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Pseudo-differential operators and Markov processes

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0. Introduction.

Given a Lévy type generator L acting on test functions on $\boldsymbol{R}^d$, there are various formulations of Markov processes associated with L. One is a weak solution of the stochastic differential equation of jump type with coefficients corresponding to the local characteristics of the operator L. Another is a Markov process whose resolvent $\{R_\lambda\}$ satisfies that $R_\lambda(\lambda-L)f=f$ for any test function f. These formulations are unified as the martingale problem for the operator L. Each probability measure P on the path space is said to solve the martingale problem for L if the process

$$f(X_t)-f(X_0)-\int_0^t Lf(X_s)ds$$

is a P-martingale for any test function f on $\boldsymbol{R}^d$. The martingale problem was introduced by Stroock and Varadhan [**15**] to prove the uniqueness of the diffusion process whose generator is a given elliptic differential operator with continuous coefficients. In the present paper we shall discuss the existence and the uniqueness of solutions of the martingale problem for a class of *non-degenerate* Lévy type generators L whose local characteristics are not always continuous, to prove the existence and the uniqueness of Markov processes with jumps having L as their generators. Grigelionis [**6**] and [**7**] gave another martingale formulation for jump type processes.

We shall say that a Lévy type generator L is non-degenerate if it is so as a pseudo-differential operator, i.e. there is a constant α, $0<\alpha\leqq 2$, and

$$e^{-ix\cdot\xi}L(e^{ix'\cdot\xi})=\psi^{(\alpha)}(x, \xi)+\phi^{(\alpha)}(x, \xi),$$

where $\psi^{(\alpha)}(x, \xi)$ is a homogeneous function in ξ with index α such that the real part of $\psi^{(\alpha)}(x, \xi)$ is strictly negative for $\xi\neq 0$, and $\phi^{(\alpha)}(x, \xi)=o(|\xi|^\alpha)$ for large $|\xi|$. In the case $\alpha=2$, the existence and the uniqueness were discussed by Komatsu [**9**] and Stroock [**14**]. So far, for $\alpha\neq 2$, they have been investigated only in the context that the real part of the principal part $\psi^{(\alpha)}(x, \xi)$ of the symbol of L is independent of the variable x. Tsuchiya [**17**] investigated the uniqueness of solution in the case where $\alpha=1$ and $\psi^{(\alpha)}(x, \xi)=-|\xi|+ib(x)\cdot\xi$. And Tsuchiya [18] studied the uniqueness in the case where $1<\alpha<2$ and $\psi^{(\alpha)}(x, \xi)=-|\xi|^{\alpha}$. Moreover Komatsu [10] discussed the uniqueness in the case where $0<\alpha<2$ and $\psi^{(\alpha)}(x, \xi)$ is not always isotropic but independent of the variable x. In this paper we shall study the general case where $\psi^{(\alpha)}(x, \xi)$ depends on x.

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1. Main theorems.

Let $\psi^{(\alpha)}(x, \xi)$ be a function on $\boldsymbol{R}^d\times\boldsymbol{R}^d$ such that, for any fixed $z\in\boldsymbol{R}^d$, the function $-\psi^{(\alpha)}(z, \xi)$ is the exponent of a stable process with index α, $0<\alpha\leqq 2$. The generator $A_z^{(\alpha)}$ of the stable process is given by

$$A_z^{(\alpha)}f(x)=\mathcal{F}^{-1}[\psi^{(\alpha)}(z, \cdot)\mathcal{F}f(\cdot)](x), \tag{1.1}$$

where $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^{-1}$, the inverse transform:

$$\mathcal{F}f(\xi)=\int e^{-ix\cdot\xi}f(x)dx, \qquad \mathcal{F}^{-1}\phi(x)=(2\pi)^{-d}\int e^{ix\cdot\xi}\phi(\xi)d\xi.$$

The function $\psi^{(\alpha)}(z, \xi)$ has the following expression (cf. Lévy [12]).

$$\begin{cases} \psi^{(\alpha)}(z, \xi)=-\int_{|\omega|=1}|\omega\cdot\xi|^{\alpha}(1-i\tan(\alpha\pi/2)\operatorname{sgn}(\omega\cdot\xi))M_0^{(\alpha)}(z, d\omega) \\ \hfill (0<\alpha<2,\ \alpha\neq 1), \\ \psi^{(1)}(z, \xi)=-\int_{|\omega|=1}\Big(|\omega\cdot\xi|+\dfrac{2}{\pi}i(\omega\cdot\xi)\log|\omega\cdot\xi|\Big)M_0^{(1)}(z, d\omega)+ib(z)\cdot\xi, \\ \psi^{(2)}(z, \xi)=-\dfrac{1}{2}\xi\cdot a(z)\xi, \end{cases} \tag{1.2}$$

where $M_0^{(\alpha)}(z, d\omega)$ is a finite measure on $\boldsymbol{S}^{d-1}=\{\omega\in\boldsymbol{R}^d\ ;\ |\omega|=1\}$, $b(z)\in\boldsymbol{R}^d$ and $a(z)=(a_{ij}(z))$, a non-negative definite matrix. It is assumed that

$$\int_{|\omega|=1}\omega M_0^{(1)}(z, d\omega)=0 \qquad (\alpha=1).$$

Define

$$M^{(\alpha)}(z, dy)=\frac{\alpha 2^{\alpha}\Gamma((1+\alpha)/2)}{\sqrt{\pi}\,\Gamma((2-\alpha)/2)}M_0^{(\alpha)}(z, d\omega)r^{-1-\alpha}dr \qquad (y=|y|\omega=r\omega).$$

Then the operator $A_z^{(\alpha)}$ is expressed as follows:

$$
(1.3)\quad \begin{cases}
A_z^{(\alpha)}f(x)=\int(f(x+y)-f(x))M^{(\alpha)}(z, dy) & (0<\alpha<1), \\
A_z^{(1)}f(x)=\int(f(x+y)-f(x)-\Theta_1[y]\cdot\partial f(x))M^{(1)}(z, dy)+b(z)\cdot\partial f(x), & \\
A_z^{(\alpha)}f(x)=\int(f(x+y)-f(x)-y\cdot\partial f(x))M^{(\alpha)}(z, dy) & (1<\alpha<2), \\
A_z^{(2)}f(x)=\frac{1}{2}\sum_{i,j}a_{ij}(z)\partial_i\partial_j f(x) &
\end{cases}
$$

where $\partial_j=\partial/\partial x_j$, $\partial=(\partial_1, \cdots, \partial_d)$ and $\Theta_1[y]=I_{(|y|\leq 1)}y$. For $\nu=(\nu_1, \cdots, \nu_d)$, $\nu_j\in \boldsymbol{Z}_+$, set $|\nu|=\nu_1+\cdots+\nu_d$ and $\partial_x^\nu=(\partial/\partial x_1)^{\nu_1}\cdots(\partial/\partial x_d)^{\nu_d}$. Throughout this paper the next assumption is maintained.

ASSUMPTION [A_1]. *In case $0<\alpha<2$, the measure $M^{(\alpha)}(z, dy)$ has the density function $m^{(\alpha)}(z, y)$ which is not identically zero with respect to the Lebesgue measure dy, and partial derivatives $\partial_y^\nu m^{(\alpha)}(z, y)$ $(|\nu|\leqq d)$ are bounded measurable on $\boldsymbol{R}^d\times\boldsymbol{S}^{d-1}$. In case $\alpha=2$, $a(z)$ is bounded measurable on $\boldsymbol{R}^d$ and positive definite.*

Let $\sigma(d\omega)$ denote the area element of the surface $\boldsymbol{S}^{d-1}$. Then

$$
\begin{aligned}
m^{(\alpha)}(z, r\omega)\sigma(d\omega)r^{d-1}dr&=m^{(\alpha)}(z, y)dy=M^{(\alpha)}(z, dy) \\
&=\text{const.}\, M_0^{(\alpha)}(z, d\omega)r^{-1-\alpha}dr \qquad \text{with} \quad y=|y|\omega=r\omega .
\end{aligned}
$$

This implies that $r^{d+\alpha}m^{(\alpha)}(z, r\omega)$ is independent of $r>0$, so the function $m^{(\alpha)}(z, y)$ is homogeneous in y with index $-d-\alpha$.

Next we shall introduce the operator $B^{(\alpha)}$:

$$
(1.4)\quad \begin{cases}
B^{(\alpha)}f(x)=\int(f(x+y)-f(x))N^{(\alpha)}(x, dy) & (0<\alpha\leqq 1), \\
B^{(\alpha)}f(x)=\int(f(x+y)-f(x)-\Theta_1[y]\cdot\partial f(x))N^{(\alpha)}(x, dy)+b^{(\alpha)}(x)\cdot\partial f(x) & \\
 & (1<\alpha\leqq 2).
\end{cases}
$$

We shall be concerned with this operator under the following assumption.

ASSUMPTION [A_2]. *$N^{(\alpha)}(x, dy)$ is a signed kernel such that $|N^{(\alpha)}(x, dy)|$ is bounded by some measure $N_*^{(\alpha)}(dy)$, independent of x, satisfying*

$$
(1.5)\qquad \int|y|^\alpha\wedge 1\, N_*^{(\alpha)}(dy)<\infty .
$$

In case $1<\alpha\leqq 2$, the vector $b^{(\alpha)}(x)$ is bounded measurable on $\boldsymbol{R}^d$. Moreover $M^{(\alpha)}(x, dy)+N^{(\alpha)}(x, dy)\geqq 0$ in case $\alpha\neq 2$, however $N^{(2)}(x, dy)\geqq 0$ in case $\alpha=2$.

From assumption [A_2] the symbol $\phi^{(\alpha)}(x, \xi)=e^{-ix\cdot\xi}B^{(\alpha)}(e^{ix'\cdot\xi})$ of the pseudo-differential operator $B^{(\alpha)}$ satisfies that $\phi^{(\alpha)}(x, \xi)=o(|\xi|^\alpha)$ as $|\xi|\to\infty$. On the other hand $\psi^{(\alpha)}(x, \xi)$ is a homogeneous function in ξ with index α. Define pseudo-differential operators $A^{(\alpha)}$ and L by

$$A^{(\alpha)}f(x)=A_x^{(\alpha)}f(x)=\psi^{(\alpha)}(x, \partial_x)f(x)=\mathcal{F}^{-1}[\psi^{(\alpha)}(x, \cdot)\mathcal{F}f](x), \tag{1.6}$$

$$L=A^{(\alpha)}+B^{(\alpha)}. \tag{1.7}$$

Let $W=D(\boldsymbol{R}_+\to\boldsymbol{R}^d)$: the space of right continuous functions having left hand limits, $X_t(w)=w(t)$ for $w\in W$, $\mathcal{W}_t=\bigcap_{\varepsilon>0}\sigma\{X_s\ ;\ s\leqq t+\varepsilon\}$ and $\mathcal{W}=\bigvee_{t>0}\mathcal{W}_t$. Let $\mathcal{D}$ be the space of test functions, i.e. smooth functions on $\boldsymbol{R}^d$ with compact supports. We shall say that a probability measure P_x on the space $(W, \mathcal{W})$ solves the martingale problem for L starting from x if the process

$$M_t^f=f(X_t)-f(X_0)-\int_0^t Lf(X_s)ds$$

is a martingale with respect to $(\mathcal{W}_t, P_x)$ such that $M_0^f=0$ for any $f\in\mathcal{D}$. Here we shall introduce a condition.

CONDITION $[C_1]$. *In case* $0<\alpha<2$, $\partial_y^\nu m^{(\alpha)}(x, y)$ $(|\nu|\leqq d)$ *are continuous on* $\boldsymbol{R}^d\times\boldsymbol{S}^{d-1}$, *moreover* $b(x)$ *is continuous in case* $\alpha=1$. *In case* $\alpha=2$, $a(x)$ *is continuous.*

One of the main results of this paper is the following existence theorem.

THEOREM 1. *Assume* $[A_1]$, $[A_2]$ *and* $[C_1]$. *Given any* x, *there exists a probability measure* P_x *solving the martingale problem associated with the operator* L *starting from* x.

Let us introduce a technical assumption for the uniqueness of solution. The assumption is not necessary in the case where $A_z^{(\alpha)}$ is independent of z (cf. [**10**]).

ASSUMPTION $[A_3]$. *In case* $0<\alpha<2$, *the function* $m^{(\alpha)}(x, y)$ *is strictly positive. Moreover, in case* $0<\alpha<1$, *the signed measure* $N^{(\alpha)}(x, dy)$ *has the density function* $n^{(\alpha)}(x, y)$ *with respect to the Lebesgue measure* dy *such that the measure*

$$N_\mu^{(\alpha)}(dy)=(\sup_x \sup_{\mu^{-1}\leqq\theta\leqq\mu} |n^{(\alpha)}(x, \theta y)|\theta^d)dy$$

satisfies

$$\int|y|^\alpha\wedge 1\, N_\mu^{(\alpha)}(dy)<\infty \qquad \textit{for any} \quad \mu\geqq 1. \tag{1.8}$$

Another main result of this paper is the following uniqueness theorem.

THEOREM 2. *Assume* $[A_1]$, $[A_2]$, $[A_3]$ *and* $[C_1]$. *Then there is at most one probability measure* P_x *on the space* $(W, \mathcal{W})$ *solving the martingale problem for the operator* L *for each starting point* x.

In case $\alpha=2$, the existence and the uniqueness of solutions hold under $[A_1]$, $[A_2]$ and $[C_1]$. These have already been proved in Komatsu [**9**] and Stroock [**14**]. The continuity condition $[C_1]$ can be relaxed to some extent.

2. Singular integrals.

Let $1<p<\infty$ and $\|\cdot\|_{L^p}$ denote the L^p-norm and $\|\cdot\|$, the norm of supremum. In our consideration the L^p-boundedness of singular integral operators plays an essential role. The following theorem is a generalization of a theorem in Hörmander [8].

THEOREM 2.1. *There is a constant C_p such that*

$$\|\sup_z|\mathcal{F}^{-1}[\phi_z\mathcal{F}f]|\|_{L^p}\leqq C_p(\sup_z\sup_{|\xi|=1}\sum_{|\nu|\leqq d}|\partial_\xi^\nu\phi_z(\xi)|)\|f\|_{L^p} \tag{2.1}$$

for any system $\{\phi_z(\xi)\}$ of homogeneous functions with index 0 and $f\in\mathcal{D}$.

PROOF. Let $\mathcal{F}^{-1}\phi_z$ denote the inverse Fourier transform of ϕ_z in the distribution sense and $\mu(\phi_z)$, the average of ϕ_z over S^{d-1}:

$$\mu(\phi_z)=\int_{|\omega|=1}\phi_z(\omega)\sigma(d\omega)\Big/\int_{|\omega|=1}\sigma(d\omega).$$

Suppose that

$$\sup_z\sup_{|\xi|=1}\sum_{|\nu|\leqq d}|\partial_\xi^\nu\phi_z(\xi)|<\infty.$$

In exactly the same way as the proof of Lemma 1.2 in [10] we can prove that the generalized function $h_z(x)=\mathcal{F}^{-1}\phi_z(x)-\mu(\phi_z)\delta(x)$ is a homogeneous function with index $-d$ and satisfies

$$\int_{|\omega|=1}h_z(\omega)\sigma(d\omega)=0, \qquad \sup_{|x|=1}|h_z(x)|\leqq c_1\sup_{|\xi|=1}\sum_{|\nu|\leqq d}|\partial_\xi^\nu\phi_z(\xi)|,$$

where c_1 denotes a constant independent of ϕ_z. It can be also proved that

$$\lim_{|y|\to 0}\sup_z\sup_{|x|=1}|h_z(x+y)-h_z(x)|=0. \tag{2.2}$$

Define singular integrals

$$h_z*f(x)=\lim_{\varepsilon\downarrow 0}\int_{|y|>\varepsilon}h_z(y)f(x-y)dy.$$

The Calderón-Zygmund inequality (cf. [2] and [3]) can be generalized to the following. As long as (2.2) is satisfied we have

$$\|\sup_z|h_z*f|\|_{L^p}\leqq c_2(\sup_z\sup_{|x|=1}|h_z(x)|)\|f\|_{L^p}, \tag{2.3}$$

where c_2 is a certain constant independent of $\{h_z\}$ and f. This inequality can be proved in a similar way to Dunford and Schwartz [4], XI-7, so the proof is omitted. Since $\mathcal{F}^{-1}[\phi_z\mathcal{F}f]=h_z*f+\mu(\phi_z)f$, we have inequality (2.1). q.e.d.

For $0<\beta<1$, let $H_\beta(f)$ denote the semi-norm

$$H_\beta(f)=\sup_{x,x'}|x-x'|^{-\beta}|f(x)-f(x')|.$$

The following is a modification of the Hölder-Kohn-Lichtenstein-Giraud inequality (cf. [1]).

LEMMA 2.1. *There is a constant* C_β *such that*

$$H_\beta(\mathcal{F}^{-1}[\phi\mathcal{F}f])\leqq C_\beta(\sup_{|\xi|=1}\sum_{|\nu|\leqq d+1}|\partial_\xi^\nu\phi(\xi)|)H_\beta(f) \tag{2.4}$$

for all $f\in\mathcal{D}$ *and homogeneous functions* ϕ *with index* 0.

PROOF. Let $\mu(\phi)$ be the average of the function ϕ over S^{d-1}. The homogeneous function $h(x)=\mathcal{F}^{-1}\phi(x)-\mu(\phi)\delta(x)$ with index $-d$ satisfies, for any $x\neq 0$ and $\nu=(\nu_1, \cdots, \nu_d)$,

$$\partial_x^\nu h(x)=\lim_{\varepsilon\downarrow 0}(2\pi)^{-d}\int_{|\omega|=1}\phi(\omega)\omega^\nu(-i)^d(d+|\nu|-1)!\ (\omega\cdot x+i\varepsilon)^{-d-|\nu|}\sigma(d\omega),$$

where $\omega^\nu=\omega_1^{\nu_1}\cdots\omega_d^{\nu_d}$. In a way similar to the proof of Lemma 1.2 in [10] we have

$$\sup_{|x|=1}(|h(x)|+|\partial h(x)|)\leqq c_1\sup_{|\xi|=1}\sum_{|\nu|\leqq d+1}|\partial_\xi^\nu\phi(\xi)|,$$

$$\int_{|\omega|=1}h(\omega)\sigma(d\omega)=0.$$

From the Hölder-Kohn-Lichtenstein-Giraud inequality, there is a constant c_2 such that

$$H_\beta(h*f)\leqq c_2\sup_{|x|=1}(|h(x)|+|\partial h(x)|)H_\beta(f).$$

Since $H_\beta(\mathcal{F}^{-1}[\phi\mathcal{F}f])\leqq H_\beta(h*f)+|\mu(\phi)|H_\beta(f)$, we have (2.4). q.e.d.

Now fix a point x_0 in $\boldsymbol{R}^d$. The λ-potential operator $G_\lambda^{(\alpha)}=G_{x_0\lambda}^{(\alpha)}$ of the stable process with the generator $A_{x_0}^{(\alpha)}$ is given by

$$\begin{aligned} G_\lambda^{(\alpha)}f(x)&=\mathcal{F}^{-1}[(\lambda-\psi^{(\alpha)}(x_0,\cdot))^{-1}\mathcal{F}f](x) \\ &=\int_0^\infty e^{-\lambda t}\mathcal{F}^{-1}[e^{t\psi^{(\alpha)}(x_0,\cdot)}\mathcal{F}f](x)dt. \end{aligned} \tag{2.5}$$

Using the Young inequality we have, for all $f\in\mathcal{D}$,

$$\|G_\lambda^{(\alpha)}f\|_{L^p}\leqq\lambda^{-1}\|f\|_{L^p},\quad \|G_\lambda^{(\alpha)}f\|\leqq\lambda^{-1}\|f\|,\quad H_\beta(G_\lambda^{(\alpha)}f)\leqq\lambda^{-1}H_\beta(f). \tag{2.6}$$

For $0<\delta<1$, let $|\partial|^\delta$ denote the pseudo-differential operator defined by

$$|\partial|^\delta f(x)=\mathcal{F}^{-1}[|\xi|^\delta\mathcal{F}f(\xi)](x)=c_\delta\int(f(x+y)-f(x))|y|^{-d-\delta}dy, \tag{2.7}$$

where $c_\delta=\sqrt{\pi}^{-\delta}\Gamma((d+\delta)/2)\Gamma(-\delta/2)^{-1}$. There are constants c_1 and c_2 for which

$$\int||y+z|^{\delta-d}-|y|^{\delta-d}|dy=c_1|z|^\delta, \tag{2.8}$$

$$f(x+z)\quad f(x)=c_2\int(|y+z|^{\delta-d}-|y|^{\delta-d})|\partial|^{\delta}f(x-y)dy \tag{2.9}$$

for all $f\in\mathcal{D}$ (cf. Lemma 2.1 in [**10**]).

Let $k_0^{(2)}(x_0)$ be the minimum eigenvalue of $a(x_0)$ and $k_1^{(2)}(x_0)$, the maximum eigenvalue of it. For $0<\alpha<2$, set

$$k_0^{(\alpha)}(x_0)=\inf_{|\xi|=1}\int_{|\omega|=1}|\xi\cdot\omega|^{\alpha}m^{(\alpha)}(x_0,\omega)\sigma(d\omega),$$

$$k_1^{(\alpha)}(x_0)=\sup_{|y|=1}\sum_{|\nu|\leqq d}|\partial_y^{\nu}m^{(\alpha)}(x_0,y)|+I_{(\alpha=1)}|b(x_0)|,$$

where $I_{(\alpha=1)}=1$ if $\alpha=1$ and $I_{(\alpha=1)}=0$ if $\alpha\neq 1$. Assumption [A_1] implies that $k_0^{(\alpha)}(x_0)>0$ and $k_1^{(\alpha)}(x_0)<\infty$. From (1.2) we see that

$$\inf_{|\xi|=1}(-\mathrm{Re}\,\psi^{(\alpha)}(x_0,\xi))\geqq\text{const.}\,k_0^{(\alpha)}(x_0)>0.$$

It can be proved in the same way as Lemma 1.1 in [**10**] that

$$\sup_{|\xi|=1}\sum_{|\nu|\leqq d+1}|\partial_\xi^{\nu}\psi^{(\alpha)}(x_0,\xi)|\leqq\text{const.}\,k_1^{(\alpha)}(x_0)<\infty.$$

Hereafter, let $k(x_0)$'s denote positive constants which continuously depend on the values $1/k_0^{(\alpha)}(x_0)$ and $k_1^{(\alpha)}(x_0)$.

LEMMA 2.2. *There is a constant $k(x_0)$ such that, for $f\in\mathcal{D}$,*

$$\begin{cases} \||\partial|^{\alpha}G_\lambda^{(\alpha)}f\|_{L^p}\leqq k(x_0)\|f\|_{L^p} & (0<\alpha<1), \\ \||\partial|^{\alpha-1}\partial_jG_\lambda^{(\alpha)}f\|_{L^p}\leqq k(x_0)\|f\|_{L^p} & (1\leqq\alpha<2), \\ \|\partial_i\partial_jG_\lambda^{(2)}f\|_{L^p}\leqq k(x_0)\|f\|_{L^p}. \end{cases} \tag{2.10}$$

For $0<\beta<1$, there is a constant $k'(x_0)$ such that, for $f\in\mathcal{D}$,

$$\begin{cases} H_\beta(|\partial|^{\alpha}G_\lambda^{(\alpha)}f)\leqq k'(x_0)H_\beta(f) & (0<\alpha<1), \\ H_\beta(|\partial|^{\alpha-1}\partial_jG_\lambda^{(\alpha)}f)\leqq k'(x_0)H_\beta(f) & (1\leqq\alpha<2), \\ H_\beta(\partial_i\partial_jG_\lambda^{(2)}f)\leqq k'(x_0)H_\beta(f). \end{cases} \tag{2.11}$$

PROOF. Suppose that $0<\alpha<1$. Then

$$\begin{aligned}|\partial|^{\alpha}G_\lambda^{(\alpha)}f&=\mathcal{F}^{-1}[|\xi|^{\alpha}(\lambda-\psi^{(\alpha)}(x_0,\xi))^{-1}\mathcal{F}f]\\&=\mathcal{F}^{-1}[(|\xi|^{\alpha}/\psi^{(\alpha)}(x_0,\xi))\mathcal{F}[\lambda G_\lambda^{(\alpha)}f-f]].\end{aligned}$$

Therefore (2.10) and (2.11) follow immediately from Theorem 2.1, Lemma 2.1 and (2.6). Similarly (2.10) and (2.11) are proved in case $1\leqq\alpha\leqq 2$, because

$$|\partial|^{\alpha-1}\partial_jG_\lambda^{(\alpha)}f=\mathcal{F}^{-1}[(i|\xi|^{\alpha-1}\xi_j/\psi^{(\alpha)}(x_0,\xi))\mathcal{F}[\lambda G_\lambda^{(\alpha)}f-f]]$$

for $1\leqq\alpha<2$ and

$$\partial_i\partial_j G_\lambda^{(2)} f = \mathcal{F}^{-1}[(-2\xi_i\xi_j/\xi\cdot a(x_0)\xi)\mathcal{F}[\lambda G_\lambda^{(\alpha)} f - f]]. \qquad \text{q.e.d.}$$

LEMMA 2.3. *Let γ be a positive constant and $0<\delta<\gamma\wedge 1$. There exists a constant $k(x_0)$ such that*

$$(2.12) \qquad (1+|x|^{d+\delta})|\mathcal{F}^{-1}[\phi e^{\psi^{(\alpha)}(x_0,\cdot)}](x)| \leqq k(x_0)(\sup_{|\xi|=1}\sum_{|\nu|\leqq d+1}|\partial_\xi^\nu \phi(\xi)|)$$

for any homogeneous function $\phi(\xi)$ with index γ.

PROOF. Set $\psi(\xi)=\psi^{(\alpha)}(x_0, \xi)$. For each ν with $|\nu|=d$, we have $\partial_\xi^\nu[\phi e^\psi]= e^\psi(\partial+\partial\psi)^\nu\phi$. Hence there are homogeneous functions $\phi_n(\xi)$ of index $n\alpha+\gamma-d$ such that

$$\sup_{n,|\xi|=1}(|\phi_n|+|\partial\phi_n|) \leqq k_1(x_0)(\sup_{|\xi|=1}\sum_{|\nu|\leqq d+1}|\partial_\xi^\nu\phi|), \qquad \partial_\xi^\nu[\phi e^\psi]=\sum_{n=0}^{d}\phi_n e^\psi .$$

It suffices to prove that

$$|x|^\delta|\mathcal{F}^{-1}[\phi_n e^\psi](x)| \leqq k_2(x_0)\sup_{|\xi|=1}(|\phi_n|+|\partial\phi_n|).$$

In case $n\alpha+\gamma>1$ the above inequality holds, because

$$|x_j\mathcal{F}^{-1}[\phi_n e^\psi](x)| \leqq (2\pi)^{-d}\int|\partial_j(\phi_n e^\psi)|d\xi \leqq k_3(x_0)\sup_{|\xi|=1}(|\phi_n|+|\partial\phi_n|).$$

Suppose that $n\alpha+\gamma\leqq 1$. Set $|\xi|=r$ and $|\xi|^{-1}\xi=\omega$. Then

$$||\partial|^\delta(\phi_n e^\psi)(\xi)| \leqq c_1 r^{n\alpha+\gamma-d-\delta}\int|\phi_n(\omega-y)e^{r^\alpha\psi(\omega-y)}-\phi_n(\omega)e^{r^\alpha\psi(\omega)}||y|^{-d-\delta}dy .$$

Set $k=2^{-\alpha}\inf\{-\mathrm{Re}\,\psi(\xi)\ ;\ |\xi|=1\}$. We have

$$\begin{aligned}
&\int_{|y|\leqq 1/2}|\phi_n(\omega-y)e^{r^\alpha\psi(\omega-y)}-\phi_n(\omega)e^{r^\alpha\psi(\omega)}||y|^{-d-\delta}dy \\
&\quad\leqq e^{-kr^\alpha}\int_{|y|\leqq 1/2}|\phi_n(\omega-y)-\phi_n(\omega)||y|^{-d-\delta}dy \\
&\qquad + |\phi_n(\omega)|r^\alpha e^{-kr^\alpha}\int_{|y|\leqq 1/2}|\psi(\omega-y)-\psi(\omega)||y|^{-d-\delta}dy \\
&\quad\leqq k_4(x_0)e^{-kr^\alpha}(1+r^\alpha)\sup_{|\xi|=1}(|\phi_n|+|\partial\phi_n|),
\end{aligned}$$

$$\int_{|y|>1/2}|\phi_n(\omega)e^{r^\alpha\psi(\omega)}||y|^{-d-\delta}dy \leqq k_5(x_0)\sup_{|\xi|=1}|\phi_n| .$$

Moreover we have

$$\begin{aligned}
&\int_{|\omega-y|\leqq 1/2}|\phi_n(\omega-y)e^{r^\alpha\psi(\omega-y)}||y|^{-d-\delta}dy \\
&\quad\leqq c_2\int_{|z|\leqq 1/2}|\phi_n(z)e^{r^\alpha\psi(z)}|dz \leqq k_6(x_0)(1\wedge r^{-n\alpha-\gamma})\sup_{|\xi|=1}|\phi_n| ,
\end{aligned}$$

$$\int_{|y|>1/2,\,|\omega-y|>1/2} |\phi_n(\omega-y)e^{r^\alpha\phi(\omega-y)}|\,|y|^{-d-\delta}dy$$

$$\leqq c_3\int_{|z|>1/2}|\phi_n(z)|e^{-k(2r|z|)^\alpha}|z|^{-d-\delta}dz\leqq k_7(x_0)(1\wedge e^{-kr^\alpha})\sup_{|\xi|=1}|\phi_n|.$$

From these inequalities we see that

$$|\,|\partial|^\delta(\phi_n e^\phi)(\xi)|\leqq k_8(x_0)r^{-d}(r^{\gamma-\delta}\wedge r^{-\delta})\sup_{|\xi|=1}(|\phi_n|+|\partial\phi_n|),$$

for $r^{n\alpha+\gamma-\delta}(1\wedge r^{-n\alpha-\gamma})\leqq r^{\gamma-\delta}\wedge r^{-\delta}$. Hence we have

$$\begin{aligned}&|x|^\delta|\mathcal{F}^{-1}[\phi_n e^\phi](x)|\\ &\leqq(2\pi)^{-d}k_8(x_0)\int|\xi|^{-d}(|\xi|^{\gamma-\delta}\wedge|\xi|^{-\delta})d\xi\sup_{|\xi|=1}(|\phi_n|+|\partial\phi_n|)\\ &\leqq k_2(x_0)\sup_{|\xi|=1}(|\phi_n|+|\partial\phi_n|).\end{aligned}$$

q.e.d.

LEMMA 2.4.

(i) *If $\alpha p>d$, then there is a constant $k(x_0)$ such that*

$$\|G_\lambda^{(\alpha)}f\|\leqq k(x_0)\lambda^{-1+d/\alpha p}\|f\|_{L^p}\qquad \textit{for}\quad f\in\mathcal{D}.$$

(ii) *If $(\alpha-1)p>d$, then there is a constant $k(x_0)$ such that*

$$\|\partial_j G_\lambda^{(\alpha)}f\|\leqq k(x_0)\lambda^{-1+1/\alpha+d/\alpha p}\|f\|_{L^p}\quad \textit{for}\quad f\in\mathcal{D}.$$

(iii) *If $0<\beta<\alpha\wedge 1$ and $(\alpha-\beta)p>d$, then there is a constant $k(x_0)$ such that*

$$H_\beta(G_\lambda^{(\alpha)}f)\leqq k(x_0)\lambda^{-1+\beta/\alpha+d/\alpha p}\|f\|_{L^p}\quad \textit{for}\quad f\in\mathcal{D}.$$

PROOF. It is easy to prove that

$$(1+|x|^d)|\mathcal{F}^{-1}[e^{\phi^{(\alpha)}(x_0,\xi)}](x)|\leqq k_1(x_0),$$

$$(1+|x|^{d+1})|\mathcal{F}^{-1}[\xi_j e^{\phi^{(\alpha)}(x_0,\xi)}](x)|\leqq k_2(x_0).$$

From Lemma 2.3 we see that

$$(1+|x|^{d+\beta/2})|\mathcal{F}^{-1}[|\xi|^\beta e^{\phi^{(\alpha)}(x_0,\xi)}](x)|\leqq k_3(x_0).$$

Define

$$g_\lambda^{(\alpha)}(x)=\int_0^\infty e^{-\lambda t}\mathcal{F}^{-1}[e^{t\phi^{(\alpha)}(x_0,\xi)}](x)dt.$$

Let $p^{-1}+q^{-1}=1$. As long as $\alpha p>d$, we have

$$\int|\lambda g_\lambda^{(\alpha)}(x)|^q dx=\int\left|\int_0^\infty\lambda e^{-\lambda t}t^{-d/\alpha p}(t^{-d/\alpha q}\mathcal{F}^{-1}[e^{\phi^{(\alpha)}(x_0,\cdot)}](t^{-1/\alpha}x))dt\right|^q dx$$

$$\leqq\left(\int_0^\infty\lambda e^{-\lambda t}t^{-d/\alpha p}dt\right)^q\int|\mathcal{F}^{-1}[e^{\phi^{(\alpha)}(x_0,\cdot)}](x)|^q dx.$$

This implies that

$$\|g_\lambda^{(\alpha)}\|_{L^q} \leqq k_1'(x_0)\lambda^{-1+d/\alpha p} \qquad \text{as long as} \quad \alpha p > d\,.$$

Similarly we have

$$\|\partial_j g_\lambda^{(\alpha)}\|_{L^q} \leqq k_2'(x_0)\lambda^{-1+1/\alpha+d/\alpha p} \qquad \text{as long as} \quad (\alpha-1)p > d\,,$$

$$\||\partial|^\beta g_\lambda^{(\alpha)}\|_{L^q} \leqq k_3'(x_0)\lambda^{-1+\beta/\alpha+d/\alpha p} \qquad \text{as long as} \quad (\alpha-\beta)p > d\,.$$

Since $G_\lambda^{(\alpha)}f = g_\lambda^{(\alpha)} * f$ and $\partial_j G_\lambda^{(\alpha)} f = (\partial_j g_\lambda^{(\alpha)}) * f$, (i) and (ii) of the present lemma follow immediately from the Hölder inequality. Using (2.8) and (2.9),

$$H_\beta(G_\lambda^{(\alpha)}f) \leqq \text{const.}\||\partial|^\beta G_\lambda^{(\alpha)} f\| \leqq \text{const.}\||\partial|^\beta g_\lambda^{(\alpha)}\|_{L^q}\|f\|_{L^p}\,,$$

which proves (iii) of the lemma. q.e.d.

In a similar way to the above proof we have

$$\|\partial_j g_\lambda^{(\alpha)}\|_{L^1} \leqq k(x_0)\lambda^{-1+1/\alpha} \qquad \text{as long as} \quad \alpha > 1\,.$$

From the Young inequality we see that, for $\alpha > 1$,

$$\left\{\begin{aligned} &\|\partial_j G_\lambda^{(\alpha)} f\|_{L^p} \leqq k(x_0)\lambda^{-1+1/\alpha}\|f\|_{L^p}\,, \\ &\|\partial_j G_\lambda^{(\alpha)} f\| \leqq k(x_0)\lambda^{-1+1/\alpha}\|f\|\,, \\ &H_\beta(\partial_j G_\lambda^{(\alpha)} f) \leqq k(x_0)\lambda^{-1+1/\alpha} H_\beta(f)\,. \end{aligned}\right. \tag{2.13}$$

LEMMA 2.5. *Let* $0 < \beta < \alpha \wedge 1$.
(i) *There is a constant* $k(x_0)$ *such that*

$$\|\mathcal{F}^{-1}[\phi\mathcal{F}[G_\lambda^{(\alpha)}f]]\| \leqq k(x_0)\lambda^{-\beta/\alpha}H_\beta(f)\Big(\sup_{|\xi|=1}\sum_{|\nu|\leqq d+1}|\partial_\xi^\nu\phi(\xi)|\Big) \tag{2.14}$$

for any $f \in \mathcal{D}$ *and homogeneous function* $\phi(\xi)$ *with index* α.
(ii) *There is a constant* $k'(x_0)$ *such that, for* $f \in \mathcal{D}$,

$$\left\{\begin{aligned} &\||\partial|^\alpha G_\lambda^{(\alpha)} f\| \leqq k'(x_0)\lambda^{-\beta/\alpha}H_\beta(f) && (0<\alpha<1)\,, \\ &\||\partial|^{\alpha-1}\partial_j G_\lambda^{(\alpha)} f\| \leqq k'(x_0)\lambda^{-\beta/\alpha}H_\beta(f) && (1\leqq\alpha<2)\,, \\ &\|\partial_i\partial_j G_\lambda^{(2)} f\| \leqq k'(x_0)\lambda^{-\beta/2}H_\beta(f)\,. \end{aligned}\right. \tag{2.15}$$

PROOF. Set $\psi = \psi^{(\alpha)}(x_0, \xi)$. Note that

$$\int \mathcal{F}^{-1}[\phi e^{t\psi}](y)dy = (2\pi)^d\langle \mathcal{F}^{-1}[\phi e^{t\psi}],\ \mathcal{F}^{-1}[\delta]\rangle = \langle \phi e^{t\psi},\ \delta\rangle = 0$$

for any homogeneous function ϕ with index α. Therefore, for $f \in \mathcal{D}$,

$$|\mathcal{F}^{-1}[\phi\mathcal{F}[G_\lambda^{(\alpha)}f]](x)| = \left|\int_0^\infty\!\!\int e^{-\lambda t}\mathcal{F}^{-1}[\phi e^{t\psi}](y)f(x-y)dt\,dy\right|$$

$$= \left|\int_0^\infty\!\!\int e^{-\lambda t}\mathcal{F}^{-1}[\phi e^{t\psi}](y)(f(x-y)-f(x))dt\,dy\right|$$

$$\leqq H_\beta(f)\int_0^\infty\int e^{-\lambda t}|y|^\beta|\mathcal{F}^{-1}[\phi e^{t\psi}](y)|\,dt\,dy$$

$$=H_\beta(f)\Gamma(\beta/\alpha)\lambda^{-\beta/\alpha}\int|y|^\beta|\mathcal{F}^{-1}[\phi e^{\psi}](y)|\,dy$$

$$\leqq H_\beta(f)\lambda^{-\beta/\alpha}k(x_0)(\sup_{|\xi|=1}\sum_{|\nu|\leqq d+1}|\partial_\xi^\nu\phi|) \qquad \text{(by Lemma 2.3).}$$

This proves (i). Applying inequality (2.14) for $\phi=|\xi|^\alpha$, $\phi=|\xi|^{\alpha-1}\xi_j$ or $\phi=\xi_i\xi_j$, we have (2.15). q.e.d.

Now define

$$(2.16)\qquad \begin{cases} \Delta^{(\alpha)}(x, z)=\sup_{|y|=1}\sum_{|\nu|\leqq d}|\partial_y^\nu(m^{(\alpha)}(x, y)-m^{(\alpha)}(z, y))| \\ \qquad +I_{(\alpha=1)}|b(x)-b(z)| \qquad (0<\alpha<2), \\ \Delta^{(2)}(x, z)=\sum_{i,j}|a_{ij}(x)-a_{ij}(z)|. \end{cases}$$

In the same way as Lemma 1.1 in [10] it can be proved that

$$(2.17)\qquad \sup_{|\xi|=1}\sum_{|\nu|\leqq d+1}|\partial_\xi^\nu(\psi^{(\alpha)}(x, \xi)-\psi^{(\alpha)}(z, \xi))|\leqq\text{const.}\,\Delta^{(\alpha)}(x, z).$$

THEOREM 2.2.
(i) *There is a constant $k_p(x_0)$ such that, for $f\in\mathcal{D}$ and $\lambda>0$,*

$$\|\sup_z|(A_{x_0}^{(\alpha)}-A_z^{(\alpha)})G_\lambda^{(\alpha)}f|\,\|_{L^p}\leqq k_p(x_0)\sup_z\Delta^{(\alpha)}(x_0, z)\|f\|_{L^p}.$$

(ii) *Let $0<\beta<\alpha\wedge 1$. There is a constant $k_\beta(x_0)$ such that, for $f\in\mathcal{D}$ and $\lambda>0$,*

$$\sup_z\|(A_{x_0}^{(\alpha)}-A_z^{(\alpha)})G_\lambda^{(\alpha)}f\|\lambda^{\beta/\alpha}+\sup_z H_\beta((A_{x_0}^{(\alpha)}-A_z^{(\alpha)})G_\lambda^{(\alpha)}f)$$
$$\leqq k_\beta(x_0)\sup_z\Delta^{(\alpha)}(x_0, z)H_\beta(f).$$

PROOF. Observe that

$$(A_{x_0}^{(\alpha)}-A_z^{(\alpha)})G_\lambda^{(\alpha)}f=\mathcal{F}^{-1}[(\psi^{(\alpha)}(z, \xi)/\psi^{(\alpha)}(x_0, \xi)-1)\mathcal{F}[f-\lambda G_\lambda^{(\alpha)}f]].$$

Hence assertion (i) follows immediately from Theorem 2.1, (2.6) and (2.17). From (2.14) with $\phi(\xi)=\psi^{(\alpha)}(x_0, \xi)-\psi^{(\alpha)}(z, \xi)$, we have

$$\|(A_{x_0}^{(\alpha)}-A_z^{(\alpha)})G_\lambda^{(\alpha)}f\|\lambda^{\beta/\alpha}$$
$$\leqq k_1'(x_0)H_\beta(f-\lambda G_\lambda^{(\alpha)}f)\sup_{|\xi|=1}\sum_{|\nu|\leqq d+1}|\partial_\xi^\nu(\psi^{(\alpha)}(z, \xi)/\psi^{(\alpha)}(x_0, \xi)-1)|$$
$$\leqq k_2'(x_0)H_\beta(f)\Delta^{(\alpha)}(x_0, z) \qquad \text{(by (2.6) and (2.17)).}$$

On the other hand, by Lemma 2.1,

$$H_\beta((A_{x_0}^{(\alpha)}-A_z^{(\alpha)})G_\lambda^{(\alpha)}f)$$

$$\leqq C_\beta H_\beta(f-\lambda G_\lambda^{(\alpha)}f)\sup_{|\xi|=1}\sum_{|\nu|\leqq d+1}|\partial_\xi^\nu(\psi^{(\alpha)}(z,\xi)/\psi^{(\alpha)}(x_0,\xi)-1)|$$

$$\leqq k_3'(x_0)H_\beta(f)\Delta^{(\alpha)}(x_0,z)\,. \qquad \text{q.e.d.}$$

3. Construction of semi-groups; special case.

In this section we shall prove Theorem 1 in the case where the principal part $A^{(\alpha)}$ of L is close to the generator of a stable process. Throughout this section [A_1] and [A_2] are assumed. Fix β and p so that $0<\beta<\alpha\wedge 1$ and $(\alpha-\beta)p>d$. Let $k_p(x_0)$ and $k_\beta(x_0)$ be the constants in Theorem 2.2. We shall construct the Feller semi-group with the pre-generator L under the following condition.

CONDITION [C_2]. $k_p(x_0)\vee k_\beta(x_0)\sup_z \Delta^{(\alpha)}(x_0,z)\leqq\dfrac{1}{4}$.

Define

$$(3.1)\quad \begin{cases} B_*^{(\alpha)}f(x)=\displaystyle\int|f(x+y)-f(x)|N_*^{(\alpha)}(dy) & (0<\alpha\leqq 1)\,, \\ B_*^{(\alpha)}f(x)=\displaystyle\int|f(x+y)-f(x)-\Theta_1[y]\cdot\partial f(x)|N_*^{(\alpha)}(dy)+\||b^{(\alpha)}|\|\cdot|\partial f(x)| & \\ & (1<\alpha\leqq 2)\,. \end{cases}$$

LEMMA 3.1 (Theorem 2 in [10]). *There is a constant $\lambda_p=\lambda_p(x_0)$ such that*

$$\|B_*^{(\alpha)}G_\lambda^{(\alpha)}f\|_{L^p}\leqq\frac{1}{4}\|f\|_{L^p} \qquad \textit{for all}\quad \lambda\geqq\lambda_p \quad \textit{and}\quad f\in\mathcal{D}\,.$$

Define $\|f\|_{C^\beta(\lambda)}=\|f\|+\lambda^{-\beta/\alpha}H_\beta(f)$.

LEMMA 3.2. *Assume that there is a constant K_β such that*

$$\int|y|^\alpha\wedge 1\,|N^{(\alpha)}(x,dy)-N^{(\alpha)}(x',dy)|+I_{(\alpha>1)}|b^{(\alpha)}(x)-b^{(\alpha)}(x')|$$
$$\leqq K_\beta|x-x'|^\beta\,.$$

Then there is a constant $\lambda_\beta=\lambda_\beta(x_0)>0$ for which

$$\|B^{(\alpha)}G_\lambda^{(\alpha)}f\|_{C^\beta(\lambda_\beta)}\leqq\frac{1}{8}\|f\|_{C^\beta(\lambda_\beta)} \qquad \textit{for all}\quad \lambda\geqq\lambda_\beta \quad \textit{and}\quad f\in\mathcal{D}\,.$$

PROOF. To simplify the proof, we suppose that $d=1$ and $1<\alpha<2$. Let $f\in\mathcal{D}$ and set $g=G_\lambda^{(\alpha)}f$. Then

$$|B^{(\alpha)}g(x)|\leqq\int|g(x+y)-g(x)-\Theta_1[y]g'(x)|N_*^{(\alpha)}(dy)+\|b^{(\alpha)}\|\cdot|g'(x)|$$

$$\leqq 2\|g\|\int_{|y|>1}N_*^{(\alpha)}(dy)+2\|g'\|\int_{\lambda^{-1}<|y|^\alpha\leqq 1}|y|N_*^{(\alpha)}(dy)$$

$$+c_1\||\partial|^{\alpha-1}g'\|\int_{|y|^\alpha\leqq\lambda^{-1}}|y|^\alpha N_*^{(\alpha)}(dy)+\|g'\|\cdot\|b^{(\alpha)}\|$$

(by (2.8) and (2.9))

$$\leqq c_2\lambda^{-1+1/\alpha}\Big(1+\int_{\lambda^{-1}<|y|^\alpha\leqq 1}|y|N_*^{(\alpha)}(dy)\Big)\|f\|$$

$$+c_3\Big(\int_{|y|^\alpha\leqq\lambda^{-1}}|y|^\alpha N_*^{(\alpha)}(dy)\Big)\lambda^{-\beta/\alpha}H_\beta(f)\quad\text{(by (2.13) and (2.15))}$$

$$\leqq c_4\Big(\lambda^{-1+1/\alpha}+\int|y|^\alpha((\lambda^{1/\alpha}|y|)^{1-\alpha}\wedge 1)N_*^{(\alpha)}(dy)\Big)\|f\|_{C^\beta(\lambda)}.$$

In the proof, all c.'s denote certain constants independent of λ and f. Let x and y be points and set $\delta=|x-z|$. Then

$$|B^{(\alpha)}g(x)-B^{(\alpha)}g(z)|$$

$$\leqq\int|g(x+y)-g(x)-\Theta_1[y]g'(x)||N^{(\alpha)}(x,dy)-N^{(\alpha)}(z,dy)|$$

$$+\int|(g(x+y)-g(x)-\Theta_1[y]g'(x))-(g(z+y)-g(z)-\Theta_1[y]g'(z))|N_*^{(\alpha)}(dy)$$

$$+|b^{(\alpha)}(x)-b^{(\alpha)}(z)||g'(x)|+|g'(x)-g'(z)|\|b^{(\alpha)}\|$$

$$\leqq\int(2\|g\|I_{(|y|>1)}+c_1\||\partial|^{\alpha-1}g'\|\cdot|y|^\alpha I_{(|y|\leqq 1)})|N^{(\alpha)}(x,dy)-N^{(\alpha)}(z,dy)|$$

$$+\delta^\beta\int(2H_\beta(g)I_{(|y|>1)}+2H_\beta(g')|y|I_{(\lambda^{-1}<|y|^\alpha\leqq 1)}$$

$$+c_1H_\beta(|\partial|^{\alpha-1}g')|y|^\alpha I_{(|y|^\alpha\leqq\lambda^{-1})})N_*^{(\alpha)}(dy)$$

$$+\|g'\||b^{(\alpha)}(x)-b^{(\alpha)}(z)|+\delta^\beta H_\beta(g')\|b^{(\alpha)}\|.$$

From the assumption, (2.6), (2.13) and (2.11), we have

$$H_\beta(B^{(\alpha)}g)\leqq K_\beta(2\|g\|+c_1\||\partial|^{\alpha-1}g'\|+\|g'\|)$$

$$+\Big(c_5\lambda^{-1+1/\alpha}+\int\{2\lambda^{-1}I_{(|y|>1)}+c_6|y|^\alpha((\lambda^{1/\alpha}|y|)^{1-\alpha}\wedge 1)\}N_*^{(\alpha)}(dy)\Big)H_\beta(f).$$

Set $\gamma=(\alpha-1)\wedge\beta$. By (2.6), (2.13) and (2.15) we have

$$H_\beta(B^{(\alpha)}g)\leqq c_7\lambda^{-1+1/\alpha}\|f\|$$

$$+c_8\Big(\lambda^{-\gamma/\alpha}+\int|y|^\alpha((\lambda^{1/\alpha}|y|)^{1-\alpha}\wedge 1)N_*^{(\alpha)}(dy)\Big)H_\beta(f).$$

Therefore, for any constant $\lambda_\beta>0$,

$$\lambda_\beta^{-\beta/\alpha}H_\beta(B^{(\alpha)}g)\leqq c_9\Big(\lambda_\beta^{-\gamma/\alpha}+\int|y|^\alpha((\lambda^{1/\alpha}|y|)^{1-\alpha}\wedge 1)N_*^{(\alpha)}(dy)\Big)\|f\|_{C^\beta(\lambda_\beta)}$$

as long as $\lambda \geqq \lambda_\beta$. Since

$$\lim_{\lambda \to \infty} \int |y|^\alpha((\lambda^{1/\alpha}|y|)^{1-\alpha} \wedge 1) N_*^{(\alpha)}(dy) = 0,$$

there is a constant λ_β such that

$$\|B^{(\alpha)} G_\lambda^{(\alpha)} f\|_{C^\beta(\lambda_\beta)} \leqq \frac{1}{8} \|f\|_{C^\beta(\lambda_\beta)} \qquad \text{for all } \lambda \geqq \lambda_\beta \text{ and } f \in \mathcal{D}. \qquad \text{q.e.d.}$$

Let λ_p be the constant in Lemma 3.1. By condition [C_2],

$$\|(A^{(\alpha)} - A_{x_0}^{(\alpha)}) G_\lambda^{(\alpha)} f\|_{L^p} \leqq \frac{1}{4} \|f\|_{L^p}.$$

Therefore, for all $\lambda \geqq \lambda_p$ and $f \in \mathcal{D}$,

$$\|(L - A_{x_0}^{(\alpha)}) G_\lambda^{(\alpha)} f\|_{L^p} \leqq \frac{1}{2} \|f\|_{L^p}. \tag{3.2}$$

Let U_λ be the closed extension of the operator $(L - A_{x_0}^{(\alpha)}) G_\lambda^{(\alpha)}$ on the space $\boldsymbol{L}^p = \{f \,;\, \|f\|_{L^p} < \infty\}$. Since the operator norm $\|U_\lambda\|_{L^p}$ is equal to or less than 1/2, the operator

$$[I - U_\lambda]^{-1} : \ \boldsymbol{L}^p \to \boldsymbol{L}^p \qquad (\lambda \geqq \lambda_p) \tag{3.3}$$

is well defined. The operator $G_\lambda^{(\alpha)}$ can be extended to the bounded operator on $\boldsymbol{L}^p$, which is also denoted by $G_\lambda^{(\alpha)}$. For $\lambda \geqq \lambda_p$, we shall define the operator

$$R_\lambda = G_\lambda^{(\alpha)} [I - U_\lambda]^{-1} : \boldsymbol{L}^p \longrightarrow G_\lambda^{(\alpha)}(\boldsymbol{L}^p). \tag{3.4}$$

From the resolvent equation:

$$G_\lambda^{(\alpha)} - G_\mu^{(\alpha)} = (\mu - \lambda) G_\lambda^{(\alpha)} G_\mu^{(\alpha)} \qquad \text{on } \boldsymbol{L}^p, \tag{3.5}$$

the space $G_\lambda^{(\alpha)}(\boldsymbol{L}^p)$ is independent of λ. Note that if $f \in \mathcal{D}$, then $(\lambda - A_{x_0}^{(\alpha)}) f \in \boldsymbol{L}^p$ and

$$(\lambda - A_{x_0}^{(\alpha)}) G_\lambda^{(\alpha)} f = G_\lambda^{(\alpha)} (\lambda - A_{x_0}^{(\alpha)}) f = f. \tag{3.6}$$

Therefore, for $\lambda \geqq \lambda_p$ and $f \in \mathcal{D}$,

$$\begin{aligned} R_\lambda(\lambda - L) f &= R_\lambda((\lambda - A_{x_0}^{(\alpha)}) - (L - A_{x_0}^{(\alpha)})) G_\lambda^{(\alpha)} (\lambda - A_{x_0}^{(\alpha)}) f \\ &= R_\lambda (I - U_\lambda)(\lambda - A_{x_0}^{(\alpha)}) f = G_\lambda^{(\alpha)} (\lambda - A_{x_0}^{(\alpha)}) f = f. \end{aligned} \tag{3.7}$$

Let $\boldsymbol{C}^k$, $k \geqq 0$, denote the completion of the space $\mathcal{D}$ by the norm

$$|f|_{C^k} = \sum_{|\nu| \leqq k} |\partial^\nu f|.$$

For $0 < \delta < 1$, set $\boldsymbol{C}^{k+\delta} = \{f \in \boldsymbol{C}^k \,;\, H_\delta(\partial^\nu f) < \infty \text{ for any } |\nu| = k\}$. From Lemma 2.4 we see that

$$G_\lambda^{(\alpha)}(\boldsymbol{L}^p) \subset \boldsymbol{C}^\beta \cap \boldsymbol{L}^p \subset \boldsymbol{C}^0 \cap \boldsymbol{L}^p. \tag{3.8}$$

It follows easily from (3.6) that

$$R_\lambda - R_\mu = (\mu-\lambda) R_\lambda R_\mu \quad \text{on } \boldsymbol{L}^p \qquad (\lambda,\ \mu \geqq \lambda_p). \tag{3.9}$$

Hence the space

$$\boldsymbol{D}(\tilde{L}) = R_\lambda (\boldsymbol{C}^0 \cap \boldsymbol{L}^p)$$

is independent of $\lambda \geqq \lambda_p$. Define an operator $\tilde{L}$ on $\boldsymbol{D}(\tilde{L})$ by

$$\tilde{L}(R_\lambda f) = \lambda R_\lambda f - f, \qquad f \in \boldsymbol{C}^0 \cap \boldsymbol{L}^p. \tag{3.10}$$

We shall show that (3.10) is well-posed. Let f and g be functions in $\boldsymbol{L}^p$ such that $R_\lambda f = R_\mu g$. Then by (3.9)

$$R_\lambda [(\lambda R_\lambda f - f) - (\mu R_\mu g - g)] = 0.$$

It suffices to prove that the operator R_λ is one to one on $\boldsymbol{L}^p$. We shall show that the operator $G_\lambda^{(\alpha)}$ is one to one on $\boldsymbol{L}^p$. Suppose that $f \in \boldsymbol{L}^p$ and $G_\lambda^{(\alpha)} f = 0$. Choose a sequence $\{f_n\} \subset \mathcal{D}$ such that $\|f_n - f\|_{L^p} \to 0$ as $n \to \infty$. Then $G_\lambda^{(\alpha)} f_n \to G_\lambda^{(\alpha)} f = 0$,

$$\|\lambda G_\lambda^{(\alpha)} f_n - f_m\|_{L^p} = \|G_\lambda^{(\alpha)} (\lambda f_n - (\lambda - A_{x_0}^{(\alpha)}) f_m)\|_{L^p} \leqq \|f_n - f_m + \lambda^{-1} A_{x_0}^{(\alpha)} f_m\|_{L^p}.$$

Taking limits $n \to \infty$ and $\lambda \to \infty$, we have $\|f_m\|_{L^p} \leqq \|f - f_m\|_{L^p}$. Let $m \to \infty$. Then $\|f\|_{L^p} = \|f - f\|_{L^p} = 0$, so $f = 0$. This shows that the operator $G_\lambda^{(\alpha)}$ is one to one, and so is the operator R_λ.

LEMMA 3.3. *The space $\boldsymbol{D}(\tilde{L})$ is dense in $\boldsymbol{C}^0$.*

PROOF. Let g be any function in $\mathcal{D}$ and set $(\lambda - L) g = f$. Choose a sequence $\{f_n\} \subset \mathcal{D}$ such that $\|f_n - f\|_{L^p} \to 0$ as $n \to \infty$. From (3.7) we have $g = R_\lambda f$. Hence by Lemma 2.4

$$\begin{aligned} \|R_\lambda f_n - g\| &= \|G_\lambda^{(\alpha)} [I - U_\lambda]^{-1} (f_n - f)\| \\ &\leqq \text{const.} \|f_n - f\|_{L^p} \to 0 \qquad \text{as} \quad n \to \infty. \end{aligned}$$

Since $\{R_\lambda f_n\} \subset \boldsymbol{D}(\tilde{L})$, the space $\mathcal{D}$ is contained in the closure of the space $\boldsymbol{D}(\tilde{L})$. Therefore the space $\boldsymbol{D}(\tilde{L})$ is dense in $\boldsymbol{C}^0$. q. e. d.

Next we claim that the operator λR_λ is a positive contraction. This proof, however, is more difficult than it looks so. Before proving, let us explain the reason besides the outline of the proof.

From the principle of a positive maximum we have

$$\|\lambda g\| \leqq \|(\lambda - L) g\| \qquad \text{for all} \quad g \in \mathcal{D}.$$

By (3.7) this implies that

$$\|\lambda R_\lambda f\| \leqq \|f\| \qquad \text{for all} \quad f \in (\lambda - L) \mathcal{D}.$$

In the case where the local characteristics of the operator L such as the diffusion coefficients, drift coefficients and the Lévy measure are continuous, it can be shown that the space $(\lambda-L)\mathcal{D}$ is dense in $\boldsymbol{C}^0$, so the operator λR_λ is contractive. However, the local characteristics of L are not always continuous in our context. Generally the space $(\lambda-L)\mathcal{D}$ is not included in the space $\boldsymbol{C}^0$ and $(\lambda-L)\mathcal{D}\cap\boldsymbol{C}^0$ is not dense in $\boldsymbol{C}^0$. Therefore we cannot conclude easily that λR_λ is contractive. To prove the contractive-ness and the positivity we approximate the operator L by a sequence $\{L^{(n)}\}$ of operators with smooth local characteristics. Let $\{R_\lambda^{(n)}\}$ be the resolvent associated with the operator $L^{(n)}$ defined similarly to (3.4). If, for each $f\in\mathcal{D}$, the function $R_\lambda^{(n)}f$ is sufficiently regular so that $L^{(n)}$ operates to the function $R_\lambda^{(n)}f$ in the usual sense, then we have from the principle of a positive maximum that

$$\|\lambda R_\lambda^{(n)}f\|\leqq\|f\| \qquad \text{for} \quad f\in\mathcal{D}\,.$$

For each function $f\in\boldsymbol{C}^0\cap\boldsymbol{L}^p$, there is a sequence $\{f_m\}\subset\mathcal{D}$ such that $\|f_m\|\leqq\|f\|$ and $\|f_m-f\|_{L^p}\to 0$ as $m\to\infty$. Therefore if $\|R_\lambda^{(n)}\phi-R_\lambda\phi\|\to 0$ as $n\to\infty$ for all $\phi\in\boldsymbol{C}^0\cap\boldsymbol{L}^p$ and if $\|R_\lambda^{(n)}\phi\|\leqq\text{const.}\|\phi\|_{L^p}$, then we can conclude that λR_λ is contractive on $\boldsymbol{C}^0\cap\boldsymbol{L}^p$. The positivity of the operator R_λ will be shown in a similar way.

Let $\rho(x)$ be a non-negative smooth function such that

$$\int\rho(x)dx=1\,, \qquad \{x\,;\,f(x)\neq 0\}=\{x\,;\,|x|<1\}\,.$$

Set $\rho_n(x)=n^d\rho(nx)$ and

$$a_n=\rho_n*a\,, \qquad b_n=\rho_n*b\,, \qquad b_n^{(\alpha)}=\rho_n*b^{(\alpha)}\,,$$

$$m_n^{(\alpha)}(x,\,y)=\int\rho_n(x-z)m^{(\alpha)}(z,\,y)dz\,, \qquad N_n^{(\alpha)}(x,\,dy)=\int\rho_n(x-z)N^{(\alpha)}(z,\,dy)dz\,.$$

Then we have, for a.a. x,

$$a_n(x)\to a(x)\,, \qquad b_n(x)\to b(x)\,, \qquad b_n^{(\alpha)}(x)\to b^{(\alpha)}(x)\,,$$

$$\sup_{|y|=1}|m_n^{(\alpha)}(x,\,y)-m^{(\alpha)}(x,\,y)|\to 0 \quad \text{and}$$

$$\int|y|^\alpha\wedge 1\,|N_n^{(\alpha)}(x,\,dy)-N^{(\alpha)}(x,\,dy)|\to 0\,,$$

as $n\to\infty$. Moreover

$$\text{(3.11)} \qquad \begin{cases} m_n^{(\alpha)}(x,\,y)|y|^{-d-\alpha}dy+N_n^{(\alpha)}(x,\,dy)\geqq 0 & (0<\alpha<2)\,, \\ a_n(x)\succ 0 \quad \text{and} \quad N_n^{(2)}(x,\,dy)\geqq 0 & (\alpha=2)\,. \end{cases}$$

Similarly to that $\phi^{(\alpha)}$, $A_z^{(\alpha)}$, $A^{(\alpha)}$, $B^{(\alpha)}$, L and $\Delta^{(\alpha)}$ are defined using the elements $\{a,\,b,\,b^{(\alpha)},\,m^{(\alpha)},\,N^{(\alpha)}\}$, we shall define $\phi_n^{(\alpha)}$, $A_z^{(n,\alpha)}$, $A^{(n,\alpha)}$, $B^{(n,\alpha)}$, $L^{(n)}$ and $\Delta_n^{(\alpha)}$

using the elements $\{a_n, b_n, b_n^{(\alpha)}, m_n^{(\alpha)}, N_n^{(\alpha)}\}$. Set

$$U_\lambda^{(n)}=(L^{(n)}-A_{x_0}^{(\alpha)})G_\lambda^{(\alpha)}\,.$$

Since $\||b_n^{(\alpha)}|\|\leqq\||b^{(\alpha)}|\|$ and $|N_n^{(\alpha)}(x, dy)|\leqq N_*^{(\alpha)}(dy)$, we have

$$\|U_\lambda^{(n)}f\|_{L^p}\leqq\frac{1}{2}\|f\|_{L^p} \qquad \text{for all} \quad \lambda\geqq\lambda_p \quad \text{and} \quad f\in\mathcal{D}\,.$$

The closed extension of the operator $U_\lambda^{(n)}$ is also denoted by $U_\lambda^{(n)}$. Define

$$R_\lambda^{(n)}=G_\lambda^{(\alpha)}[I-U_\lambda^{(n)}]^{-1}: \quad \boldsymbol{L}^p\to\boldsymbol{C}^\beta\cap\boldsymbol{L}^p\,.$$

LEMMA 3.4. *For each $\lambda\geqq\lambda_p$ and $f\in\boldsymbol{L}^p$,*

$$\lim_{n\to\infty}\|R_\lambda^{(n)}f-R_\lambda f\|=0\,.$$

PROOF. Let $f\in\boldsymbol{L}^p$. By Lemma 2.4

$$\|R_\lambda^{(n)}f-R_\lambda f\|\leqq\text{const.}\|[I-U_\lambda^{(n)}]^{-1}f-[I-U_\lambda]^{-1}f\|_{L^p}$$

Note that

$$\begin{aligned}[I-U_\lambda^{(n)}]^{-1}-[I-U_\lambda]^{-1}&=\sum_{k=0}^{\infty}((U_\lambda^{(n)})^{k+1}-(U_\lambda)^{k+1})\\&=\sum_{k=0}^{\infty}\sum_{j=0}^{k}(U_\lambda^{(n)})^{k-j}(U_\lambda^{(n)}-U_\lambda)(U_\lambda)^j\,.\end{aligned}$$

Then we have

$$\|[I-U_\lambda^{(n)}]^{-1}f-[I-U_\lambda]^{-1}f\|_{L^p}\leqq\sum_{k=0}^{\infty}\sum_{j=0}^{k}\left(\frac{1}{2}\right)^{k-j}\|(U_\lambda^{(n)}-U_\lambda)(U_\lambda)^jf\|_{L^p}\,.$$

Since

$$\|(U_\lambda^{(n)}-U_\lambda)(U_\lambda)^jf\|_{L^p}\leqq\left(\frac{1}{2}\right)^j\|f\|_{L^p}\,,\qquad \sum_{k=0}^{\infty}\sum_{j=0}^{k}\left(\frac{1}{2}\right)^{k-j}\left(\frac{1}{2}\right)^j<\infty\,,$$

it suffices to prove that

$$\lim_{n\to\infty}\|(U_\lambda^{(n)}-U_\lambda)(U_\lambda)^jf\|_{L^p}=0 \qquad \text{for each } j\,.$$

Considering that the space $\mathcal{D}$ is dense in $\boldsymbol{L}^p$ and the operator norms $\|U_\lambda^{(n)}-U_\lambda\|_{L^p}$ are bounded, it is sufficient to prove that

$$\|(U_\lambda^{(n)}-U_\lambda)f\|_{L^p}\to 0 \quad \text{as } n\to\infty \quad \text{for each } f\in\mathcal{D}\,. \tag{3.12}$$

We shall prove (3.12) only for $0<\alpha<1$. Other cases are proved similarly. Let $f\in\mathcal{D}$ and set $g=G_\lambda^{(\alpha)}f$. Note that $\partial^\nu g\in\boldsymbol{L}^p$ for any ν. Therefore

$$\left\|\int|g(x+y)-g(x)||y|^{-d-\alpha}dy\right\|_{L^p}$$

$$\leqq (2\|g\|_{L^p}+\sum_j \|\partial_j g\|_{L^p})\int (|y|\wedge 1)|y|^{-d-\alpha}dy<\infty .$$

Since

$$|(A^{(n,\alpha)}-A^{(\alpha)})g(x)|\leqq \text{const.}\int |g(x+y)-g(x)||y|^{-d-\alpha}dy ,$$

and since

$$|(A^{(n,\alpha)}-A^{(\alpha)})g(x)|$$

$$\leqq \int |y|\wedge 1 |m_n^{(\alpha)}(x, y)-m^{(\alpha)}(x, y)||y|^{-d-\alpha}dy\cdot[\sup_y (|y|\wedge 1)^{-1}|g(x+y)-g(x)|]$$

$$\to 0 \quad \text{a.a. } x \quad \text{as } n\to\infty ,$$

we see that $\|(A^{(n,\alpha)}-A^{(\alpha)})g\|_{L^p}\to 0$. Similarly we have $\|(B^{(n,\alpha)}-B^{(\alpha)})g\|_{L^p}\to 0$. Hence

$$\|(U_\lambda^{(n)}-U_\lambda)f\|_{L^p}=\|(L^{(n)}-L)g\|_{L^p}\to 0 \qquad \text{as } n\to\infty . \qquad \text{q.e.d.}$$

LEMMA 3.5. *If* $f\in\mathcal{D}$ *and* $\lambda\geqq\lambda_p$, *then* $[I-U_\lambda^{(n)}]^{-1}f\in \boldsymbol{C}^\beta\cap \boldsymbol{L}^p$.

PROOF. (*Step* 1) First we shall show that there is a constant μ_n such that, for $\lambda\geqq\mu_n$ and $f\in\mathcal{D}$,

$$\|U_\lambda^{(n)}f\|_{C^\beta(\mu_n)}\leqq\frac{1}{2}\|f\|_{C^\beta(\mu_n)} . \tag{3.13}$$

Obviously the elements $b_n^{(\alpha)}$, $N_n^{(\alpha)}$ satisfy the assumption of Lemma 3.2, and so

$$\|B^{(n,\alpha)}G_\lambda^{(\alpha)}f\|_{C^\beta(\mu'_n)}\leqq\frac{1}{8}\|f\|_{C^\beta(\mu'_n)} \qquad \text{for } \lambda\geqq\mu'_n$$

as long as the constant μ'_n is sufficiently large. Considering inequality (2.14) for the function $\phi=\phi_n^{(\alpha)}(z', \xi)-\phi_n^{(\alpha)}(z, \xi)$, we have

$$\|(A_{z'}^{(n,\alpha)}-A_z^{(n,\alpha)})G_\lambda^{(\alpha)}f\|\leqq k(x_0)\lambda^{-\beta/\alpha}H_\beta(f)\Delta_n^{(\alpha)}(z', z) .$$

There is a constant K'_n such that $\Delta_n^{(\alpha)}(z_1, z_2)\leqq K'_n|z_1-z_2|^\beta$ for all $z_1, z_2\in\boldsymbol{R}^d$. Since

$$|(A^{(n,\alpha)}-A_{x_0}^{(\alpha)})G_\lambda^{(\alpha)}f(z')-(A^{(n,\alpha)}-A_{x_0}^{(\alpha)})G_\lambda^{(\alpha)}f(z)|$$

$$\leqq\|(A_{z'}^{(n,\alpha)}-A_z^{(n,\alpha)})G_\lambda^{(\alpha)}f\|+|z'-z|^\beta H_\beta((A_z^{(n,\alpha)}-A_{x_0}^{(\alpha)})G_\lambda^{(\alpha)}f) ,$$

we have

$$H_\beta((A^{(n,\alpha)}-A_{x_0}^{(\alpha)})G_\lambda^{(\alpha)}f)\leqq k(x_0)K'_n\lambda^{-\beta/\alpha}H_\beta(f)+\sup_z H_\beta((A_z^{(\alpha)}-A_{x_0}^{(\alpha)})G_\lambda^{(\alpha)}f) .$$

Let μ_0 be a certain positive constant. For any $\lambda\geqq\mu_0$,

$$\|(A^{(n,\alpha)}-A_{x_0}^{(\alpha)})G_\lambda^{(\alpha)}f\|_{C^\beta(\mu_0)}\leqq k(x_0)K'_n\mu_0^{-2\beta/\alpha}H_\beta(f)$$

$$+\sup_z\|(A_z^{(\alpha)}-A_{x_0}^{(\alpha)})G_\lambda^{(\alpha)}f\|+\sup_z H_\beta((A_z^{(\alpha)}-A_{x_0}^{(\alpha)})G_\lambda^{(\alpha)}f)\mu_0^{-\beta/\alpha}$$

$$\leqq [k(x_0)K_n'\mu_0^{-\beta/\alpha}+k_\beta(x_0)\sup_z \Delta^{(\alpha)}(x_0, z)]\mu_0^{-\beta/\alpha}H_\beta(f) \qquad \text{(by Theorem 2.2)}$$

$$\leqq \left[k(x_0)K_n'\mu_0^{-\beta/\alpha}+\frac{1}{4}\right]\|f\|_{C^\beta(\mu_0)}.$$

Let μ_n'' be a constant such that $k(x_0)K_n'(\mu_n'')^{-\beta/\alpha}\leqq 1/8$ and set $\mu_n=\mu_n'\vee\mu_n''$. Then inequality (3.13) holds.

(*Step* 2) Let $\mu=\lambda_p\vee\mu_n$. From (3.13) we see that

$$[I-U_\mu^{(n)}]^{-1}(\boldsymbol{C}^\beta\cap\boldsymbol{L}^p)\subset\boldsymbol{C}^\beta\cap\boldsymbol{L}^p.$$

From (3.5) we have $U_\lambda^{(n)}-U_\mu^{(n)}-(\mu-\lambda)U_\mu^{(n)}G_\lambda^{(\alpha)}=0$ for $\lambda\geqq\lambda_p$. Therefore

$$(I-U_\mu^{(n)})-(I-U_\lambda^{(n)})-(\mu-\lambda)G_\lambda^{(\alpha)}+(\mu-\lambda)(I-U_\mu^{(n)})G_\lambda^{(\alpha)}=0.$$

Multiplying this equality by $[I-U_\mu^{(n)}]^{-1}$ from the left side and by $[I-U_\lambda^{(n)}]^{-1}$ from the right side, we have

$$[I-U_\lambda^{(n)}]^{-1}=[I-U_\mu^{(n)}]^{-1}(I+(\mu-\lambda)R_\lambda^{(n)})-(\mu-\lambda)R_\lambda^{(n)}.$$

Let $f\in\mathcal{D}$. Then $f+(\mu-\lambda)R_\lambda^{(n)}f\in\boldsymbol{C}^\beta\cap\boldsymbol{L}^p$ and $R_\lambda^{(n)}f\in\boldsymbol{C}^\beta\cap\boldsymbol{L}^p$. Hence

$$[I-U_\lambda^{(n)}]^{-1}f\in\boldsymbol{C}^\beta\cap\boldsymbol{L}^p. \qquad \text{q.e.d.}$$

Let $\boldsymbol{B}$ denote the space of bounded Borel measurable functions on $\boldsymbol{R}^d$. Lemma 3.4 and Lemma 3.5 are used to prove the following lemma.

LEMMA 3.6. *Let* $\lambda\geqq\lambda_p$.

(i) $\|R_\lambda f\|\leqq\lambda^{-1}\|f\|$ *for all* $f\in\boldsymbol{B}\cap\boldsymbol{L}^p$.

(ii) *If* $f\in\boldsymbol{B}\cap\boldsymbol{L}^p$ *and* $f\geqq 0$, *then* $R_\lambda f\geqq 0$.

(iii) *There is a sequence* $\{f_n\}\subset\mathcal{D}$ *such that* $0\leqq f_n\uparrow 1$ *and* $R_\lambda f_n\uparrow 1$.

PROOF. (i) Let $f\in\boldsymbol{B}\cap\boldsymbol{L}^p$. Choose a sequence $\{f_m\}\subset\mathcal{D}$ so that $\|f_m-f\|_{L^p}\to 0$ and $\|f_m\|\leqq\|f\|$. From (2.11), (2.15) and Lemma 3.5, we see that

$$R_\lambda^{(n)}f_m=G_\lambda^{(\alpha)}[I-U_\lambda^{(n)}]^{-1}f_m\in\boldsymbol{C}^{\alpha+\beta'} \qquad \text{for any } \beta'<\beta.$$

Hence $L^{(n)}$ operates to $R_\lambda^{(n)}f_m$ in the usual sense. Note that, for any function g in $\boldsymbol{C}^\beta\cap\boldsymbol{L}^p$, $(L^{(n)}-A_{x_0}^{(\alpha)})(G_\lambda^{(\alpha)}g)=U_\lambda^{(n)}g$. Therefore

$$(\lambda-L^{(n)})R_\lambda^{(n)}f_m=((\lambda-A_{x_0}^{(\alpha)})-(L^{(n)}-A_{x_0}^{(\alpha)}))G_\lambda^{(\alpha)}[I-U_\lambda^{(n)}]^{-1}f_m$$

$$=(I-U_\lambda^{(n)})[I-U_\lambda^{(n)}]^{-1}f_m=f_m.$$

From the maximum principle and (3.11) it can be easily proved that

$$\|R_\lambda^{(n)}f_m\|\leqq\frac{1}{\lambda}\|f_m\|\leqq\frac{1}{\lambda}\|f\|.$$

Since $\|R_\lambda^{(n)}(f_m-f)\|\leqq\text{const.}\|f_m-f\|_{L^p}\to 0$ as $m\to\infty$, we have $\|R_\lambda^{(n)}f\|\leqq\lambda^{-1}\|f\|$. By Lemma 3.4, we have $\|R_\lambda^{(n)}f-R_\lambda f\|\to 0$ as $n\to\infty$, so that $\|R_\lambda f\|\leqq\lambda^{-1}\|f\|$.

(ii) Let $f\in \boldsymbol{B}\cap \boldsymbol{L}^p$ and $f\geqq 0$. Choose a sequence $\{f_m\}\subset\mathcal{D}$ so that $f_m\geqq 0$ and $\|f_m-f\|_{L^p}\to 0$. From the maximum principle we have $R_\lambda^{(n)}f_m\geqq 0$. Let $m\to\infty$ and $n\to\infty$. Then we have $R_\lambda f\geqq 0$.

(iii) Let $\rho(t)$ be a smooth decreasing function on $\boldsymbol{R}_+$ such that $\rho(t)=1$ for $t\leqq 1$ and $\rho(t)=0$ for $t\geqq 2$. Set $f_n(x)=\rho(|x|/n)$. Then $f_n\uparrow 1$ and $R_\lambda f_n\uparrow$. It is easy to show that $Lf_n\in \boldsymbol{B}\cap \boldsymbol{L}^p$ and $\|Lf_n\|\to 0$ as $n\to\infty$. From (i) we have

$$\|R_\lambda(Lf_n)\|\leqq \lambda^{-1}\|Lf_n\|\to 0\,.$$

By (3.7) we see that $f_n=R_\lambda(\lambda-L)f_n=\lambda R_\lambda f_n-R_\lambda(Lf_n)$. Hence

$$\lim_{n\to\infty}\lambda R_\lambda f_n=\lim_{n\to\infty}(f_n+R_\lambda(Lf_n))=1\,. \qquad \text{q.e.d.}$$

Using the semi-group theory (cf. Gihman and Skorohod [**5**]), the following theorem can be proved immediately from Lemma 3.3 and Lemma 3.6.

THEOREM 3.1. *Assume* [A_1] *and* [A_2]. *Under condition* [C_2] *there exists a Feller semi-group* $(T_t)_{t\geqq 0}$ *on the space* $\boldsymbol{C}^0$ *whose generator is the closed extension of* $(\tilde{L}, \boldsymbol{D}(\tilde{L}))$.

Let $\{W, \mathcal{W}, \mathcal{W}_t, P_x\,;\, X_t\}$ be the Markov process on $\boldsymbol{R}^d$ associated with the Feller semi-group $(T_t)_{t\geqq 0}$. Let $E_x[\cdot]$ denote the expectation by P_x. Then, for any f in $\boldsymbol{C}^0\cap \boldsymbol{L}^p$ and $\lambda\geqq\lambda_p$,

$$R_\lambda f(x)=E_x\left[\int_0^\infty e^{-\lambda t}f(X_t)dt\right]. \tag{3.14}$$

THEOREM 3.2. *Under* [A_1], [A_2] *and* [C_2], *there is a solution* P_x *of the martingale problem associated with the operator* L *for any starting point* x.

PROOF. Let $\{X_t, P_x\}$ be the above process. Clearly equality (3.14) holds for any $f\in \boldsymbol{B}\cap \boldsymbol{L}^p$ and $\lambda\geqq\lambda_p$. Let g be an arbitrary function in $\mathcal{D}$ and set $f=(\lambda-L)g\in \boldsymbol{B}\cap \boldsymbol{L}^p$. Then $g=R_\lambda f$ by (3.7). Set

$$M_t^{\lambda,g}=e^{-\lambda t}g(X_t)-g(X_0)+\int_0^t e^{-\lambda\tau}(\lambda-L)g(X_\tau)d\tau\,.$$

From the Markov property, for $s<t$,

$$\begin{aligned}
&E_x[M_t^{\lambda,g}-M_s^{\lambda,g}\,|\,\mathcal{W}_s]\\
&=E_x\left[e^{-\lambda t}R_\lambda f(X_t)-\int_t^\infty e^{-\lambda\tau}f(X_\tau)d\tau\,\middle|\,\mathcal{W}_s\right]\\
&\quad -E_x\left[e^{-\lambda s}R_\lambda f(X_s)-\int_s^\infty e^{-\lambda\tau}f(X_\tau)d\tau\,\middle|\,\mathcal{W}_s\right]\\
&=e^{-\lambda t}E_x\left[R_\lambda f(X_t)-E_{X_t}\left[\int_0^\infty e^{-\lambda\tau}f(X_\tau)d\tau\right]\middle|\,\mathcal{W}_s\right]\\
&\quad -e^{-\lambda s}\left(R_\lambda f(X_s)-E_{X_s}\left[\int_0^\infty e^{-\lambda\tau}f(X_\tau)d\tau\right]\right)=0\,.
\end{aligned}$$

Therefore $M_t^{\lambda,g}$ is a P_x-martingale for any $\lambda \geqq \lambda_p$ and $g \in \mathcal{D}$. Hence the probability P_x solves the martingale problem associated with the operator L starting from x. q. e. d.

4. Uniqueness of solution; special case.

In this section the constant p is chosen so that $p > d/\alpha$ in case $0 < \alpha \leqq 1$, and that $p > d/(\alpha - 1)$ in case $1 < \alpha \leqq 2$. We shall prove the uniqueness of solution of the martingale problem for L under [A₁], [A₂], [A₃] and the following condition.

CONDITION [C₃]. $k_p(x_0) \sup_z \Delta^{(\alpha)}(x_0, z) \leqq \frac{1}{4}$,

where k_p and $\Delta^{(\alpha)}$ are the same objects as in Theorem 2.2. *And*

$$\inf_{x, |y|=1} m^{(\alpha)}(x, y) > 0 \quad (0 < \alpha < 2), \qquad \inf_{x, |\xi|=1} \xi \cdot a(x) \xi > 0 \quad (\alpha = 2). \tag{4.1}$$

The proof of the uniqueness theorem is an improvement of that in [**10**] where the principal part $A^{(\alpha)}$ was the generator of a stable process. The proof is based on the following lemma, which is slightly different from the one in Stroock and Varadhan [**16**], Section 6.2.

LEMMA 4.1 (Lemma 3.1 in [**10**]). *Let P^1 and P^2 be probability measures on $(W, \mathcal{W})$ such that $P^1[X_0 \in dx] = P^2[X_0 \in dx]$. If, for any $s \geqq 0$, $\lambda \geqq \lambda_0$ and $f \in \boldsymbol{C}^0 \cap \boldsymbol{L}^p$, there is a function $g \in \boldsymbol{C}^0$ such that*

$$E^i\left[\int_0^\infty e^{-\lambda t} f(X_{s+t}) dt \,\middle|\, \mathcal{W}_s\right] = g(X_s) \qquad P^i\text{-a.e.} \quad (i = 1, 2),$$

then we have $P^1 = P^2$ on $\mathcal{W}$, where λ_0 is a certain constant and $E^i[\cdot|\mathcal{W}_s]$ denotes the conditional expectation by P^i.

Let $\{X_t, P_x\}$ be a process solving the martingale problem associated with L starting from x. For a moment let $\phi \in \mathcal{D}$. Though the support of the function $G_\lambda^{(\alpha)} \phi$ is not always compact, it is clear that the process

$$G_\lambda^{(\alpha)} \phi(X_t) - G_\lambda^{(\alpha)} \phi(x) - \int_0^t L G_\lambda^{(\alpha)} \phi(X_\tau) d\tau$$

is a P_x-martingale. Hence the process

$$e^{-\lambda t} G_\lambda^{(\alpha)} \phi(X_t) - G_\lambda^{(\alpha)} \phi(x) + \int_0^t e^{-\lambda \tau} (\lambda - L) G_\lambda^{(\alpha)} \phi(X_\tau) d\tau$$

is a P_x-martingale with mean 0 for any $\lambda \geqq \lambda_p$. Therefore

$$\begin{aligned} G_\lambda^{(\alpha)} \phi(X_s) &= E_x\left[\int_s^\infty e^{-\lambda(\tau - s)} (\lambda - L) G_\lambda^{(\alpha)} \phi(X_\tau) d\tau \,\middle|\, \mathcal{W}_s\right] \\ &= E_x\left[\int_0^\infty e^{-\lambda t} [I - U_\lambda] \phi(X_{s+t}) dt \,\middle|\, \mathcal{W}_s\right], \end{aligned}$$

where $E_x[\cdot|\mathcal{W}_s]$ denotes the conditional expectation by P_x. Recall that $U_\lambda=(L-A_{x_0}^{(\alpha)})G_\lambda^{(\alpha)}$ and $G_\lambda^{(\alpha)}=R_\lambda[I-U_\lambda]$. Therefore the equality

$$E_x\left[\int_0^\infty e^{-\lambda t}f(X_{s+t})dt\middle|\mathcal{W}_s\right]=R_\lambda f(X_s) \qquad P_x\text{-a. e.} \tag{4.2}$$

holds for any function f in $[I-U_\lambda]\mathcal{D}$. Note that $[I-U_\lambda]\mathcal{D}$ is dense in $\boldsymbol{L}^p$ and $\|R_\lambda f\|\leqq\text{const.}\|f\|_{L^p}$. Hence equality (4.2) holds for any function f in $\boldsymbol{C}^0\cap\boldsymbol{L}^p$ if there is a constant c_λ such that

$$\left|E_x\left[\int_0^\infty e^{-\lambda t}f(X_{s+t})dt\middle|\mathcal{W}_s\right]\right|\leqq c_\lambda\|f\|_{L^p} \qquad P_x\text{-a. e.} \tag{4.3}$$

for all $f\in\boldsymbol{C}^0\cap\boldsymbol{L}^p$, where (s, x) is fixed. By Lemma 4.1, it suffices for the uniqueness theorem to prove that (4.3) holds for each function f in $\boldsymbol{C}^0\cap\boldsymbol{L}^p$.

We shall explain the outline of the proof, for it is rather long and complicated. A process is said to be a *piecewise α-stable process* if there exists a partition: $0=t_1<t_2<\cdots<t_n\uparrow\infty$ of the time space $\boldsymbol{R}_+$ and, on each time interval $[t_k, t_{k+1})$, the process is an α-stable process with respect to the conditional probability $P_x[\cdot|\mathcal{W}_{t_k}]$. Now suppose that $1<\alpha\leqq 2$. If a process Z_t is a piecewise α-stable process with perturbations of drift and infrequent jumps, then the $\boldsymbol{L}^p$-estimate similar to (4.3) for the process Z_t can be easily proved. Hence, if there is a sequence $\{Z_t^n\}$ of such processes which approximates the process X_t and if the $\boldsymbol{L}^p$-estimates for the processes Z_t^n are uniform in n, then the $\boldsymbol{L}^p$-estimate for the process X_t holds also. In case $\alpha=2$, it is easy to construct such a sequence $\{Z_t^n\}$ (cf. Theorem 4.2 in [**9**]). However, in case $\alpha\neq 2$, it is generally impossible to construct such a sequence $\{Z_t^n\}$ of processes on the space $(W, \mathcal{W}, P_x)$. By the change of sizes of jumps:

$$\Delta X_s\to m^{(\alpha)}(X_s,\ |\Delta X_s|^{-1}\Delta X_s)^{-1/\alpha}\Delta X_s\,,$$

it can be essentially reduced to the case where $m^{(\alpha)}(x, y)=1$, so we shall consider here this simple case. This case was considered in [**10**]. The sequence $\{Z_t^n\}$ of processes can be constructed in the following way. Let Y_t be an α-stable process with the generator $A^{(\alpha)}$ which is independent of the process X_t. To realize such a situation, it is necessary to take X_t for the process defined on a direct product space $(W\times W, \mathcal{W}\times\mathcal{W}, P_x\times Q)$. Cut off the jumps $\{\Delta X_s\ ;\ |\Delta X_s|\leqq 1/n,\ s\leqq t\}$ from the process X_t, and add the jumps $\{\Delta Y_s\ ;\ |\Delta Y_s|\leqq 1/n,\ s\leqq t\}$ to it. Then the obtained process Z_t^n is an isotropic α-stable process with perturbations of drift and infrequent jumps. Obviously the sequence $\{Z_t^n\}$ approximates the process X_t. In the general case, processes Z_t^n are constructed similarly, but the usual Calderón-Zygmund inequality is useless for the proof of the uniform L^p-estimates for the processes Z_t^n. In the proof, (i) of Theorem 2.2 plays an essential role.

Hereafter we assume that $0<\alpha<2$ unless otherwise stated. Define

$$J_X(dt,\ dy)=\#\{s\in dt\ ;\ \Delta X_s=X_s-X_{s-}\in dy\setminus\{0\}\}\ ,$$
$${}^cJ_X(dt,\ dy)=J_X(dt,\ dy)-(M^{(\alpha)}(X_t,\ dy)+N^{(\alpha)}(X_t,\ dy))dt\ .$$

By Theorem 2.1 in [**9**] (see also Grigelionis [**6**]), ${}^cJ_X(dt,\ dy)$ is a P_x-martingale measure, i.e. for each non-negative measurable function $h(t,x,y)$ and each stopping time T,

$$E_x\left[\int_0^T\int h(t,\ X_t,\ y)J_X(dt,\ dy)\right]$$
$$=E_x\left[\int_0^T\int h(t,\ X_t,\ y)(M^{(\alpha)}(X_t,\ dy)+N^{(\alpha)}(X_t,\ dy))dt\right].$$

The process $\{X_t,\ P_x\}$ is expressed as follows:

$$(4.4)\quad \begin{cases} X_t=x+\int_0^t\int yJ_X(ds,\ dy) & (0<\alpha<1)\,, \\ X_t=x+\int_0^t\int y[J_X(ds,\ dy)-I_{(|y|\leq 1)}M^{(1)}(X_s,\ dy)ds]+\int_0^t b(X_s)ds & (\alpha=1)\,, \\ X_t=x+\int_0^t\int y\{J_X(ds,\ dy)-M^{(\alpha)}(X_s,\ dy)ds-I_{(|y|\leq 1)}N^{(\alpha)}(X_s,\ dy)ds\} & \\ \qquad +\int_0^t b^{(\alpha)}(X_s)ds & (1<\alpha<2)\,. \end{cases}$$

Define

$$\pi(m,\ t\ ;\ \tau)=t+k2^{-m}\qquad \text{if}\quad k2^{-m}<\tau-t\leqq(k+1)2^{-m}\,.$$

Using the same argument as in §7 of Tsuchiya [**17**], we have the following lemma.

LEMMA 4.2. *There exist a point $t_0\in[0,\ 1)$ and a subsequence $\{m_n\}$ such that, for each $T<\infty$,*

$$\int_0^T\int_{|\omega|=1}|m^{(\alpha)}(X_{\pi(m_n,t_0;\tau)},\ \omega)^{1/\alpha}-m^{(\alpha)}(X_\tau,\ \omega)^{1/\alpha}|\sigma(d\omega)d\tau$$
$$+I_{(\alpha=1)}\int_0^T|b(X_{\pi(m_n,t_0;\tau)})-b(X_\tau)|d\tau\ \to\ 0$$

in probability as $n\to\infty$.

Set $t_k^{(n)}=(t_0+k2^{-m_n})\vee 0$ and

$$(4.5)\qquad \pi(n,\ t)=t_k^{(n)}\qquad \text{if}\quad t_k^{(n)}<t\leqq t_{k+1}^{(n)}\,.$$

Let $\{W,\ \mathcal{W},\ \mathcal{W}_t,\ Q\ ;\ X_t\}$ be a stable process such that

$$\int J_X(dt,\ dy)Q(dw)=|y|^{-d-\alpha}dy\ dt\,.$$

Set $\tilde{W}=W\times W$, $\tilde{\mathcal{W}}=\mathcal{W}\times\mathcal{W}$, $\tilde{\mathcal{W}}_t=\mathcal{W}_t\times\mathcal{W}_t$ and $\tilde{P}_x=P_x\times Q$. For $\tilde{w}=(w_1, w_2)\in\tilde{W}$, let $X_t(\tilde{w})=w_1(t)$ and $Y_t(\tilde{w})=w_2(t)$. It is convenient for the arguments in this section to use the same symbol X_t for the mappings on W and $\tilde{W}$. Let $J_X(dt, dy)$, ${}^cJ_X(dt, dy)$ be the same random measure as before, and set

$$J_Y(dt, dy)=\#\{s\in dt\ ;\ \Delta Y_s=Y_s-Y_{s-}\in dy\setminus\{0\}\}\,,$$
$${}^cJ_Y(dt, dy)=J_Y(dt, dy)-|y|^{-d-\alpha}dy\,dt\,.$$

Define

$$F(z, y)=m^{(\alpha)}(z, |y|^{-1}y)^{1/\alpha}y\,, \qquad \Omega(z, x\ ;\ y)=(m^{(\alpha)}(z, y)/m^{(\alpha)}(x, y))^{1/\alpha}y\,. \tag{4.6}$$

Note that, for any non-negative function ϕ on $\boldsymbol{R}^d$,

$$\int\phi(\Omega(z, x\ ;\ y))M^{(\alpha)}(x, dy)=\int\phi(F(z, y))|y|^{-d-\alpha}dy=\int\phi(y)M^{(\alpha)}(z, dy)\,.$$

Set $\Theta_n[y]=I_{(|y|\leq 1/n)}y$ and $\Theta_n^c[y]=I_{(|y|>1/n)}y$. We shall consider a sequence $\{Z_t^n\}$ of processes on the product space $\{\tilde{W}, \tilde{\mathcal{W}}, \tilde{\mathcal{W}}_t, \tilde{P}_x\}$ defined by

$$\tag{4.7}\left\{\begin{aligned}
Z_t^n&=x+\int_0^t\!\!\int\Theta_n^c[\Omega(X_{\pi(n,\tau)}, X_\tau\ ;\ y)]J_X(d\tau, dy)\\
&\quad+\int_0^t\!\!\int\Theta_n[F(X_{\pi(n,\tau)}, y)]J_Y(d\tau, dy) \qquad (0<\alpha<1)\,,\\
Z_t^n&=x+\int_0^t\!\!\int\Theta_n^c[\Omega(X_{\pi(n,\tau)}, X_\tau\ ;\ y)]J_X(d\tau, dy)+\int_0^t b(X_{\pi(n,\tau)})d\tau\\
&\quad+\int_0^t\!\!\int\Theta_n[F(X_{\pi(n,\tau)}, y)]\,{}^cJ_Y(d\tau, dy) \qquad (\alpha=1)\,,\\
Z_t^n&=x+\int_0^t\!\!\int\Theta_n^c[\Omega(X_{\pi(n,\tau)}, X_\tau\ ;\ y)]\\
&\quad\times\{J_X(d\tau, dy)-M^{(\alpha)}(X_\tau, dy)d\tau-I_{(|y|\leq 1)}N^{(\alpha)}(X_\tau, dy)d\tau\}\\
&\quad+\int_0^t b^{(\alpha)}(X_\tau)d\tau+\int_0^t\!\!\int\Theta_n[F(X_{\pi(n,\tau)}, y)]\,{}^cJ_Y(d\tau, dy) \qquad (1<\alpha<2)\,.
\end{aligned}\right.$$

LEMMA 4.3. *$Z_t^n\to X_t$ in probability ($\tilde{P}_x$) for any $t\geq 0$.*

PROOF. Since the proofs in these cases are similar to each other, we shall give the proof only for the case $0<\alpha<1$. From (4.4) and (4.7) we see that

$$\begin{aligned}
Z_t^n-X_t=&\int_0^t\!\!\int\{\Omega(X_{\pi(n,\tau)}, X_\tau\ ;\ y)-y\}J_X(d\tau, dy)\\
&-\int_0^t\!\!\int\Theta_n[\Omega(X_{\pi(n,\tau)}, X_\tau\ ;\ y)]J_X(d\tau, dy)+\int_0^t\!\!\int\Theta_n[F(X_{\pi(n,\tau)}, y)]J_Y(d\tau, dy)\,.
\end{aligned}$$

It is easy to show that the second and the third terms of the right hand tend to 0 in probability as $n\to\infty$. Note that

$$\tilde{E}_x\left[\int_0^t\int\{|\Omega(X_{\pi(n,\tau)},\ X_\tau\ ;\ y)-y|\wedge N\}J_X(d\tau,\ dy)\right]$$

$$=\tilde{E}_x\left[\int_0^t\int\{|F(X_{\pi(n,\tau)},\ y)-F(X_\tau,\ y)|\wedge N\}|y|^{-d-\alpha}dyd\tau\right]$$

$$=\tilde{E}_x\left[\int_0^t d\tau\int_0^\infty dr\int_{|\omega|=1}\{|m^{(\alpha)}(X_{\pi(n,\tau)},\ \omega)^{1/\alpha}\right.$$

$$\left.-m^{(\alpha)}(X_\tau,\ \omega)^{1/\alpha}|r\wedge N\}r^{-1-\alpha}\sigma(d\omega)\right].$$

Therefore we have

$$\int_0^t\int\{|\Omega(X_{\pi(n,\tau)},\ X_\tau\ ;\ y)-y|\wedge N\}J_X(d\tau,\ dy)\rightarrow 0$$

in probability, for fixed N. On the other hand

$$\sup_n\int_{|y|>N}|\Omega(X_{\pi(n,\tau)},\ X_\tau\ ;\ y)-y|J_X(d\tau,\ dy)$$

$$\leqq\text{const.}\int_{|y|>N}|y|J_X(d\tau,\ dy)\rightarrow 0\qquad\text{in probability as}\quad N\rightarrow\infty.$$

This completes the proof. q.e.d.

From the above lemma we see that, for any $f\in\boldsymbol{C}^0$,

$$\tilde{E}_x\left[\int_0^\infty e^{-\lambda t}f(X_{t+s})dt\,\Big|\,\mathcal{W}_s\right]=\lim_{n\to\infty}\tilde{E}_x\left[\int_0^\infty e^{-\lambda t}f(Z_{t+s}^n)dt\,\Big|\,\mathcal{W}_s\right]\tag{4.8}$$

in $\boldsymbol{L}^1(\tilde{W},\ \mathcal{W},\ \tilde{P}_x)$. Fix $(s,\ x)\in\boldsymbol{R}_+\times\boldsymbol{R}^d$, and define

$$V_\lambda^n f(\tilde{w})=\tilde{E}_x\left[\int_0^\infty e^{-\lambda t}f(Z_{s+t}^n)dt\,\Big|\,\mathcal{W}_s\right].\tag{4.9}$$

LEMMA 4.4. *There exists a constant* c_λ^n *such that*

$$|V_\lambda^n f(\tilde{w})|\leqq c_\lambda^n\|f\|_{L^p}\qquad \tilde{P}_x\text{-}a.e.\tag{4.10}$$

for all $f\in\boldsymbol{B}\cap\boldsymbol{L}^p$.

PROOF. We shall prove the lemma only for the case $0<\alpha<1$. Other cases are proved similarly. Let $f\in\mathcal{D}$ and set

$$g(z,\ x)=\mathcal{F}^{-1}[(\lambda-\psi^{(\alpha)}(z,\ \xi))^{-1}\mathcal{F}f(\xi)](x).$$

By Lemma 2.4 and condition [C_3], there exists a constant $c(\lambda)$ independent of z and f such that

$$\|g(z,\ \cdot)\|\leqq c(\lambda)\|f\|_{L^p}.$$

Set $t_k=t_k^{(n)}$ and $t_{k+1}=t_{k+1}^{(n)}$. Using the formula of change of variables of semimartingales (cf. Kunita and Watanabe [11], Meyer [13]), as long as $t_k<t\leqq t_{k+1}$,

we have

$$e^{-\lambda t}g(X_{t_k}, Z_t^n)-e^{-\lambda t_k}g(X_{t_k}, Z_{t_k}^n)+\int_{t_k}^t e^{-\lambda\tau}f(Z_\tau^n)d\tau$$

$$=\int_{t_k}^t e^{-\lambda\tau}\int\{g(X_{t_k}, Z_\tau^n+\Theta_n^c[\Omega(X_{t_k}, X_\tau ; y)])-g(X_{t_k}, Z_\tau^n)\}N^{(\alpha)}(X_\tau, dy)d\tau$$

$$+\{\text{a martingale with mean 0 on the time interval } (t_k, t_{k+1}]\}.$$

Therefore

$$\left|\tilde{E}_x\left[\int_{t_k\vee s}^{t_{k+1}\vee s} e^{-\lambda\tau}f(Z_\tau^n)d\tau\middle|\mathcal{W}_s\right]\right|$$

$$\leqq 2\|g(X_{t_k}, \cdot)\|e^{-\lambda(t_k\vee s)}\Big(1+\lambda^{-1}\int_{|y|>1/n\mu}N_\mu^{(\alpha)}(dy)\Big),$$

where $\mu=\sup\{|\Omega(z, x ; y)| ; z, x, |y|=1\}$ and $N_\mu^{(\alpha)}(dy)$ is the measure in assumption [A_3]. Hence we have

$$\left|\tilde{E}_x\left[\int_s^\infty e^{-\lambda\tau}f(Z_\tau^n)d\tau\middle|\mathcal{W}_s\right]\right|$$

$$\leqq 2c(\lambda)\|f\|_{L^p}\Big(\sum_{k:t_{k+1}\geqq s}e^{-\lambda t_k}\Big)\Big(1+\lambda^{-1}\int_{|y|>1/n\mu}N_\mu^{(\alpha)}(dy)\Big).$$

This implies that there is a constant c_λ^n for which (4.10) holds for any $f\in\mathcal{D}$. Since V_λ^n is a positive bounded operator on $\boldsymbol{B}$, by the Egorov theorem, it is easily proved that (4.10) holds for any $f\in\boldsymbol{B}\cap\boldsymbol{L}^p$. q.e.d.

For a moment, let $f\in\mathcal{D}$ and $g=G_\lambda^{(\alpha)}f=\mathcal{F}^{-1}[(\lambda-\psi^{(\alpha)}(x_0, \xi))^{-1}\mathcal{F}f(\xi)]$. Using the formula of change of variables of semi-martingales, we see that

$$(4.11)\quad\left\{\begin{aligned}
e^{-\lambda t}g(Z_t^n)&=g(x)-\int_0^t e^{-\lambda\tau}f(Z_\tau^n)d\tau+\int_0^t e^{-\lambda\tau}(A_{X_{\pi(n,\tau)}}^{(\alpha)}-A_{x_0}^{(\alpha)})g(Z_\tau^n)d\tau\\
&\quad+\int_0^t\int e^{-\lambda\tau}(g(Z_\tau^n+\Theta_n^c)-g(Z_\tau^n))N^{(\alpha)}(X_\tau, dy)d\tau\\
&\quad+\{\text{a } P_x\text{-martingale with mean 0}\}\qquad (0<a\leqq 1),\\
e^{-\lambda t}g(Z_t^n)&=g(x)-\int_0^t e^{-\lambda\tau}f(Z_\tau^n)d\tau+\int_0^t e^{-\lambda\tau}(A_{X_{\pi(n,\tau)}}^{(\alpha)}-A_{x_0}^{(\alpha)})g(Z_\tau^n)d\tau\\
&\quad+\int_0^t\int e^{-\lambda\tau}\{g(Z_\tau^n+\Theta_n^c)-g(Z_\tau^n)-I_{(|y|\leqq 1)}\Theta_n^c\cdot\partial g(Z_\tau^n)\}N^{(\alpha)}(X_\tau,dy)d\tau\\
&\quad+\int_0^t e^{-\lambda\tau}b^{(\alpha)}(X_\tau)\cdot\partial g(Z_\tau^n)d\tau\\
&\quad+\{\text{a } P_x\text{-martingale with mean 0}\}\qquad (1<\alpha<2),
\end{aligned}\right.$$

where $\Theta_n^c=\Theta_n^c[\Omega(X_{\pi(n,\tau)}, X_\tau ; y)]$. Set $\mu=\sup\{|\Omega(z, x ; y)| ; z, x, |y|=1\}$. Since

$$\Theta_n^c[\Omega(X_{\pi(n,\tau)}, X_\tau ; y)]\in\{0\}\cup\{\theta y ; \mu^{-1}\leqq\theta\leqq\mu\},$$

it follows from [A_2], [A_3] and (4.11) that

$$
(4.12)\quad
\begin{cases}
|V_\lambda^n f| \leqq \|g\| + V_\lambda^n(\sup_z |(A_z^{(\alpha)} - A_{x_0}^{(\alpha)})g|) \\
\qquad + V_\lambda^n\Big(\int |g(\cdot+y) - g(\cdot)| N_\mu^{(\alpha)}(dy)\Big) & (0<\alpha<1), \\
|V_\lambda^n f| \leqq \|g\|\Big(1 + 2\lambda^{-1}\int_{\lambda|y|>1} N_*^{(1)}(dy)\Big) + V_\lambda^n(\sup_z |(A_z^{(1)} - A_{x_0}^{(1)})g|) \\
\qquad + V_\lambda^n\Big(\int_0^\mu d\theta \int_{\lambda|y|\leqq 1} |y\cdot\partial g(\cdot+\theta y)| N_*^{(1)}(dy)\Big) & (\alpha=1), \\
|V_\lambda^n f| \leqq \|g\|\Big(1 + 2\lambda^{-1}\int_{|y|>1} N_*^{(\alpha)}(dy)\Big) + V_\lambda^n(\sup_z |(A_z^{(\alpha)} - A_{x_0}^{(\alpha)})g|) \\
\qquad + V_\lambda^n\Big(\int_0^\mu d\theta \int_{|y|\leqq 1} |y\cdot(\partial g(\cdot+\theta y) - \partial g(\cdot))| N_*^{(\alpha)}(dy) \\
\qquad\qquad + \||b^{(\alpha)}|\|\cdot|\partial g(\cdot)|\Big) & (1<\alpha<2).
\end{cases}
$$

LEMMA 4.5. *As long as λ_0 is sufficiently large, for $\lambda \geqq \lambda_0$, there is a constant c_λ such that*

$$
(4.13)\qquad |V_\lambda^n f(\tilde{w})| \leqq c_\lambda \|f\|_{L^p} \qquad \tilde{P}_x\text{-}a.e.
$$

for all $f \in \boldsymbol{B}\cap\boldsymbol{L}^p$.

PROOF. (*Step* 1) From Lemma 4.3, the constant

$$
c_\lambda^n = \inf\{c\ ;\ \tilde{P}_x[|V_\lambda^n f| > c\|f\|_{L^p}] = 0 \quad \text{for all } f \in \boldsymbol{B}\cap\boldsymbol{L}^p\}
$$

is finite. Let $f \in \mathcal{D}$ and $g = G_\lambda^{(\alpha)} f$. By Theorem 2.2,

$$
V_\lambda^n(\sup_z |(A_z^{(\alpha)} - A_{x_0}^{(\alpha)})g|) \leqq c_\lambda^n \|\sup_z |(A_z^{(\alpha)} - A_{x_0}^{(\alpha)})g|\|_{L^p} \leqq \frac{1}{4} c_\lambda^n \|f\|_{L^p}.
$$

Set

$$
(4.14)\quad
\begin{cases}
S_\lambda^{(\alpha)} f(x) = \int |g(x+y) - g(x)| N_\mu^{(\alpha)}(dy) & (0<\alpha<1), \\
S_\lambda^{(1)} f(x) = \int_0^\mu d\theta \int_{\lambda|y|\leqq 1} |y\cdot\partial g(x+\theta y)| N_*^{(1)}(dy) & (\alpha=1), \\
S_\lambda^{(\alpha)} f(x) = \int_0^\mu d\theta \int_{|y|\leqq 1} |y\cdot(\partial g(x+\theta y) - \partial g(x))| N_*^{(\alpha)}(dy) \\
\qquad + \||b^{(\alpha)}|\|\cdot|\partial g(x)| & (1<\alpha<2).
\end{cases}
$$

From (4.12) and Lemma 2.4, we see that there is a constant c_λ independent of n and f such that

$$
|V_\lambda^n f| \leqq \frac{1}{2} c_\lambda \|f\|_{L^p} + c_\lambda^n\Big(\frac{1}{4}\|f\|_{L^p} + \|S_\lambda^{(\alpha)} f\|_{L^p}\Big).
$$

Suppose that there is a constant λ_0 independent of f satisfying

$$\|S_\lambda^{(\alpha)} f\|_{L^p} \leqq \frac{1}{4}\|f\|_{L^p} \qquad \text{for} \quad \lambda \geqq \lambda_0 . \tag{4.15}$$

Then we have

$$|V_\lambda^n f| \leqq \frac{1}{2}(c_\lambda + c_\lambda^n)\|f\|_{L^p} .$$

Since V_λ^n is a positive bounded operator on $\boldsymbol{B}$, using the Egorov theorem, it can be proved that the above inequality holds for all f in $\boldsymbol{B} \cap \boldsymbol{L}^p$. Therefore we have

$$c_\lambda^n \leqq \frac{1}{2}(c_\lambda + c_\lambda^n)$$

as long as $\lambda \geqq \lambda_0$, which implies that $c_\lambda^n \leqq c_\lambda$.

(*Step* 2) It suffices to prove (4.15). Let $1<\alpha<2$. From (2.8), (2.9) and (2.10) we have

$$\begin{aligned}&\|\partial_j G_\lambda^{(\alpha)} f(\cdot+\theta y) - \partial_j G_\lambda^{(\alpha)} f(\cdot)\|_{L^p} \\ &\qquad \leqq \text{const.}\,|\theta y|^{\alpha-1}\|\,|\partial|^{\alpha-1}\partial_j G_\lambda^{(\alpha)} f\|_{L^p} \leqq \text{const.}\,|y|^{\alpha-1}\|f\|_{L^p} .\end{aligned}$$

Since $\|\partial_j G_\lambda^{(\alpha)} f\|_{L^p} \leqq \text{const.}\,\lambda^{-1+1/\alpha}\|f\|_{L^p}$, we have

$$\begin{aligned}&\left\|\int_0^\mu d\theta \int_{|y|\leqq 1} |y\cdot(\partial G_\lambda^{(\alpha)} f(\cdot+\theta y) - \partial G_\lambda^{(\alpha)} f(\cdot))|\,N_*^{(\alpha)}(dy)\right\|_{L^p} \\ &\leqq \left(2\mu \int_{\lambda^{-1}<|y|^\alpha \leqq 1} |y|\,N_*^{(\alpha)}(dy)\right)\|\,|\partial G_\lambda^{(\alpha)} f|\,\|_{L^p} \\ &\qquad + \text{const.}\left(\int_{|y|^\alpha \leqq \lambda^{-1}} |y|^\alpha N_*^{(\alpha)}(dy)\right)\|f\|_{L^p} \\ &\leqq \text{const.}\left(\int |y|^\alpha ((\lambda^{1/\alpha}|y|)^{1-\alpha} \wedge 1) N_*^{(\alpha)}(dy)\right)\|f\|_{L^p} .\end{aligned}$$

Hence we have inequality (4.15) for sufficiently large λ_0. In the case $0<\alpha\leqq 1$, the proof of (4.15) is much easier. q. e. d.

THEOREM 4.1. *Let $0<\alpha\leqq 2$, and assume* [A_1], [A_2], [A_3] *and* [C_3]. *Then, for any $x \in \boldsymbol{R}^d$, there is at most one solution P_x of the martingale problem associated with L starting from x.*

PROOF. Let $0<\alpha<2$. By (4.8) and (4.13) we have

$$\left|\tilde{E}_x\left[\int_0^\infty e^{-\lambda t} f(X_{s+t})dt \,\middle|\, \tilde{\mathcal{W}}_s\right]\right| \leqq c_\lambda \|f\|_{L^p} \qquad \tilde{P}_x\text{-a. e.}$$

for any $f \in \boldsymbol{C}^0 \cap \boldsymbol{L}^p$ as long as $\lambda \geqq \lambda_0$. This implies that

$$\left|E_x\left[\int_0^\infty e^{-\lambda t} f(X_{s+t})dt \,\middle|\, \mathcal{W}_s\right]\right| \leqq c_\lambda \|f\|_{L^p} \qquad P_x\text{-a. e.},$$

because $\boldsymbol{P}_x$ is the direct product of P_x and Q. This completes the proof for the case $0<\alpha<2$. As for the case $\alpha=2$, the theorem was proved in Komatsu [**9**] assuming the continuity of the coefficient $a(x)$ (cf. Theorem 5.2 in [**9**]). This additional assumption can be removed considering a suitable step function $\pi(n, t)$ similar to (4.5). The detailed proof is omitted. q.e.d.

5. Connection of solutions; proof of main theorems.

Throughout this section $[A_1]$, $[A_2]$ and $[C_1]$ are assumed. Fix p and β so that $0<\beta<\alpha$ and $p>d/(\alpha-\beta)$ in case $0<\alpha\leqq 1$, and that $0<\beta<1$ and $p>d/(\alpha-1)$ in case $1<\alpha\leqq 2$. Let $k_p(x_0)$ and $k_\beta(x_0)$ be the functions in Theorem 2.2. Hereafter we write z for x_0, because x_0 must be taken for a variable in this section.

From condition $[C_1]$ there exists a positive measurable function $r(z)$ such that

$$k_p(z)\vee k_\beta(z) \sup_{|z'-z|\leqq 2r(z)} \Delta^{(\alpha)}(z', z)\leqq \frac{1}{4} \qquad \text{for all} \quad z\in \boldsymbol{R}^d . \tag{5.1}$$

Moreover we can assume that $1/r(z)$ is locally bounded, for $k_p(z)$ and $k_\beta(z)$ are continuous under condition $[C_1]$ by the remark precedent to Lemma 2.2. Let $\rho(t)$ be a smooth function on $\boldsymbol{R}_+$ such that $0\leqq\rho(t)\leqq 1$, $\rho(t)=1$ for $t\leqq 1$ and $\rho(t)=0$ for $t\geqq 2$. Set $\rho_z(x)=\rho(r(z)^{-1}|x-z|)$ and

$$L^{[z]}=A_z^{(\alpha)}+\rho_z(L-A_z^{(\alpha)}) . \tag{5.2}$$

Let $\{X_t, P_x^{[z]}\}$ denote the Markov process associated with the operator $L^{[z]}$ constructed in Theorem 3.1. Set

$$Q_x=P_x^{[x]} .$$

Since $P_x^{[z]}$ is measurable in (x, z), Q_x is also measurable in x. Define

$$S=\inf\{t>0 \ ; \ |X_t-X_0|>r(X_0)\} . \tag{5.3}$$

Let $\{S(n)\}$ be a sequence of stopping times defined by $S(0)=0$ and

$$S(n+1)=S(n)+S\circ\theta_{S(n)} ,$$

where θ_s is the shift operator: $X_t\circ\theta_s=X_{s+t}$. It is possible to construct a sequence $\{P_x^n\}$ of probabilities such that

$$P_x^{n+1}=P_x^n \qquad \text{on} \quad \mathcal{W}_{S(n)} ,$$

$$P_x^{n+1}[\theta_{S(n)}^{-1}(\Gamma)|\mathcal{W}_{S(n)}]=Q_{X_{S(n)}}[\Gamma] \qquad \text{for } \Gamma\in\mathcal{W} \qquad (n\geqq 0) .$$

LEMMA 5.1. *The probability P_x^n solves the martingale problem for the operator L starting from x on the time interval $[0, S(n)]$.*

PROOF. For the sake of simplicity, we shall prove only for $n=2$. Let $f\in\mathcal{D}$ and T be any bounded stopping time. Then

$$
\begin{aligned}
&E_x^2\left[f(X_{T\wedge S(2)})-f(x)-\int_0^{T\wedge S(2)} Lf(X_\tau)d\tau\right] \\
&=E_x^2\left[f(X_{T\wedge S(2)})-f(X_{T\wedge S(1)})-\int_{T\wedge S(1)}^{T\wedge S(2)} Lf(X_\tau)d\tau\right] \\
&\quad+E_x^2\left[f(X_{T\wedge S(1)})-f(x)-\int_0^{T\wedge S(1)} Lf(X_\tau)d\tau\right] \\
&=E_x^2\left[f(X_{(T\wedge S(2))\vee S(1)})-f(X_{S(1)})-\int_{S(1)}^{(T\wedge S(2))\vee S(1)} L^{[X_{S(1)}]}f(X_\tau)d\tau\right] \\
&\quad+E_x^2\left[f(X_{T\wedge S(1)})-f(x)-\int_0^{T\wedge S(1)} L^{[x]}f(X_\tau)d\tau\right] \\
&=E_x^1\left[\int\left\{f(X_{T'}(w')-f(X_{S(1)}(w))\right.\right. \\
&\qquad\left.\left.-\int_0^{T'} L^{[X_{S(1)}(w)]}f(X_\tau(w'))d\tau\right\}Q_{X_{S(1)}(w)}(dw')\right] \\
&\quad+\int\left\{f(X_{T\wedge S(1)})-f(x)-\int_0^{T\wedge S(1)} L^{[x]}f(X_\tau)d\tau\right\}Q_x(dw) \\
&=0,
\end{aligned}
$$

where $T'(w')$ is a certain bounded stopping time. This implies that P_x^2 solves the martingale problem associated with L on the time interval $[0, S(2)]$.

q. e. d.

LEMMA 5.2. $\lim_{n\to\infty} P_x^n[S(n)\leqq t]=0$ *for any* $t<\infty$.

PROOF. For any $\varepsilon>0$, there is a constant R such that

$$\sup_n P_x^n[\sup_{\tau\leqq S(n)\wedge t}|X_\tau|>R]<\varepsilon.$$

Let $r_0=\inf\{r(z)\ ;\ |z|\leqq R\}$ and $f(x)=\rho(|x|/r_0)$. Since

$$\{S(n)\leqq t,\ \sup_{\tau\leqq S(n)\wedge t}|X_\tau|\leqq R\}\subset\left\{\sum_{k=1}^n f(X_{S(k)\wedge t}-X_{S(k-1)\wedge t})\geqq n\right\},$$

we have

$$
\begin{aligned}
P_x^n[S(n)\leqq t]&\leqq\varepsilon+\frac{1}{n}\sum_{k=1}^n E_x^n[f(X_{S(k)\wedge t}-X_{S(k-1)\wedge t})] \\
&=\varepsilon+\frac{1}{n}\sum_{k=1}^n E_x^n\left[\int_{S(k-1)\wedge t}^{S(k)\wedge t} L(f(\cdot-X_{S(k-1)\wedge t}))(X_\tau)d\tau\right] \\
&\leqq\varepsilon+\frac{t}{n}\sup_z\|L(f(\cdot-z))\|.
\end{aligned}
$$

This completes the proof. q. e. d.

PROOF OF THEOREM 1. Let P_x denote the probability on the space $(W, \vee \mathcal{W}_{S(n)})$ such that $P_x = P_x^n$ on $\mathcal{W}_{S(n)}$. From Lemma 5.1 we see that P_x solves the martingale problem for L on the time interval $[0, \lim S(n))$. From Lemma 5.2 we have

$$\lim_{n\to\infty} S(n) = \infty \qquad P_x\text{-a.e.}$$

Hence P_x solves the martingale problem for L. q.e.d.

PROOF OF THEOREM 2. (*Step* 1) Hereafter assume $[\mathrm{A}_1]$, $[\mathrm{A}_2]$, $[\mathrm{A}_3]$ and $[\mathrm{C}_1]$. Then the operator $L^{[z]}$ satisfies the assumption of Theorem 4.1 for each $z \in \boldsymbol{R}^d$. Let $P_x^{[z]}$ be the same probability as before. Let P_x be any solution of the martingale problem for L starting from x. Obviously P_x solves the martingale problem for L on the time interval $[0, S]$, where S is the stopping time given by (5.3). There is a probability $\tilde{Q}_x$ on the space $(W, \mathcal{W})$ such that

$$\tilde{Q}_x = P_x \quad \text{on } \mathcal{W}_S = \mathcal{W}_{S(1)}, \qquad \tilde{Q}_x[\theta_S^{-1}(\Gamma) \mid \mathcal{W}_S] = P_{X_S}^{[x]}(\Gamma) \quad \text{for } \Gamma \in \mathcal{W},$$

where θ_s is the shift operator. It can be proved in the same way as Lemma 5.1 that $\tilde{Q}_x$ is a solution of the martingale problem for $L^{[x]}$ starting from x. From Theorem 4.1 we know that $\tilde{Q}_x$ is uniquely determined. Therefore P_x is uniquely determined on the σ-field $\mathcal{W}_{S(1)}$.

(*Step* 2) Let Q^w be the regular conditional probability of P_x with respect to the σ-field $\mathcal{W}_{S(1)}$. It is easy to show that, for almost all $w\,(P_x)$, Q^w solves the martingale problem for L starting from $X_{S(1)}$ at time $S(1)$ (cf. Theorem 6.1.3 in Stroock and Varadhan [**16**]). That is, for any w except elements of a P_x-null set N, processes $M_t^f \circ \theta_{S(1)}$ $(f \in \mathcal{D})$ are Q^w-martingales, where

$$M_t^f = f(X_t) - f(X_0) - \int_0^t Lf(X_s)ds\,.$$

Set

$$\tilde{M}_t^f = f(X_t) - f(X_0) - \int_0^t L^{[X_0]} f(X_s)ds\,.$$

Then, for $w \notin N$, processes $\tilde{M}_t^f \circ \theta_{S(1)}$ $(f \in \mathcal{D})$ are Q^w-martingale on the time interval $[0, S]$. Namely $Q^w \circ \theta_{S(1)}^{-1}$ solves the martingale problem for $L^{[X_0]}$ on the time interval $[0, S]$ for all $w \notin N$. By the same argument as in step 1, the probability $Q^w \circ \theta_{S(1)}^{-1}$ is uniquely determined on $\mathcal{W}_S$, so that Q^w is uniquely determined on the σ-field $\theta_{S(1)}^{-1}(\mathcal{W}_S)$. Since

$$\mathcal{W}_{S(2)} = \mathcal{W}_{S(1)} \vee \theta_{S(1)}^{-1}(\mathcal{W}_S)\,,$$

the probability P_x is uniquely determined on $\mathcal{W}_{S(2)}$. Using such arguments repeatedly, we know that P_x is uniquely determined on the σ-field $\vee \mathcal{W}_{S(n)}$. From Lemma 5.2 we have $\lim S(n) = \infty$, so $\vee \mathcal{W}_{S(n)} = \mathcal{W}$. q.e.d.

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