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On some representations of continuous additive functionals locally of zero energy

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Introduction.

Many fruitful studies have been produced before 1980 to generalize the classical Ito formula for Ito processes and for C^2-functions to more general processes than Ito processes (H. Kunita - S. Watanabe [**9**], P. A. Meyer [**12**] and so on) or to more general functions than C^2-functions (for example Tanaka's formula [**10**], [**12**]). These generalizations can be characterized as specific realizations of semimartingale decomposition due to J. L. Doob and P. A. Meyer; Semimartingale=martingale+process of bounded variation.

At the end of 1970's, noting that the square integrable martingale of zero quadratic variation is identically zero, M. Fukushima has introduced a new point of view where the Ito formula can be conceived as a decomposition into the sum;

(0.1) Martingale+process of zero quadratic variation, or into the sum;

(0.1′) Martingale+continuous additive functional (CAF) of zero energy.

In this conception, he has established a unique decomposition ([**2**], [**3**], [**5**]);

$$u(X_t)-u(X_0)=M_t^{[u]}+N_t^{[u]},\quad M_t^{[u]}\in\mathring{\mathcal{M}}_{\mathrm{loc}},\quad N_t^{[u]}\in\mathcal{N}_{\mathrm{loc}} \tag{0.2}$$

for the symmetric Markov process X_t and for any function $u\in\mathcal{F}_{\mathrm{loc}}$ where $u\in\mathcal{F}_{\mathrm{loc}}$ means that u belongs locally to the Dirichlet space associated with X_t. In (0.2) $\mathring{\mathcal{M}}_{\mathrm{loc}}$ denotes the family of martingale additive functionals locally of finite energy and $\mathcal{N}_{\mathrm{loc}}$ is the family of CAF's locally of zero energy.

In this direction, M. Yor [**21**] and the second author of the present paper [**19**] produced several concrete realizations of the decomposition of the type (0.1′) and gave some applications to the local time of one dimensional Brownian path. Some related topics have been discussed in [**7**] and in [**20**].

Once the decomposition (0.2) has been established for $u\in\mathcal{F}_{\mathrm{loc}}$, it is quite natural to ask if the decompositions for $u\in\mathcal{F}_{\mathrm{loc}}$ exhaust all possible decompositions of the form (0.2): In other words the question is to ask if for any given $N_t\in\mathcal{N}_{\mathrm{loc}}$ there exists $u\in\mathcal{F}_{\mathrm{loc}}$ such that $u(X_t)-u(X_0)-N_t\in\mathring{\mathcal{M}}_{\mathrm{loc}}$ holds.

In § 2 of this paper, we will attempt to answer the question in the affirmative in the case where the Dirichlet space associated with X_t is assumed to be regular as well as to have the local property. In § 3 we will treat some examples where representations of CAF's locally of zero energy can be given in more concrete or simpler forms than those given in the general case in § 2.

During the writing of this paper, the authors were enlightened by stimulating discussions with N. Ikeda. M. Fukushima gave us some valuable suggestions. We wish to express our gratitude to them.

§ 1. Preliminaries.

Let E be a locally compact separable Hausdorff space and m a positive Radon measure such that $\mathrm{supp}[m]=E$; i.e., m is a non-negative Borel measure on E which is finite on compact sets and strictly positive on each non-empty open set.

Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet space on $L^2(E; m)$. We assume the following conditions on $(\mathcal{E}, \mathcal{F})$

(C.1) $(\mathcal{E}, \mathcal{F})$ is regular,

(C.2) $(\mathcal{E}, \mathcal{F})$ posseses the local property and no killing measure.

On the canonical path space Ω we consider the associated Hunt process $X=(\Omega, \mathcal{M}, \mathcal{M}_t, X_t, \zeta, P_x)$ on E which is m-symmetric:

$$\int_E P_t f(x) g(x) m(dx) = \int_E f(x) P_t g(x) m(dx),$$

$f, g \in \mathcal{B}^+(E)$, $t>0$, where P_t is the transition function of X and $\mathcal{B}^+(E)$ is the family of all non-negative Borel functions on E.

By the condition (C.2), the process X is of continuous sample paths (diffusion) and without killing inside for q.e. starting point, i.e., there exists a properly exceptional set N such that

$$P_x(X_{\zeta-}=\Delta/\zeta<\infty)=1 \qquad \text{holds for } x\in E-N,$$

where $E\cup\{\Delta\}$ means a one point compactification of the space E (Cf. Chap. 4 of [**3**]).

Let G be a finely open set of E. Then it is known that there exists the part X^G of the process X on G whose Dirichlet space is the part $(\mathcal{E}^G, \mathcal{F}^G)$ on $L^2(G; m)$ of the space $(\mathcal{E}, \mathcal{F})$. (Cf. [**3**] and [**15**]).

Let A_t be a continuous additive functional (CAF) of the process X where we use the same definition of CAF as proposed in Fukushima's book ([**3**]).

We set

$$e(A)=\lim_{t\downarrow 0}\frac{1}{2t}E_m[A_t^2]=\lim_{t\downarrow 0}\frac{1}{2t}\int_E E_x[A_t^2]m(dx)$$

when the limit exists. $e(A)$ is called the energy of A.

We introduce two kinds of Markov times σ_B and τ_B of a set B by $\sigma_B(w)=\sigma_B=\inf\{t>0; X_t\in B\}$ and

$$\tau_B(w)=\sigma_{E-B} \qquad \text{where } \inf\{\emptyset\} \text{ means } \infty .$$

Here, we introduce several families of CAF's.

(I) Martingale additive functionals (MAF). We say that M is MAF if M is CAF such that for each $t>0$ $E_x[M_t^2]<\infty$ and $E_x(M_t)=0$ q.e. x. Put $\mathcal{M}=\{M; M \text{ is an MAF}\}$.

(II) Local martingale additive functionals (LMAF). We say that M is LMAF if there exist an increasing sequence of relatively compact finely open sets G_n and a sequence of MAF's $M^{(n)}\in\mathcal{M}$ such that $\lim_{n\to\infty}\tau_{G_n}=\zeta$ and $M_t=M_t^{(n)}$ for $0\leqq t<\tau_{G_n}$ a.s. (P_x) for q.e. x. We denote the family of LMAF's by $\mathcal{M}_{\text{loc}}$.

(III) MAF of finite energy. We say that M is MAF of finite energy if $M\in\mathcal{M}$ such that $e(M)<+\infty$. Put

$$\mathring{\mathcal{M}}=\{M; M\in\mathcal{M},\ e(M)<+\infty\}.$$

(IV) MAF locally of finite energy. An MAF M is called locally of finite energy if there exist an increasing sequence of relatively compact finely open sets G_n and a sequence of $M^{(n)}\in\mathring{\mathcal{M}}$ such that $\lim_{n\to\infty}\tau_{G_n}=\zeta$ and $M_t=M_t^{(n)}$ for $0\leqq t<\tau_{G_n}$ a.s. (P_x) for q.e. x. Put

$$\mathring{\mathcal{M}}_{\text{loc}}=\{M;\ M \textit{ is an MAF locally of finite energy}\}.$$

(V) CAF of zero energy. A CAF N is called CAF of zero energy if $e(N)=0$ holds. Put

$$\mathcal{N}=\{N;\ N \textit{ is a CAF of zero energy}\}.$$

(VI) CAF locally of zero energy. We say that N is CAF locally of zero energy if there exist an increasing sequence of relatively compact finely open sets G_n and a sequence of CAF's $N^{(n)}\in\mathcal{N}$ such that $\lim_{n\to\infty}\tau_{G_n}=\zeta$ and $N_t=N_t^{(n)}$ for $0\leqq t<\tau_{G_n}$ a.s. (P_x) for q.e. x. Put

$$\mathcal{N}_{\text{loc}}=\{N;\ N \textit{ is a CAF locally of zero energy}\}.$$

We adopt a slightly modified definition of the function space $\mathcal{F}_{\text{loc}}$ of Fukushima (Cf. [3]). "$u\in\mathcal{F}_{\text{loc}}$" means that there exist an increasing sequence (G_n) of relatively compact finely open sets such that $\lim_{n\to\infty}\tau_{G_n}=\zeta$ a.s. (P_x) for q.e. x and a sequence (u_n) of functions of $\mathcal{F}$ such that $u=u_n$ m-a.e. on G_n.

By the result of M. Fukushima ([2], [3]) we know that for any $u\in\mathcal{F}_{\text{loc}}$

there exist $M^{[u]}\in \mathcal{M}_{\mathrm{loc}}$ and $N^{[u]}\in \mathcal{N}_{\mathrm{loc}}$ such that the CAF $u(X_t)-u(X_0)$ can be expressed uniquely as

$$u(X_t)-u(X_0)=M_t^{[u]}+N_t^{[u]}.$$

§ 2. Representation of CAF's locally of zero energy — general case —.

The main result of this section is the following;

Theorem 2.1. *Let N_t belong to $\mathcal{N}_{\mathrm{loc}}$ and G be a relatively compact open set such that*

$$P_x(\tau_G<\zeta)>0 \qquad q.e.\ x\in G. \tag{2.1}$$

Then there exist an increasing sequence $\{G_n\}$ $(G_n\subset G)$ of relatively compact finely open sets and a sequence $\{u_n\}$ of functions of $\mathcal{F}_{\mathrm{loc}}$ such that

$$\lim_{n\to\infty}\tau_{G_n}=\tau_G \qquad a.s.\ (P_x)\quad q.e.\ x\in G \qquad and \tag{2.2}$$

$$N_t=N_t^{[u_n]} \qquad 0\leqq t<\tau_{G_n}\quad a.s.\ (P_x)\quad q.e.\ x\in G_n.$$

If $\{G'_n\}$ and $\{u'_n\}$ are other sequences satisfying (2.2), then $u_n-u'_n$ is harmonic on $G_n\cap G'_n$.

Preparatory lemmas; The following chain of lemmas shall lead up to the Theorem 2.1.

Lemma 2.1. *Let $N_t\in\mathcal{N}$ and G be a relatively compact finely open set such that*

$$q.e.\ \sup_{x\in G}E_x[\bar{N}_{\tau_G}]<+\infty^{(1)} \tag{2.3}$$

and $P_x(\tau_G<\zeta)>0$ q.e. $x\in G$, where $\bar{N}_{\tau_G}=\max_{0\le t\le\tau_G}|N_t|$. Put $u(x)=-E_x[N_{\tau_G}]$ and $M_t=u(X_t)-u(X_0)-N_t$. Then $M_{t\wedge\tau_G}\in\mathcal{M}^G$ holds, where $\mathcal{M}^G$ means the family of MAF's with respect to the part X^G of the process X on G.

Proof. The proof of this lemma will be broken in several steps.

(1°) In this step it will be shown that

$$P_x(\tau_G\circ\theta_{\tau_G}=0\,;\ \tau_G<\zeta)=P_x(\tau_G<\zeta) \qquad \text{q.e. } x\in G \text{ holds.} \tag{2.4}$$

By the strong Markov property we have

$$P_x(\tau_G\circ\theta_{\tau_G}=0\,;\ \tau_G<\zeta)=E_x[P_{X_{\tau_G}}(\tau_G=0)\,;\ \tau_G<\zeta] \tag{2.5}$$

holds. Let A^r be the set of regular points for A. Then we see that

$$X_{\tau_G}\in(E-G)\cup(E-G)^r \qquad \text{a.s. } (P_x)\quad \text{q.e. } x\in G \tag{2.6}$$

holds as well as that $(E-G)\setminus(E-G)^r$ is semi polar. (Cf. Th. 1. 11.4, Prop. 2.3.3

in [1]). In view of the fact that any semi polar set is exceptional (Cf. Th. 4.2.3. in [3]), we have from (2.6) that

$$P_x(X_{\tau_G}\in(E-G)\cup(E-G)^r\,;\ \tau_G<\zeta) \tag{2.7}$$

$$=P_x(X_{\tau_G}\in(E-G)^r\,;\ \tau_G<\zeta)=P_x(\tau_G<\zeta) \qquad \text{q.e. } x \text{ on } G$$

holds.

Pay attention to the fact that for any $y\in(E-G)^r$

$$P_y(\tau_G=0)=1 \tag{2.8}$$

holds. Then combining (2.7) with (2.5), we obtain the desired equality

$$P_x(\tau_G\circ\theta_{\tau_G}=0\,;\ \tau_G<\zeta)=P_x(\tau_G<\zeta) \qquad \text{q.e. } x\in G\,. \tag{2.4}$$

The relation (2.4) yields immediately

$$\tau_G\circ\theta_{\tau_G\wedge t}=\tau_G-\tau_G\wedge t \qquad \text{a.s. } (P_x) \ \text{ q.e. } x\in G\,. \tag{2.9}$$

(2°) Here, we will show that $M_{t\wedge\tau_G}$ is an MAF of the part process X^G. In the following of this step we put $\tau=\tau_G$ for abbreviation.

First, we will see several simple relations; For $t+s<\tau$

$$N_{(t+s)\wedge\tau}=N_t+N_s\circ\theta_t=N_{t\wedge\tau}+N_{s\wedge\tau}\circ\theta_{t\wedge\tau} \tag{2.10}$$

and

$$u(X_{(t+s)\wedge\tau})=u(X_{s\wedge\tau}\circ\theta_{t\wedge\tau}) \tag{2.11}$$

holds. Let $t<\tau<t+s$. Then by (2.9) we see that $\tau-t=\tau\circ\theta_t\leqq s$. Thus we get for $t<\tau\leqq t+s$,

$$N_{(t+s)\wedge\tau}=N_\tau=N_{\tau\wedge t}+\tau\circ\theta_{\tau\wedge t}=N_{\tau\wedge t}+N_{\tau\wedge s}\circ\theta_{\tau\wedge t}\,, \tag{2.12}$$

and

$$u(X_{(t+s)\wedge\tau})=u(X_\tau)=u(X_{s\wedge\tau}\circ\theta_{\tau\wedge t})\,. \tag{2.13}$$

If $\tau\leqq t$, then we have

$$N_{(t+s)\wedge\tau}=N_\tau=N_{\tau\wedge t}+N_{\tau\wedge s}\circ\theta_{\tau\wedge t} \tag{2.14}$$

and

$$u(X_{(t+s)\wedge\tau})=u(X_\tau)=u(X_{s\wedge\tau}\circ\theta_{\tau\wedge t})\,. \tag{2.15}$$

Thus the above relations show that $M_{t\wedge\tau}$ is a CAF with respect to the part process X^G.

Next we shall show

$$E_x[M^2_{t\wedge\tau}]<+\infty \qquad \text{for any } t>0 \ \text{ q.e. } x\in G\,. \tag{2.16}$$

By the definition of M_t we have

$$E_x[M_{t\wedge\tau}^2] \leqq 3E_x[u^2(X_{t\wedge\tau})] + 3E_x[u^2(X_0)] + 3E[N_{t\wedge\tau}^2] \tag{2.17}$$
$$= 3I_1 + 3I_2 + 3I_3 \quad \text{say.}$$

By the assumption (2.3) supposed on N_t, we get for I_3

$$I_3 \leqq E_x[\bar{N}_\tau^2] < +\infty\,. \tag{2.18}$$

For I_1 we see that

$$I_1 = E_x[(E_{X_{t\wedge\tau}}[N_\tau])^2] \leqq E_x[E_{X_{t\wedge\tau}}[N_\tau^2]] = E_x[(N_\tau - N_{t\wedge\tau})^2] \tag{2.19}$$
$$\leqq 2\{E_x[N_\tau^2] + E_x[N_{t\wedge\tau}^2]\} < +\infty$$

holds.

For I_2 it is easy to see

$$I_2 = E_x[u^2(X_0)] < +\infty\,. \tag{2.20}$$

Thus the relations (2.17), (2.18), (2.19) and (2.20) yield the relation (2.16).

(3°) Finally we will show that $E_x[M_{t\wedge\tau}] = 0$ q.e. $x \in G$. We have

$$E_x[M_{t\wedge\tau}] = -E_x[N_{t\wedge\tau}] - E_x[E_{X_{t\wedge\tau}}[N_\tau]] + E_x[E_{X_0}[N_\tau]]$$
$$= -E_x[N_{t\wedge\tau}] - E_x[N_\tau - N_{t\wedge\tau}] + E_x[N_\tau] = 0 \qquad \text{q.e. } x \in G\,.$$

Hence we can conclude that $M_{t\wedge\tau} \in \mathcal{M}^G$. Q.E.D.

The next lemma can be found in Fukushima ([3]).

Lemma 2.2. *Let G be a relatively compact finely open set. Then*

$$\mathcal{M}_{\text{loc}}^G = \mathcal{\mathring{M}}_{\text{loc}}^G \tag{2.21}$$

holds where

$\mathcal{M}_{\text{loc}}^G = \{M\,;\ M$ *is an LMAF with respect to the process* $X^G\}$

and

$\mathcal{\mathring{M}}_{\text{loc}}^G = \{M\,;\ M$ *is an MAF locally of finite energy with respect to the process* $X^G\}$ *respectively.*

In the following, the resolvent of order $\alpha > 0$ of the part process will be denoted by R_α^G and that of order 0 by R^G. P_t^G means the transition function of X^G.

Let M_t be the LMAF which has been introduced in Lemma 2.1. Choose a strictly positive Borel function f on G.

We set $F_n = \{x \in G_n\,;\ R^{G_n} f > 1/n\}$, where $\{G_n\}$ is the sequence of finely open sets associated with M_t. Then F_n is a finely open set whose $\mathcal{E}^G$-capacity is finite.

LEMMA 2.3. *Let G be a relatively compact and finely open set satisfying (2.1). Then*

$$\lim_{t\downarrow 0}\int_G R^{F_n}\phi(x)(1-P_t^{G_n}1)(x)m(dx)=0 \tag{2.22}$$

holds for any non-negative bounded Borel function ϕ on G, where we put $R^{F_n}\phi(x)=0$ when $x\in F_n^c$.

PROOF. Let $B_t=t\wedge\tau_{G_n}$ and A_t be the 1-sweeping out of B_t on the set F_n^c (Cf. [3]). Then we can see that A_t is a positive CAF of which smooth measure has its support in the set $(F_n^c)^r$.

Put

$$V_{B,A}^{p,q}f(x)=E_x\left[\int_0^{\tau_{G_n}}e^{-pB_t-qA_t}f(X_t)dA_t\right]$$

and

$$V_{A,B}^{q,p}f(x)=E_x\left[\int_0^{\tau_{G_n}}e^{-qA_t-pB_t}f(X_t)dB_t\right].$$

Then we get

$$V_{A,B}^{0,q}f(x)=E_x\left[\int_0^{\tau_{G_n}}e^{-qt}f(X_t)dt\right]=R_q^{G_n}f(x), \qquad x\in G_n.$$

The following two equalities can be checked easily;

$$R^{F_n}\phi(x)=\lim_{p\to\infty}V_{A,B}^{p,0}\phi(x) \tag{2.23}$$

$$V_{A,B}^{p,0}-V_{A,B}^{0,q}+pV_{B,A}^{0,p}V_{A,B}^{0,q}-qV_{A,B}^{p,0}V_{A,B}^{0,q}=0. \tag{2.24}$$

From these equalities, it follows that

$$q\int_{G_n}R^{F_n}\phi(x)(1-qV_{A,B}^{0,q}1)(x)m(dx) \tag{2.25}$$

$$=\lim_{p\to\infty}q\int_{G_n}V_{A,B}^{p,0}\phi(x)(1-qV_{A,B}^{0,q}1)(x)m(dx)$$

(By the fact that the kernel $V_{A,B}^{p,0}$ is symmetric)

$$=\lim_{p\to\infty}q\int_{G_n}\phi(x)V_{A,B}^{p,0}(1-qV_{A,B}^{0,q}1)(x)m(dx)$$

$$=q\int_{G_n}\phi(x)(V_{A,B}^{0,q}1-H_{F_n^c}^{G_n}V_{A,B}^{0,q}1)(x)m(dx)$$

where

$$H_{F_n^c}^{G_n}f(x)=E_x[f(X_{\sigma_{F_n^c}}^{G_n})].$$

Noticing that $H^{G_n}_{F^c_n}1(x)=P_x(\sigma_{F^c_n}<\tau_{G_n})=1$, $x\in G_n$ we get from (2.25) that

$$\lim_{q\to\infty} q\int_{G_n} R^{F_n}\phi(x)(1-qV^{0,q}_{A,B}1)(x)m(dx) \tag{2.26}$$

$$=\lim_{q\to\infty} q\int_{G_n}\phi(x)(V^{0,q}_{A,B}1-H^{G_n}_{F^c_n}V^{0,q}_{A,B}1)(x)m(dx)$$

$$=\int_{G_n}\phi(x)(1-H^{G_n}_{F^c_n}1)(x)m(dx)=0$$

holds.

On the other hand, we can see

$$\lim_{q\to\infty} q\int_{G_n} R^{F_n}\phi(x)(1-qV^{0,q}_{A,B}1)(x)m(dx) \tag{2.27}$$

$$=\lim_{q\to\infty} q\int_{G_n} R^{F_n}\phi(x)(1-qR^{G_n}_q1)(x)m(dx)$$

$$=\lim_{t\downarrow 0}\frac{1}{t}\int_{G_n} R^{F_n}\phi(x)(1-P^{G_n}_t1)(x)m(dx).$$

Combine (2.27) with (2.26). Then we get (2.22). Q.E.D.

Let us put $K_n=\{x\in F_n,\ R^{F_n}f>1/n\}$ and $\mathcal{L}_{K_n}=\{v\in\mathcal{F}^{F_n},\ v\geqq 1$ on K_n a.e. $(m)\}$, where $(\mathcal{E}^{F_n}, \mathcal{F}^{F_n})$ stands for the Dirichlet space which corresponds to the part $X^{F_n}_t$.

It is well known that $\mathcal{F}^{F_n}\subset\mathcal{F}^G$ holds i.e.; if $v\in\mathcal{F}^{F_n}$ and v is extended on G such that $v(x)=0$ for $x\in G-F_n$ then $v\in\mathcal{F}^G$.

The following lemmas (from lemma 2.4 to lemma 2.8) are essentially due to Fukushima. In these lemmas G will be assumed to be relatively compact finely open set satisfying (2.1). Under this condition X^G_t is transient. Hence we shall state the 0-th order version of Fukushima's result.

LEMMA 2.4. *There exists a function* $e_n\in\mathcal{L}_{K_n}$ *such that*

$$\mathcal{E}^{F_n}(e_n, e_n)=\inf\{\mathcal{E}^{F_n}(v, v),\ v\in\mathcal{L}_{K_n}\}=\alpha.$$

PROOF. Let $v_k\in\mathcal{L}_{K_n}$ be a sequence such that

$$\lim_{k\to\infty}\mathcal{E}^{F_n}(v_k, v_k)=\alpha.$$

By the following equality

$$\mathcal{E}^{F_n}\left(\frac{v_k-v_m}{2}, \frac{v_k-v_m}{2}\right)=\frac{1}{2}\mathcal{E}^{F_n}(v_k, v_k)+\frac{1}{2}\mathcal{E}^{F_n}(v_m, v_m)$$

$$-\mathcal{E}^{F_n}\left(\frac{v_k+v_m}{2}, \frac{v_k+v_m}{2}\right),$$

we can observe that

$$\lim_{\substack{k\to\infty\\ m\to\infty}} \mathcal{E}^{F_n}\left(\frac{v_k-v_m}{2}, \frac{v_k-v_m}{2}\right) \leqq \alpha-\alpha=0 .$$

Let e_n be the limit element in $\mathcal{F}^{F_n}$ of the $\mathcal{E}^{F_n}$-Cauchy sequence $\{v_k\}$. By the definition of e_n we see that $\mathcal{E}^{F_n}(e_n, e_n)=\alpha$.

On the other hand it is easy to check $e_n \geqq 1$ on K_n a.e. (m). Thus $e_n \in \mathcal{L}_{K_n}$ holds. Q.E.D.

LEMMA 2.5. *e_n is the unique element of $\mathcal{F}^{F_n}$ which satisfies the following;*

$$e_n=1 \quad \text{on } K_n \qquad a.e.\ (m) \tag{2.28}$$

and

$$\mathcal{E}^{F_n}(e_n, v) \geqq 0 \qquad \text{for any } v \in \mathcal{F}^{F_n} \text{ such that } v \geqq 0 \text{ on } K_n \text{ a.e. } (m). \tag{2.29}$$

The proof can be shown in essentially the same manner as employed in that of Lemma 3.1.1 of [3]. So we omit it.

LEMMA 2.6. *e_n is an excessive function with respect to the part process $X_t^{F_n}$.*

PROOF (Cf. Th. 3.2.1 of [3]). Let us put $\mathcal{K}=\{w \in \mathcal{F}^{F_n}, w \geqq e_n \text{ a.e. } (m)\}$. Then by Lemma 2.5, we observe that $\mathcal{K} \subset \mathcal{L}_{K_n}$ and $e_n \in \mathcal{K}$. Hence

$$\mathcal{E}^{F_n}(e_n, e_n)=\inf_{w \in \mathcal{K}} \mathcal{E}^{F_n}(w, w) .$$

Since $|e_n| \in \mathcal{K}$ and $\mathcal{E}^{F_n}(|e_n|, |e_n|) \leqq \mathcal{E}^{F_n}(e_n, e_n)$ we see that $e_n=|e_n| \geqq 0$ a.e. (m).

On the other hand,

$$\begin{aligned}(e_n-P_t^{F_n}e_n, v)_{F_n}{}^{(2)} &=(e_n, v-P_t^{F_n}v)_{F_n}\\ &=\mathcal{E}^{F_n}(e_n, R^{F_n}v-R^{F_n}P_t^{F_n}v) \geqq 0\end{aligned}$$

holds for any $v \in \mathcal{F}^{F_n}$ such that $v \geqq 0$ a.e. (m), because

$$R_1^{F_n}v-R^{F_n}P_t^{F_n}v=\int_0^t P_s^{F_n}v\,ds$$

is non-negative. Hence e_n is excessive. Q.E.D.

For the proof of the next lemma, see Lemma 3.3.2 in [3].

LEMMA 2.7. *Let u_1, u_2 be excessive functions. Suppose that $u_1 \leqq u_2$ a.e. (m) and $u_2 \in \mathcal{F}^{F_n}$. Then $u_1 \in \mathcal{F}^{F_n}$ and $\mathcal{E}^{F_n}(u_1, u_1) \leqq \mathcal{E}^{F_n}(u_2, u_2)$ hold.*

LEMMA 2.8. *Let us put $H_n^{F_n}f(x)=E_x[f(X_{\sigma_{K_n}}); \sigma_{K_n}<\tau_{F_n}]$. Then*

$$H_n^{F_n}1(x)=e_n(x) \tag{2.30}$$

holds on F_n a.e. (m).

PROOF (Cf. Lemma 4.3.1 in [3]). First we will show that

$$H_n^{F_n}1(x) \leqq e_n(x) \quad \text{a.e. } (m) \tag{2.31}$$

holds.

Take a Borel modification $\tilde{e}_n$ of e_n such that $\tilde{e}_n=1$ on K_n. Put $Y_t(\omega)=\tilde{e}_n(X_t^{F_n})$. Then $(Y_t, \mathcal{F}_t^0, P_{h\cdot m}^{F_n})$(3) is a supermartingale, where $\mathcal{F}_t^0=\sigma\{X_s^{F_n}; s\leqq t\}$ and h is a non-negative function such that $\int_{F_n} h(x)m(dx)=1$.

Let S be a finite set of $(0, \infty)$ with $\min S=a$ and $\max S=b$.

Put $\sigma(S; n)=\inf\{t\in S, X_t^{F_n}\in K_n\}$. If the set in the braces is empty we put $\sigma(S, n)=b$.

By the optional sampling theorem, we can see that

$$P_{h\cdot m}^{F_n}(\sigma(S, n)<b) \leqq E_{h\cdot m}^{F_n}[Y_{\sigma(S;n)}] \leqq E_{h\cdot m}^{F_n}[Y_a] \leqq (h, e_n)_{F_n}.$$

Letting S increase to a countable dense set in $(0, b)$ and then b tend to infinity in the above inequalities, we get

$$P_{h\cdot m}^{F_n}(\sigma_{K_n}<\tau_{F_n}) \leqq (h, e_n)_{F_n}.$$

Hence we can see that (2.31) holds.

On the other hand we know that $H_n^{F_n}1(x)$ is excessive with respect to $P_t^{F_n}$. Hence in view of Lemmas 2.6 and 2.7, we can conclude from (2.31) that (2.30) holds. Q.E.D.

LEMMA 2.9. *Let u be the function which has been introduced in Lemma 2.1. Then u belongs to $\mathcal{F}_{\text{loc}}^G$.*

PROOF. Set $\|u\|_\infty=\sup_{x\in G}|u(x)|$. By Theorem 3.3.3 in [3], we observe that

$$u(x)e_n(x) \leqq \|u\|_\infty e_n(x) \leqq \|u\|_\infty n R^{F_n} f(x) \tag{2.32}$$

on F_n because $e_n(x)\leqq nR^{F_n}f$ on K_n and $nR^{F_n}f$ is excessive.

Note that

$$\begin{aligned}
\mathcal{E}_{G_n}^{(t)}(ue_n, ue_n) &= \frac{1}{t}\int_{G_n}(ue_n)(x)(1-P_t^{G_n})(ue_n)(x)m(dx) \\
&= \frac{1}{2t}\int_{G_n\times G_n}\int \{(ue_n)(y)-(ue_n)(x)\}^2 P_t^{G_n}(x, dy)m(dx) \\
&\quad + \frac{1}{t}\int_{G_n}(ue_n)^2(x)(1-P_t^{G_n}1)(x)m(dx).
\end{aligned}$$

Then, using Lemma 2.3, we have

$$\lim_{t\downarrow 0}\frac{1}{t}\mathcal{E}_{G_n}^{(t)}(ue_n,\ ue_n) \tag{2.33}$$

$$=\lim_{t\downarrow 0}\frac{1}{2t}\iint\{(ue_n)(y)-(ue_n)(x)\}^2 P_t^{G_n}(x,\ dy)m(dx)$$

$$\leqq\lim_{t\downarrow 0}\frac{1}{t}\|e_n\|_\infty^2\iint(u(x)-u(y))^2 P_t^{G_n}(x,\ dy)m(dx)$$

$$+\lim_{t\downarrow 0}\frac{1}{t}\|u\|_\infty^2\iint(e_n(x)-e_n(y))^2 P_t^{G_n}(x,\ dy)m(dx)$$

$$=\lim_{t\downarrow 0}\{I_1+I_2\},\qquad \text{say.}$$

From the definition of u, we have for I_1

$$I_1=\frac{1}{t}\|e_n\|_\infty^2\int_{G_n}E_x[(u(X_t)-u(X_0))^2;\ t<\tau_{G_n}]m(dx) \tag{2.34}$$

$$\leqq\frac{2}{t}\|e_n\|_\infty^2\int E_x[N_t^2;\ t<\tau_{G_n}]m(dx)$$

$$+\frac{2}{t}\|e_n\|_\infty^2\int E_x[M_t^2;\ t<\tau_{G_n}]m(dx)\,.$$

Recalling the fact that $N_t\in\mathcal{N}$, $M_t=M_t^{(n)}$ $(0\leqq t<\tau_{G_n})$ and $M^{(n)}\in\mathcal{M}^G$ we can see from (2.34) that

$$\lim_{t\downarrow 0}I_1<+\infty \tag{2.35}$$

holds.

Since $e_n\in\mathcal{F}^{F_n}\subset\mathcal{F}^{G_n}$, we have for I_2 that

$$\lim_{t\downarrow 0}I_2<+\infty \tag{2.36}$$

holds.

By (2.33), (2.35) and (2.36), we can conclude that $\lim_{t\downarrow 0}\mathcal{E}_{G_n}^{(t)}(ue_n,\ ue_n)<+\infty$ holds. Hence $ue_n\in\mathcal{F}^{G_n}\subset\mathcal{F}^G$.

By Lemma 2.5 $ue_n(x)=u(x)$ on K_n holds.

On the other hand, by the definition of F_n and K_n, we have

$$E_x\left[\int_{\tau_{K_n}}^{\tau_{G_n}}f(X_t)dt\right]=E_x\left[\int_{\tau_{K_n}}^{\tau_{F_n}}f(X_t)dt+\int_{\tau_{F_n}}^{\tau_{G_n}}f(X_t)dt\right]$$

$$=E_x[R^{F_n}(X_{\tau_{K_n}})]+E_x[R^{G_n}(X_{\tau_{F_n}})]\leqq\frac{2}{n}\,.$$

This yields immediately $\lim_{n\to\infty}\tau_{K_n}=\lim_{n\to\infty}\tau_{G_n}=\tau_G$ a.s. (P_x) q.e. $x\in G$. Thus we have shown that $u\in\mathcal{F}^G_{\mathrm{loc}}$. Q.E.D.

Following Walsh [18], we shall say that two elements ω and ω' of Ω are t-equivalent if $\omega(s)=\omega'(s)$ for all $s\leqq t$. Let $\gamma_t\omega$ be the class of elements ω' satisfying

$$\omega'(s)=\begin{cases}\omega(t-s) & \text{if } s<t\leqq\zeta(\omega)\\ \Delta & \text{for all } s \text{ if } \zeta(\omega)<t\,.\end{cases}$$

For $N\in\mathfrak{N}$ define $\hat{N}$ by

$$\hat{N}_t(\omega)=\begin{cases}N_t\circ\gamma_t\omega & \text{if } t\leqq\zeta(\omega) \text{ or } \zeta(\omega)=0\\ \hat{N}_{\zeta(\omega)} & \text{if } t>\zeta(\omega)>0\,.\end{cases}$$

Then, by [18] it is known that $\hat{N}$ is a CAF of X and

$$E_m[Y\,;\zeta\geqq t]=E_m[Y\circ\gamma_t\,;\zeta\geqq t] \tag{2.37}$$

for all $\mathcal{F}^0_t$-measurable function Y.

LEMMA 2.10. *Let N be an element of $\mathfrak{N}$ satisfying $\sup_{x\in G}E_x[N^2_{\tau_G}]<+\infty$ and $\sup_{x\in G}[\hat{N}^2_{\tau_G}]<\infty$. Then $N_{t\wedge\tau_G}$ belongs to $\mathfrak{N}^G_{\mathrm{loc}}$.*

PROOF. By analogous arguments employed in Lemma 2.3, Lemma 2.4 and Lemma 2.5, there exists a function $e^G_{F_n}$ which satisfies the following;

$$e^G_{F_n}(x)=1 \qquad \text{on } F_n \quad \text{a.e. } (m) \tag{2.38}$$

$$e^G_{F_n}\in\mathcal{F}^G \tag{2.39}$$

and

$$\mathcal{E}^G(e^G_{F_n},v)\geqq 0 \qquad \text{for } v\in\mathcal{F}^G \text{ such that } v\geqq 0 \text{ on } F_n \quad \text{a.e. } (m)\,. \tag{2.40}$$

We put $e(x)=e^G_{F_n}(x)$ for abbreviation.

By an obvious modification of the proof of Theorem 5.1.1 of [3], we can show that there exists a positive CAF $A^{[e]}_t$ of X^G such that $e(x)=E_x[A^{[e]}_{\tau_G}]$ q.e. $x\in G$. The CAF$-A^{[e]}$ is equal to $N^{[e]}$ which appears in Fukushima's decomposition of $e(X^G_t)-e(X^G_0)$, that is;

$$e(X^G_t)-e(X^G_0)=M^{[e]}_t-A^{[e]}_t\,, \qquad 0\leqq t\leqq\tau_G$$

where $M^{[e]}\in\mathring{\mathcal{M}}_G$ (see Lemma 5.3.1 in [3]).

Set

$$(N_e)_t=N_{t\wedge\tau_G}e(X^G_t)-\int_0^t N_s dM^{[e]}_s+\int_0^{t\wedge\tau_G}N_s dA^{[e]}_s\,. \tag{2.41}$$

Then $(N_e)_t$ is a CAF with respect to X^G_t.

We will divide the following part of the proof in several steps.

(1°) Here we will show that

$$\lim_{t\downarrow 0}\frac{1}{t}\int_G E_x[(N_{t\wedge\tau_G}e(X_t^G))^2]m(dx)=0 \tag{2.42}$$

holds.

Since $e(X_{\tau_G})=0$ a.e. (P_x) q.e. x on G, we have $N_{t\wedge\tau_G}e(X_t^G)=N_t e(X_{t\wedge\tau_G})$. Hence

$$\lim_{t\downarrow 0}\frac{1}{t}\int_G E_x[(N_{t\wedge\tau_G}e(X_{t\wedge\tau_G}))^2]m(dx)\leqq\lim_{t\downarrow 0}\frac{1}{t}\int_E E_x[(N_t)^2]m(dx)=0\,.$$

Thus (2.42) has been proved.

(2°) In this step we will show that

$$\lim_{t\downarrow 0}\frac{1}{t}E_m\left[\left(\int_0^{t\wedge\tau_G}N_s dM_s^{[e]}\right)^2\right]=0\,. \tag{2.43}$$

Put $C_t=\langle M^{[e]}, M^{[e]}\rangle_t$ for abbreviation. Then we have

$$E_m\left[\left(\int_0^{t\wedge\tau_G}N_s dM_s^{[e]}\right)^2\right]=E_m\left[\int_0^{t\wedge\tau_G}N_s^2 dC_s\right]. \tag{2.44}$$

Now we set $t/n=\delta$. Then by (2.37) we have

$$\begin{aligned}
&E_m\left[\int_0^{t\wedge\tau_G}N_s^2 dC_s\right] \\
&=\lim_{n\to\infty}\sum_{k=0}^{n-1}E_m[N_{k\delta}^2(C_{(k+1)\delta}-C_{k\delta})\,;\,(k+1)\delta<\tau_G] \\
&=\lim_{n\to\infty}\sum_{k=0}^{n-1}E_m[N_{k\delta}^2 E_{X_{k\delta}}[C_\delta\,;\,\delta<\tau_G]\,;\,k\delta<\tau_G] \\
&=\lim_{n\to\infty}\sum_{k=0}^{n-1}E_m[N_{k\delta}^2\circ\gamma_{k\delta}E_{X_{k\delta}\circ\gamma_{k\delta}}[C_\delta\,;\,\delta<\tau_G]\,;\,k\delta<\tau_G\circ\gamma_{k\delta}] \\
&=\lim_{n\to\infty}\sum_{k=0}^{n-1}E_m[N_{k\delta}^2\circ\gamma_{k\delta}E_{X(0)}[C_\delta\,;\,\delta<\tau_G]\,;\,k\delta<\tau_G] \\
&=\lim_{\delta\to 0}\frac{1}{\delta}\int_G E_x[C_\delta\,;\,\delta<\tau_G]E_x\left[\int_0^{t\wedge\tau_G}\hat{N}_s^2 ds\right]m(ds) \\
&\quad+\lim_{\delta\to 0}\int_G\left\{E_x\left[\sum_{k=0}^{n-1}\hat{N}_{k\delta\wedge\tau_G}^2\delta-\int_0^t\hat{N}_s^2 ds\,;\,t<\tau_G\right]\right\}\frac{1}{\delta}E_x[C_\delta\,;\,\delta<\tau_G]m(dx) \\
&=\lim_{\delta\to 0}\{J_1+J_2\}\qquad\text{say.}
\end{aligned} \tag{2.45}$$

Let μ be the smooth measure which corresponds to $\langle M^{[e]}, M^{[e]}\rangle_t=C_t$. Then, by Fukushima's result (Cf. Chap. 5 in [3]), we have

$$\frac{1}{2}\mu(G)=\varepsilon^G(e,\ e)<+\infty\,. \tag{2.46}$$

Now we will evaluate J_1. Using Lemma 5.1.4 of [3], we get

$$\begin{aligned}
&\frac{1}{\delta}\int_G E_x[C_\delta\,;\,\delta<\tau_G]E_x\Big[\int_0^{t\wedge\tau_G}\hat{N}_s^2ds\Big]m(dx) \\
&=\frac{1}{\delta}\int_0^\delta\int_G P_u^G\Big(E.\Big[\int_0^{t\wedge\tau_G}\hat{N}_s^2ds\Big]\Big)(x)\mu(dx)du \\
&=\int_G\frac{1}{\delta}\int_0^\delta E_x\Big[\int_0^{t\wedge\tau_G\circ\theta_u}\hat{N}_s^2(\theta_u\omega)ds\,;\,u<\tau_G\Big]du\,\mu(dx) \\
&=\int_G\frac{1}{\delta}\int_0^\delta E_x\Big[\int_0^{t\wedge(\tau_G-u)}\hat{N}_s^2(\theta_u\omega)ds\,;\,u<\tau_G\Big]du\,\mu(dx)\,.
\end{aligned} \tag{2.47}$$

Note that for $u<\tau_G$ and $s<\tau_G-u$, $\lim_{u\to0}\hat{N}_s(\theta_u\omega)=\hat{N}_s$, $|\hat{N}_s(\theta_u\omega)|=|\hat{N}_{s+u}(\omega)-\hat{N}_u(\omega)|\leqq2\bar{\hat{N}}_{\tau_G}(\omega)$ and that $E_x[\bar{N}_{\tau_G}^2(\omega)]$ is bounded on G. Then we have by Lebesgue's convergence Theorem

$$\begin{aligned}
\lim_{\delta\to0}J_1&=\lim_{\delta\to0}\frac{1}{\delta}\int_G E_x[C_\delta\,;\,\delta<\tau_G]E_x\Big[\int_0^{t\wedge\tau_G}\hat{N}_s^2ds\Big]m(dx) \\
&=\int_G E_x\Big[\int_0^{t\wedge\tau_G}\hat{N}_s^2ds\Big]\mu(dx)\,.
\end{aligned} \tag{2.48}$$

Set

$$\phi_\delta(x)=E_x\Big[\sum_{k=0}^{n-1}\hat{N}_{k\delta\wedge\tau_G}^2\cdot\delta-\int_0^{t\wedge\tau_G}\hat{N}_s^2ds\,;\,t<\tau_G\Big]\,.$$

Then we have for J_2

$$\begin{aligned}
\lim_{\delta\to0}J_2&=\lim_{\delta\to0}\int_G\phi_\delta(x)\frac{1}{\delta}E_x[C_\delta\,;\,\delta<\tau_G]m(dx) \\
&=\lim_{\delta\to0}\int_G\frac{1}{\delta}\int_0^\delta P_u^G\phi_\delta(x)\mu(dx)du\,.
\end{aligned}$$

Recalling that $P_u^G\phi_\delta(x)\to0$ q.e. x on G as $\delta\to0$, we observe that

$$\lim_{\delta\to0}J_2=0 \tag{2.49}$$

holds.

In view of (2.44), (2.45), (2.48) and (2.49), we get

$$\lim_{t\downarrow0}\frac{1}{t}E_m\Big[\Big(\int_0^{t\wedge\tau_G}N_s dM_s^{[e]}\Big)^2\Big]=\lim_{t\downarrow0}\frac{1}{t}E_m\Big[\int_0^{t\wedge\tau_G}N_s^2dC_s\Big] \tag{2.50}$$

$$=\lim_{t\downarrow 0}\frac{1}{t}E_\mu\left[\int_0^{t\wedge\tau_G}\hat{N}_s^2 ds\right].$$

Recalling that $\sup_{x\in G}E_x[\bar{\hat{N}}^2_{\tau_G}]<+\infty$ and (2.46) $\mu(G)<+\infty$, we get from (2.50)

(2.51) $$\lim_{t\downarrow 0}\frac{1}{t}E_m\left[\left(\int_0^{t\wedge\tau_G}N_s dM_s^{[e]}\right)^2\right]=0.$$

(3°) In this step, we will show

(2.52) $$\lim_{t\downarrow 0}\frac{1}{t}E_m\left[\left(\int_0^{t\wedge\tau_G}N_s dA_s^{[e]}\right)^2\right]=0.$$

First, we observe that

$$E_m\left[\left(\int_0^{t\wedge\tau_G}N_s dA_s^{[e]}\right)^2\right]\leqq E_m\left[\left(\int_0^{t\wedge\tau_G}\bar{N}_s dA_s^{[e]}\right)^2\right]$$

$$=2E_m\left[\int_0^{t\wedge\tau_G}\bar{N}_s dA_s^{[e]}\int_s^{t\wedge\tau_G}\bar{N}_u dA_u^{[e]}\right]$$

$$=2E_m\left[\int_0^{t\wedge\tau_G}\bar{N}_s dA_s^{[e]}\int_0^{(t-s)\wedge\tau_G}\bar{N}_{s+u} d_u A_{s+u}^{[e]}\right]$$

$$\leqq 2E_m\left[\int_0^{t\wedge\tau_G}\bar{N}_s dA_s^{[e]}\int_0^{(t-s)\wedge\tau_G}(\bar{N}_s+\bar{N}_u(\theta_s\omega))dA_u^{[e]}(\theta_s\omega)\right]$$

$$\leqq 2E_m\left[\int_0^{t\wedge\tau_G}\bar{N}_s^2 E_{X_s}[A^{[e]}_{(t-s)\wedge\tau_G}]dA_s^{[e]}\right]$$

$$+2E_m\left[\int_0^{t\wedge\tau_G}\bar{N}_s E_{X_s}\left[\int_0^{(t-s)\wedge\tau_G}\bar{N}_u dA_u^{[e]}\right]dA_s^{[e]}\right]=t(L_1+L_2);\quad \text{say}.$$

Let ν be the smooth measure associated with the increasing process $A^{[e]}$. Then

$$\nu(G)=\lim_{t\to 0}\frac{1}{t}E_m[A_t^{[e]};t<\tau_G]=\lim_{t\to 0}\frac{1}{t}E_m[e(X_0^G)-e(X_t^G)]$$

$$=\lim_{t\to 0}\frac{1}{t}(1,(I-P_t^G)e)_G=\lim_{t\to 0}\frac{1}{t}(1-P_t^G1,e)_G$$

$$\leqq\lim_{t\to 0}\frac{1}{t}(1-P_t^G1,nR^Gf)_G=\lim_{t\to 0}\frac{n}{t}(1,(I-P_t^G)R^Gf)_G$$

$$=\lim_{t\to 0}\frac{n}{t}\left(1,\int_0^t P_s^G f ds\right)_G=n(1,f)_G<\infty.$$

The relations $\lim_{t\to 0}L_1=0$ and $\lim_{t\to 0}L_2=0$ can be shown in exactly the same manner employed in the last step. Hence we can get (2.52).

(4°) By (2.41), (2.42), (2.43) and (2.52), we can conclude that $(N_e)_t\in\mathcal{N}^G$ holds. On the other hand, we know that

$$(N_e)_t = N_t \quad 0 \leqq t \leqq \tau_{F_n} \quad \text{and} \quad \tau_{F_n} \uparrow \tau_G \quad \text{a.s. } (P_x) \quad \text{q.e. } x \text{ on } G.$$

Hence we see that $N_{t\wedge\tau_G} \in \mathfrak{N}^G_{\text{loc}}$. Q.E.D.

We are now in a position to state the following.

PROPOSITION 2.1. *Let N belong to $\mathfrak{N}$ and G be a relatively compact finely open set such that*

$$q.e. \sup_{x\in G} E_x[\bar{N}^2_{\tau_G}] < +\infty, \quad q.e. \sup_{x\in G} E_x[\bar{\hat{N}}^2_{\tau_G}] < \infty \quad and$$

$$P_x(\tau_G < \infty) > 0, \quad q.e. \ x.$$

Put $u(x) = -E_x[N_{\tau_G}]$. Then u belongs to $\mathcal{F}^G_{\text{loc}}$ and $N_t = N^{[u]}_t$, for $0 \leqq t < \tau_G$ holds.

PROOF. By Lemma 2.9, u belongs to $\mathcal{F}^G_{\text{loc}}$. By the definition of M_t in Lemma 2.1, we have $u(X^G_t) - u(X^G_0) = M_{t\wedge\tau_G} + N_{t\wedge\tau_G}$ where $M_{t\wedge\tau_G} \in \mathcal{M}^G_{\text{loc}}$ (by Lemma 2.2) and $N_{t\wedge\tau_G} \in \mathfrak{N}^G_{\text{loc}}$ (by Lemma 2.10).

On the other hand, it is known by Fukushima's result that for $u \in \mathcal{F}^G_{\text{loc}}$, $u(X^G_t) - u(X^G_0) = M^{[u]}_{t\wedge\tau_G} + N^{[u]}_{t\wedge\tau_G}$ holds where $M^{[u]} \in \mathcal{M}^G_{\text{loc}}$ and $N^{[u]} \in \mathfrak{N}^G_{\text{loc}}$, moreover the decomposition is unique.

Hence we can conclude that $N_t = N^{[u]}_t$ for $0 \leqq t < \tau_G$. Q.E.D.

To prove Theorem 2.1 we require still an auxiliary lemma which is essentially due to H. P. McKean and H. Tanaka. (Cf. [**6**]).

LEMMA 2.11 (H. P. McKean and H. Tanaka). *Let $N \in \mathfrak{N}_{\text{loc}}$ and G be a relatively compact finely open set satisfying* (2.1).

Then there exists an increasing sequence $\{U_n\}$ of finely open sets such that

(i) $U_n \subset G$, $\lim_{n\to\infty} \tau_{U_n} = \tau_G$ *a.s.* (P_x) *q.e.* x *on* G,

(ii) $\sup_{x\in U_n} E_x[\bar{N}^2_{\tau_{U_n}}] < +\infty$

and

(iii) $\sup_{x\in U_n} E_x[\bar{\hat{N}}^2_{\tau_{U_n}}] < +\infty$.

PROOF. (Cf. [**6**]).

Put

$$\eta(\omega) = \max_{0\leq t\leq\tau_G} |N_{\tau_G} - N_t|.$$

Then, we have

$$\eta \leqq 2\bar{N}_{\tau_G} \tag{2.53}$$

and

$$\bar{N}_{\tau_G} \leqq 2\eta\,. \tag{2.54}$$

Set $e_\lambda(x) = E_x[e^{-\lambda\eta}]$ and $\bar{e}_\lambda(x) = E_x[e^{-\lambda\bar{N}_{\tau_G}}]$ for $\lambda > 0$. Then it follows from (2.53) and (2.54) that

(2.55) $$e_{2\lambda}(x) \leqq \bar{e}_{\lambda}(x) \leqq e_{\lambda/2}(x).$$

Observe that

$$E_x[(1-e_\lambda)(X_t)\,;\, t<\tau_G] = E_x[1-e^{-\lambda \max_{t \leq t+s \leq \tau_G} |N_{\tau_G} - N_{s+t}|}\,;\, t<\tau_G] \uparrow 1-e_\lambda(x)$$

on G as $t \downarrow 0$.

Hence $1-e_\lambda(x)$ is excessive with respect to the part process X^G.

Put $V_n=\{x\,;\, x\in G,\, e_\lambda(x)>1/n\}$. Then V_n is finely open such that $V_n \subset G$ and $V_n \subset V_{n+1}$. If $x \in V_n$, then by (2.55) we have

$$\frac{1}{n} \leqq e_\lambda(x) \leqq \bar{e}_{\lambda/2}(x) \leqq P_x(\bar{N}_{\tau_G} \leqq T) + e^{-\lambda T/2} = 1 - P_x(\bar{N}_{\tau_G} > T) + e^{-\lambda T/2}.$$

From the above inequalities, we get

$$P_x(\bar{N}_{\tau_{V_n}} > T) \leqq P_x(\bar{N}_{\tau_G} > T) \leqq 1 - \frac{1}{n} + e^{-\lambda T/2}.$$

Hence letting T be sufficiently large, we can choose a number C such that

(2.56) $$P_x(\bar{N}_{\tau_{V_n}} > T) \leqq C < 1 \qquad \text{for } x \in V_n.$$

Now, put $\eta_k = \inf\{t>0,\, \bar{N}_t = kT\}$. Observe that $\eta_k \geqq \eta_{k-1} + \eta_1 \circ \theta_{\eta_{k-1}}$. Then we get from (2.56) that

(2.57)
$$\begin{aligned} P_x(\bar{N}_{\tau_{V_n}} > kT) &= P_x(\eta_k < \tau_{V_n}) \\ &\leqq E_x[P_{X_{\eta_{k-1}}}(\eta_1 < \tau_{V_n})\,;\, \eta_{k-1} < \tau_{V_n}] \\ &\leqq C P_x(\eta_{k-1} < \tau_{V_n}) \leqq C^k. \end{aligned}$$

By (2.57), one can choose two positive constants $C_1>0$, $C_2>0$ such that $P_x(\bar{N}_{\tau_{V_n}} > t) \leqq C_1 e^{-C_2 t}$ $x \in V_n$ holds.

This inequality yields immediately $\sup_{x\in V_n} E_x[\bar{N}^2_{\tau_{V_n}}] < +\infty$.

On the other hand, by (2.1), we know that $e_\lambda(x)>0$ q.e. x on the fine closure of G. Hence we have

$$\inf_{0 \leq t \leq \tau_G} e_\lambda(X_t) > 0 \qquad \text{a.s. } (P_x) \qquad \text{q.e. } x \text{ on } G.$$

Then $\tau_{V_n}(\omega) = \tau_G(\omega)$ holds for sufficiently large number n which may depend on ω. Thus we get $P_x(\lim_{n\to\infty} \tau_{V_n} = \tau_G) = 1$ q.e. x on G.

Similarly there exists an increasing sequence $\{W_n\}$ of finely open subsets of G such that $\sup_{x\in W_n} E_x[\bar{\hat{N}}^2_{\tau_{W_n}}] < +\infty$ and $\tau_{W_n}(\omega) = \tau_G(\omega)$ for all sufficiently large n. Taking $U_n = V_n \cap W_n$ we obtain the desired result. Q.E.D.

After the long chain of lemmas, we are at last in a position to prove Theorem 2.1.

PROOF OF THEOREM 2.1. By the definition of the CAF locally of zero energy, there exist an increasing sequence $\{\tilde{G}_n\}$ of relatively compact finely open sets and a sequence $N^{(n)}$ of CAF's of zero energy such that

$$\lim_{n\to\infty}\tau_{\tilde{G}_n}=\zeta,\quad \bigcup_n \tilde{G}_n=E \quad\text{and}\quad N_t=N_t^{(n)} \qquad \text{for}\quad 0\leqq t\leqq\tau_{\tilde{G}_n}.$$

Put $G_n=\tilde{G}_n\cap G\cap U_n$, where U_n is the set introduced in Lemma 2.11. Then we have

(i) $N_{t\wedge\tau_{G_n}}=N_t^{(n)} \qquad 0\leqq t\leqq\tau_{G_n}$,

(ii) $\sup_{x\in G_n} E_x[\bar{N}^2_{\tau_{G_n}}]<+\infty$,

(iii) $\sup_{x\in G_n} E_x[\bar{\hat{N}}^2_{\tau_{G_n}}]<+\infty$

and

(iv) $\lim_{n\to\infty}\tau_{G_n}=\tau_G \qquad$ a.s. (P_x) q.e. x on G.

In view of Proposition 2.1, there exists a sequence $\{u_n\}$ of functions such that $u_n\in\mathcal{F}^{G_n}_{\text{loc}}$, $N_t^{[u_n]}=N_t \quad 0\leqq t<\tau_{G_n}$ a.s. P_x q.e. x on G.

Now we will give the proof of the last part of the theorem. Suppose that there exist finely open sets $G_n\subset G$, $G'_n\subset G$ and functions $u_n\in\mathcal{F}^G_{\text{loc}}$, $u'_n\in\mathcal{F}^{G'_n}_{\text{loc}}$ such that

$$N_t^{[u_n]}=N_t \qquad 0\leqq t<\tau_{G_n},$$
$$N_t^{[u'_n]}=N_t \qquad 0\leqq t<\tau_{G'_n}, \qquad \text{a.s. } (P_x).$$

Then

$$(u_n-u'_n)(X_t)-(u_n-u'_n)(X_0)=M_t^{[u_n-u'_n]} \qquad 0\leqq t<\tau_{G_n\cap G'_n} \quad \text{a.s. } (P_x).$$

Since $(M^{[u_n-u'_n]}(t\wedge\tau_{G_n\cap G'_n}))$ belongs to $\mathcal{M}^{G_n\cap G'_n}_{\text{loc}}$, for q.e. $x\in G_n\cap G'_n$ there exists a finely open neighbourhood $U\subset G_n\cap G'_n$ of x such that $(M^{[u_n-u'_n]}_{t\wedge\tau_U})$ is a square integrable P_x-martingale. In particular, for all finely open neighbourhood V of x such that $V\subset U$ we have $E_x[M^{[u_n-u'_n]}_{\tau_V}]=0$, that is $(u_n-u'_n)(x)=E_x[(u_n-u'_n)(X_{\tau_V})]$. This shows that $u_n-u'_n$ is a harmonic function on $G_n\cap G'_n$.

Q.E.D.

§ 3. Examples.

In this section we will discuss two cases where the representation of CAF locally of zero energy can be given in more precise expression than that is formulated in Theorem 2.1.

EXAMPLE 1. Uniformly elliptic case.

Let $E=\boldsymbol{R}^d$ and m be a Radon measure on $\boldsymbol{R}^d$. Let $\mathcal{E}$ be a symmetric form on $L^2(\boldsymbol{R}^d; m)$ satisfying the following;

(i) $\mathcal{E}$ is defined on $C_0^\infty \times C_0^\infty$, where

$$C_0^\infty = \{u\,;\, u \in C^\infty,\ \mathrm{supp}[u] \text{ is compact}\},$$

(ii) $\mathcal{E}$ has the local property,

(iii) $\mathcal{E}$ is closable.

Then, in view of Beurling-Deny theorem (§ 2.2 in [**3**]), $\mathcal{E}$ has the following expression for $u, v \in C_0^\infty$;

$$\mathcal{E}(u, v) = \frac{1}{2}\sum_{i,j}\int_{R^d} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\nu_{ij}(dx) \tag{3.1}$$

with some Radon measures ν_{ij}.

Further, we assume the following

Assumption. There exist two positive constants; $0 < K_1 < K_2 < \infty$ such that

$$K_1 \sum_{i=1}^d \xi_i^2 d\nu_{ii} \leqq \sum_{i,j=1}^d \xi_i \xi_j d\nu_{ij} \leqq K_2 \sum_{i=1}^d \xi_i^2 d\nu_{ii} \tag{3.2}$$

holds for any $(\xi_1, \cdots, \xi_d) \in \boldsymbol{R}^d$.

We denote the smallest closed extension of $(\mathcal{E}, C_0^\infty)$ in $L^2(\boldsymbol{R}^d\,;\, m)$ by $(\mathcal{E}, \mathcal{F})$. Since, by (3.2),

$$K_1 \sum_{i=1}^d \int \Big(\frac{\partial u_n}{\partial x_i} - \frac{\partial u_k}{\partial x_j}\Big)^2 d\nu_{ii} \leqq 2\mathcal{E}(u_n - u_k,\ u_n - u_k)$$

$$\leqq K_2 \sum_{i=1}^d \int \Big(\frac{\partial u_n}{\partial x_i} - \frac{\partial u_k}{\partial x_j}\Big)^2 d\nu_{ii},$$

$\{u_n\}$ is a Cauchy sequence relative to $\mathcal{E}$ if and only if $\{\partial u_n/\partial x_i\}$ are Cauchy sequences in $L^2(\boldsymbol{R}^d\,;\, \nu_{ii})$ for all i. Suppose that $u \in \mathcal{F}$. Then there exists an $\mathcal{E}$-Cauchy sequence $\{u_n\}$ of C_0^∞ functions such that $u_n \to u$ in $L^2(\boldsymbol{R}^d\,;\, m)$. Hence by the above remark, $\lim_{n\to\infty}(\partial u_n/\partial x_i)$ exists in $L^2(\boldsymbol{R}^d\,;\, \nu_{ii})$. We shall denote it by $\partial u/\partial x_i$.

Theorem 3.1. *Let N_t be a CAF locally of zero energy and G be a bounded finely open set such that $P.(\tau_G < \infty) > 0$ q.e. on G. Then, there exist an increasing sequence of finely open sets $\{G_n\}$ and a sequence of functions $\{u_n\}$ such that*

(i) *$G_n \subset G$ and $\tau_{G_n} \uparrow \tau_G$ a.s. (P_x) q.e. $x \in G$,*

(ii) *$u_n \in \mathcal{F}_{\mathrm{loc}}$,*

(iii)

$$u_n(X_t) - u_n(X_0) = \sum_{i=1}^d \int_0^t \frac{\partial u_n}{\partial x_i}(X_s)\, dM_s^{[x_i]} + N_t, \quad 0 \leqq t < \tau_{G_n}; \tag{3.3}$$

where $M_s^{[x_i]}$, $i = 1, \cdots, d$ are MAF's which appear in the decomposition

$$X_t^i - X_0^i = M_t^{[x_i]} + N_t^{[x_i]}, \qquad 1 \leqq i \leqq d, \tag{3.4}$$

$M_t^{[x_i]} \in \mathcal{M}_{\mathrm{loc}}$, $N_t^{[x_i]} \in \mathcal{N}_{\mathrm{loc}}$.

For the proof we require two preparatory lemmas which are essentially due to Fukushima (Theorem 2 [4]).

LEMMA 3.1.

$$\mathring{\mathcal{M}} = \{\sum_{i=1}^d f_i \cdot M^{[x_i]}\ ;\ f_i \in L^2(\boldsymbol{R}^d\ ; \nu_{ii}),\ 1 \leqq i \leqq d\} \tag{3.5}$$

and

$$e(\sum_{i=1}^d f_i \cdot M^{[x_i]}) = \frac{1}{2} \sum_{i,j} \int_{\boldsymbol{R}^d} f_i(x) f_j(x) d\nu_{ij} \tag{3.6}$$

hold, where $f \cdot M$ *stands for* $\int_0^{\cdot} f(X_s) dM_s$.

PROOF. Since (3.6) is well known, it suffices to show (3.5). First, we shall show that the family $\{\sum_{i=1}^d f_i \cdot M^{[x_i]}; f_i \in L^2(\boldsymbol{R}^d\ ; \nu_{ii}),\ 1 \leqq i \leqq d\}$ is a closed subset of $(\mathring{\mathcal{M}}, e)$. Let $M_n = \sum_{i=1}^d f_i^{(n)} \cdot M^{[x_i]}$, $f_i \in L^2(\boldsymbol{R}^d\ ; \nu_{ii})$, $1 \leqq i \leqq d$, $n=1, 2, \cdots$ and $M \in \mathring{\mathcal{M}}$, such that $e(M_n - M)$ tends to zero as $n \to \infty$. Then, we have

$$\begin{aligned} e(M_n - M_k) &= \frac{1}{2} \sum_{i,j} \int_{\boldsymbol{R}^d} (f_i^{(n)}(x) - f_i^{(k)}(x))(f_j^{(n)}(x) - f_j^{(k)}(x)) d\nu_{ij} \\ &\geqq \frac{1}{2} K_1 \sum_i \int_{\boldsymbol{R}^d} (f_i^{(n)}(x) - f_i^{(k)}(x))^2 d\nu_{ii}, \end{aligned}$$

where we have utilized the relation (3.2). Hence, there exist functions $f_i \in L^2(\boldsymbol{R}^d\ ; \nu_{ii})$ $(1 \leqq i \leqq d)$ such that $f_i^{(n)}$ converges to f_i in $L^2(\boldsymbol{R}^d\ ; \nu_{ii})$, $1 \leqq i \leqq d$. Put $M = \sum_{i=1}^d f_i \cdot M^{[x_i]}$. Then, we observe that $e(M) = \frac{1}{2} \sum_{i,j} \int_{\boldsymbol{R}^d} f_i(x) f_j(x) d\nu_{ij}$ and

$$\begin{aligned} e(M_n - M) &= \frac{1}{2} \sum_{i,j} \int_{\boldsymbol{R}^d} (f_i^{(n)}(x) - f_i(x))(f_j^{(n)}(x) - f_j(x)) d\nu_{ij} \\ &\leqq \frac{1}{2} K_2 \sum_i \int_{\boldsymbol{R}^d} (f_i^{(n)}(x) - f_i(x))^2 d\nu_{ii} \end{aligned}$$

hold, where we utilized the relation (3.2). This yields immediately $\lim_{n \to \infty} e(M_n - M) = 0$. Hence, the family discussed is closed in $(\mathring{\mathcal{M}}, e)$.

On the other hand, in view of Lemma 5.4.5 in [3] it is known that the family $\{f \cdot M^{[u]};\ f \in C_0^1,\ u \in C_0^1\}$ is a dense subset in $(\mathring{\mathcal{M}}, e)$. Further we know by Theorem 5.4.4 in [3] that $f \cdot M^{[u]} = \sum_{i=1}^d f(\partial u / \partial x_i) \cdot M^{[x_i]}$ and $f \cdot (\partial u / \partial x_i) \in L^2(\boldsymbol{R}^d\ ; \nu_{ii})$ $(1 \leqq i \leqq d)$ hold. Thus, we can conclude that (3.5) holds. Q.E.D.

LEMMA 3.2. *Let* $M \in \mathcal{M}_{\mathrm{loc}}$. *Then there exist an increasing sequence of bounded* *finely open sets* $\tilde{G}_n$ *and functions* $\{f_i\}$ $1 \leqq i \leqq d$ *such that*

(i) $\tau_{\tilde{G}_n} \uparrow \zeta$ *a.s.* (P_x) *for q.e.* x,

(ii) $f_i \in L^2(\tilde{G}_n; \nu_{ii})$ *for any* $1 \leqq i \leqq d$ *and* n,

(iii) $M_t = \sum_{i=1}^{d} \int_0^t f_i(X_s) dM_s^{[x_i]}$.

PROOF. By Lemma 3.2, $\mathcal{M}_{\mathrm{loc}} = \mathring{\mathcal{M}}_{\mathrm{loc}}$. Hence, there exist an increasing sequence of bounded finely open sets $\{\tilde{G}_n\}$ and a sequence of MAF's, $M^{(n)} \in \mathring{\mathcal{M}}$ such that $\tau_{\tilde{G}_n} \uparrow \zeta$ and $M_t = M_t^{(n)}$ for $0 \leqq t < \tau_{\tilde{G}_n}$ a.s. (P_x) q.e. x. Utilizing Lemma 3.1, we can choose $f_i^{(n)} \in L^2(\boldsymbol{R}^d; \nu_{ii})$ such that $M_t^{(n)} = \sum_{i=1}^{d} \int_0^t f_i^{(n)}(X_s) dM_s^{[x_i]}$. Put $g_i = f_i^{(n)} - f_i^{(n+1)}$. Then it suffices to show that $g_i = 0$ a.e. (ν_{ii}) on $\tilde{G}_n$. We see that

$$\sum_{i=1}^{d} \int_0^t g_i(X_s) dM_s^{[x_i]} = 0 \qquad 0 \leqq t < \tau_{\tilde{G}_n} \text{ a.s. } (P_x) \text{ q.e. } x\,.$$

Noticing that the smooth measure associated with the above CAF is $\sum_{i,j} g_i(x) g_j(x) d\nu_{ij}$, we have, (by Lemma 5.1.5 in [3])

$$\begin{aligned}
&\int_{\tilde{G}_n} f(x) \sum_{i,j} g_i(x) g_j(x) d\nu_{ij} \\
&\leqq \int_{\{x;\ \text{irregular point for } \tilde{G}_n^c\}} f(x) \sum_{i,j} g_i(x) g_j(x) d\nu_{ij} \\
&= \lim_{\alpha \to \infty} \alpha E_m \left[\int_0^{\tau_{G_n}} e^{-\alpha t} f(X_t) \sum_{i,j} g_i(X_t) g_j(X_t) d\langle M^{[x_i]}, M^{[x_j]} \rangle_t \right] \\
&= 0\,.
\end{aligned}$$

Hence, by (3.2) we have

$$K_1 \int_{\tilde{G}_n} \sum_{i=1}^{d} f(x) g_i(x)^2 d\nu_{ii} = 0\,.$$

Thus, we get $g_i = 0$ a.e. (ν_{ii}) on $\tilde{G}_n$. Q.E.D.

PROOF OF THE THEOREM. By Theorem 2.1, we know that there exist $\{G_n\}$ and $\{u_n\}$ satisfying

(i) $G_n \subset G$ and $\tau_{G_n} \uparrow \tau_G$ a.s. (P_x) q.e. $x \in G$,

(ii) $u_n \in \mathcal{F}_{\mathrm{loc}}$,

(iii)′

$$u_n(X_t) - u_n(X_0) = M_t^{[u_n]} + N_t \qquad 0 \leqq t \leqq \tau_{G_n'} \tag{3.7}$$

where $M_t^{[u_n]} \in \mathcal{M}_{\mathrm{loc}}$. By Lemma 3.2,

$$M_t^{[u_n]} = \sum_{i=1}^{d} \int_0^t f_i(X_s) dM_s^{[x_i]}\,,$$

$f_i \in L^2(\tilde{G}_k ; \nu_{ii})$ for any i and k. On the other hand, it is known that $\partial u_n/\partial x_i \in L^2_{\text{loc}}(\boldsymbol{R}^d ; \nu_{ii})$ $1 \leqq i \leqq d$. Put

$$u_n(X_t)-u_n(X_0)=\sum_{i=1}^{d}\int_0^t \frac{\partial u_n}{\partial x_i}(X_s)dM_s^{[x_i]}+\tilde{N}_t .$$

Then $\sum_{i=1}^{d}\int_0^t \frac{\partial u_n}{\partial x_i}(X_s)dM_s^{[x_i]} \in \mathcal{M}_{\text{loc}}$ and $\tilde{N}_t \in \mathcal{N}_{\text{loc}}$. By the uniqueness of the decomposition of the CAF $u_n(X_t)-u_n(X_0)$, we observe that

$$M_t^{[u_n]}=\sum_{i=1}^{d}\int_0^t f_i(X_s)dM_s^{[x_i]}=\sum_{i=1}^{d}\int_0^t \frac{\partial u_n}{\partial x_i}(X_s)dM_s^{[x_i]}$$

$0 \leqq t \leqq \tau_{G_n}$ and $\tilde{N}_t=N_t$ $0 \leqq t \leqq \tau_{G_n}$. Hence, it suffices to show $\partial u_n/\partial x_i=f_i$ a.e. (ν_{ii}) on every G_n. Noting that $\partial u_n/\partial x_i-f_i \in L^2(G_n \cap \tilde{G}_k ; \nu_{ii})$, we have

$$
\begin{aligned}
0&=\lim_{t\to 0}\frac{1}{t}E_m\Big[\Big\{\sum_{i=1}^{d}\int_0^{t\wedge\tau_{G_n\cap\tilde{G}_k}}\Big(\frac{\partial u_n}{\partial x_i}(X_s)-f_i(X_s)\Big)dM_s^{[x_i]}\Big\}^2\Big]\\
&=\lim_{t\to 0}\frac{1}{t}E_m\Big[\sum_{i,j}\int_0^{t\wedge\tau_{G_n\cap\tilde{G}_k}}\Big(\frac{\partial u_n}{\partial x_i}-f_i\Big)\Big(\frac{\partial u_n}{\partial x_j}-f_j\Big)(X_s)d\langle M^{[x_i]}, M^{[x_j]}\rangle_s\Big]\\
&=\sum_{i,j}\int_{G_n\cap\tilde{G}_k}\Big(\frac{\partial u_n}{\partial x_i}-f_i\Big)\Big(\frac{\partial u_n}{\partial x_j}-f_j\Big)d\nu_{ij} .
\end{aligned}
$$

Utilizing (3.2), we observe that $f_i=\partial u_n/\partial x_i$ a.e. (ν_{ii}) on $G_n \cap \tilde{G}_k$. Letting k tend to infinity, we get $f_i=\partial u_n/\partial x_i$ a.e. (ν_{ii}) on G_n. Q.E.D.

EXAMPLE 2. One dimensional Brownian motion.

Let X_t be a one dimensional Brownian motion. Then we have the following theorem:

THEOREM 3.2. *Let $N_t \in \mathcal{N}_{\text{loc}}$. Then there exists a function $u \in \mathcal{F}_{\text{loc}}=\{v$; absolutely continuous, $dv/dx \in L^2_{\text{loc}}(\boldsymbol{R}^1 ; dx)\}$ such that*

$$u(X_t)-u(X_0)=\int_0^t \frac{du}{dx}(X_s)dX_s+N_t . \tag{3.8}$$

PROOF. We know that the fine topology with respect to the one dimensional Brownian motion coincides with the Euclidean topology on $\boldsymbol{R}^1$. Hence, by Theorem 3.1, we have for N_t an increasing sequence of compact intervals $I_n=(a_n, b_n)\uparrow(-\infty, \infty)$, and a sequence of functions $u_n \in \mathcal{F}_{\text{loc}}$ such that

$$N_t=u_n(X_t)-u_n(X_0)-\int_0^t \frac{du_n}{dx}(X_s)dX_s \quad 0 \leqq t \leqq \tau_{I_n} \quad \text{a.s. } (P_x),\ x \in \boldsymbol{R}^1 . \tag{3.9}$$

On the other hand, H. Tanaka has shown that there exist a continuous function u and a Borel function $g \in L^2_{\text{loc}}(\boldsymbol{R}^1 ; dx)$ [16], such that

(3.10) $$N_t=u(X_t)-u(X_0)-\int_0^t g(X_s)dX_s\,,\qquad 0\leqq t<\infty \quad \text{a.s. } (P_x)\,.$$

Put $A_t^{[u_n-u]}=(u_n-u)(X_t)-(u_n-u)(X_0)$. Then we have, by (3.9) and (3.10) that

$$A_t^{[u_n-u]}=\int_0^t\Big(\frac{du_n}{dx}-g\Big)(X_s)dX_s\,,\qquad 0\leqq t<\tau_{I_n}\,.$$

Noting that $du_n/dx-g\in L^2(I_n\,;\,dx)$, we have

$$e_{I_n}(A_{t\wedge\tau_{I_n}}^{[u_n-u]})=e_{I_n}\Big(\int_0^{t\wedge\tau_{I_n}}\Big(\frac{du_n}{dx}-g\Big)(X_s)dX_s\Big)$$

$$=\frac{1}{2}\int_{I_n}\Big(\frac{du_n}{dx}-g\Big)^2(x)dx<\infty\,,$$

where $e_{I_n}(A)$ stands for the energy of A_t with respect to the part of X_t on I_n. Hence, we can conclude that the function u_n-u belongs to $\mathcal{F}^{I_n}=\{v\,;\, v$ is absolutely continuous on I_n, $dv/dx\in L^2(I_n\,;\,dx)\}$. Since $u_n\in\mathcal{F}_{\text{loc}}$, we observe that u is absolutely continuous on I_n and $du/dx\in L_2(I_n\,;\,dx)$. Since I_n increases to $(-\infty,\,\infty)$, one can see that $u\in\mathcal{F}_{\text{loc}}$. Put

(3.11) $$u(X_t)-u(X_0)=\int_0^t\frac{du}{dx}(X_s)dX_s+\tilde{N}_t\,.$$

Then $\int_0^t\frac{du}{dx}(X_s)dX_s\in\mathcal{M}_{\text{loc}}$ and $\tilde{N}_t\in\mathcal{N}_{\text{loc}}$ hold. Combine (3.11) with (3.10) and utilize the uniqueness of the decomposition of $A_t^{[u]}=u(X_t)-u(X_0)$. Then we get

$$N_t=u(X_t)-u(X_0)-\int_0^t\frac{du}{dx}(X_s)dX_s\,,\qquad 0\leqq t<\infty\quad \text{a.s. } (P_x)\,.$$

Q.E.D.

REMARK. So far, we are concerned with a representation theorem of CAF's locally of zero energy. In general, the CAF $N^{[u]}$ corresponding to $u\in\mathcal{F}$ belongs to $\mathcal{N}$ (Theorem 5.2.2 in [**3**]). But it is not necessarily true that the function u corresponding to a given $N\in\mathcal{N}$ belongs to $\mathcal{F}$. We shall give such an example.

Let X_t be a one dimensional Brownian motion and N_t be its local time at 0. We shall first show that $N_t\in\mathcal{N}$. Let $\sigma=\sigma_{\{0\}}$. Then

$$E_x[N_t^2]=E_x[N_{t-\sigma(\omega)}^2(\omega_\sigma^+)\,;\, t\geqq\sigma]$$

$$=E_x[E_{X_\sigma}[N_{t-s}^2]_{s=\sigma}\,;\, t\geqq\sigma]=\int_0^t E_0[N_{t-s}^2]P_x(\sigma\in ds)$$

$$=\frac{1}{4}\int_0^t E_0[(N_{t-s}^+)^2]P_x(\sigma\in ds)\,,$$

where $N_t^+=2N_t$ is the local time of the reflecting Brownian motion. Since

$$P_0[N_{t-s}^+\in du]=\frac{2}{\sqrt{2\pi(t-s)}}\exp\left\{-\frac{u^2}{2(t-s)}\right\}du,$$

by an Ito-McKean's result (p. 45, Problem 3 in [8]), we have

$$E_0[(N_{t-s}^+)^2]=t-s.$$

Also, we know (Cf. p. 25 of [8])

$$P_x[\sigma\in ds]=\frac{x}{\sqrt{2s^3}}\exp\left(-\frac{x^2}{2s}\right)ds.$$

Hence

$$E_x[N_t^2]=\frac{1}{4}\int_0^t(t-s)\frac{x}{\sqrt{2s^3}}\exp\left(-\frac{x^2}{2s}\right)ds$$

holds. Integrating by the speed measure $m(dx)=2dx$, we have

$$E_m[N_t^2]=\frac{2}{3}\sqrt{\frac{2}{\pi}}t\sqrt{t},$$

which implies that

$$e(N)=\lim_{t\to 0}\frac{1}{2t}E_m[N_t^2]=0,$$

that is, $N\in\mathcal{N}$. On the other hand, by Tanaka's formula ([10], [12])

$$N_t=X_t^+-X_0^+-\int_0^t I_{(0,\infty)}(X_s)dX_s.$$

This implies that N_t is the CAF of zero energy associated with $x^+\in\mathcal{F}_{\mathrm{loc}}$.

Notes.

(1) q.e. $\sup_{x\in G}$ stands for the supremum on $G-\{\text{a negligible set}\}$.

(2) $(u, v)_{F_n}$ stands for $\int_{F_n}u(x)v(x)m(dx)$.

(3) $P_{h\cdot m}^{F_n}$ stands for $\int_{F_n}P_x^{F_n}h(x)m(dx)$.

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