,;X)$ by $(D\xi)(s\,;X)=\nabla^{\perp}_{\dot{X}}\xi(s\,;\dot{\gamma})$ where $\nabla^{\perp}$ is the covariant differential operator with respect to the normal connection. Then we have $(D\xi)(s\,;X)=\zeta'(s\,;X)$. Let $\xi(X)=\xi(L\,;X)$ and $(D\xi)(X)=(D\xi)(L\,;X)$. In terms of h, k, l, $\xi(X)$ and $(D\xi)(X)$, $\xi(s\,;X)$ and $\zeta(s\,;X)$ are given by

(1.10) $$\xi(s\,;X)=h(s)H(X,X)+k(s)\xi(X)\,,$$

(1.11) $$\zeta(s\,;X)=l(s)(D\xi)(X)\,.$$

It follows that (1.4) becomes

(1.12) $$\tau(s)=F(s)x+f_1(s)X+h(s)H(X,X)+k(s)\xi(X)+l(s)(D\xi)(X)\,.$$

Here we notice the geometric meanings of $\xi(X)$ and $(D\xi)(X)$ as follows. Let $y=\gamma(L)$ be the cut-point of x. Then (1.4) shows $y=bx+\xi(X)$. Also (1.4) shows $\dot{\tau}(L)=aX+(D\xi)(X)$. The tangent space $T_x\mathrm{Cut}(y)$ of the cut-locus of y and $\mathcal{H}_x(X)$ are eigenspaces of the second fundamental tensor $A_{\xi(X)}$ corresponding to $\xi(X)$, i.e.,

(1.13)
$$T_x\mathrm{Cut}(y)=\{Y: A_{\xi(X)}Y=bY\}\,,$$
$$\mathcal{H}_x(X)=\{Z: A_{\xi(X)}Z=(b-a)Z\}\,,$$

so that $b=ea/n$.

It is easily verified that $a<0$ is equivalent with $f_1>0$ on $(0, L)$. If $a>0$, then we have

THEOREM 1.1 ([**11**]). *Let $f: M\to S(1)$ be a helical minimal imbedding of order 4 of a compact Riemannian manifold M into $S(1)$. Assume that $a>0$. Then $a=(e+2)/(n+2)$ and M is isometric to one of $\boldsymbol{RP}^n$, $\boldsymbol{CP}^m$, $\boldsymbol{QP}^m$ $(m\geqq 2)$ and $\boldsymbol{Cay}P^2$ where $m=n/e$. If $M=\boldsymbol{RP}^n$, then the sectional curvature is equal to $n/4(n+3)$. If $M=\boldsymbol{CP}^m$, $\boldsymbol{QP}^m$ or $\boldsymbol{Cay}P^2$, then the maximal curvature is equal to $n/(n+e+2)$. Moreover f is equivalent to the second standard minimal imbedding.*

Therefore, in the sequel, *we shall assume* $a<0$. Under this condition, we shall prove $a=-1$ which implies $a_3'=0$.

§ 2. Equations satisfied by the second fundamental form.

We shall use the following normal vectors:

$$\xi_X(s\,;V)=\|V\|\frac{d}{d\omega}\xi\Big(s\,;\cos\omega X+\sin\omega\frac{V}{\|V\|}\Big)\Big|_{\omega=0},$$

$$(D\xi)(s\,;V\,;X)=\nabla_V^{\perp}\xi(s\,;X^*)$$

where $X\in UM$ (unit sphere bundle), $V\in\{X\}^{\perp}$ and X^* is the local field extending X such that $\nabla X^*=0$ at the origin of X. In the same way, we define $\zeta_X(s\,;V)$ and $(D\zeta)(s\,;V\,;X)$. Clearly we have

$$(D\zeta)'(L\,;V\,;X)=\nabla_V^{\perp}(D\xi)(X^*)=(D^2\xi)(V\,;X)\,.$$

Let γ be the unit speed geodesic such that $\gamma(0)=x$ and $\dot\gamma(0)=X$. Let J_V and J_V^* be Jacobi fields along γ such that $J_V(0)=0$, $\nabla_X J_V=V\in\{X\}^{\perp}$ and $J_V^*(0)=V$, $\nabla_X J_V^*=0$ respectively. Then they are given by

$$J_V(s)=f_1(s)V+\xi_X(s\,;V)+\zeta_X(s\,;V)\,, \tag{2.1}$$

$$\begin{aligned}J_V^*(s)=&F(s)V-A_{\xi(s;X)}V-A_{\zeta(s;X)}V\\&+f_1(s)H(V,\,X)+(D\xi)(s\,;V\,;X)+(D\zeta)(s\,;V\,;X)\end{aligned} \tag{2.2}$$

which were computed in [11] (see also [10]). Notice that $A_{\xi(s;X)}V$ and $A_{\zeta(s;X)}V$ are orthogonal to X (cf. Lemma 3.3 [9]). Since $\gamma(s)$ is not a conjugate point of x for every $s\in(0,\,L)$, $\{J_V(s): V\in\{X\}^{\perp}\}$ spans $\{\dot\gamma(s)\}^{\perp}$. Thus there exists $W\in\{X\}^{\perp}$ such that $J_V^*(s)=J_W(s)$. Define the symmetric transformation $S_X(s)$ on $\{X\}^{\perp}$ by

$$S_X(s)=\frac{1}{f_1(s)}\{F(s)I-A_{\xi(s;X)}-A_{\zeta(s;X)}\} \tag{2.3}$$

where I denotes the identity transformation. Equations (2.1) and (2.2) show that $W=S_X(s)V$. Furthermore we have

$$\xi_X(s\,;W)+\zeta_X(s\,;W)=f_1(s)H(V,\,X)+(D\xi)(s\,;V\,;X)+(D\zeta)(s\,;V\,;X)\,.$$

It follows that

$$\begin{aligned}&f_1(s)(D\xi)(s\,;V\,;X)\\&\quad=-\xi_X(s\,;A_{\zeta(s;X)}V)+F(s)\zeta_X(s\,;V)-\zeta_X(s\,;A_{\xi(s;X)}V)\,,\end{aligned} \tag{2.4}$$

$$\begin{aligned}&f_1(s)^2H(V,\,X)+f_1(s)(D\zeta)(s\,;V\,;X)\\&\quad=F(s)\xi_X(s\,;V)-\xi_X(s\,;A_{\xi(s;X)}V)-\zeta_X(s\,;A_{\zeta(s;X)}V)\end{aligned} \tag{2.5}$$

(cf. [11]). If we compute the first and the third derivatives of (2.4) at $s=L$, then we get

LEMMA 2.1 (cf. Lemmas 5.3 and 5.5 [**11**]). *Let* $Z \in \mathcal{H}_x^*(X)$, $Y \in T_x\mathrm{Cut}(y)$ $(y=\gamma(L))$ *and* $c=(a+2)\nu^2/2-1$. *Then we have*

$$\left(c+\frac{\nu^2}{2}\right)(D\xi)_X(Z)-\frac{\nu^2}{2a}\xi_X(A_{(D\xi)(X)}Z) \tag{2.6}$$

$$=2H(X, A_{(D\xi)(X)}Z)+(D\xi)_X(A_{H(X,X)}Z),$$

$$c(D\xi)_X(Y)-\frac{\nu^2}{2a}\xi_X(A_{(D\xi)(X)}Y) \tag{2.7}$$

$$=2H(X, A_{(D\xi)(X)}Y)+(D\xi)_X(A_{H(X,X)}Y),$$

where $(D\xi)_X(Z)=\zeta'_X(L\,;Z)$, $\xi_X(A_{(D\xi)(X)}Z)=\xi_X(L\,;A_{(D\xi)(X)}Z)$ *and so on.*

Next we shall compute the second and the fourth derivatives of (2.5) at $s=L$ and $s=0$ respectively. Let $\eta(X)=\xi''(L\,;X)$. From (1.10) we have

$$\eta(X)=H(X, X)-\nu^2\xi(X). \tag{2.8}$$

LEMMA 2.2. *For* $Y \in T_x\mathrm{Cut}(y)$ *we obtain*

$$aH(X, Y)+(D^2\xi)(Y\,;X) \tag{2.9}$$

$$=-\frac{1}{2}\xi_X\left(Y+\frac{1}{a}A_{\eta(X)}Y\right)-\frac{1}{a}(D\xi)_X(A_{(D\xi)(X)}Y),$$

$$-2cH(X, Y)-\frac{\nu^2}{2a}c\xi_X(Y)+\frac{\nu^2}{2a}\xi_X(A_{H(X,X)}Y) \tag{2.10}$$

$$+2H(X, A_{H(X,X)}Y)+\frac{\nu^2}{a}(D\xi)_X(A_{(D\xi)(X)}Y)=0.$$

PROOF. Calculate the second derivative of the both hand side of (2.5) at $s=L$. We have

$$2a^2H(V, X)+2a(D^2\xi)(V\,;X)$$

$$=-a\xi_X(V)+b\eta_X(V)-\eta_X(A_{\xi(X)}V)$$

$$-\xi_X(A_{\eta(X)}V)-2(D\xi)_X(A_{(D\xi)(X)}V)$$

where we have used equations $f_1(L)=f_3(L)=f_1''(L)=f_3''(L)=0$. Let $V=Y$. In virtue of (1.13), we obtain (2.9). Substitute

$$(D\zeta)(s\,;V\,;X)=l(s)(D^2\xi)(V\,;X) \qquad \text{(cf. (1.11))},$$

$$\xi_X(s\,;V)=2h(s)H(X, V)+k(s)\xi_X(V) \qquad \text{(cf. (1.10))},$$

$$\zeta_X(s\,;V)=l(s)(D\xi)_X(V),$$

(1.10) and (1.11) into (2.5). Then (2.5) becomes

$$
\begin{aligned}
f_1(s)^2 H(X, V) &+ f_1(s) l(s)(D^2\xi)(V\,;X) \\
&= F(s)\{2h(s)H(X, V)+k(s)\xi_X(V)\} \\
&\quad -2h(s)H(X, h(s)A_{H(X,X)}V+k(s)A_{\xi(X)}V) \\
&\quad -k(s)\xi_X(h(s)A_{H(X,X)}V+k(s)A_{\xi(X)}V) \\
&\quad -l(s)^2(D\xi)_X(A_{(D\xi)(X)}V)\,.
\end{aligned}
$$

Letting $V=Y$ and making use of (1.13), (2.8) and (2.9), we have

$$
\begin{aligned}
&(f_1^2-2hF+2bhk-alf_1)H(X, Y) \\
&\quad +\left\{bk^2-kF-\frac{1}{2a}(1-\nu^2)lf_1\right\}\xi_X(Y) \\
&\quad +\left(hk-\frac{1}{2a}lf_1\right)\xi_X(A_{H(X,X)}Y)+2h^2H(X, A_{H(X,X)}Y) \\
&\quad +\left(l^2-\frac{1}{a}lf_1\right)(D\xi)_X(A_{(D\xi)(X)}Y)=0\,.
\end{aligned}
$$

In order to compute the fourth derivatives of coefficients at $s=0$, we use (1.7), (1.8) and (1.9). The following are easily verified:

$$
\begin{aligned}
&(f_1^2)^{(4)}(0)=-4\nu^2(3a+5), \qquad (hF)^{(4)}(0)=-2(2\nu^2+3), \\
&(hk)^{(4)}(0)=0, \qquad (lf_1)^{(4)}(0)=-6\nu^2, \qquad (h^2)^{(4)}(0)=6, \\
&(l^2)^{(4)}(0)=0, \qquad (k^2)^{(4)}(0)=0, \qquad (kF)^{(4)}(0)=\frac{3}{2}\nu^4\,.
\end{aligned}
$$

Thus we have (2.10). Q.E.D.

LEMMA 2.3. *For any $Y, V \in T_x\mathrm{Cut}(y)$ we get*

(2.11) $$\frac{1}{2}\left\langle \xi_X\left(Y+\frac{1}{a}A_{\eta(X)}Y\right), \xi_X(V)\right\rangle=\langle Y, V\rangle\,,$$

(2.12) $$\langle (D\xi)_X(Y), \xi_X(V)\rangle=-\frac{4}{\nu^2}\langle A_{(D\xi)(X)}Y, V\rangle\,.$$

PROOF. Consider Jacobi field $K_{\tilde{Y}}$ along γ such that $K_{\tilde{Y}}(0)=Y\in T_x\mathrm{Cut}(y)$, $K_{\tilde{Y}}(L)=0$ and $\nabla_{\dot{\gamma}}K_{\tilde{Y}}(L)=\tilde{Y}\in T_y\mathrm{Cut}(x)$. If we put $W=\nabla_X K_{\tilde{Y}}$, then $K_{\tilde{Y}}=J_Y^*+J_W$ (cf. Theorem 3.4 [**11**]). Using (2.1) and (2.2), we have

$$\tilde{Y}=-A_{(D\xi)(X)}Y+aW+aH(X, Y)+(D^2\xi)(Y\,;X)+(D\xi)_X(W)\,.$$

Since $T_y\mathrm{Cut}(x)=\{\xi_X(Y)\,;Y\in T_x\mathrm{Cut}(y)\}$ (cf. Lemma 3.2 [**11**]), we see that $W=(1/a)A_{(D\xi)(X)}Y$ and so

$$\tilde{Y}=aH(X, Y)+(D^2\xi)(Y\,;X)+\frac{1}{a}(D\xi)_X(A_{(D\xi)(X)}Y)\,.$$

From (2.9) it follows that $\tilde{Y}=-(1/2)\xi_X(Y+(1/a)A_{\eta(X)}Y)$. Since $\langle K_{\tilde{Y}}, \nabla_{\dot{\gamma}}J_V\rangle-\langle\nabla_{\dot{\gamma}}K_{\tilde{Y}}, J_V\rangle$=constant along γ, we have $\langle Y, V\rangle=-\langle\tilde{Y}, \xi_X(V)\rangle$, which shows (2.11). Next we prove (2.12). Observe

$$\frac{d^2}{ds^2}K_{\tilde{Y}}=R(\dot{\gamma}, K_{\tilde{Y}})\dot{\gamma}-A_{H(\dot{\gamma}, K_{\tilde{Y}})}\dot{\gamma}+2H(\dot{\gamma}, \nabla_{\dot{\gamma}}K_{\tilde{Y}})+(DH)(\dot{\gamma}, \dot{\gamma}, K_{\tilde{Y}})$$

where R denotes the curvature tensor of M. Thus we have

$$\frac{d^2}{ds^2}K_{\tilde{Y}}(L)=2H(\dot{\gamma}(L), \tilde{Y}).$$

On the other hand, equations (2.1), (2.2) show

$$\frac{d^2}{ds^2}(J_Y^*+J_W)(L)=-aY-A_{\eta(X)}Y+(D\eta)(Y\,;X)+\eta_X(W)$$

where $(D\eta)(Y\,;X)=\nabla_Y^{\perp}\eta(X^*)$. As $\xi_X(V)$ is tangent to M at y, we obtain $\langle(D\eta)(Y\,;X)+\eta_X(W), \xi_X(V)\rangle=0$. Therefore it suffices to show

$$(2.13)\qquad (D\eta)(Y\,;X)+\eta_X(W)=-\frac{1}{2}\nu^2(D\xi)_X(Y)+2H\Big(X, \frac{1}{a}A_{(D\xi)(X)}Y\Big)$$

because from (1.13) we have

$$\begin{aligned}\langle H(X, U), \xi_X(V)\rangle&=-\langle H(V, U), \xi(X)\rangle+\langle H(X, X), \xi(X)\rangle\langle U, V\rangle\\&=-a\langle V, U\rangle\end{aligned}$$

for every $U\in\{X\}^{\perp}$. Equation (2.8) gives

$$(D\eta)(Y\,;X)=(DH)(Y, X, X)-\nu^2(D\xi)(Y\,;X),$$

$$\eta_X(W)=2H(X, W)-\nu^2\xi_X(W).$$

By using (3.4) [11], we obtain $(D\xi)(Y\,;X)+\xi_X(W)=0$. Moreover the definition of $\zeta(s\,;X)$ shows

$$(D\xi)(U)=\zeta'(L\,;U)=a_3'(\lambda_1\lambda_2)^{-1}(DH)(U^3)=-\frac{2}{3\nu^2}(DH)(U^3)$$

for every $U\in U_xM$ where we have used (1.6). It follows that

$$\begin{aligned}(DH)(Y, X, X)&=\frac{1}{3}\{(DH)(Y, X, X)+(DH)(X, Y, X)+(DH)(X, X, Y)\}\\&=-\frac{\nu^2}{2}(D\xi)_X(Y).\end{aligned}$$

Therefore we have proved (2.13). Q.E.D.

The symmetric transformation $S_X(s)$ has nice properties stated in the following two lemmas.

LEMMA 2.4 [12]. *Let* g_s $(s\in(0, L))$ *be the Riemannian metric induced on*

U_xM by the map $U_xM \to$(geodesic sphere with center x and radius s) sending V to $\exp_x sV$. The derivative $S'_X(s)$ satisfies

$$g_s(S'_X(s)V, W) = -\langle V, W\rangle$$

for every $V, W \in \{X\}^\perp$ and $s \in (0, L)$, where we note that

$$g_s(V, W) = \langle J_V(s), J_W(s)\rangle .$$

LEMMA 2.5. *Define $\phi_{X,s}\colon \{X\}^\perp \to \{\dot\gamma(s)\}^\perp$ by*

$$\phi_{X,s}(V) = J_{S'_X(s)V}(s) .$$

Then we have

$$\mathcal{H}^*_{\dot\gamma(s)}(\dot\gamma(s)) = \phi_{X,s}(\mathrm{Ker}(S_X(s+L) - S_X(s)))$$

for each $s \in (0, L)$.

PROOF. Let $s \in (0, L)$ be arbitrarily fixed. Let $\mathcal{J}$ be a Jacobi field such that $\mathcal{J}(s)=0$ and $\mathcal{J} \perp \dot\gamma$ on γ. Let $W = \mathcal{J}(0)$ and $V = \nabla_X \mathcal{J}(0)$. Then this Jacobi field can be written as

$$\mathcal{J} = J^*_W + J_V .$$

Since $\mathcal{J}(s)=0$, we have from (2.1) and (2.2)

$$f_1(s)S_X(s)W + f_1(s)V = 0 .$$

The assumption $a<0$ is equivalent to $f_1>0$ on $(0, L)$ because of (1.7). Thus $V = -S_X(s)W$. We shall compute $\nabla_{\dot\gamma(s)}\mathcal{J}$. Since

$$\begin{aligned}\mathcal{J}(s+t) &= J^*_W(s+t) + J_V(s+t)\\ &= f_1(s+t)(S_X(s+t) - S_X(s))W\\ &\quad + \xi_X(s+t\,;\,(S_X(s+t) - S_X(s))W)\\ &\quad + \zeta_X(s+t\,;\,(S_X(s+t) - S_X(s))W) ,\end{aligned}$$

we obtain

$$\begin{aligned}\nabla_{\dot\gamma}\mathcal{J}(s) &= \frac{d}{dt}\mathcal{J}(s+t)\Big|_{t=0}\\ &= f_1(s)S'_X(s)W + \xi_X(s\,;\,S'_X(s)W) + \zeta_X(s\,;\,S'_X(s)W)\\ &= \phi_{X,s}(W) .\end{aligned}$$

Noting that $\mathcal{J}(s+L) = \xi_{\dot\gamma(s)}(\phi_{X,s}(W))$ and using Lemma 3.1 [**11**], we see that $\phi_{X,s}(W) \in \mathcal{H}^*_{\dot\gamma(s)}(\dot\gamma(s))$ if and only if $\mathcal{J}(s+L)=0$. Thus we have proved $\phi_{X,s}(W) \in \mathcal{H}^*_{\dot\gamma(s)}(\dot\gamma(s))$ if and only if $(S_X(s+L) - S_X(s))W = 0$. Q.E.D.

COROLLARY 2.6. *Let $\mathcal{Z} \in \{\dot\gamma(L/2)\}^\perp$ and $\mathcal{Z} = \phi_{X,L/2}(W)$. Then $\mathcal{Z} \in \mathcal{H}^*_{\dot\gamma(L/2)}(\dot\gamma(L/2))$ if and only if $F(L/2)W - A_{\xi(L/2;X)}W = 0$.*

PROOF. By Lemma 2.5 we see that $\mathcal{Z}\in\mathcal{H}^*_{\gamma(L/2)}(\dot{\gamma}(L/2))$ if and only if $(S_X(3L/2)-S_X(L/2))W=0$. It is easily verified that

$$S_X(3L/2)-S_X(L/2)=-2\{F(L/2)I-A_{\xi(L/2;X)}\}/f_1(L/2). \qquad \text{Q.E.D.}$$

REMARK. Equations (2.1), (2.2), (2.4), (2.5), (2.9), (2.11), Lemmas 2.4, 2.5 and Corollary 2.6 hold for any order helical minimal imbedding of a compact Riemannian manifold into a unit sphere.

§ 3. A geodesic from $\gamma(L/2)$ to $\gamma(3L/2)$.

As before, let γ be the unit speed geodesic such that $\gamma(0)=x$ and $\dot{\gamma}(0)=X$. Let σ be a unit speed geodesic such that $\sigma(0)=\gamma(L/2)$ and $\dot{\sigma}(0)=\mathcal{Z}\in\mathcal{H}^*_{\gamma(L/2)}(\dot{\gamma}(L/2))\cap U_{\gamma(L/2)}M$. Then $\sigma(L)=\gamma(3L/2)$. Let $\mathcal{Z}$ be given by $\mathcal{Z}=\phi_{X,L/2}(W)$ where $W\in\{X\}^{\perp}$. Using (1.8)~(1.10) and Corollary 2.6, we have

$$A_{H(X,X)}W=\left\{\frac{1}{2}(a-3)+2\nu^2\right\}W-\frac{\nu^2}{2}A_{\xi(X)}W. \tag{3.1}$$

Moreover $\|\mathcal{Z}\|=1$ implies that

$$\langle S'_X(L/2)W,\, W\rangle=-1 \tag{3.2}$$

since $\|\mathcal{Z}\|^2=g_{L/2}(S'_X(L/2)W,\, S'_X(L/2)W)=-\langle S'_X(L/2)W,\, W\rangle$ in virtue of Lemma 2.4.

LEMMA 3.1. *We have*

$$S'_X(L/2)W=\frac{2\nu}{(1-a)^2}A_{(D\xi)(X)}W-\left(W+\frac{1}{1-b}A_{\xi(X)}W\right), \tag{3.3}$$

$$S''_X(L/2)W=\frac{4}{1-b}\left[\frac{\nu}{1-a}\{(b-a)W-A_{\xi(X)}W\}+\frac{1+a}{(1-a)^2}A_{(D\xi)(X)}W\right]. \tag{3.4}$$

PROOF. Differentiating the both hand sides of

$$f_1(s)S_X(s)W=F(s)W-h(s)A_{H(X,X)}W-k(s)A_{\xi(X)}W-l(s)A_{(D\xi)(X)}W,$$

at $s=L/2$ and using (1.7)~(1.9), we have

$$-\nu(1+a)S_X(L/2)W+(1-a)S'_X(L/2)W$$
$$=-(1-a)W-\nu^2A_{\xi(X)}W+\nu A_{(D\xi)(X)}W.$$

Since $S_X(L/2)W=-(l/f_1)(L/2)A_{(D\xi)(X)}W=A_{(D\xi)(X)}W/(1-a)$, we obtain (3.3). If we make use of (3.1), then we have (3.4) in a similar way. Q.E.D.

LEMMA 3.2. *The unit tangent vector $\dot{\sigma}(L)$ is given by*

$$\dot{\sigma}(L)=a\mathcal{Z}+(D\xi)(\mathcal{Z})$$
$$=-\psi_{-X,L/2}(W).$$

PROOF. The first equality is an immediate consequence of (1.12). As in the preceding section, let $\mathcal{J}$ be the Jacobi field along γ such that $\mathcal{J}(0)=W$, $\mathcal{J}(L/2)=0$ and $\nabla_{\dot{\gamma}}\mathcal{J}(L/2)=\mathcal{Z}$. Taking account of (2.1), we see that $\nabla_{\dot{\gamma}}\mathcal{J}(3L/2)=a\mathcal{Z}+(D\xi)_{\dot{\gamma}(L/2)}(\mathcal{Z})$. Since $(D\xi)_{\dot{\gamma}(L/2)}(\mathcal{Z})=(D\xi)(\mathcal{Z})$ (cf. (3.5) in [**11**]), we have $\dot{\sigma}(L)=\nabla_{\dot{\gamma}}\mathcal{J}(3L/2)$. Differentiating the both hand sides of

$$\begin{aligned}\mathcal{J}(L/2+t)=&f_1(L/2+t)(S_X(L/2+t)-S_X(L/2))W\\&+\xi_X(L/2+t\,;(S_X(L/2+t)-S_X(L/2))W)\\&+\zeta_X(L/2+t\,;(S_X(L/2+t)-S_X(L/2))W)\end{aligned}$$

at $t=L$, we obtain

$$\begin{aligned}\nabla_{\dot{\gamma}}\mathcal{J}(3L/2)=&\frac{d}{dt}\mathcal{J}(L/2+t)\Big|_{t=L}\\=&f_1(3L/2)S'_X(3L/2)W+\xi_X(3L/2\,;S'_X(3L/2)W)\\&+\zeta_X(3L/2\,;S'_X(3L/2)W).\end{aligned}$$

By the definition (2.3) we easily have $S'_X(3L/2)W=S'_{-X}(L/2)W$. It follows that

$$\begin{aligned}\dot{\sigma}(L)=&-f_1(L/2)S'_{-X}(L/2)W-\xi_{-X}(L/2\,;S'_{-X}(L/2)W)\\&-\zeta_{-X}(L/2\,;S'_{-X}(L/2)W)\\=&-\psi_{-X,L/2}(W).\end{aligned}$$

Q.E.D.

Let $s(t)=\delta(x,\sigma(t))$ and $V(t)\in U_xM$ the unit tangent vector of the geodesic from x to $\sigma(t)$. Notice that $V(t)$ is unique for each $t\in[0, 2L)$. In fact, suppose $\sigma(t_0)\in \mathrm{Cut}(x)$, $t_0\in(0, L)$. If $t_0\leqq L/2$, then length $(\gamma|_{[0,L/2]})+\mathrm{length}(\sigma|_{[0,t_0]})\leqq L$. Thus $\dot{\gamma}(L/2)=\mathcal{Z}$ which contradicts $\dot{\gamma}(L/2)\perp\mathcal{Z}$. If $t_0\geqq L/2$, then it suffices to consider the curves $\sigma|_{[t_0,L]}$ and $\gamma|_{[3L/2,2L]}$. Decompose W as

$$W=Z_0+Y_0,\qquad Z_0\in\mathcal{H}^*_x(X),\qquad Y_0\in T_x\mathrm{Cut}(y).$$

LEMMA 3.3. *We see that $s(t)$ and $V(t)$ satisfy*

$$F(s(t))=F(L/2)+\frac{a}{1-b}\|Y_0\|^2h(t),\tag{3.5}$$

$$\begin{aligned}(f_1(s(t))/f_1(L/2))V(t)=&\{F(t)-(1+b)k(t)+\mu(X, W)h(t)\}X\\&+\{f_1(t)-al(t)\}S'_X(L/2)W\\&-l(t)S'_{-X}(L/2)W+\nu^2h(t)B(X, W)\end{aligned}\tag{3.6}$$

with $s(0)=s(L)=L/2$, $s'(0)=0$, $V(0)=X$ *and* $V'(0)=S'_X(L/2)W$, *where* $\mu(X, W)=1-\|S'_X(L/2)W\|^2+(2a(1+a)/(1-a)(1-b)^2)\|Y_0\|^2$ *and* $B(X, W)$ *is a certain tangent vector orthogonal to* X.

PROOF. Let N_xM denote the normal space of M at x in $S(1)$. We may write $\sigma(t)$ as

$$\sigma(t)\equiv F(s(t))x+f_1(s(t))V(t) \qquad \mathrm{mod}\, N_xM.$$

By (1.12), we may also write

$$\sigma(t)=F(t)\gamma(L/2)+f_1(t)\mathcal{Z}+h(t)H(\mathcal{Z}, \mathcal{Z}) + k(t)\xi(\mathcal{Z})+l(t)(D\xi)(\mathcal{Z}). \tag{3.7}$$

Let $\widetilde{W}=S'_X(L/2)W/\|S'_X(L/2)W\|$. Then we have $J_{\widetilde{W}}(L/2)=\mathcal{Z}/\|S'_X(L/2)W\|$. Let $\alpha(\theta)$ be the curve on M defined by

$$\alpha(\theta)=F(L/2)x+f_1(L/2)U(\theta)+\xi(L/2\,;U(\theta))+\zeta(L/2\,;U(\theta)),$$

where $U(\theta)=\cos\theta X+\sin\theta \widetilde{W}$. We find

$$\dot\alpha(0)=J_{\widetilde{W}}(L/2)=\mathcal{Z}/\|S'_X(L/2)W\|,$$

$$\ddot\alpha(0)\equiv -f_1(L/2)X \qquad \mathrm{mod}\, N_xM.$$

It follows that

$$\begin{aligned}-f_1(L/2)X&\equiv\bar\nabla_{\dot\alpha}\dot\alpha(0)+\langle\ddot\alpha(0), \alpha(0)\rangle\alpha(0)\\ &=\nabla_{\dot\alpha}\dot\alpha(0)+H(\dot\alpha(0), \dot\alpha(0))-\|\dot\alpha(0)\|^2\gamma(L/2)\\ &=\nabla_{\dot\alpha}\dot\alpha(0)+\|S'_X(L/2)W\|^{-2}(H(\mathcal{Z}, \mathcal{Z})-\gamma(L/2))\end{aligned}$$

$\mathrm{mod}\, N_xM$, $\bar\nabla$ being the covariant differential operator on $S(1)$. Decompose $\nabla_{\dot\alpha}\dot\alpha(0)$ as

$$\nabla_{\dot\alpha}\dot\alpha(0)=J_{A(X,W)}(L/2)+\langle\nabla_{\dot\alpha}\dot\alpha(0), \dot\gamma(L/2)\rangle\dot\gamma(L/2),$$

where $A(X, W)\in\{X\}^{\perp}$. Then we have

$$\begin{aligned}H(\mathcal{Z}, \mathcal{Z})\equiv&\{F(L/2)+f_1(L/2)\|S'_X(L/2)W\|^2\langle\nabla_{\dot\alpha}\dot\alpha(0), \dot\gamma(L/2)\rangle\}x\\ &+\{f_1(L/2)-\|S'_X(L/2)W\|^2f_1(L/2)-f'_1(L/2)\|S'_X(L/2)W\|^2\\ &\cdot\langle\nabla_{\dot\alpha}\dot\alpha(0), \dot\gamma(L/2)\rangle\}X-f_1(L/2)\|S'_X(L/2)W\|^2A(X, W)\end{aligned}$$

$\mathrm{mod}\, N_xM$ because of (1.4) and (2.1). We next prove

$$\langle\nabla_{\dot\alpha}\dot\alpha(0), \dot\gamma(L/2)\rangle=-\frac{1}{2}\langle S''_X(L/2)W, W\rangle/\|S'_X(L/2)W\|^2. \tag{3.8}$$

Let Γ denote the variation $(s, \theta)\mapsto \exp_x sU(\theta)$. Then

$$\langle \nabla_{\dot\alpha}\dot\alpha(0), \dot\gamma(L/2)\rangle = -\left\langle \frac{\partial \Gamma}{\partial \theta}, \nabla_{\partial\Gamma/\partial\theta}\partial\Gamma/\partial s \right\rangle\Big|_{(s,\theta)=(L/2,0)}$$

$$= -\left\langle \frac{\partial \Gamma}{\partial \theta}, \nabla_{\partial\Gamma/\partial s}\partial\Gamma/\partial \theta \right\rangle\Big|_{(s,\theta)=(L/2,0)}$$

$$= -\frac{1}{2}\frac{d}{ds}\|J_{\tilde W}\|^2\Big|_{s=L/2}.$$

Furthermore using Lemma 2.4 we have

$$\frac{d}{ds}\|J_{\tilde W}\|^2\Big|_{s=L/2} = \frac{d}{ds} g_s(\widetilde W, \widetilde W)\Big|_{s=L/2}$$

$$= -\frac{d}{ds}\langle S'_X(s)^{-1}S'_X(L/2)W, S'_X(L/2)W\rangle\Big|_{s=L/2}\Big/\|S'_X(L/2)W\|^2$$

$$= \langle S''_X(L/2)W, W\rangle / \|S'_X(L/2)W\|^2.$$

Therefore we have shown (3.8). Thus

$$H(\mathcal{Z}, \mathcal{Z}) \equiv \left\{F(L/2) - \frac{1}{2}f_1(L/2)\langle S''_X(L/2)W, W\rangle\right\}x + \left\{f_1(L/2)(1 - \|S'_X(L/2)W\|^2) + \frac{1}{2}f'_1(L/2)\langle S''_X(L/2)W, W\rangle\right\}X + f_1(L/2)\nu^2 B(X, W) \mod N_x M, \tag{3.9}$$

where $B(X, W) = -\|S'_X(L/2)W\|^2 A(X, W)/\nu^2$. Since $\gamma(3L/2) = b\gamma(L/2) + \xi(\mathcal{Z})$, we see that

$$\xi(\mathcal{Z}) \equiv (1-b)F(L/2)x - (1+b)f_1(L/2)X \mod N_x M. \tag{3.10}$$

Moreover Lemma 3.2 implies $(D\xi)(\mathcal{Z}) = -\phi_{-X, L/2}(W) - a\phi_{X, L/2}(W)$, so that

$$(D\xi)(\mathcal{Z}) \equiv -f_1(L/2)\{S'_{-X}(L/2)W + aS'_X(L/2)W\} \mod N_x M. \tag{3.11}$$

Substituting (3.9)~(3.11) into (3.7) and noting that $F(t) + h(t) + (1-b)k(t) = 1$, we obtain

$$\sigma(t) \equiv \left\{F(L/2) - \frac{1}{2}f_1(L/2)\langle S''_X(L/2)W, W\rangle h(t)\right\}x + f_1(L/2)[\{F(t) - (1+b)k(t) + \mu(X, W)h(t)\}X + \{f_1(t) - al(t)\}S'_X(L/2)W \tag{3.12}$$

$$-l(t)S'_{-X}(L/2)W+\nu^2h(t)B(X, W)]$$

mod N_xM, where $\mu(X, W)=1-\|S'_X(L/2)W\|^2+(f_1'(L/2)/2f_1(L/2))\langle S''_X(L/2)W, W\rangle$. Finally we shall prove $(f_1'(L/2)/2f_1(L/2))\langle S''_X(L/2)W, W\rangle=(2a(1+a)/(1-a)(1-b)^2)\cdot\|Y_0\|^2$. Consider geodesics $\tilde{\gamma}(s)=\gamma(2L-s)$ and $\tilde{\sigma}(t)=\sigma(L-t)$. We have $\dot{\tilde{\gamma}}(0)=-X$ and $\dot{\tilde{\sigma}}(0)=\phi_{-X,L/2}(W)$ (cf. Lemma 3.2). Thus $\tilde{\sigma}$ satisfies (3.12) in which X is replaced by $-X$. Since $\langle\sigma(L-t), x\rangle=\langle\tilde{\sigma}(t), x\rangle$ and $h(L-t)=h(t)$, we get $\langle S''_X(L/2)W, W\rangle=\langle S''_{-X}(L/2)W, W\rangle$. It follows from (3.4) that

$$\langle A_{(D\xi)(X)}W, W\rangle=0\,, \tag{3.13}$$

and so, using (1.13),

$$\langle S''_X(L/2)W, W\rangle=\frac{-4a\nu}{(1-b)(1-a)}\|Y_0\|^2\,.$$

From (1.7) we obtain the assertion. Q.E.D.

LEMMA 3.4. *We obtain*

$$\left\|W+\frac{1}{1-b}A_{\xi(X)}W\right\|^2=\nu^2+\frac{a}{(1-b)^2}\|Y_0\|^2\,, \tag{3.14}$$

$$\|A_{(D\xi)(X)}W\|^2=\frac{a(1-a)^2(3a+1)}{4(1-b)}\|Y_0\|^2\,. \tag{3.15}$$

PROOF. Substituting (3.3) into (3.2) and making use of (3.13), we have

$$\left\langle W+\frac{1}{1-b}A_{\xi(X)}W, W\right\rangle=1\,.$$

If follows from (1.13) that

$$\nu^2\|Z_0\|^2+\frac{1}{1-b}\|Y_0\|^2=1\,.$$

Thus we see that

$$\left\|W+\frac{1}{1-b}A_{\xi(X)}W\right\|^2=\left\|\nu^2Z_0+\frac{1}{1-b}Y_0\right\|^2$$

$$=\nu^2+\frac{a}{(1-b)^2}\|Y_0\|^2\,.$$

Next we prove (3.15). As in the proof of Lemma 3.3, we consider geodesics $\tilde{\gamma}(s)=\gamma(2L-s)$ and $\tilde{\sigma}(t)=\sigma(L-t)$. Since $\langle\tilde{\sigma}(L/2), X\rangle=\langle\sigma(L/2), X\rangle$, (3.12) for σ and $\tilde{\sigma}$ give

$$2\{F(L/2)-(1+b)k(L/2)\}+\{\mu(X, W)+\mu(-X, W)\}h(L/2)=0\,.$$

Using (1.8) and (1.9), it is easily shown

$$\{F(L/2)-(1+b)k(L/2)\}/h(L/2)=\nu^2-1\,.$$

Furthermore using (3.3) and (3.14) we have

$$\mu(X, W)+\mu(-X, W)$$

$$=2-\|S'_X(L/2)W\|^2-\|S'_{-X}(L/2)W\|^2+\frac{4a(1+a)}{(1-a)(1-b)^2}\|Y_0\|^2$$

$$=2\left\{1-\nu^2-\frac{4\nu^2}{(1-a)^4}\|A_{(D\xi)(X)}W\|^2+\frac{a(3a+1)}{(1-a)(1-b)^2}\|Y_0\|^2\right\}.$$

Therefore we obtain (3.15). Q.E.D.

LEMMA 3.5. *Vectors* $A_{(D\xi)(X)}W$, $W+A_{\xi(X)}W/(1-b)$ *and* $B(X, W)$ *are orthogonal.*

PROOF. From (1.7), (1.9) and (3.3) we see that

$$\{f_1(t)-al(t)\}S'_X(L/2)W-l(t)S'_{-X}(L/2)W \tag{3.16}$$

$$=\frac{1}{(1-a)^2}\sin 2\nu t\, A_{(D\xi)(X)}W-\frac{1}{\nu}\sin\nu t\left(W+\frac{1}{1-b}A_{\xi(X)}W\right).$$

Thus (3.6) can be written as

$$(f_1(s(t))/f_1(L/2))V(t)$$

$$=\{F(t)-(1+b)k(t)+\mu(X, W)h(t)\}X+\nu^2h(t)B(X, W)$$

$$+\frac{1}{(1-a)^2}\sin 2\nu t\, A_{(D\xi)(X)}W-\frac{1}{\nu}\sin\nu t\left(W+\frac{1}{1-b}A_{\xi(X)}W\right).$$

Since $V(t)$ is a unit vector, we get

$$(f_1(s(t))/f_1(L/2))^2=2\left(\frac{\nu}{1-a}\right)^2h(t)\sin 2\nu t\langle A_{(D\xi)(X)}W, B(X, W)\rangle$$

$$-2\nu h(t)\sin\nu t\left\langle W+\frac{1}{1-b}A_{\xi(X)}W, B(X, W)\right\rangle$$

$$+(\text{even function}).$$

Taking account of the fact that F is monotone decreasing on $(0, L)$ and $h(t)$ is an even function into (3.5), we obtain $s(t)=s(-t)$. Therefore the above equation implies that $B(X, W)$ is orthogonal to $A_{(D\xi)(X)}W$ and $W+A_{\xi(X)}W/(1-b)$. We next prove $A_{(D\xi)(X)}W$ is orthogonal to $W+A_{\xi(X)}W/(1-b)$. Apply (2.6) (resp. (2.7)) to Z_0 (resp. Y_0) and add (2.6) to (2.7). Then the result is

$$c(D\xi)_X(W)+\frac{\nu^2}{2}(D\xi)_X(Z_0)-\frac{\nu^2}{2a}\xi_X(A_{(D\xi)(X)}W)$$

$$=2H(X, A_{(D\xi)(X)}W)+(D\xi)_X(A_{H(X,X)}W).$$

Noting that

$$\langle \xi_X(U),\ H(X,\ V)\rangle = -\langle \xi(X),\ H(U,\ V)\rangle + (b-a)\langle U,\ V\rangle\,,$$

$$2\langle H(X,\ U),\ H(X,\ V)\rangle = -\langle H(X,\ X),\ H(U,\ V)\rangle + \kappa_1^2\langle U,\ V\rangle\,,$$

$$\langle (D\xi)_X(U),\ H(X,\ V)\rangle = -\langle (D\xi)(X),\ H(U,\ V)\rangle$$

for every $U,\ V \in \{X\}^{\perp}$ (cf. Corollary 3.5 [**9**]), from (3.13) we have

$$\langle A_{(D\xi)(X)}W,\ a\nu^2 Z_0 - \nu^2 A_{\xi(X)}W - 4aA_{H(X,X)}W\rangle = 0\,.$$

Using (1.13) and (3.1), it follows that $\langle A_{(D\xi)(X)}W,\ Z_0\rangle = 0$, from which we obtain the assertion. Q.E.D.

LEMMA 3.6. *Equation* (3.6) *reduces to*

$$(f_1(s(t))/f_1(L/2))V(t) = \cos\nu t\, X - \frac{1}{\nu}\sin\nu t\Big(W + \frac{1}{1-b}A_{\xi(X)}W\Big)$$

$$+\frac{1}{(1-a)^2}\sin 2\nu t\, A_{(D\xi)(X)}W + \frac{1}{4}(1-\cos 2\nu t)B(X,\ W)\,.$$

PROOF. By (3.3), Lemmas 3.4 and 3.5, we find

$$\|S'_X(L/2)W\|^2 = \nu^2 + \frac{2a(1+a)}{(1-a)(1-b)^2}\|Y_0\|^2$$

from which $\mu(X,\ W) = 1-\nu^2$. Hence the straightforward computation shows

$$F(t) - (1+b)k(t) + \mu(X,\ W)h(t) = \cos\nu t\,.$$

The second and the third terms have already been computed as (3.16). Q.E.D.

LEMMA 3.7. *We have* $Y_0 = A_{(D\xi)(X)}W = B(X,\ W) = 0$.

PROOF. Since

$$\langle f_1'(s(t))s'(t)V(t) + f_1(s(t))V'(t),\ f_1(s(t))V(t)\rangle$$

$$= -f_1'(s(t))(F(s(t)))'\,,$$

we easily see from (1.7)$\sim$(1.9) and (3.5) that

$$\langle (f_1(s(t))V(t))',\ f_1(s(t))V(t)\rangle / (f_1(L/2))^2$$

$$= -\frac{a\nu}{(1-a)^2(1-b)}\|Y_0\|^2 \sin 2\nu t$$

$$\cdot\{(1-a)\cos\nu s(t) + (1+a)\cos 2\nu s(t)\}\,.$$

On the other hand, Lemmas 3.5 and 3.6 implies that L.H.S. of the above equation is equal to

$$\sin 2\nu t\Big\{-\frac{\nu}{2}+\frac{1}{2\nu}\Big\|W+\frac{1}{1-b}A_{\xi(X)}W\Big\|^2$$

$$+\frac{2\nu}{(1-a)^4}\cos 2\nu t\,\|A_{(D\xi)(X)}W\|^2+\frac{\nu}{8}(1-\cos 2\nu t)\|B(X, W)\|^2\Big\}.$$

Using (3.14) and (3.15), we thus have

$$(3.17)\qquad -2a\|Y_0\|^2\{(1-a)\cos\nu s(t)+(1+a)\cos 2\nu s(t)\}$$
$$=a(1-a)\|Y_0\|^2+G+\{a(3a+1)\|Y_0\|^2-G\}\cos 2\nu t$$

where $G=(1-a)^2(1-b)\|B(X, W)\|^2/4$. Equation (3.5) is equivalent to

$$(3.18)\qquad 4(1-a)\cos\nu s(t)+(1+a)\cos 2\nu s(t)$$
$$=-(1+a)+\frac{2a}{1-b}\|Y_0\|^2(1-\cos 2\nu t).$$

If we eliminate $(1+a)\cos 2\nu s(t)$, then (3.17) and (3.18) give

$$(3.19)\qquad \|Y_0\|^2\cos\nu s(t)=G^*(1-\cos 2\nu t)$$

where G^* is defined by

$$G^*=\frac{1}{6a(1-a)}\Big[G+\Big\{\frac{4a^2}{1-b}\|Y_0\|^2-a(3a+1)\Big\}\|Y_0\|^2\Big].$$

Assume that $Y_0\neq 0$. Noting that $\cos 2\nu s(t)=2\cos^2\nu s(t)-1$ and substituting (3.19) into (3.18), we obtain

$$2(1+a)G^{*2}(1-\cos 2\nu t)+\Big\{4(1-a)G^*-\frac{2a}{1-b}\|Y_0\|^4\Big\}\|Y_0\|^2=0$$

for every t. Therefore we get $G^*=0$ and so $\|Y_0\|=0$, which is a contradiction. We have proved $Y_0=0$. Equations (3.15) and (3.17) show $A_{(D\xi)(X)}W=0$ and $B(X, W)=0$ respectively. Q.E.D.

COROLLARY 3.8. *We see that* $\mathrm{Ker}(S_X(3L/2)-S_X(L/2))=\mathcal{H}_x^*(X)$, $A_{(D\xi)(X)}\mathcal{H}_x^*(X)=0$ *and, for* $Z\in\mathcal{H}_x^*(X)$,

$$(3.20)\qquad A_{H(X,X)}Z=\Big(c+\frac{\nu^2}{2}\Big)Z,$$

c being defined in Lemma 2.1.

PROOF. The first and second assertion are derived from Lemmas 2.5 and 3.7. Equation (3.20) is derived from (1.13) and (3.1). Q.E.D.

COROLLARY 3.9. *For every* $s\in(0, L)$ *we have*

$$\mathrm{Ker}(S_X(s+L)-S_X(s))=\mathcal{H}_x^*(X).$$

PROOF. Since the dimension of $\operatorname{Ker}(S_X(s+L)-S_X(s))$ coincides with that of $\mathcal{H}_x^*(X)$ (cf. Lemma 2.5), it suffices to show $(S_X(s+L)-S_X(s))Z=0$ for every $s\in(0, L)$ and $Z\in\mathcal{H}_x^*(X)$. In virtue of (1.7)~(1.11), (1.13) and Corollary 3.8, we easily obtain

$$f_1(s+L)\{F(s)Z-A_{\xi(s;X)}Z-A_{\zeta(s;X)}Z\}$$
$$=\frac{1}{8\nu}\sin 2\nu s\{(1+a)^2\cos^2\nu s-(1-a)^2\}Z\,.$$

The right hand side of the above equation is a periodic function with period L. Hence the definition (2.3) shows the assertion. Q.E.D.

§4. Theorem.

Let $x\in M$ and $X\in U_xM$ be arbitrarily fixed. Let γ be the unit speed geodesic such that $\gamma(0)=x$ and $\dot\gamma(0)=X$. In the preceding section, we have shown that $A_{H(X,X)}$ leaves $\mathcal{H}_x^*(X)$ invariant (cf. (3.20)). Thus $A_{H(X,X)}$ also leaves $T_x\operatorname{Cut}(y)$ invariant. At first we prove

LEMMA 4.1. *Suppose $A_{H(X,X)}Y=vY$ for $Y\in T_x\operatorname{Cut}(y)$ where $y=\gamma(L)$. If $(D\xi)_X(Y)\neq 0$, then $v\leqq c$. If $(D\xi)_X(Y)=0$, then $A_{(D\xi)(X)}Y=0$.*

PROOF. From (2.7) we have

$$(v-c)(D\xi)_X(Y)+2H(X, A_{(D\xi)(X)}Y)+\frac{\nu^2}{2a}\xi_X(A_{(D\xi)(X)}Y)=0\,.$$

Taking the inner product with $(D\xi)_X(Y)$, we obtain

$$(v-c)\|(D\xi)_X(Y)\|^2-2\|A_{(D\xi)(X)}Y\|^2$$
$$+\frac{\nu^2}{2a}\langle\xi_X(A_{(D\xi)(X)}Y),\ (D\xi)_X(Y)\rangle=0\,.$$

Apply (2.12) to the last term. Our assumption was $-1<a<0$. It follows that

$$(v-c)\|(D\xi)_X(Y)\|^2=2\frac{1+a}{a}\|A_{(D\xi)(X)}Y\|^2\leqq 0\,,$$

completing the proof. Q.E.D.

LEMMA 4.2. *Suppose $A_{H(X,X)}Y=vY$ for $Y\in T_x\operatorname{Cut}(y)$. Assume $v>c$. Then we have $v=c+(1+a)\nu^2$.*

PROOF. From Lemma 4.1 we see that $(D\xi)_X(Y)=0$ and hence $A_{(D\xi)(X)}Y=0$. Using (2.10), we have

$$H(X, Y)+\frac{\nu^2}{4a}\xi_X(Y)=0\,.$$

It follows that if $\|Y\|=1$, then

$$\|H(X, Y)\|^2=\nu^2/4\,, \tag{4.1}$$

from which we obtain (cf. (1.1) and (1.6))

$$\begin{aligned}\langle H(X, X), H(Y, Y)\rangle&=\kappa_1^2-2\|H(X, Y)\|^2\\ &=c+(1+a)\nu^2\,.\end{aligned}$$

Q.E.D.

LEMMA 4.3. *For every $Y\in T_x\mathrm{Cut}(y)$, we have*

$$A_{H(X,X)}Y=cY\,. \tag{4.2}$$

PROOF. Firstly we shall prove that if v is any eigenvalue of $A_{H(X,X)}|T_x\mathrm{Cut}(y)$, then $v\leqq c$. Assume that $v>c$ for some eigenvalue v. Let $Y\in T_x\mathrm{Cut}(y)$ be its eigenvector such that $\|Y\|=1$. By Lemmas 4.1 and 4.2 we see that $(D\xi)_X(Y)=0$, $A_{(D\xi)(X)}Y=0$ and $v=c+(1+a)\nu^2$. Also we have (4.1). Taking the inner product of (2.9) with $H(X, Y)$, we get

$$\begin{aligned}a\|H(X, Y)\|^2-\langle(D\xi)(X), (DH)(X, Y, Y)\rangle\\ =\frac{a}{2}\left\langle Y+\frac{1}{a}A_{\eta(X)}Y, Y\right\rangle,\end{aligned} \tag{4.3}$$

where we have used the fact that $\langle(D\xi)(V), H(V, U)\rangle=0$ for every $V, U\in UM$ satisfying $U\perp V$ (cf. Lemma 3.3 [9]). Since $\langle(D\xi)(U), (DH)(U, U, V)\rangle=0$ for every $U, V\in UM$ such that $U\perp V$, we have

$$\begin{aligned}&2\langle(D\xi)(X), (DH)(X, Y, Y)\rangle\\ &=-\langle(D\xi)_X(Y), (DH)(X, X, Y)\rangle+\langle(D\xi)(X), (DH)(X^3)\rangle\\ &=-\frac{2}{3\nu^2}\|(DH)(X^3)\|^2\end{aligned}$$

(cf. the proof of Lemma 2.3). Using (1.2) and (1.6), the second term of the left hand side of (4.3) is equal to $3\nu^2(1-a^2)/4$. Furthermore (2.8) implies that if $A_{H(X,X)}Y=vY$, then

$$Y+\frac{1}{a}A_{\eta(X)}Y=\frac{1}{a}(a+v-b\nu^2)Y\,. \tag{4.4}$$

Thus the right hand side of (4.3) is equal to $(a+v-b\nu^2)/2$. It follows that (4.3) becomes

$$\frac{\nu^2}{4}\{a+3(1-a^2)\}=\frac{1}{2}\left\{\frac{\nu^2}{2}(3a+4)-1+a-b\nu^2\right\}.$$

Since $\nu^2(1-b)=1-a$, we have $(1+a)(1-3a)=0$ which contradicts $-1<a<0$. Secondly we shall prove every eigenvalue v of $A_{H(X,X)}|T_x\mathrm{Cut}(y)$ is greater than c. By virtue of Lemma 2.5 and Corollary 3.8, we see that $J_Z(s)$ is proportional to $\psi_{X,s}(Z)$ for every $s\in(0, L)$ and $Z\in\mathcal{H}_x^*(X)$. Moreover Lemma 2.5 and

Corollary 3.9 show $J_Z(s)\in\mathcal{H}^*_{\dot\gamma(s)}(\dot\gamma(s))$ for each $s\in(0, L)$. Since $\langle J_Y, J_Z\rangle=0$ on $(0, L)$ for $Y\in T_x\mathrm{Cut}(y)$ and $Z\in\mathcal{H}^*_x(X)$ because of Lemma 2.4 and (3.20), it follows that $J_Y(s)\in T_{\gamma(s)}\mathrm{Cut}(\gamma(s+L))$. We have proved

$$T_{\gamma(s)}\mathrm{Cut}(\gamma(s+L))=\mathrm{Span}\{J_Y(s)\,;\ Y\in T_x\mathrm{Cut}(y)\}$$

for each $s\in(0, L)$. The base point x and vector X are arbitrarily chosen. Thus we see that $\langle A_{H(\dot\gamma(s),\dot\gamma(s))}\mathcal{Y}, \mathcal{Y}\rangle\leqq c$ for every $\mathcal{Y}\in T_{\gamma(s)}\mathrm{Cut}(\gamma(s+L))\cap U_{\gamma(s)}M$. By Causs equation, the sectional curvature $K(\mathcal{Y}, \dot\gamma(s))$ of the section spanned by $\mathcal{Y}$ and $\dot\gamma(s)$ is given by

$$K(\mathcal{Y}, \dot\gamma)=1+\langle H(\dot\gamma, \dot\gamma), H(\mathcal{Y}, \mathcal{Y})\rangle-\|H(\dot\gamma, \mathcal{Y})\|^2 .$$

Noting that

$$2\|H(\dot\gamma, \mathcal{Y})\|^2=-\langle H(\dot\gamma, \dot\gamma), H(\mathcal{Y}, \mathcal{Y})\rangle+\kappa_1^2$$

(cf. [7]), we have

$$\begin{aligned} K(\mathcal{Y}, \dot\gamma)&=1-\frac{1}{2}\kappa_1^2+\frac{3}{2}\langle H(\dot\gamma, \dot\gamma), H(\mathcal{Y}, \mathcal{Y})\rangle \\ &\leqq 1-\frac{1}{2}\kappa_1^2+\frac{3}{2}c \\ &=\frac{\nu^2}{4} . \end{aligned}$$

Consider an n-dimensional sphere of curvature $\nu^2/4$ and use Rauch's comparison theorem (cf. [2], [4]). We get $\|J_Y(L)\|^2\geqq 4/\nu^2$ for every $Y\in T_x\mathrm{Cut}(y)\cap U_xM$. Since (2.1) gives $J_Y(L)=\xi_X(Y)$, it follows that $\|\xi_X(Y)\|^2\geqq 4/\nu^2$. On the other hand, by (2.1) and (4.4) we have $\|\xi_X(Y)\|^2=2a/(a+v-b\nu^2)$. Since $a<0$, we conclude $v\geqq\nu^2 a/2-a+b\nu^2$. The right hand side is equal to c. Q. E. D.

THEOREM 4.4. *Let $f: M\to S(1)$ be a helical minimal imbedding of order 4 of a compact Riemannian manifold M into a unit sphere $S(1)$. Then M is isometric to one of* $\boldsymbol{RP}^n$, $\boldsymbol{CP}^m$, $\boldsymbol{QP}^m$ $(m\geqq 2)$ *and* $\boldsymbol{Cay}P^2$ *where $m=n/e$ (the maximal curvature is given in Theorem 1.1). Moreover f is equivalent to the second standard minimal imbedding.*

PROOF. From (3.20) and (4.2) we find

$$\begin{aligned} \mathrm{Trace}\,A_{H(X,X)}&=\kappa_1^2+(e-1)\Big(c+\frac{\nu^2}{2}\Big)+(n-e)c \\ &=\frac{\nu^2}{2}\{(n+2)a+2(n+1)+e\}-n . \end{aligned}$$

Since f is minimal, $\mathrm{Trace}\,A_{H(X,X)}=0$. Using $\nu^2=(1-\mathrm{a})/(1-b)$ and $b=ea/n$, we obtain

$$(1+a)\{(n+2)a-(e+2)\}=0 ,$$

which contradicts the assumption $-1<a<0$. We have proved $a>0$. From Theorem 1.1, the assertion follows. Q.E.D.

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