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The asymptotic formulas for the number of bound states in the strong coupling limit

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Introduction.

The problem we want to discuss in the present paper is that of the asymptotic number of bound state energies (negative eigenvalues) of the Schrödinger operator $-\Delta+\lambda V$, $\lambda>0$, in the strong coupling limit $\lambda\rightarrow\infty$. This problem has been already discussed by many authors. See, for example, Birman-Borzov [**5**], Kac [**10**], Lieb [**13**], Martin [**14**], Reed-Simon [**17**], Rosenbljum [**19**] and the references quoted there. Roughly speaking, in the case of 3-dimensional space $\boldsymbol{R}_x^3$, the result obtained by these authors can be formulated as follows: Assume that $V(x)$ is real and $V(x)\in L^{3/2}(\boldsymbol{R}_x^3)$, $L^p(\boldsymbol{R}_x^3)$ being the Lebesgue space, and denote by $N(\lambda)$ the number of bound state energies of $-\Delta+\lambda V$. Then $N(\lambda)$ obeys the asymptotic formula

$$N(\lambda)=(6\pi^2)^{-1}\int|V_-(x)|^{3/2}dx\,\lambda^{3/2}(1+o(1)),\qquad \lambda\rightarrow\infty,$$

where $V_-(x)$ denotes the attractive part of $V(x)$; $V_-(x)=\min(0, V(x))$, and the integration is taken over the whole space $\boldsymbol{R}_x^3$. (Here and in what follows, integration with no domain attached is taken over the whole space.) For the proof, [**5**], [**14**] and [**19**] use the min-max principle combined with a technique of Dirichlet-Neumann bracketing, while [**10**], [**13**] and [**17**] use the Feynman-Kac formula. The aim of the present paper is to derive a similar asymptotic formula with the improved remainder estimate $O(\lambda^{-1/2})$ under rather restrictive assumptions on $V(x)$.

We shall formulate the main theorem precisely. We work in the 3-dimensional space and consider only a class of attractive potentials, so it is convenient in the discussion below to write the Schrödinger operator as $-\Delta-\lambda V$, $V>0$, without using the standard notation $-\Delta+\lambda V$. Furthermore, the class of potentials we consider admits finite singularities. For brevity, we confine ourselves to potentials having singularities at the origin only. Such potentials are important in a physical application.

First, we make the assumptions on $V(x)$, which specify the behavior of $V(x)$ as $|x|\rightarrow0$ and as $|x|\rightarrow\infty$. To describe these assumptions, we follow the standard multi-index notations and write $\langle x\rangle=(1+|x|^2)^{1/2}$. We also use the notation $f\sim g$ for positive functions f and g defined over the same region when both f/g and g/f are uniformly bounded.

ASSUMPTION (V). (V.0) $V(x)>0$ and $V(x)$ is C^∞-smooth in $\boldsymbol{R}_x^3\setminus\{0\}$.
(V.1) For $|x|\leqq 1$, $V(x)\sim|x|^{-d}$ for some d, $0\leqq d<2$, and

$$|\partial_x^\alpha V(x)|\leqq C_\alpha V(x)|x|^{-|\alpha|} \qquad \text{for all} \quad \alpha.$$

(V.2) For $|x|\geqq 1$, $V(x)\leqq C\langle x\rangle^{-m}$ for some m, $m>2$, and there exists a constant l, $1\geqq l>3-m$, such that

$$|\partial_x^\alpha V(x)|\leqq C_\alpha V(x)\langle x\rangle^{-l|\alpha|} \qquad \text{for all} \quad \alpha.$$

One of typical examples is the Yukawa potential $\exp(-kr)/r$ $(r=|x|)$, $k>0$, for which we can take $d=1$, $m\gg 1$ (large enough) and $l=0$. Throughout the entire discussion we fix the constants d, m and l with the meanings ascribed in Assumption (V). The choice for l, which seems to be rather peculiar, requires an explanation. The motivation will be made clear in section 2 where we define a class of pseudodifferential operators. The choice for l is closely related to a pair of weight functions which defines a symbol class of pseudodifferential operators.

Now, we can state the main theorem. Under Assumption (V), $V\in L^{3/2}(\boldsymbol{R}_x^3)$ and the operator $-\Delta-\lambda V$ admits a unique self-adjoint realization in $L^2(\boldsymbol{R}_x^3)$. We denote it by $H(\lambda; V)$.

THEOREM. *Assume Assumption* (V) *and denote by* $N(\lambda; V)$ *the number of bound state energies of* $H(\lambda; V)$. *Then*

$$N(\lambda; V)=(6\pi^2)^{-1}\int V(x)^{3/2}dx\ \lambda^{3/2}(1+O(\lambda^{-1/2})), \qquad \lambda\to\infty. \tag{1}$$

The theorem above gives the semi-classical asymptotic formula for the number of bound state energies of $-h^2\Delta-V$, $0<h\ll 1$. We denote by $n(h; V)$ this number. Then $n(h; V)=N(h^{-2}; V)$ and hence

$$n(h; V)=(6\pi^2)^{-1}\int V(x)^{3/2}dx\ h^{-3}(1+O(h)), \qquad h\to 0, \tag{2}$$

if the potential $V(x)$ satisfies Assumption (V).

A semi-classical asymptotic formula similar to (2) has been obtained by Colin de Verdière [6] and Helffer-Robert [7] for general elliptic (pseudo) differential operators. The results in these works, for example, apply to the number of bound state energies less than $-\kappa$ of $-h^2\Delta-V$, $\kappa>0$ being fixed, if $V(x)$ is smooth and if κ is not a critical value of $V(x)$. However, they do not apply to the case $\kappa=0$, because the energy level $\{(x,\xi):|\xi|^2=V(x)\}$ has $(\infty, 0)$ as a critical point.

Finally we note that the theorem has one interesting aspect from a standpoint of the spectral asymptotics. The leading term in formula (1) is associated to the phase space volume of the region $\{(x, \xi): |\xi|^2<\lambda V(x)\}$. This region is of finite volume but is not bounded, which is related to the above-mentioned fact that the energy level has $(\infty, 0)$ as a critical point. In the usual problems such as those for elliptic operators with coefficients growing unboundedly as $|x|\to\infty$ (for example, the Schrödinger operator with growing potentials), the corresponding regions in the phase space are bounded. This makes it difficult to apply directly the existing methods (the min-max principle or the tauberian arguments). In the final section we will make further comments on the main theorem.

§1. Sketch of proof.

The proof of the main theorem is rather long. We first give a sketch for the proof.

Let $\sigma=5/(12-6d)>0$, and define

$$\phi(x;\lambda)=\rho(x;\lambda)\langle x\rangle^{l-1}, \tag{1.1}$$

where $\rho(x;\lambda)=(|x|^2+\lambda^{-2\sigma})^{1/2}$. The first step toward the proof is to approximate $V(x)$ by a smooth potential behaving like $\rho(x;\lambda)^{-d}$ in a neighborhood of the origin. By Assumption (V), we can decompose $V(x)=V_1(x;\lambda)+V_2(x;\lambda)$, where $V_1(x;\lambda)$ has the following properties:

(V_λ-0) $V_1(x;\lambda)>0$ is smooth in $\boldsymbol{R}_x^3$;

(V_λ-1) $V_1\sim\rho(x;\lambda)^{-d}$ for $|x|<1$ and $V_1=O(\langle x\rangle^{-m})$ for $|x|>1$;

(V_λ-2) $|\partial_x^\alpha V_1(x;\lambda)|\leqq C_\alpha V_1(x;\lambda)\phi(x;\lambda)^{-|\alpha|}$

for C_α independent of $\lambda\gg1$, while $V_2(x;\lambda)\geqq0$ has support in $|x|<\lambda^{-\sigma}$, so that

$$\int V_2(x;\lambda)^{3/2}dx=O(\lambda^{-5/4}). \tag{1.2}$$

We also have by the Hölder inequality that

$$\int V_1(x;\lambda)^{3/2}dx=\int V(x)^{3/2}dx+O(\lambda^{-5/6}). \tag{1.3}$$

The main theorem follows from the next result.

PROPOSITION 1. *Assume that* $V(x;\lambda)$ *satisfies* (V_λ-0)-(V_λ-2), *and denote by* $N(\lambda;V_\lambda)$ *the number of bound state energies of* $-\Delta-\lambda V(x;\lambda)$. *Then*

$$N(\lambda;V_\lambda)=(6\pi^2)^{-1}\int V(x;\lambda)^{3/2}dx\,\lambda^{3/2}(1+O(\lambda^{-1/2})),\qquad \lambda\to\infty.$$

If, in addition, a family of potentials $V(x\,;\,\lambda,\,\varepsilon)$ *with parameter* ε *satisfies* (V_λ-0) -(V_λ-2) *uniformly in* ε, *then the remainder estimate above is uniform in* ε.

We shall complete the proof of the main theorem, accepting Proposition 1 as proved.

PROOF OF THEOREM. Recall the decomposition $V=V_1(x\,;\,\lambda)+V_2(x\,;\,\lambda)$. For $\kappa\geqq 0$, let $N_\kappa(\lambda\,;\,V)$ be the number of bound state energies less than $-\kappa$ of $H(\lambda\,;\,V)$; $N_0(\lambda\,;\,V)=N(\lambda\,;\,V)$. We use a similar notation $N_\kappa(\lambda\,;\,V_{j\lambda})$, $1\leqq j\leqq 2$, for the potential $V_j(x\,;\,\lambda)$.

Since $V(x)\geqq V_1(x\,;\,\lambda)$,

$$N(\lambda\,;\,V)\geqq N(\lambda\,;\,V_{1\lambda}). \tag{1.4}$$

This gives a lower bound for $N(\lambda\,;\,V)$. By the Birman-Schwinger principle (Birman [4], Schwinger [20]), $N_\kappa(\lambda\,;\,V)$, $\kappa>0$, coincides with the number of positive eigenvalues greater than 1 of

$$T_\kappa(\lambda)=\lambda(-\Delta+\kappa)^{-1/2}V(-\Delta+\kappa)^{-1/2}.$$

We decompose $T_\kappa(\lambda)=T_{1\kappa}(\lambda)+T_{2\kappa}(\lambda)$, where

$$T_{j\kappa}(\lambda)=\lambda(-\Delta+\kappa)^{-1/2}V_j(x\,;\,\lambda)(-\Delta+\kappa)^{-1/2},$$

and use the Weyl inequality or the Fan inequality ([17], p. 383). Then we have

$$N_\kappa(\lambda\,;\,V)\leqq N_\kappa(\lambda\,;\,(1-2\varepsilon)^{-1}V_{1\lambda})+N_\kappa(\lambda\,;\,\varepsilon^{-1}V_{2\lambda})+1 \tag{1.5}$$

for any ε, $0<\varepsilon\ll 1$. (A similar argument to obtain (1.5) can be found in Tamura [22], pp. 178-181.) By the Cwikel-Lieb-Rosenbljum bound ([17], Theorem XIII. 12) and by (1.2),

$$N_\kappa(\lambda\,;\,\varepsilon^{-1}V_{2\lambda})=\varepsilon^{-3/2}O(\lambda^{1/4}) \tag{1.6}$$

uniformly in κ. We now let $\kappa\to 0$ and take $\varepsilon=\lambda^{-1/2}$. Then Proposition 1, together with (1.3), (1.4) and (1.6), proves the theorem. □

The remainder of the paper is devoted to the proof of Proposition 1. Let $V(x\,;\,\lambda)$ be as in Proposition 1 and let $N_\kappa(\lambda\,;\,V_\lambda)$, $\kappa>0$, be the number of bound state energies less than $-\kappa$ of $-\Delta-\lambda V(x\,;\,\lambda)$. The proof is done by taking the limit $\kappa\to 0$ of $N_\kappa(\lambda\,;\,V_\lambda)$. For technical reasons, we consider $-\Delta+\kappa$ rather than $-\Delta$ itself, because $(-\Delta)^{-1}$ is not well-defined as a bounded operator on $L^2(\boldsymbol{R}_x^3)$. The choice for κ made actually in the proof is a little more technical.

Now, we define the operator $A_\kappa(\lambda)$ by

$$A_\kappa(\lambda)=V(x\,;\,\lambda)^{-1/2}(-\Delta+\kappa)V(x\,;\,\lambda)^{-1/2}. \tag{1.7}$$

$A_\kappa(\lambda)$ is a positive self-adjoint operator with a compact inverse. We denote by $M_\kappa(\mu\,;\,V_\lambda)$ the number of eigenvalues less than μ of $A_\kappa(\lambda)$. By the Birman-Schwinger principle, $M_\kappa(\lambda\,;\,V_\lambda)=N_\kappa(\lambda\,;\,V_\lambda)$. Thus the proof of Proposition 1 is reduced to the study on the asymptotic distribution of eigenvalues of $A_\kappa(\lambda)$.

Let $\{\mu_j\}_{j=1}^{\infty}$, $\mu_j=\mu_j(\lambda, \kappa)$, be a system of eigenvalues of $A_\kappa(\lambda)$ and let $\{u_j\}_{j=1}^{\infty}$, $u_j=u_j(x ; \lambda, \kappa)$, be an orthonormal system of the corresponding eigenfunctions. For given symbol $\omega(x, \xi)$, we define $M_\kappa(\mu ; V_\lambda, \omega)$ by

$$M_\kappa(\mu ; V_\lambda, \omega)=\sum_{\mu_j<\mu}\|\omega(x, D_x)u_j\|^2,$$

where $\|\ \|$ denotes the L^2 norm. Proposition 1 is verified by combining the two lemmas below.

LEMMA 1.1. *Fix θ, $l>\theta>3-m$, close enough to $3-m$, and take δ, $\delta>0$, so small that $\delta(m-2\theta)<m+\theta-3$. Let $\omega(x ; \lambda)$ be a real symbol (independent of ξ) with support in*

$$\Omega_0(\lambda)=\{(x, \xi): \lambda^{1-\delta}\langle x\rangle^{2\theta}V(x ; \lambda)<2\},$$

and assume that

$$|\partial_x^\alpha\omega(x ; \lambda)|\leqq C_\alpha\phi(x ; \lambda)^{-|\alpha|}, \tag{1.8}$$

ϕ being as in (1.1). *Then there exists $\kappa_0(\lambda)$ such that $M_\kappa(\lambda ; V_\lambda, \omega)=O(\lambda)$ uniformly in κ, $0<\kappa<\kappa_0(\lambda)$.*

LEMMA 1.2. *Let θ, δ and $\kappa_0(\lambda)$ be as in Lemma* 1.1. *Let $\omega(x ; \lambda)$ be a real symbol with support in*

$$\Sigma_0(\lambda)=\{(x, \xi): \lambda^{1-\delta}\langle x\rangle^{2\theta}V(x ; \lambda)>1\},$$

and assume that $\omega(x ; \lambda)$ satisfies (1.8). *Then*

$$M_\kappa(\lambda ; V_\lambda, \omega)=(6\pi^2)^{-1}\iint\omega(x ; \lambda)^2V(x ; \lambda)^{3/2}dx\,\lambda^{3/2}+O(\lambda)$$

uniformly in κ, $0<\kappa<\kappa_0(\lambda)$.

Lemma 1.1 is a key lemma which controls the contribution from the critical point $x=\infty$. The proof uses the Feynman-Kac formula. This idea is due to Lieb [**13**]. On the other hand, the proof of Lemma 1.2 is standard and it uses the tauberian argument and the theory of oscillatory integral operators.

§2. Class of oscillatory integral operators.

We begin by fixing a pair of weight functions. Let $V(x ; \lambda)$ be as in Proposition and let $\phi(x ; \lambda)$ be as in (1.1). We define

$$F(x ; \lambda)=\lambda V(x ; \lambda)+\lambda^\delta\langle x\rangle^{-2\theta}$$

for θ and δ as in Lemma 1.1, and fix the pair of weight functions $\{\Phi, \phi\}$ as

$$\Phi=\Phi(x, \xi ; \lambda)=(|\xi|^2+F(x ; \lambda))^{1/2}, \qquad \phi=\phi(x ; \lambda).$$

By definition and by (V_λ-1)

$$F(x\,;\,\lambda)^{1/2}\phi(x\,;\,\lambda)\sim\lambda^{1/2}\rho(x\,;\,\lambda)^{(2-d)/2}\sim\lambda^{1/2-\sigma(2-d)/2}\sim\lambda^{1/12}$$

for $|x|<1$, and

$$F(x\,;\,\lambda)^{1/2}\phi(x\,;\,\lambda)\geqq\lambda^{\delta/2}\langle x\rangle^{l-\theta}$$

for $|x|>1$. By assumption, $l>\theta>3-m$ and hence there exists $c>0$ small enough such that

$$\Phi(x,\,\xi\,;\,\lambda)\phi(x\,;\,\lambda)\geqq\lambda^{c}\langle x\rangle^{c}\langle\xi\rangle^{c}. \tag{2.1}$$

This is required in order that classes of pseudodifferential operators with weight pair $\{\Phi,\,\phi\}$ are closed under calculus of composition and adjunction, and also this is the motivation by which we take the constant l as in (V.2). For later references, we further note that

$$(|\xi|^{2}+\lambda V(x\,;\,\lambda))^{1/2}\sim\Phi(x,\,\xi\,;\,\lambda) \tag{2.2}$$

and

$$V(x\,;\,\lambda)^{-1}(|\xi|^{2}+\lambda V(x\,;\,\lambda))\geqq C\lambda^{c}\langle x\rangle^{c}\langle\xi\rangle^{c} \tag{2.3}$$

with another c, if $(x,\,\xi)\in\Sigma_{0}(\lambda)$, $\Sigma_{0}(\lambda)$ being as in Lemma 1.2. (2.3) is proved in a way similar to (2.1).

2.1. Class of pseudodifferential operators.

DEFINITION 2.1. We denote by $L_{p,q}(\lambda)$, $\lambda\gg1$, the set of all $a(x,\,\xi\,;\,\lambda)$ such that

$$|\partial_{\xi}^{\alpha}\partial_{x}^{\beta}a(x,\,\xi\,;\,\lambda)|\leqq C_{\alpha\beta}\Phi(x,\,\xi\,;\,\lambda)^{-p-|\alpha|}\phi(x\,;\,\lambda)^{-q-|\beta|}$$

for $C_{\alpha\beta}$ independent of λ. We also say that a family of symbols $a(x,\,\xi\,;\,\lambda,\,\varepsilon)$ with parameter ε belongs to $L_{p,q}(\lambda)$ uniformly in ε if the constants $C_{\alpha\beta}$ above are taken independently of ε.

A pseudodifferential operator $a(x,\,D_{x}\,;\,\lambda)$ with symbol $a(x,\,\xi\,;\,\lambda)$ of class $L_{p,q}(\lambda)$ is defined as

$$a(x,\,D_{x}\,;\,\lambda)f(x)=\int e^{ix\cdot\xi}a(x,\,\xi\,;\,\lambda)\hat{f}(\xi)d\xi,\quad d\xi=(2\pi)^{-3/2}d\xi,$$

where $\hat{f}(\xi)$ is the Fourier transform of f; $\hat{f}=\int e^{-ix\cdot\xi}f(x)dx$. We denote by $\mathrm{OPL}_{p,q}(\lambda)$ the class of such operators. By the standard arguments (Beals [**3**], Kumano-go [**12**], Treves [**24**]), we can prove that the classes $\mathrm{OPL}_{p,q}(\lambda)$ are closed under calculus of composition and adjunction. We omit the detailed statements on such a calculus. (For the details, see, for example, Theorems 4.1 and 4.2 in [**3**].) We can also prove that operators of class $\mathrm{OPL}_{0,0}(\lambda)$ are L^{2}-bounded uniformly in λ. The proof is done in exactly the same way as the L^{2}-boundedness of the Hörmander class $S^0_{\rho,\delta}$, $0\leqq\delta<\rho\leqq 1$, is proved.

LEMMA 2.1. *For $k\geqq 0$ integer, there exists $N=N(k)$ such that $a(x, D_x; \lambda)$ of class $\mathrm{OPL}_{N,N}(\lambda)$: $L^2(\boldsymbol{R}^3_x)\to H^{2k}(\boldsymbol{R}^3_x)$ is bounded uniformly in λ, where $H^s(\boldsymbol{R}^3_x)$ denotes the Sobolev space of order s.*

PROOF. Since k is an integer, $(-\Delta)^k\in\mathrm{OPL}_{-2k,0}(\lambda)$, and hence $(-\Delta)^k a(x, D_x; \lambda)$ $\in\mathrm{OPL}_{0,0}(\lambda)$ by (2.1), if N is large enough. This proves the lemma. □

2.2. Class of Fourier integral operators. The proof of Lemma 1.2 uses the theory of Fourier integral operators. We here introduce a class of such operators.

DEFINITION 2.2. We denote by $A_{p,q}(\lambda)$, $\lambda\gg 1$, the set of all $a(x, \xi, y; \lambda)$ such that

$$|\partial_\xi^\alpha\partial_x^\beta\partial_y^\gamma a(x, \xi, y; \lambda)|\leqq C_{\alpha\beta\gamma}\Phi(x, \xi; \lambda)^{-p-|\alpha|}\phi(x; \lambda)^{-q-|\beta|}\phi(y; \lambda)^{-|\gamma|}$$

for $C_{\alpha\beta\gamma}$ independent of λ.

Next, we describe the conditions to be imposed on a phase function (cf. Kumano-go [**11**]).

CONDITION (Ψ). We say that a real-valued smooth function $\psi(x, \xi, y; \lambda)$, $\lambda\gg 1$, satisfies (Ψ), if

(Ψ.0) $\psi_1=\psi-(x-y)\cdot\xi=O(|\xi|\,|x-y|^2)$ as $|\xi|\to\infty$ and $x\to y$;

(Ψ.1) $|\partial_\xi^\alpha\partial_x^\beta\partial_y^\gamma\psi_1|\leqq C_{\alpha\beta\gamma}|\xi|\Phi(x, \xi; \lambda)^{-|\alpha|}\phi(x; \lambda)^{1-|\beta|}\phi(y; \lambda)^{-|\gamma|}$

for $C_{\alpha\beta\gamma}$ independent of λ;

(Ψ.2) $|\nabla_x\psi_1|<|\xi|/100$ and $\|\nabla_x\nabla_\xi\psi_1\|<1/100$,

where $\nabla_x\nabla_\xi\psi_1$ is the 3×3 matrix with components $(\partial^2/\partial x_j\partial\xi_k)\psi_1$, $1\leqq j, k\leqq 3$, and $\|\ \|^2$ is the matrix norm defined by taking the square summation of all components. (The numerical constant 1/100 above has no special meaning, and it means only that the quantities in (Ψ.2) are sufficiently small.)

For later reference, we here introduce the notation

$$\tilde{\nabla}_x\psi(x, x', \xi, y; \lambda)=\int_0^1\nabla_x\psi(x'+s(x-x'), \xi, y; \lambda)ds. \tag{2.4}$$

Now, we define Fourier integral operators. Let $a(x, \xi, y; \lambda)$ be of class $A_{p,q}(\lambda)$ and let $\psi(x, \xi, y; \lambda)$ satisfy (Ψ). Furthermore, let $H(\xi, y)$ be a smooth function (independent of x) with compact support. Then our Fourier integral operator is defined as

$$I_\psi(aH)f(x)=\iint e^{i\psi(x,\xi,y;\lambda)}a(x, \xi, y; \lambda)H(\xi, y)f(y)dyd\xi.$$

It is not hard to see that $I_\psi(aH)$ is well-defined as an operator from $\mathcal{S}$ into itself,

$\mathcal{S}$ being the Schwartz space. As is seen from Condition (Ψ), the phase function ϕ takes the form perturbed from $(x-y)\cdot\xi$ and hence the operator $I_\phi(aH)$ has many properties similar to those of pseudodifferential operators. For example, the following composition formula can be proved in almost the same way as in the case of pseudodifferential operators ([**3**], [**11**], [**24**]).

LEMMA 2.2. *Let* $b(x, \xi; \lambda)$ *be of class* $L_{p_1,q_1}(\lambda)$ *and let* $a(x, \xi, y; \lambda)$ *be of class* $A_{p_2,q_2}(\lambda)$. *Then:*

(i) *The composition* $b(x, D_x; \lambda)I_\phi(aH)$ *is represented as* $I_\phi(cH)$ *with* $c(x, \xi, y; \lambda)$ *of class* $A_{p,q}(\lambda)$, *where* $p=p_1+p_2$ *and* $q=q_1+q_2$.

(ii) *The amplitude function c admits the following asymptotic expansion: For any N*

$$c=\sum_{|\alpha|=0}^{N-1} L_\alpha(x, \nabla_x\phi; b)a+r_N$$

with r_N *of class* $L_{p+N,q+N}(\lambda)$, *where* $L_\alpha: A_{p,q}(\lambda)\to A_{p_\alpha,q_\alpha}(\lambda)$ $(p_\alpha=p+|\alpha|, q_\alpha=q+|\alpha|)$ *is the mapping defined by*

$$L_\alpha(x, \nabla_x\phi; b)a=(\alpha!)^{-1}D_u^\alpha\{\partial_\xi^\alpha b(x, \tilde{\nabla}_x\phi; \lambda)a(x+u, \xi, y; \lambda)\}_{u=0},$$

$\tilde{\nabla}_x\phi=\tilde{\nabla}_x\phi(x, x+u, \xi, y; \lambda)$ *being as in* (2.4).

§ 3. Parametrices of resolvents.

Let $A_\kappa(\lambda)$ be the self-adjoint operator defined by (1.7). The aim in the present section is to construct microlocal parametrices of the resolvents of $A_\kappa(\lambda)$ by use of pseudodifferential operators of class $\mathrm{OPL}_{p,q}(\lambda)$.

We begin by specifying the constant κ. Let $\Sigma_0(\lambda)$ be as in Lemma 1.2 and define

$$\kappa_0(\lambda)=\inf_{(x,\xi)\in\Sigma_0(\lambda)} \Phi(x, \xi; \lambda)/\phi(x; \lambda).$$

Roughly speaking, as $\lambda\to\infty$, $\kappa_0(\lambda)>0$ behaves like $\kappa_0(\lambda)\to 0$ for $m<3$ and like $\kappa_0(\lambda)\to\infty$ for $m>3$. This will be easily seen, if $V(x)\sim\langle x\rangle^{-m}$. From now on we assume that κ ranges over the interval $(0, \kappa_0(\lambda))$. If κ is as above, then by (2.2)

$$\kappa(|\xi|^2+\lambda V(x; \lambda))^{-1}\leqq C\Phi(x, \xi; \lambda)^{-1}\phi(x; \lambda)^{-1} \tag{3.1}$$

in $\Sigma_0(\lambda)$. We now write the total symbol of $A_\kappa(\lambda)+\lambda$ as

$$(A_0(x, \xi; \lambda)+\lambda)(1+B_0(x, \xi; \lambda, \kappa))$$

where $A_0(x, \xi; \lambda)=V(x; \lambda)^{-1}|\xi|^2$. If $\omega(x, \xi; \lambda)$ $(\in L_{0,0}(\lambda))$ has support in $\Sigma_0(\lambda)$, then it follows from (3.1) and (2.2) that $\omega(x, \xi; \lambda)B_0(x, \xi; \lambda, \kappa)$ belongs to $L_{1,1}(\lambda)$ uniformly in κ.

Now, let $\omega(x, \xi; \lambda)$ be as above. We construct a parametrix of $(A_\kappa(\lambda)+\lambda)^{-1}$ $\omega(x, D_x; \lambda)$. We formally set

$$(A_\kappa(\lambda)+\lambda)^{-1}\omega(x, D_x ; \lambda)=\sum_{j=0}^{\infty} p_j(x, D_x ; \lambda).$$

For notational brevity, we occasionally drop the parameter κ, the dependence on κ being always uniform. The symbols p_j are determined in the usual way and a simple inductive argument shows that p_j is supported in $\Sigma_0(\lambda)$ and takes the form

$$p_j=(A_0(x, \xi ; \lambda)+\lambda)^{-1}a_j(x, \xi ; \lambda)$$

with $a_j \in L_{j,j}(\lambda)$. We fix N large enough and define

$$P_0(x, \xi ; \lambda)=\sum_{j=0}^{N-1} p_j(x, \xi ; \lambda).$$

Then by construction

$$(A_\kappa(\lambda)+\lambda)^{-1}\omega=P_0+(A_\kappa(\lambda)+\lambda)^{-1}R_N$$

with some $R_N \in \mathrm{OPL}_{N,N}(\lambda)$. The symbol $R_N(x, \xi ; \lambda)$ has support in $\Sigma_0(\lambda)$. This follows from the fact that the operator $A_\kappa(\lambda)$ under consideration is a differential (not pseudodifferential) operator.

The same reasoning as above applies to the iterated power $A_\kappa(\lambda)^k$. We have

$$(3.2) \qquad (A_\kappa(\lambda)^k+\lambda^k)^{-1}\omega=P_{k0}+(A_\kappa(\lambda)^k+\lambda^k)^{-1}R_N,$$

where $R_N \in \mathrm{OPL}_{N,N}(\lambda)$, and

$$P_{k0}(x, \xi ; \lambda)=(A_0(x, \xi ; \lambda)^k+\lambda^k)^{-1}\sum_{j=0}^{N-1} a_j(x, \xi ; \lambda)$$

with another $a_j \in L_{j,j}(\lambda)$.

The next lemma is used together with the Agmon kernel theorem (Agmon [**1**], Theorem 3.1), when we estimate kernel functions of integral operators.

LEMMA 3.1. *Let $\omega(x, \xi ; \lambda)$ be of class $L_{0,0}(\lambda)$ and be supported in $\Sigma_0(\lambda)$. Define*

$$G(\lambda)=\omega(x, D_x ; \lambda)^*(A_\kappa(\lambda)^k+\lambda^k)^{-1}.$$

If k is taken large enough, then $G(\lambda): L^2(\boldsymbol{R}_x^3) \to H^4(\boldsymbol{R}_x^3)$ is bounded uniformly in κ and λ.

PROOF. The proof uses the relation (3.2). If k is large enough, then by (2.3) P_{k0} is of class $\mathrm{OPL}_{N',N'}(\lambda)$ for N' large enough, and hence so is P_{k0}^*. Thus, taking the adjoint of (3.2) and using Lemma 2.1 prove the lemma. □

Let k and $\omega(x, \xi ; \lambda)$ be as in Lemma 3.1. For example, Lemma 3.1 is applied to estimate a kernel of an operator of the following form:

$$S(\lambda)=\omega(x, D_x ; \lambda)^*(A_\kappa(\lambda)^k+\lambda^k)^{-1}R_N(x, D_x ; \lambda),$$

where $R_N(x, \xi ; \lambda)$ is assumed to be supported in $\Sigma_0(\lambda)$ and to be of class $L_{N,N}(\lambda)$ for N large enough.

LEMMA 3.2. *Let* $S(\lambda)$ *be as above and denote by* $K_S(x, y; \lambda)$ *the kernel of* $S(\lambda)$. *If* N *is taken large enough, then*

$$|K_S(x, x; \lambda)| \leqq C\lambda^{-k}\langle x\rangle^{-4}.$$

PROOF. Take $\omega_0(x; \lambda)$ ($\in L_{0,0}(\lambda)$) with support in $\Sigma_0(\lambda)$ in such a way that

$$\omega_0(x; \lambda)R_N(x, \xi; \lambda) = R_N(x, \xi; \lambda)$$

and decompose $S(\lambda)$ as

$$S(\lambda) = [\omega(x, D_x; \lambda)^*(A_\kappa(\lambda)^k + \lambda^k)^{-1}\omega_0]R_N(x, D_x; \lambda).$$

We denote by $T(\lambda)$ the operator in bracket and by $K_T(x, y; \lambda)$ the kernel of $T(\lambda)$. Then Agmon's kernel theorem combined with Lemma 3.1 proves that $K_T(x, y; \lambda) = O(1)$. On the other hand, if N is large enough, then the kernel $K_R^{(N)}(x, y; \lambda)$ of R_N is estimated as

$$|K_R^{(N)}(x, y; \lambda)| \leqq C\lambda^{-k}\langle x\rangle^{-4}\langle x-y\rangle^{-4}$$

by making use of (2.1) and by integrating by parts. Hence the composite kernel K_S obeys the estimate in the lemma. □

§4. Proof of Lemma 1.1.

The present section is devoted to the proof of Lemma 1.1.

LEMMA 4.1. *Under the same assumptions as in Lemma* 1.1,

$$M_\kappa(\lambda; V_\lambda, \omega) \leqq C_1\int(\omega(x; \lambda)V(x; \lambda))^{3/2}dx\, \lambda^{3/2} + C_2\lambda$$

for C_j, $1 \leqq j \leqq 2$, *independent of* κ *and* λ.

We first complete the proof of Lemma 1.1, accepting the lemma above as proved.

PROOF OF LEMMA 1.1. We may assume that $0 \leqq \omega \leqq 1$. Hence, for the proof, it is sufficient to prove that

$$\int_{\lambda^{1-\delta}\langle x\rangle^{2\theta}V(x;\lambda)<2} V(x; \lambda)^{3/2}dx = O(\lambda^{-1/2}).$$

Recall that θ, $1 > \theta > 3-m$, is close enough to $3-m$ and that

$$0 < \delta < (m+\theta-3)/(m-2\theta) \ll 1.$$

Set

$$\varepsilon_0 = m+\theta-3-(m-2\theta)\delta/(2-2\delta), \qquad 0 < \varepsilon_0 \ll 1,$$

and $\nu = \varepsilon_0/(m-2\theta) + 1/(2-2\delta)$, $\nu > 0$. We write

$$V(x; \lambda) = [\langle x\rangle^{2\theta}V(x; \lambda)]^\nu U(x; \lambda),$$

where $U(x;\lambda)=\langle x\rangle^{-2\theta\nu}V(x;\lambda)^{3/2-\nu}$, and we apply the Hölder inequality with $(p, q)=(3/\varepsilon_0, 3/(3-\varepsilon_0))$ to the decomposition above. By $(V_\lambda-1)$, we can assume that $V(x;\lambda)=O(\langle x\rangle^{-m})$ in the integral domain, and hence $U(x;\lambda)^q=O(\langle x\rangle^{-3q})$, $q>1$, is integrable. Thus the integral under consideration is estimated by

$$C\left[\int_{\lambda^{1-\delta}\langle x\rangle^{2\theta}V(x;\lambda)<2}(\langle x\rangle^{2\theta}V(x;\lambda))^{3\nu/\varepsilon_0}dx\right]^{\varepsilon_0/3}. \tag{4.1}$$

We now define

$$m(s;\lambda)=\mathrm{meas}(\{x:\langle x\rangle^{-2\theta}V(x;\lambda)^{-1}<s\}).$$

Then, by $(V_\lambda-1)$, $m(s;\lambda)=O(s^{3/(m-2\theta)})$, $s\to\infty$, and the integral (4.1) is estimated as

$$\left[\int_{\lambda^{1-\delta}/2}^{\infty}s^{-3\nu/\varepsilon_0}dm(s;\lambda)\right]^{\varepsilon_0/3}=O(\lambda^{-1/2}).$$

This proves the lemma. □

PROOF OF LEMMA 4.1. We again drop the parameter κ in the proof, and assume that $0\leqq\omega\leqq1$. We further assume that $\omega=1$ on $\boldsymbol{R}_x^3\times\boldsymbol{R}_\xi^3\setminus\Sigma_0(\lambda)$, which loses no generality. Estimate (1.8) implies that $\omega\in L_{0,0}(\lambda)$. We denote by V_λ the multiplication by $V(x;\lambda)$. Set $Q(\lambda)=(-\Delta+\kappa+\lambda V_\lambda)^{-1}$, so that

$$(A_\kappa(\lambda)+\lambda)^{-1}=V_\lambda^{1/2}Q(\lambda)V_\lambda^{1/2}. \tag{4.2}$$

Furthermore, recall the notations $\{\mu_j\}_{j=1}^\infty$ and $\{u_j\}_{j=1}^\infty$ (see section 1), and denote by $(\ ,\ )$ the scalar product in $L^2(\boldsymbol{R}_x^3)$.

We start with the following inequality:

$$M_\kappa(\lambda;V_\lambda,\omega)\leqq4\lambda^2\sum_{j=1}^\infty(T(\lambda)u_j, u_j)=4\mathrm{Trace}(\lambda^2T(\lambda)),$$

where $T(\lambda)=V_\lambda^{1/2}Q(\lambda)\omega^2V_\lambda Q(\lambda)V_\lambda^{1/2}$. This inequality can be easily obtained by making use of the trivial estimate $\mu_j+\lambda<2\lambda$ for $\mu_j<\lambda$ and of the relation

$$u_j=(\mu_j+\lambda)V_\lambda^{1/2}Q(\lambda)V_\lambda^{1/2}u_j,$$

and also it is easy to see that $T(\lambda)$ is of trace class. We write

$$Q(\lambda)\omega=\omega Q(\lambda)+Q(\lambda)r(\lambda)Q(\lambda),$$

where $r(\lambda)=\omega(-\Delta)-(-\Delta)\omega$, and use the cyclicity of trace. Then $\mathrm{Trace}(\lambda^2T(\lambda))$ can be decomposed into two terms;

$$I_1(\lambda)=\mathrm{Trace}(\lambda W_\lambda Q(\lambda)(\lambda W_\lambda)Q(\lambda)),$$

$$I_2(\lambda)=\lambda^2\mathrm{Trace}(V_\lambda^{1/2}Q(\lambda)r(\lambda)Q(\lambda)\omega V_\lambda Q(\lambda)V_\lambda^{1/2}),$$

where $W_\lambda=\omega V_\lambda$ $(W(x;\lambda)=\omega(x;\lambda)V(x;\lambda)\leqq V(x;\lambda))$. We assert that:

$$I_1(\lambda)\leqq C\int(\omega(x;\lambda)V(x;\lambda))^{3/2}dx\ \lambda^{3/2}, \tag{4.3}$$

$$I_2(\lambda)=O(\lambda). \tag{4.4}$$

The assertions above complete the proof at once. □

PROOF OF (4.3). The proof uses the idea in the proof of Theorem XIII.12, Reed-Simon [**17**], to which the Feynman-Kac formula is elegantly applied. Set $P(\lambda)=(-\Delta+\kappa+\lambda W_\lambda)^{-1}$ and denote by $K_Q(x, y; \lambda)$ and $K_P(x, y; \lambda)$ the kernels of $Q(\lambda)$ and $P(\lambda)$, respectively. Then, by the maximum principle for elliptic operators of second order, $0\leqq K_Q(x, y; \lambda)\leqq K_P(x, y; \lambda)$, and hence

$$I_1(\lambda)\leqq \mathrm{Trace}(\lambda W_\lambda P_0(\lambda W_\lambda)P(\lambda)),$$

where $P_0=(-\Delta+\kappa)^{-1}$. The term on the right side can be estimated in exactly the same way as in the proof of Theorem XIII.12, and we obtain (4.3). □

PROOF OF (4.4). The proof is based on the theory of pseudodifferential operators of class $\mathrm{OPL}_{p,q}(\lambda)$. By (4.2)

$$I_2(\lambda)=\lambda^2\sum_{j=1}^{\infty}((A_\kappa(\lambda)+\lambda)^{-1}u_j,\ q(\lambda)(A_\kappa(\lambda)+\lambda)^{-1}u_j),$$

where $q(\lambda)=\omega(A_\kappa(\lambda)+\lambda)^{-1}V_\lambda^{-1/2}r(\lambda)^*V_\lambda^{-1/2}$. Since $\omega=1$ on $\boldsymbol{R}_x^3\times\boldsymbol{R}_\xi^3\setminus\Sigma_0(\lambda)$ by assumption, the symbol of the differential operator $p(\lambda)=V_\lambda^{-1/2}r(\lambda)^*V_\lambda^{-1/2}$ has support in $\Sigma_0(\lambda)$. We follow the arguments in section 3 to construct a parametrix of $q(\lambda)$. More precisely, the arguments there should be slightly modified, because $p(\lambda)$ is not of class $\mathrm{OPL}_{0,0}(\lambda)$ but of class $\mathrm{OPL}_{-N,-N}(\lambda)$ for N large enough. However, such a modification can be easily justified. In any case, $q(\lambda)$ is represented in the form

$$q(\lambda)=a_1+\omega(A_\kappa(\lambda)+\lambda)^{-1}a_N,$$

where $a_N\in\mathrm{OPL}_{N,N}(\lambda)$ for N large enough, while $a_1\in\mathrm{OPL}_{1,1}(\lambda)$ and its symbol is supported in $\Sigma_0(\lambda)$.

LEMMA 4.2. *Let $B(\lambda): L^2(\boldsymbol{R}_x^3)\to L^2(\boldsymbol{R}_x^3)$ be bounded uniformly in λ and let $\omega_N(x, \xi; \lambda)$ be of class $L_{N,N}(\lambda)$. If N is taken large enough, then*

$$\sum_{j=1}^{\infty}|(\omega_N u_j,\ B(\lambda)u_j)|=O(1), \qquad \lambda\to\infty.$$

LEMMA 4.3. *Let $\omega_1(x, \xi; \lambda)$ be of class $L_{1,1}(\lambda)$ with support in $\Sigma_0(\lambda)$. Then*

$$\sum_{j=1}^{\infty}|((A_\kappa(\lambda)+\lambda)^{-1}u_j,\ \omega_1(A_\kappa(\lambda)+\lambda)^{-1}u_j)|=O(\lambda^{-1}).$$

COMPLETION OF PROOF OF (4.4). We use Lemma 4.2 with $B(\lambda)=(A_\kappa(\lambda)+\lambda)^{-1}\omega$. Then Lemmas 4.2 and 4.3 complete the proof of (4.4) at once. □

PROOF OF LEMMA 4.2. We decompose $\omega_N(x, D_x; \lambda)$ as

$$\omega_N=\langle x\rangle^{-4}\langle D_x\rangle^{-4}[\langle D_x\rangle^4\langle x\rangle^4\omega_N].$$

The operator $\langle x\rangle^{-4}\langle D_x\rangle^{-4}$ is of Hilbert-Schmidt class and so is the operator in bracket, if N is large enough. This proves the lemma. □

PROOF OF LEMMA 4.3. By assumption, there exists a non-negative symbol $\omega_{1/2}(x, \xi; \lambda)$ $(\in L_{1/2,1/2}(\lambda))$ with support in $\Sigma_0(\lambda)$ such that

$$\omega_1(x, \xi; \lambda)=\omega_{1/2}(x, \xi; \lambda)^2\omega_0(x, \xi; \lambda)$$

with $\omega_0 \in L_{0,0}(\lambda)$. Hence we can write

$$\omega_1(x, D_x; \lambda)=\omega_{1/2}(x, D_x; \lambda)\omega_0\omega_{1/2}(x, D_x; \lambda)^*+\omega_N$$

with another $\omega_0 \in \mathrm{OPL}_{0,0}(\lambda)$, where $\omega_N \in \mathrm{OPL}_{N,N}(\lambda)$ for $N \gg 1$. Thus we have only to prove that

$$\|(A_\kappa(\lambda)+\lambda)^{-1}\omega_{1/2}\|_{\mathrm{H.S}}=O(\lambda^{-1/2}),$$

where $\| \ \|_{\mathrm{H.S}}$ denotes the norm of Hilbert-Schmidt class. We again follow the arguments in section 3 to construct a parametrix of $(A_\kappa(\lambda)+\lambda)^{-1}\omega_{1/2}$, and we obtain

$$(A_\kappa(\lambda)+\lambda)^{-1}\omega_{1/2}=p_0(x, D_x; \lambda)+(A_\kappa(\lambda)+\lambda)^{-1}\omega_N$$

with another $\omega_N \in \mathrm{OPL}_{N,N}(\lambda)$, where $p_0(x, \xi; \lambda)$ takes the form

$$p_0=(A_0(x, \xi; \lambda)+\lambda)^{-1}a_{1/2}(x, \xi; \lambda), \qquad a_{1/2} \in L_{1/2,1/2}(\lambda),$$

and we have again written $A_0(x, \xi; \lambda)=|\xi|^2 V(x; \lambda)^{-1}$. It can be easily verified that:

$$\|(A_\kappa(\lambda)+\lambda)^{-1}\omega_N\|_{\mathrm{H.S}} \leqq \lambda^{-1}\|\omega_N\|_{\mathrm{H.S}}=O(\lambda^{-1});$$

$$\|p_0\|_{\mathrm{H.S}}^2 \leqq C\iint (A_0(x, \xi; \lambda)+\lambda)^{-2}a_{1/2}(x, \xi; \lambda)^2 d\xi dx=O(\lambda^{-1}).$$

This completes the proof. □

As an immediate consequence of Lemma 4.3, we obtain

LEMMA 4.4. *Let $\omega_1(x, \xi; \lambda)$ be as in Lemma 4.3. Then*

$$\sum_{\mu_j<\lambda} |(\omega_1 u_j, u_j)|=O(\lambda).$$

This lemma is used for the proof of Lemma 1.2.

§5. Proof of Lemma 1.2.

The present and next sections are devoted to the proof of Lemma 1.2. We first break up $\Sigma_0(\lambda)$ into two regions;

$$\Omega_1(\lambda)=\{(x, \xi)\in\Sigma_0(\lambda): |\xi|^2<\lambda V(x; \lambda)/2 \quad \text{or} \quad |\xi|^2>2\lambda V(x; \lambda)\},$$

$$\Omega_2(\lambda)=\{(x, \xi)\in\Sigma_0(\lambda): \lambda V(x; \lambda)/3<|\xi|^2<3\lambda V(x; \lambda)\}.$$

LEMMA 5.1. *Let a real symbol $\omega(x, \xi; \lambda)$ be of class $L_{0,0}(\lambda)$ and be supported in $\Omega_1(\lambda)$. Then*

$$(5.1)\qquad M_\kappa(\lambda\,;\, V_\lambda,\, \omega)=(2\pi)^{-3}\iint_{|\xi|^2<\lambda V(x;\,\lambda)} \omega(x,\, \xi\,;\, \lambda)^2 d\xi dx+O(\lambda).$$

We should note that the symbol ω specified above really exists. The lemma above estimates the contribution from the outside of the energy level $\{(x, \xi): |\xi|^2=\lambda V(x\,;\,\lambda)\}$. The proof is based on the "resolvent method", which is more or less standard in the study on the spectral asymptotics (Agmon [2], Robert [18]). So we give only a sketch for the proof. (For the details, see the proof of Theorem 1.1, [22].)

SKETCH OF PROOF OF LEMMA 5.1. (1) First, fix an integer k so large that $A_\kappa(\lambda)^{-k}$ is of trace class. Let $\zeta_0=\lambda^k(1+i\lambda^{-1/2})$, and introduce an oriented curve $c(\zeta_0)$ from $\bar{\zeta}_0$ to ζ_0, not intersecting the positive axis, such that: (i) the length of $c(\zeta_0)$ is bounded by $C\lambda^k$; (ii)

$$(5.2)\qquad |A_0(x,\, \xi\,;\, \lambda)^k-\zeta|\geqq C(A_0(x,\, \xi\,;\, \lambda)^k+\lambda^k)$$

for $(x, \xi)\in\Sigma_0(\lambda)$ and $\zeta\in c(\lambda_0)$. Let $\omega(x, \xi\,;\, \lambda)$ be as in the lemma. We define

$$E(\zeta,\, \lambda)=\omega^*(A_\kappa(\lambda)^k-\zeta)^{-1}\omega$$

and denote by $K_E(x, y\,;\, \zeta, \lambda)$ its kernel.

(2) Trace$[E(\zeta, \lambda)]$ can be represented as the Stieltjes transform of $M_\kappa(\mu\,;\, V_\lambda, \omega^*)$;

$$\text{Trace}[E(\zeta,\, \lambda)]=\int_0^\infty(\mu-\zeta)^{-1}dM_\kappa(\mu^{1/k}\,;\, V_\lambda,\, \omega^*).$$

Hence, by the inversion formula due to Pleijel [16],

$$|M_\kappa(\lambda\,;\, V_\lambda,\, \omega^*)-(2\pi i)^{-1}\int_{c(\zeta_0)}\text{Trace}[E(\zeta,\, \lambda)]d\zeta|\leqq C\lambda^{k-1/2}|\text{Trace}[E(\zeta_0,\, \lambda)]|.$$

(3) (5.2) enables us to construct a parametrix of $E(\zeta, \lambda)$ by the arguments in section 3 and hence we can approximate $K_E(x, x\,;\, \zeta, \lambda)$ with the error estimate $O(\lambda^{-k}\langle x\rangle^{-4})$. (For the proof, Lemma 3.2 is used.) We can prove

$$\text{Trace}[E(\zeta,\, \lambda)]=O(\lambda^{-k+3/2})$$

and by a simple application of Cauchy's integral formula

$$(2\pi i)^{-1}\int_{c(\zeta_0)}\text{Trace}[E(\zeta,\, \lambda)]d\zeta=k_0(\lambda)+O(\lambda),$$

where $k_0(\lambda)$ denotes the leading term in formula (5.1). Thus

$$M_\kappa(\lambda\,;\, V_\lambda,\, \omega^*)=k_0(\lambda)+O(\lambda),$$

from which formula (5.1) follows by Lemmas 4.2 and 4.4. □

§ 6. Completion of proof of Lemma 1.2.

In this section we work in a neighborhood of the energy level $\{(x, \xi): |\xi|^2 = \lambda V(x; \lambda)\}$.

LEMMA 6.1. *Let $\Omega_2(\lambda)$ be as in section 5. Let a real symbol $\omega(x, \xi; \lambda)$ be of class $L_{0,0}(\lambda)$ and be supported in $\Omega_2(\lambda)$. Then $M_\kappa(\lambda; V_\lambda, \omega)$ obeys the same asymptotic formula as in* (5.1).

The lemma above can be also verified in almost the same way as in the proof of Theorem 4.1, Tamura [**23**]. So we give only a sketch for the proof. The proof is based on the construction of microlocal parametrices of $\exp(-itA_\kappa(\lambda))$ and on the tauberian arguments due to Hörmander [**8**].

SKETCH OF PROOF. (0) We begin by fixing several notations. We define

$$G_x(z, \varepsilon, \lambda) = \{x : |x-z| < \varepsilon\phi(z; \lambda)\}$$

for z such that $\lambda^{1-\delta}\langle z\rangle^{2\theta} V(z; \lambda) > 1$, and

$$\Pi(h) = \{\tilde{\xi} \in S^2 : |\tilde{\xi} - \xi_0| < h\}$$

for fixed $\xi_0 \in S^2$, S^2 being the unit sphere. We denote by X the pair (S, h), $S>1$. and further define

$$\Gamma_\xi(z, X, \lambda) = \{\xi : \lambda V(z; \lambda)/S < |\xi|^2 < S\lambda V(z; \lambda),\ \xi/|\xi| \in \Pi(h)\},$$

$$\Theta(z, \varepsilon, X, \lambda) = G_x(z, \varepsilon, \lambda) \times \Gamma_\xi(z, X, \lambda) \times G_y(z, \varepsilon, \lambda).$$

Throughout the proof z is assumed to be fixed as above.

(1) Let $\omega(x, \xi; \lambda)$ be as in the lemma. For the moment we assume that ω is supported in

$$\{(x, \xi) \in \Omega_2(\lambda) : \xi/|\xi| \in \Pi(h_0)\}$$

for h_0 small enough. We set $\omega(\xi; \lambda, z) = \omega|_{x=z}$, so that $\omega(\xi; \lambda, z)$ has support in $\Gamma_\xi(z, X_3, \lambda)$ with $X_3 = (3, h_0)$. Let $\chi(x; \lambda, z)$ $(\in L_{0,0}(\lambda))$ be a cut-off function with support in $G_x(z, \varepsilon_0, \lambda)$, ε_0 being fixed small enough, such that $\chi = 1$ and $\partial_x^\alpha \chi = 0$, $|\alpha| \geqq 1$, at $x = z$. We define

$$b_0(x, \xi; \lambda, z) = \chi(x; \lambda, z)\omega(\xi; \lambda, z).$$

By definition, b_0 is supported in $G_x(z, \varepsilon_0, \lambda) \times \Gamma_\xi(z, X_3, \lambda)$ and we may assume that b_0 belongs to $L_{0,0}(\lambda)$ uniformly in z.

We now construct a parametrix of

$$U(t; z, \lambda) = \exp(-itA_\kappa(\lambda)) b_0(x, D_x; \lambda, z)$$

by the usual W. K. B. method. We formally set

$$U(t; z, \lambda) = \sum_{j=0}^{\infty} I_\phi(p_j H_t),$$

where $H_t=H_t(\xi, y; \lambda)=\exp(-itA_0(y, \xi; \lambda))$. As is seen from the discussion below, all the amplitude functions p_j has compact support. Thus we may assume that H_t is compactly supported by multiplying a cut-off function, if necessary.

(2) First, we have to determine a phase function ϕ. Let $X_4=(4, 2h_0)$ be fixed. Then there exists $\phi(x, \xi, y; \lambda, z)$ such that: (i) ϕ satisfies Condition (Ψ); (ii) ϕ solves the time-independent Hamilton-Jacobi equation

$$A_0(x, \nabla_x\phi; \lambda)=A_0(y, \xi; \lambda), \tag{6.1}$$
$$\phi=0 \quad \text{when} \quad (x-y)\cdot\xi=0 \quad \text{and} \quad \nabla_x\phi=\xi \quad \text{at} \quad x=y$$

in $\mathcal{O}(z, 2\varepsilon_0, X_4, \lambda)$. This is proved by exactly the same way as in the proof of Theorem 3.1, [**23**].

Let b_0 be as in step (1) and let ϕ be as above. Then there exists $a_0(x, \xi, y; \lambda, z)$ of class $A_{0,0}(\lambda)$ such that: (i) a_0 has support in $\mathcal{O}(z, \varepsilon_0, X_4, \lambda)$; (ii) $a_0|_{x=y=z}=\omega(z, \xi; \lambda)$; (iii) for $f(x)$ with support in $G_x(z, \varepsilon_0/2, \lambda)$

$$b_0(x, D_x; \lambda, z)f(x)=I_\phi(a_0)f(x).$$

(For the proof, see Lemma 4.1, [**23**].) We take a_0 to be an initial condition of the transport equation for p_0.

(3) Next, we have to determine the amplitude functions $p_j(t, x, \xi, y; \lambda, z)$ by solving the transport equations. The phase function ϕ solves (6.1) in $\mathcal{O}(z, 2\varepsilon_0, X_4, \lambda)$. If $x\in G_x(z, 2\varepsilon_0, \lambda)$ and $\xi\in\Gamma_\xi(z, X_4, \lambda)$, then $|\nabla_\xi A_0(x, \xi; \lambda)|\sim \lambda^{1/2}V(z; \lambda)^{-1/2}$, which implies that a classical particle starting from z at $t=0$ moves at a speed proportional to $\lambda^{1/2}V(z; \lambda)^{-1/2}$. Hence it remains in $G_x(z, 2\varepsilon_0, \lambda)$ for $|t|<\rho\tau_0(z; \lambda)$, $\rho\ll 1$, where $\tau_0=\lambda^{-1/2}V(z; \lambda)^{1/2}\phi(z; \lambda)$. Thus, it is convenient to introduce the following class of functions with values in $A_{p,q}(\lambda)$.

DEFINITION. Let $I(z, \lambda)=\{t: |t|<\rho\tau_0(z; \lambda)\}$. We denote by $S(I(z; \lambda); A_{p,q}(\lambda))$ the set of all $a(t, x, \xi, y; \lambda, z)$ such that for all j, $\tau_0(z; \lambda)^j\partial_t^j a\in A_{p,q}(\lambda)$ uniformly in z and $t\in I(z, \lambda)$.

The transport equations for p_j are given inductively by use of the composition formula in Lemma 2.2, and the initial conditions for p_j are chosen as $p_0|_{t=0}=a_0$ and $p_j|_{t=0}=0$, $j\geqq 1$. Then, for N, $N\gg 1$, there exists an interval $I_0(z, \lambda)$ such that we can determine p_j, $0\leqq j\leqq N-1$, with the following properties: (i) $p_j\in S(I_0(z, \lambda); A_{j,j}(\lambda))$; (ii) p_j has support in $\mathcal{O}(z, 2\varepsilon_0, X_4, \lambda)$ for $t\in I_0(z, \lambda)$. (For the detailed discussion, see pp. 95-96, [**23**].) Thus we can construct parametrices of $U(t; z, \lambda)$ for $t\in I_0(z, \lambda)$.

(4) Let $E(t; z, \lambda)=b_0^*U(t; z, \lambda)$ and denote by $K_E(t, x, y; z, \lambda)$ its kernel. By making use of Lemma 2.2 again and of the Duhamel principle, we have

$$E(t; z, \lambda)f=I_\phi(qH_t)f+T_N(t; z, \lambda)f$$

for f as in step (2), where

$$T_N=\int_0^t b_0^* \exp(-i(t-s)A_\kappa(\lambda))I_\phi(r_N H_s)ds$$

and $r_N \in S(I_0(z, \lambda);\ A_{N,N}(\lambda))$, $N \gg 1$, has support in $\mathcal{O}(z, 2\varepsilon_0, X_4, \lambda)$, while $q \in S(I_0(z, \lambda);\ A_{0,0}(\lambda))$ takes the form

$$q=b_0(x, \nabla_x\phi;\ \lambda, z)p_0(t, x, \xi, y;\ \lambda, z)+q_1$$

with $q_1 \in S(I_0(z, \lambda);\ A_{1,1}(\lambda))$. In particular, at $t=0$ and $x=y=z$,

$$q=\omega(z, \xi;\ \lambda)^2+O(\Phi(z, \xi;\ \lambda)^{-1}\phi(z;\ \lambda)^{-1}).$$

We have to estimate the kernel $K_T^{(N)}(t, x, y;\ z, \lambda)$ of T_N. Set $K_T^{(N)}(t, z;\ \lambda)=K_T^{(N)}|_{x=y=z}$. Then

$$|\partial_t^j K_T^{(N)}(t, z;\ \lambda)|=O(\lambda^{-k}\langle z\rangle^{-4}), \quad k \gg 1, \quad 0 \leqq j \leqq 2.$$

This is proved by making use of Lemma 3.2. Indeed, take $\omega_0(x;\ \lambda)$ $(\in L_{0,0}(\lambda))$ with support in $G_x(z, 2\varepsilon_0, \lambda)$ in such a way that $\omega_0 r_N=r_N$, and decompose T_N as

$$T_N=\int_0^t F(t, s;\ z, \lambda)S_N(s;\ z, \lambda)ds,$$

where

$$F=b_0^* \exp(-i(t-s)A_\kappa(\lambda))(A_\kappa(\lambda)+\lambda)^{-k}\omega_0,$$

$$S_N=(A_\kappa(\lambda)+\lambda)^k I_\phi(r_N H_s).$$

Then the argument in the proof of Lemma 3.2 is applied to the decomposition above. Thus we can approximate $K_E(t, z;\ \lambda)=K_E|_{x=y=z}$ with the error estimate $O(\lambda^{-k}\langle z\rangle^{-4})$.

(5) We represent $K_E(t, z;\ \lambda)$ as the Fourier transform of the spectral function. Set $v_j(z;\ \lambda)=(b_0^* u_j)(x;\ \lambda)|_{x=z}$ for the normalized eigenfunction u_j and define

$$e(\mu;\ z, \lambda)=\sum_{\mu_j<\mu}|v_j(z;\ \lambda)|^2.$$

Then

$$K_E(t, z;\ \lambda)=\int e^{-it\mu}de(\mu;\ z, \lambda).$$

By exactly the same tauberian arguments as in [8], we obtain

$$e(\lambda;\ z, \lambda)=(2\pi)^{-3}\int_{|\xi|^2<\lambda V(z;\ \lambda)}\omega(z, \xi;\ \lambda)^2 d\xi(1+O(\lambda^{-1}\tau_0(z;\ \lambda)^{-1})). \tag{6.2}$$

For the derivation, we must carefully look at the contribution which the time interval $I_0(z, \lambda)$ makes to the remainder estimate.

(6) We now recall the definition of b_0. The symbol $b_0^\#$ of b_0^* is written in the form $b_0^\#=b_0+b_1$ with $b_1 \in L_{1,1}(\lambda)$. If we set $\omega_N(z, \xi;\ \lambda)=b_1|_{x=z}$, regarding z as variables, then $\omega_N \in L_{N,N}(\lambda)$ for $N \gg 1$, because $\partial_x^\alpha b_0|_{x=z}=0$ for $|\alpha| \geqq 1$. Thus

we can write

$$v_j(z;\lambda)=\{\omega(z, D_z;\lambda)+\omega_N(z, D_z;\lambda)\}u_j(z;\lambda).$$

Hence, we integrate (6.2) with respect to z and use Lemma 4.2 to obtain the desired formula for ω specified in step (1). A partition of unity, together with Lemma 4.4, gives the formula for ω as in the lemma. □

PROOF OF LEMMA 1.2. Once Lemmas 5.1 and 6.1 are established, the proof is easy. Indeed, by a partition of unity, the lemma follows immediately from Lemma 4.4. □

§7. Proof of Proposition 1.

PROOF OF PROPOSITION 1. In the proof of Lemma 1.1, we have proved that

$$\int_{\lambda^{1-\delta}\langle x\rangle^{2\theta}V(x;\lambda)<1} V(x;\lambda)^{3/2}dx=O(\lambda^{-1/2}).$$

Combining this with Lemmas 1.1 and 1.2 and taking the limit $\kappa\to 0$ give the asymptotic formula for $N(\lambda; V_\lambda)$, and also the second assertion is clear from the proof. □

§8. Concluding remarks.

We shall make several comments.

(1) The restriction that the space dimension $n=3$ is not essential. The result can be extended to the case $n\geqq 3$. (See Theorem XIII.12, [**17**].)

(2) The assumption that $V(x)$ is strictly positive is also not essential. Indeed, the proof of Lemma 1.1 does not use this assumption essentially, and in the proof of Lemma 1.2 we work exclusively in $\Sigma_0(\lambda)$ where $V(x)>0$. The proof for general $V(x)$ depends on the choice for a nice pair of weight functions. Let $\Omega=\{x: V(x)>0\}$ and let $d(x, \partial\Omega)$ be the distance from x to the boundary $\partial\Omega$. Such a pair will be determined using the distance function $d(x, \partial\Omega)$. Thus several assumptions should be made on the behavior of $V(x)$ in a neighborhood of $\partial\Omega$.

(3) The result is closely related to the works of Seeley [**21**], Merlose [**15**] and Ivrii [**9**] on the asymptotic distribution of eigenvalues of $-\Delta$ in bounded domains. Let Ω be a bounded domain in $\boldsymbol{R}_x^3$ with smooth boundary and let $V(x)=\chi_\Omega(x)$ the characteristic function of Ω. (Such a $V(x)$ is called a square-well potential.) Then, by the min-max principle

$$N(\lambda; V)=(6\pi^2)^{-1}\mathrm{vol}(\Omega)\lambda^{3/2}(1+O(\lambda^{-1/2})).$$

Unfortunately, our result cannot cover this important case.

(4) The author does not know whether or not it is possible to determine

the second term in formula (1). As is well known, the second term problem requires the information on $\exp(-itA_{\kappa}(\lambda))$ for t large and hence the information on the global behavior in t of the Hamilton phase trajectory plays an essential role. In the present problem, the phase trajectory is defined as solutions to the Hamilton equation

$$\dot{x}=\nabla_{\xi}A_0(x, \xi; \lambda), \qquad \dot{\xi}=-\nabla_x A_0(x, \xi; \lambda).$$

This trajectory goes to infinity ($|x(t)|\to\infty$) in a finite time, which will be easily seen, for example, in the one dimensional case. This makes the second term problem very difficult.

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